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# Location Problem for Compressor Stations in Pipeline Networks

Martin Gugat <sup>\*</sup>      Michael Schuster <sup>\*†</sup>      Jan Sokolowski <sup>‡</sup>

**Abstract.** In the operation of pipeline networks, compressors play a crucial role in ensuring the network's functionality for various scenarios. In this contribution we address the important question of finding the optimal location of the compressors. This problem is of a novel structure, since it is related with the gas dynamics that governs the network flow. That results in non-convex mixed integer stochastic optimization problems with probabilistic constraints.

Using a steady state model for the gas flow in pipeline networks including compressor control and uncertain loads given by certain probability distributions, the problem of finding the optimal location for the control on the network, s.t. the control cost is minimal and the gas pressure stays within given bounds, is considered.

In the deterministic setting, explicit bounds for the pipe length and the inlet pressure, s.t. a unique optimal compressor location with minimal control cost exists, are presented. In the probabilistic setting, an existence result for the optimal compressor location is presented and the uniqueness of the solution is discussed depending on the probability distribution. For Gaussian distributed loads a uniqueness result for the optimal compressor location is presented.

Further the problem of finding the optimal compressor locations on networks including the number of compressor stations as variable is considered. Results for the existence of optimal locations on a graph in both, the deterministic and the probabilistic setting, are presented and the uniqueness of the solutions is discussed depending on probability distributions and graph topology. The paper concludes with an illustrative example demonstrating that the compressor locations determined using a steady state approach are also admissible in transient settings.

**Keywords:** Gas Networks; Compressor Control; Compressor Location; Weber Problem; Optimal Location; Uncertain Boundary Data; Non-convex Mixed Integer Stochastic Problem

**Subject Classification:** 49J55, 90B15.

## 1 Introduction and Motivation

Gas transport in pipeline networks forms an essential part of the energy infrastructure in particular for the transport of hydrogen that will be an important energy carrier in the near future worldwide. On the other hand natural gas is still highly used in households and industry in Europe. Transport in pipelines is a cost-efficient method also for large distances, but due to the high transport distances there is a significant pressure drop along the pipelines caused by friction. Compressor stations are used to counteract this pressure drop. Since the construction and operation of compressor stations is cost intensive, a natural question is to ask for the best location. The problem of the optimal location of compressors in a gas pipeline network is a special type of Weber problem (see e.g. [9] for a Weber problem in planes and [8] for Weber problems for transportation systems), where the location cost is related to the dynamics in a

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<sup>\*</sup>Chair in Dynamics, Control, Machine Learning and Numerics (Alexander von Humboldt-Professorship), Department of Mathematics, Friedrich-Alexander Universität Erlangen-Nürnberg FAU), Cauerstr. 11, 91058 Erlangen, Germany

<sup>†</sup>Correspondence should be addressed to [michi.schuster@fau.de](mailto:michi.schuster@fau.de)

<sup>‡</sup>Institut Élie Cartan de Lorraine, CNRS, UMR 7502, Université de Lorraine, B.P. 70239, 54506 Vandoeuvre-lès-Nancy Cedex, France; Systems Research Institute of the Polish Academy of Sciences, ul. Newelska 6, 01-447 Warszawa, Poland; Department of Scientific Computing, Informatics Center, Federal University of Paraíba, Brazil

transportation network. Weber problems where the value of the objective functional is linked to a transportation network have also been considered in [33], but in contrast to [33] in the current contribution the compressor station is an active element in the transportation network. Even if compressor stations already exist for the European natural gas network, there will always be pipeline network expansion, e.g., due to new industry areas and due to new hydrogen pipelines. The aspect of network expansion (not just for gas networks) was discussed in [15]. For hydrogen transport a new pipeline network is even planned in Germany [20] and Europe [2]. From an industrial perspective this topic was discussed e.g., in [3, 39].

Gas transport and other network flow problems have been the topic of many studies in the last decades. The gas transport in pipeline networks is modeled by hyperbolic balance laws as the isothermal Euler equations (see [6, 31, 32, 40]) and simplifications like stationary models (see e.g. [22, 23, 25, 29]). In [14] and [37] the authors give an excellent overview about gas network models and network components. In [5] the authors discuss coupling conditions at the network junctions. Optimal control problems for the gas transport on networks were often considered (see e.g. [4, 10, 11, 24, 30]) but optimal control problems on gas networks including the compressor location have not been considered yet.

In this paper we consider problems of finding the optimal location for compressor stations on arbitrary stationary gas networks including bounds for the pressure and uncertain gas demand. Motivated by the application, the pressure in the pipes is neither allowed to be too high, nor too low, which is modeled by state constraints for the pressure. The survey [19] gives an excellent overview about optimal control under state constraints and related topics. Also motivated by the application, the consumers gas demand is not known exactly a priori, i.e., it is uncertain. Uncertain gas demand in the context of mathematical gas transport means, that the boundary data is random and probabilistic constraints are an excellent modelling tool for the uncertainty in the boundary data. Gas networks under uncertainty have been analyzed e.g., in [22, 29, 43], a model for a power network with uncertain demand has been analyzed in [21] and the distributionally robustness of uncertain demands in control problems was analyzed in [45].

Altogether this leads to the analysis of probabilistic constrained optimal control location problems for the stationary gas transport on arbitrary networks with state constraints, which, to our best knowledge, have not been considered yet. Our studies are related to the supply chain models that were analyzed e.g. in [12] and in [13].

This paper is structured as follows: In Section 2 we analyze the optimal location of a compressor station on a single pipe. We first consider only a lower bound for the pressure, then we consider a problem with a lower and an upper bound for the pressure. In both cases we state results about the existence of an optimal compressor location with minimal control cost for deterministic and uncertain boundary conditions (gas outflow). We also give sufficient conditions for the uniqueness of an optimal compressor location depending on the inlet pressure and the pipe length.

In Section 3 we analyze the optimal compressor location with minimal control cost on arbitrary graphs with pressure bounds for both, a fix and a variable number of compressor stations. Based on the results of Section 2, we state sufficient conditions for the existence of optimal compressor locations in all problems with and without uncertain boundary data (gas outflow). We also discuss the uniqueness of solutions depending on the topology of the graph. Further we discuss the question of the minimal number of compressor stations needed on a graph to satisfy bounds for the pressure. We present an example where the minimal number of compressor stations in the deterministic case is infeasible in the probabilistic case. In Section 4 we present a transient simulation based on the results of the previous sections and we discuss the robustness of the stationary solution in the transient setting. All examples were solved using the *AMPL* software (see [18]) and *MATLAB*<sup>®</sup> 2019a. All pictures were created in *MATLAB*<sup>®</sup> 2019a.

## 2 Optimal Compressor Location on a Single Pipe

The isothermal Euler equations are a well-known model for the gas flow through a network, see e.g., [6, 31, 32, 40]. They consist in a  $2 \times 2$  system of hyperbolic partial differential equations

$$\begin{aligned} \rho_t + q_x &= 0, \\ q_t + \left( p + \frac{q^2}{\rho} \right)_x &= -\frac{\lambda}{2D} \frac{q|q|}{\rho}, \end{aligned}$$

where  $\rho = \rho(t, x)$  is the gas density,  $p = p(t, x)$  is the gas pressure,  $q = q(t, x)$  is the gas flow,  $\lambda > 0$  is the pipe friction coefficient and  $D > 0$  is the pipe diameter. For long time horizon planning, this model is often replaced by a static model, that represents the steady states. Considering ideal gas, in a single pipe with length  $L > 0$  this approach (similar as in [38]) yields the stationary isothermal Euler equations (see e.g., [22, 23, 25, 29])

$$\begin{aligned} q_x(x) &= 0, \\ p_x(x) &= -\frac{\lambda}{2D} R_S T \frac{q(x)|q(x)|}{p(x)}. \end{aligned} \tag{1}$$

Here,  $R_S$  is the specific gas constant of natural gas and  $T$  is the gas temperature. With inlet pressure  $p(0) = p_0$  and gas outflow (load)  $q(L) = b \geq 0$ , the stationary isothermal Euler equations (1) have the solution

$$p^2(x) = p_0^2 - \phi b |b| x \quad \text{with} \quad \phi = \frac{\lambda}{D} R_S T. \tag{2}$$

We consider a compressor station at  $x = x_C \in [0, L]$  on the pipe as it is shown in *Figure 1*. Compressor stations often are necessary in gas networks since due to pipe friction the pressure decreases along the pipe.

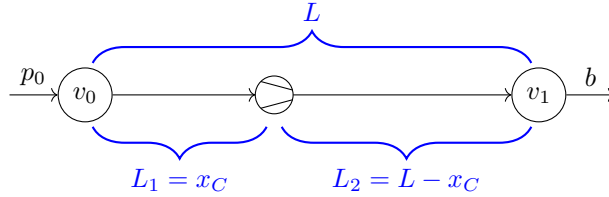


Figure 1: Single edge with a compressor station

Let  $p_1(x)$  be the pressure between  $v_0$  and the compressor station and let  $p_2(x)$  be the pressure between the compressor station and  $v_1$ . The compressor station increases the pressure by a control  $u$  and is modeled by

$$\frac{p_2^2(0)}{p_1^2(L_1)} = u,$$

with  $u \geq 1$ . From (2) we have

$$p_1^2(x) = p_0^2 - \phi b |b| x \quad x \in [0, L_1] = [0, x_C], \tag{3}$$

and

$$\begin{aligned} p_2^2(x) &= u p_1^2(L_1) - \phi b |b| x \\ &= u p_0^2 - \phi b |b| (u L_1 + x) \quad x \in [0, L_2] = [0, L - x_C]. \end{aligned}$$

For the pressure at node  $v_1$ , i.e., for  $x = L - x_C$  we have

$$p_2^2(L_2) = u p_0^2 - \phi b |b| ((u - 1) x_C + L). \tag{4}$$

In the following sections we introduce the uncertain gas demand and the bounds for the pressure. We define our optimal control problems including the compressor location and present results about the existence and uniqueness of optimal controls for the deterministic and the probabilistic setting.

## 2.1 Lower Pressure Bound at the End of the Pipe

The pipe friction causes a decrease for the pressure along the pipe. Therefore we request the pressure to satisfy a lower bound  $p_{\min} > 0$  at node  $v_1$ , i.e. we demand the inequality state constraint

$$p_2(L_2) \geq p_{\min} \quad (5)$$

In applications the gas demand  $b$  is a priori unknown. In the European gas market, traders have to determine individual gas nominations hours before the technical system operators (TSO) actually transports the gas through the network. Thus the TSO have to deal with possible renominations which can be seen as uncertainty in the loads. We assume random gas outflow at node  $v_1$ , i.e. we consider a Gaussian distributed random variable

$$\xi \sim \mathcal{N}(\mu, \sigma^2), \quad (6)$$

with mean value  $\mu > 0$  and standard deviation  $\sigma > 0$  on the appropriate probability space  $(\Omega, \mathcal{A}, \mathbb{P})$ . We identify the random gas outflow  $b^\omega$  with the image  $\xi(\omega)$  for  $\omega \in \Omega$  on this probability space. In [22] and in [37, Chapter 13] the authors explain why a Gaussian distribution is a good choice for random loads. Note that due to the random loads also the pressures are random variables. In order to avoid an additional index, we write  $p$  and  $b$  instead of  $p^\omega$  and  $b^\omega$ . Due to the structure of this paper it is always clear, where pressures and flows are deterministic and where random.

Since the load is random we cannot demand (5) for all  $\omega \in \Omega$  anymore but for  $\alpha \in (0, 1)$  we demand the probabilistic constraint

$$\mathbb{P}(p_2(L_2, \xi) \geq p_{\min}) \geq \alpha,$$

which is equivalent to

$$\mathbb{P}(p_2^2(L_2, \xi) \geq p_{\min}^2) \geq \alpha. \quad (7)$$

Note that in [35] the authors provide an explicit gradient formula for probabilistic constraints with Gaussian distributions. As from now we will neglect the dependency on the random variable since it is obvious while discussing probabilities. We assume that  $p_2(L_2)$  and  $p_2^2(L_2)$  have absolutely continuous probability distribution functions with probability density functions  $\rho_{p_2(L_2)}$  and  $\rho_{p_2^2(L_2)}$ . Consider the probabilistic constrained optimization problem

$$(OPT\ 0) \quad \begin{cases} \min_{u, x_C} & u^2, \\ \text{s.t.} & \mathbb{P}(p_2^2(L_2) \geq p_{\min}^2) \geq \alpha, \\ & u \geq 1, \\ & x_C \in [0, L]. \end{cases}$$

Note that  $p_2^2(L_2)$  also depends on  $u$  and  $x_C$  as it is stated in (4). For the reader's convenience we do not explicitly mention the dependency. The probability (7) can be computed by integrating the probability density function from  $p_{\min}$  resp.  $p_{\min}^2$  to infinity, i.e., we have

$$\mathbb{P}(p_2^2(L_2) \geq p_{\min}^2) = \int_{p_{\min}^2}^{\infty} \rho_{p_2^2(L_2)}(z) dz.$$

The following lemma contains a result about the existence of optimal solutions of (OPT 0). In the next sections this result will be extended to gas networks with a finite number of compressor stations and upper and lower bounds on every pipe.

**Lemma 1.** *The probabilistic constrained optimization problem (OPT 0) either has a unique solution  $(u^*, x_C^*)$  with  $x_C^* \in \{0, L\}$  or there exists  $u^* \geq 1$ , s.t.  $(u^*, x_C)$  is a solution of (OPT 0) for every  $x_C \in [0, L]$ .*

**Remark 2.** *When every point  $(u, x_C)$  with  $u = 1$  is a solution of (OPT 0) this means that the compressor station is not needed for this pipe. Thus it can be placed anywhere at the pipe since the pressure is not increased anywhere. If and only if  $p_0 < p_{\min}$  and*

$$\int_{-\infty}^0 \rho_\xi(z) dz = \alpha,$$

where  $\rho_\xi$  is the probability density function of the random variable  $\xi$ , then there exists  $u^* > 1$ , s.t. every point  $(u^*, x_C)$  with  $x_C \in [0, L]$  is a solution of (OPT 0).

*Proof. Proof of Lemma 1.* We first show that there exist admissible pairs  $(u, x_C)$  for (OPT 0). For  $u \geq 1$  and  $x_C \in [0, L]$  define the admissible set

$$M(u, x_C) := \{ b \in \mathbb{R} \mid p_2^2(L_2) \geq p_{\min}^2 \},$$

where  $b = b^\omega = \xi(\omega)$  is as in (6). As mentioned before we only write  $b$  instead of  $\xi(\omega)$  for  $\xi \sim \mathcal{N}(\mu, \sigma^2)$ . The probability  $\mathbb{P}(b \in M(u, x_C))$  is equal to the probability  $\mathbb{P}(p_2^2(L_2) \geq p_{\min}^2)$  and it can be computed by

$$\mathbb{P}(b \in M(u, x_C)) = \int_{M(u, x_C)} \rho_b(z) dz. \quad (8)$$

The probability density function  $\rho_b$  is given by the Gaussian distribution (6) but the set  $M(u, x_C)$  is not known explicitly. Due to (4) for a given  $x_C \in [0, L]$  we can enlarge the set  $M$  by increasing the control  $u$ . For a monotonously increasing sequence  $(u_k)_{k \geq 1} \subseteq \mathbb{R}_{\geq 1}$ , we have  $M(u_k, x_C) \subseteq M(u_{k+1}, x_C)$ . If  $\lim_{k \rightarrow \infty} u_k = \infty$ , we have  $\lim_{k \rightarrow \infty} M(u_k, x_C) = \mathbb{R}$ , and this implies

$$\mathbb{P}(b \in M(u, x_C)) \rightarrow 1 \quad \text{for } u \rightarrow \infty. \quad (9)$$

So for all  $x_C \in [0, L]$  we can find  $u^* \geq 1$ , s.t.

$$\mathbb{P}(b \in M(u, x_C)) \geq \alpha \quad \text{for all } u \geq u^*,$$

and thus the set of admissible controls and compressor locations is nonempty.

Next we show the second statement in *Lemma 1* for  $u = 1$  (the case  $u > 1$  follows later). Due to (4) we have

$$p_2^2(L_2) = p_0^2 - \phi b |b| L,$$

and thus the pressure at the end of the pipe and also  $M$  are independent of the compressor location. So if

$$\mathbb{P}(b \in M(1, x_C)) \geq \alpha,$$

every point  $(u, x_C)$  with  $u = 1$  and  $x_C \in [0, L]$  is a solution of (OPT 0).

Now we consider the first statement in *Lemma 1*. If

$$\mathbb{P}(b \in M(1, x_C)) < \alpha,$$

then we need to show that there exists a unique solution of (OPT 0) with  $x_C \in \{0, L\}$ . Obviously we have  $u > 1$  in this case. We first show that in this case the probabilistic constraint is always active. For the set  $M(u, x_C)$ , we fix an admissible pair  $(u, x_C)$ . Then we can find  $\bar{b} \in \mathbb{R}$ , s.t.

$$\begin{aligned} u p_0^2 - \phi \bar{b} |\bar{b}| ((u-1)x_C + L) &= p_{\min}^2, \\ \text{and } u p_0^2 - \phi b |b| ((u-1)x_C + L) &\geq p_{\min}^2 \quad \forall b \leq \bar{b}. \end{aligned}$$

From this it follows that

$$M(u, x_C) = (-\infty, \bar{b}].$$

Then due to (8) and (9) it follows that if  $(u^*, x_C^*)$  is a solution of (OPT 0) then we have

$$\mathbb{P}(b \in M(u^*, x_C^*)) = \alpha, \quad (10)$$

and thus we can find  $\beta^*$ , s.t.,

$$\int_{-\infty}^{\beta^*} \rho_b(z) dz = \alpha.$$

Now we can show the first statement in *Lemma 1* by cases. For  $\beta^* > 0$  consider the admissible points  $(u_1, 0)$ ,  $(u_2, \varepsilon)$  with  $\varepsilon > 0$  and

$$\mathbb{P}( b \in M(u_1, 0) ) = \mathbb{P}( b \in M(u_2, \varepsilon) ) = \alpha.$$

Then with (4) we have

$$u_1 p_0^2 - \phi \beta^* |\beta^*| L = u_2 p_0^2 - \underbrace{\phi \beta^* |\beta^*| (u_2 - 1) \varepsilon}_{>0} - \phi \beta^* |\beta^*| L,$$

and it follows  $u_2 > u_1$ . That means that the compressor located at the beginning of the pipe is the most efficient if  $\beta^* > 0$  and thus  $(u^*, 0)$  is a unique solution of (OPT 0) with  $u^*$  s.t. (10) holds.

For  $\beta^* < 0$  consider the admissible points  $(u_1, L)$ ,  $(u_2, L - \varepsilon)$  with  $\varepsilon > 0$  and

$$\mathbb{P}( b \in M(u_1, L) ) = \mathbb{P}( b \in M(u_2, L - \varepsilon) ) = \alpha.$$

Then with (4) we have

$$u_1 (p_0^2 - \phi \beta^* |\beta^*| L) = u_2 (p_0^2 - \phi \beta^* |\beta^*| L) + \underbrace{\phi \beta^* |\beta^*| (u_2 - 1) \varepsilon}_{<0},$$

and it follows  $u_2 > u_1$ . That means that the compressor located at the end of the pipe is the most efficient if  $\beta^* < 0$  and thus,  $(u^*, L)$  is a unique solution of (OPT 0) with  $u^*$  s.t. (10) holds.

For  $\beta^* = 0$  due to (4) the pressure at the end of the pipe does not depend on the compressor location, so we can find a unique  $u^* > 1$ , s.t.  $(u^*, x_C)$  is a solution of (OPT 0) for every  $x_C \in [0, L]$ . This completes the proof of *Lemma 1*. □

If the inlet pressure is larger than the lower pressure bound then there always exist positive loads s.t. the pressure at the end of the pipe is also larger than the lower pressure bound. This implies  $\beta^* > 0$  and we can formulate a more precise statement:

**Corollary 3.** *Let  $p_0 > p_{min}$  be given. If*

$$\mathbb{P}( p_0^2 - \phi b |b| L \geq p_{min}^2 ) \geq \alpha,$$

*then the pair  $(1, x_C)$  is a solution of (OPT 0) for every  $x_C \in [0, L]$ . Otherwise  $(u^*, 0)$  is the unique solution of (OPT 0), where  $u^*$  can be determined by (10).*

**Remark 4.** *Even if negative loads usually do not occur in real world applications a Gaussian distribution is a reasonable choice for the random loads as it is mentioned above. When mean value  $\mu$  and standard deviation  $\sigma$  are chosen close to real world application<sup>1</sup> it always follows that  $\mathbb{P}( b \leq 0 )$  is very small. In order to completely avoid negative loads one can consider a probability distribution for the load vector that does not allow negative values like e.g., a truncated Gaussian distribution or a logarithmic distribution. For these distributions the statement of Corollary 3 even holds for  $p_0 \leq p_{min}$ .*

## 2.2 Upper and Lower Pressure Bound for the whole Pipe

In this section we consider pressure bounds  $p_{min}, p_{max} > 0$  with  $p_{min} < p_{max}$  for the whole pipe, i.e., we demand the state constraints

$$\begin{aligned} p_1(x) &\in [p_{min}, p_{max}] \quad \forall x \in [0, L_1], \\ \text{and} \quad p_2(x) &\in [p_{min}, p_{max}] \quad \forall x \in [0, L_2], \end{aligned} \tag{11}$$

In this setting it is reasonable to assume  $p_0 \in [p_{min}, p_{max}]$  because if  $p_0 > p_{max}$  then (11) cannot be satisfied at all and if  $p_0 < p_{min}$  then the compressor station needs to be located at the beginning of the

<sup>1</sup>Real world data can be found on <https://gaslib.zib.de/>

edge, s.t.  $x_C = L_1 = 0$ . Since the pressure is monotonously decreasing along the pipe, for  $p_0 \in [p_{\min}, p_{\max}]$  we have

$$(11) \iff \begin{cases} p_1(L_1) \geq p_{\min}, \\ p_2(0) \leq p_{\max}, \\ p_2(L_2) \geq p_{\min}. \end{cases}$$

We first consider the deterministic optimization problem

$$(OPT\ 1) \quad \begin{cases} \min_{u, x_C} u^2, \\ \text{s.t. } p_1(L_1) \geq p_{\min}, \\ p_2(0) \leq p_{\max}, \\ p_2(L_2) \geq p_{\min}, \\ u \geq 1, \\ x_C \in [0, L]. \end{cases}$$

The existence of a solution depends on the pipe length and the inlet pressure. If the pipe is sufficiently short a compressor station might not be needed. When the pipe length exceeds a certain limit the compressor control generates a state that always violates at least one of the pressure bounds, as it is shown in *Figure 2*.

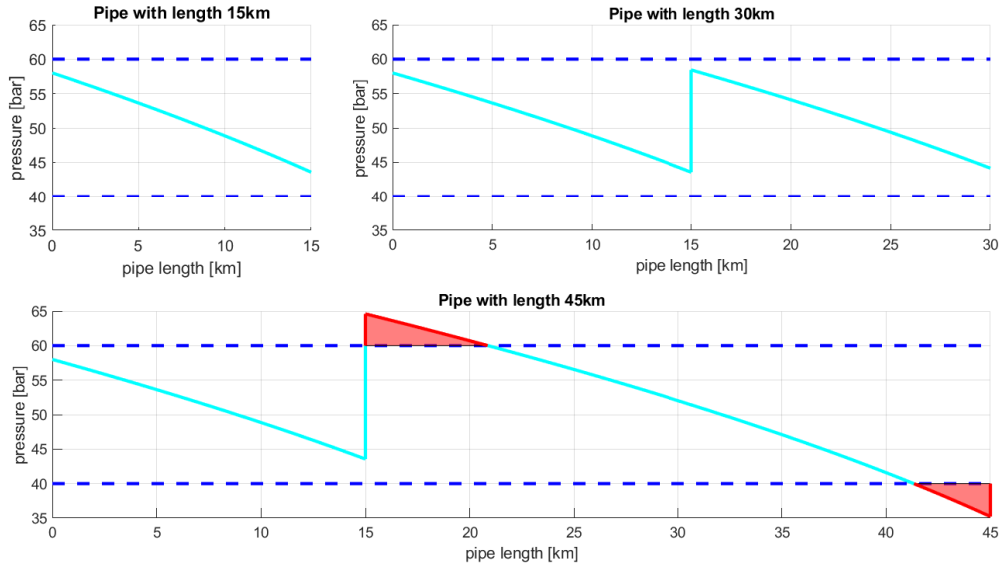


Figure 2: Pressure loss in a single pipe with length  $L \in \{15\text{km}, 30\text{km}, 45\text{km}\}$ , compressor location  $x_C = 15\text{km}$ , control  $u \in \{1, 1.8, 2.2\}$ , inlet pressure  $p_0 = 58\text{bar}$ , outflow  $b = 180 \frac{\text{kg}}{\text{m}^2\text{s}}$ ,  $\phi = 3.0179 \cdot 10^4 \frac{\text{m}}{\text{s}^2}$  and pressure bounds  $[p_{\min}, p_{\max}] = [40\text{bar}, 60\text{bar}]$ .

**Lemma 5.** Let  $p_0 \in [p_{\min}, p_{\max}]$  and  $b > 0$  be given.

(i) For

$$L \leq \frac{p_0^2 - p_{\min}^2}{\phi b |b|} \quad (12)$$

every point  $(u, x_C)$  with  $u = 1$  and  $x_C \in [0, L]$  is a solution of the optimization problem (OPT 1).

(ii) For

$$\frac{p_0^2 - p_{\min}^2}{\phi b |b|} < L \leq \frac{p_0^2 + p_{\max}^2 - 2 p_{\min}^2}{\phi b |b|} \quad (13)$$

the optimization problem (OPT 1) has a unique solution  $(u^*, x_C^*)$  with  $u^* > 1$  and  $x_C \in [0, L]$ .



(iii) For

$$L > \frac{p_0^2 + p_{\max}^2 - 2 p_{\min}^2}{\phi b |b|} \quad (14)$$

the optimization problem (OPT 1) does not have a solution.

*Proof. Proof of Lemma 5.* Since we assumed  $b > 0$ , for  $i \in \{1, 2\}$  and  $x_1, x_2 \in [0, L_i]$  with  $x_1 < x_2$  we always have

$$p_i^2(x_2) < p_i^2(x_1). \quad (15)$$

(i) Consider a pair  $(1, x_C)$  with  $x_C \in [0, L]$ . The condition (12) is equivalent to

$$p_0^2 - \phi b |b| L \geq p_{\min}^2.$$

Since  $L_1 \leq L$  we have

$$p_1^2(L_1) = p_0^2 - \phi b |b| L_1 \geq p_{\min}^2.$$

Due to  $u = 1$  we have

$$p_2^2(L_2) = p_0^2 - \phi b |b| L \geq p_{\min}^2,$$

and together with (15) it follows

$$p_2^2(0) = p_1^2(L_1) \leq p_1^2(0) = p_0^2 \leq p_{\max}^2,$$

and thus the pair  $(1, x_C)$  is a solution of (OPT 1) for every  $x_C \in [0, L]$ .

(iii) Assume that there exists a solution  $(u^*, x_C^*)$  of (OPT 1) with  $u^* \geq 1$  and  $x_C^* \in [0, L]$ . Then due to (4) we have

$$p_2^2(L_2) = u^* p_0^2 - u^* \phi b |b| x_C^* - \phi b |b| L + \phi b |b| x_C^*. \quad (16)$$

By reformulating (16) and by adding  $p_0^2$  we have

$$\underbrace{p_2^2(L_2)}_{\geq p_{\min}^2} + \underbrace{p_0^2 - \phi b |b| x_C^*}_{= p_1^2(L_1) \geq p_{\min}^2} + \phi b |b| L = p_0^2 + \underbrace{u^* (p_0^2 - \phi b |b| x_C^*)}_{= p_2^2(0) \leq p_{\max}^2}.$$

Thus it follows

$$2 p_{\min}^2 + \phi b |b| L \leq p_0^2 + p_{\max}^2,$$

which is a contradiction to (14) and  $(u^*, x_C^*)$  cannot be a solution of (OPT 1).

(ii) Since the constraints of (OPT 1) are not strict, the admissible set (defined in the proof of Lemma 1) is compact. If the admissible set is non empty a solution of (OPT 1) exists. Suppose that the admissible set is empty, i.e. there is no pair  $(u, x_C)$  that satisfies the constraints of (OPT 1). We construct the maximal pipe length  $L_1 + L_2$  for admissible points. For  $p_0 \in [p_{\min}, p_{\max}]$  we choose  $L_1$  s.t.

$$p_1^2(L_1) = p_0^2 - \phi b |b| L_1 = p_{\min}^2.$$

The largest possible control is to raise the pressure from the lower to the upper bound, i.e., we have

$$u = \frac{p_{\max}^2}{p_{\min}^2},$$

thus we have  $p_2^2(0) = p_{\max}^2$ . Then we find the maximal admissible  $L_2$  by solving

$$p_2^2(L_2) = p_{\max}^2 - \phi b |b| L_2 = p_{\min}^2.$$

Since we have no admissible point, either  $L_1$  or  $L_2$  must be larger, so we have

$$L = L_1 + L_2 > \frac{p_0^2 - p_{\min}^2}{\phi b |b|} + \frac{p_{\max}^2 - p_{\min}^2}{\phi b |b|} = \frac{p_0^2 + p_{\max}^2 - 2 p_{\min}^2}{\phi b |b|},$$

which is a contradiction to (13). So the admissible set is non empty, i.e., a solution of (OPT 1) must exist.

Assume that the pairs  $(u_1, x_{C,1})$  and  $(u_2, x_{C,2})$  are solutions of (OPT 1). Due to the objective function we have  $u^* := u_1 = u_2$ . Assume w.l.o.g.  $x_{C,1} < x_{C,2}$ . Since  $b > 0$ , we have

$$u^* p_0^2 - \phi b |b| ((u^* - 1) x_{C,1} + L) > u^* p_0^2 - \phi b |b| ((u^* - 1) x_{C,2} + L) \geq p_{\min}^2,$$

and thus we can find  $u < u^*$ , s.t.  $(u, x_{C,1})$  is admissible. This contradicts the optimality of  $(u^*, x_{C,i})$  ( $i = \{1, 2\}$ ). Hence the solution  $(u^*, x_C^*)$  is unique. The optimal control  $u^*$  is strictly larger than 1 because otherwise, if  $u^* = 1$ , we have

$$p_2^2(L_2) = p_0^2 - \phi b |b| L \geq p_{\min}^2,$$

which is a contradiction to the lower bound of  $L$  in (13) and the lemma is proven.  $\square$

$\square$

The *Proof of Lemma 5 (ii)* shows that smaller compressor locations  $x_C$  are more efficient, i.e., the compressor cost is lower. This fact allows us to state the following necessary optimality conditions:

**Corollary 6.** *Let  $(u^*, x_C^*)$  be a unique solution of (OPT 1). Then we have*

$$p_2(0; (u^*, x_C^*)) = p_{\max} \quad \text{and} \quad p_2(L_2; (u^*, x_C^*)) = p_{\min}.$$

In the applications often the pipe length is given a priori, so another interesting question is the following: For given pipe length and gas demand, can we find an appropriate inlet pressure s.t. the optimization problem (OPT 1) has a (unique) solution? The answer directly follows from *Lemma 5*. From *Lemma 5 (i)* it follows that if the pipe is sufficiently short then we can find suitable inlet pressures  $p_0$  s.t. a compressor station is not needed. From *Lemma 5 (iii)* it follows that if the pipe length exceeds a certain limit, then every pressure increase violates the pressure bounds for any  $p_0 \in [p_{\min}, p_{\max}]$ . Consequently from *Lemma 5 (ii)* it follows that if the pipe is neither too short nor too long then we can find a suitable  $p_0$  s.t. a compressor station is needed to guarantee the bounds for the pressure.

**Corollary 7.** *Let  $L > 0$  and  $b > 0$  be given.*

(i) *If*

$$\mathcal{P}_1 := [p_{\min}^2 + \phi b |b| L, \infty) \cap [p_{\min}^2, p_{\max}^2] \neq \emptyset,$$

*then for all  $p_0 \in \mathcal{P}_1$ , every point  $(u, x_C)$  with  $u = 1$  and  $x_C \in [0, L]$  is a solution of (OPT 1).*

(ii) *If*

$$\mathcal{P}_2 := [2 p_{\min}^2 - p_{\max}^2 + \phi b |b| L, \infty) \cap [p_{\min}^2, p_{\max}^2] \neq \emptyset,$$

*then for all  $p_0 \in \mathcal{P}_2 \setminus \mathcal{P}_1$  the optimization problem (OPT 1) has a unique solution  $(u^*, x_C^*)$  with  $u^* > 1$  and  $x_C \in [0, L]$ .*

(iii) *If  $\mathcal{P}_2 = \emptyset$  then for all  $p_0 \in [p_{\min}, p_{\max}]$  the optimization problem (OPT 1) has no solution.*

Note that  $\mathcal{P}_1 \subseteq \mathcal{P}_2$ . In the end of this section we consider an example based on the pipe with length 30km from *Figure 2* for both the deterministic and the probabilistic case.

In the next part of this section we consider random gas outflow at the end of the pipe. As in the last section we consider the Gaussian distributed random variable (6) with mean value  $\mu > 0$  and standard deviation  $\sigma > 0$  on an appropriate probability space  $(\Omega, \mathcal{A}, \mathbb{P})$ . We identify the random gas outflow  $b^\omega$  with the image  $\xi(\omega)$  for  $\omega \in \Omega$  on this probability space. Thus the corresponding pressures  $p^\omega$  are also random. We mention again, that in order to prevent the use of additional indices, we will refer to  $p^\omega$  as  $p$  and  $b^\omega$  as  $b$  henceforth.

In contrast to the deterministic optimization problem (OPT 1), the Gaussian distribution load  $b$  could take negative values. Even if this is not possible in applications, we need to consider this fact in our analysis. Negative loads increase the pressure, s.t. we also need to make sure that

$$p_2(L_2) \leq p_{\max}. \tag{17}$$

Note that this constraint can be neglected for probability distributions that allow only positive values for  $b$  (cf. *Remark 4*).

**Remark 8.** Due to possible negative loads in the probabilistic setting, even for  $p_0 < p_{\max}$  we could have  $p_1(L_1) > p_{\max}$ , but due to  $u \geq 1$  this would imply  $p_2(0) > p_{\max}$ , which is already covered by a constraint.

For  $\alpha \in (0, 1)$  the probabilistic constrained optimization problem corresponding to the deterministic problem (OPT 1) is given by

$$(OPT\ 2) \quad \left\{ \begin{array}{l} \min_{u, x_C} u^2, \\ \text{s.t. } \mathbb{P} \left( \begin{array}{l} p_1(L_1) \geq p_{\min} \\ p_2(0) \leq p_{\max} \\ p_2(L_2) \geq p_{\min} \\ p_2(L_2) \leq p_{\max} \end{array} \right) \geq \alpha, \\ u \geq 1, \\ x_C \in [0, L]. \end{array} \right.$$

The probabilistic problem (OPT 2) shows a similar behaviour as the deterministic problem (OPT 1) but due to the uncertainty we cannot state sharp bounds for the length  $L$  or the inlet pressure  $p_0$  as we did in Lemma 5 and Corollary 7. We first state a necessary optimality condition:

**Lemma 9.** If  $(u^*, x_C^*)$  with  $u^* > 1$  is a solution of (OPT 2), then we have

$$\mathbb{P} \left( \begin{array}{l} p_1(L_1) \geq p_{\min} \\ p_2(0) \leq p_{\max} \\ p_2(L_2) \geq p_{\min} \\ p_2(L_2) \leq p_{\max} \end{array} \right) = \alpha.$$

*Proof. Proof of Lemma 9.* We use a proof by contradiction here. For a solution  $(u^*, x_C^*)$  of (OPT 2) assume that

$$\mathbb{P} \left( \begin{array}{l} p_1^2(L_1) \geq p_{\min}^2 \\ p_2^2(0) \leq p_{\max}^2 \\ p_2^2(L_2) \geq p_{\min}^2 \\ p_2^2(L_2) \leq p_{\max}^2 \end{array} \right) = \alpha + \varepsilon_1,$$

with  $\varepsilon_1 > 0$ . Note that since  $0 < p_{\min} < p_{\max}$  we can use pressures and squared pressures equivalently. We define the set

$$M(u^*, x_C^*) := \left\{ b \in \mathbb{R} \left| \begin{array}{l} p_1^2(L_1) \geq p_{\min}^2 \\ p_2^2(0) \leq p_{\max}^2 \\ p_2^2(L_2) \geq p_{\min}^2 \\ p_2^2(L_2) \leq p_{\max}^2 \end{array} \right. \right\},$$

so we have

$$\mathbb{P} \left( \begin{array}{l} p_1^2(L_1) \geq p_{\min}^2 \\ p_2^2(0) \leq p_{\max}^2 \\ p_2^2(L_2) \geq p_{\min}^2 \\ p_2^2(L_2) \leq p_{\max}^2 \end{array} \right) = \mathbb{P}(b \in M(u^*, x_C^*)),$$

and the probability can be computed by

$$\mathbb{P}(b \in M(u^*, x_C^*)) = \int_{M(u^*, x_C^*)} \rho_b(z) dz,$$

where  $\rho_b$  is the probability density function of the Gaussian distributed random load  $b$ . Due to

$$\begin{aligned} p_1^2(x) &= p_0^2 - \phi b |b| x, \\ \text{and } p_2^2(x) &= u p_0^2 - \phi b |b| (u L_1 + x), \end{aligned} \tag{18}$$

the set  $M(u^*, x_C^*)$  is given by the interval  $[\underline{b}^*, \bar{b}^*]$ , where the lower bound  $\underline{b}^*$  is given by the larger solution of  $p_2^2(0) = p_{\max}^2$  and  $p_2^2(L_2) = p_{\max}^2$ , and the upper bound  $\bar{b}^*$  is given by the lower solution of  $p_1^2(L_1) = p_{\min}^2$  and  $p_2^2(L_2) = p_{\min}^2$ . Note that due to  $p_0 \in [p_{\min}, p_{\max}]$  and due to  $u^* > 1$ , we have  $\bar{b}^* \geq 0$

but  $\underline{b}^*$  can be negative (depending on the control  $u^*$ ). Further due to  $\alpha \in (0, 1)$  we have  $\underline{b}^* \neq \bar{b}^*$  because otherwise it follows  $\mathbb{P}(b \in M(u^*, x_C^*)) = 0 \neq \alpha$ . Consider  $\varepsilon_2 > 0$ , s.t.  $u^* - \varepsilon_2 \geq 1$ , due to (18) we have

$$M(u^* - \varepsilon_2, x_C^*) = [\underline{b}, \bar{b}] \quad \text{with} \quad \underline{b} < \underline{b}^* \quad \text{and} \quad \bar{b} \leq \bar{b}^*.$$

Thus we have

$$\mathbb{P}(b \in M(u^* - \varepsilon_2, x_C^*)) = \int_{M(u^* - \varepsilon_2, x_C^*)} \rho_b(z) dz = \alpha + \varepsilon_1 + \nu.$$

Due to the continuity of  $\rho_b$  and due to the continuity of (18) for all  $\varepsilon_1 > 0$  we can find  $\varepsilon_2 > 0$  sufficiently small, s.t.  $|\nu|$  is sufficiently small, s.t.

$$\alpha + \varepsilon_1 + \nu > \alpha.$$

Thus  $(u^*, x_C^*)$  cannot be a solution of (OPT 2) and the proof is complete.  $\square$

$\square$

The next statement is an existence result for (OPT 2) and it guarantees that if the admissible set is nonempty, then a solution of (OPT 2) exists.

**Theorem 10.** Consider the probabilistic constrained optimization problem (OPT 2) and let  $p_0 \in [p_{min}, p_{max}]$  be given.

- (i) If there exists a pair  $(u, x_C)$  with  $u = 1$  and  $x_C \in [0, L]$ , that satisfies the constraints of (OPT 2), then every pair  $(u, x_C)$  with  $u = 1$  and  $x_C \in [0, L]$  is a solution of (OPT 2).
- (ii) If there exist a pair  $(u, x_C)$ , that satisfies the constraints of (OPT 2) and if  $(u, x_C)$  with  $u = 1$  is infeasible for at least one  $x_C \in [0, L]$ , then there exists at least one solution  $(u^*, x_C^*)$  of (OPT 2) with  $u^* > 1$  and  $x_C^* \in [0, L]$ .

Apart from the assumption that the admissible set is nonempty, the statements (i) and (ii) in Theorem 10 match the statements (i) and (ii) of Lemma 5 and Corollary 7.

*Proof. Proof of Theorem 10.*

(i) It is easy to see that if we find an admissible pair  $(u, x_C)$  with  $u = 1$  that satisfies the constraints of (OPT 2) then a compressor station is not needed to guarantee that the pressure stays within the pressure bounds with probability  $\alpha$ . Thus every point  $(1, x_C)$  with  $x_C \in [0, L]$  is a solution of (OPT 2).

(ii) We know that if  $(1, x_C)$  is infeasible for one  $x_C \in [0, L]$  then  $(1, x_C)$  is infeasible for all  $x_C \in [0, L]$  because otherwise this contradicts to case (i), so we have  $u > 1$ . That means that a compressor station is needed due to the pressure loss caused by pipe friction.

As in the proof of Lemma 9, we define the set

$$M(u, x_C) := \left\{ b \in \mathbb{R} \left| \begin{array}{l} p_1^2(L_1) \geq p_{min}^2 \\ p_2^2(0) \leq p_{max}^2 \\ p_2^2(L_2) \geq p_{min}^2 \\ p_2^2(L_2) \leq p_{max}^2 \end{array} \right. \right\}.$$

The probability in (OPT 2) can be written as  $\mathbb{P}(b \in M(u, x_C))$  and it can be evaluated by

$$\mathbb{P}(b \in M(u, x_C)) = \int_{M(u, x_C)} \rho_b(z) dz, \quad (19)$$

where  $\rho_b$  is the probability density function of the Gaussian distribution. We mention again that we write  $b$  instead of  $b^\omega$  as a realization of  $\xi \sim \mathcal{N}(\mu, \sigma)$ . The admissible set  $\mathcal{A}$  is given by

$$\mathcal{A} = \{ (u, x_C) \mid u \geq 1, x_C \in [0, L], \mathbb{P}(b \in M(u, x_C)) \geq \alpha \}.$$

The set is nonempty by assumption. We now show that  $\mathcal{A}$  is compact.

If  $\mathcal{A}$  contains a finite number of elements the existence of a solution is trivial. Otherwise we first show, that  $\mathcal{A}$  is bounded. From the proof of *Lemma 9* we know, that for  $(u, x_C) \in \mathcal{A}$ , the set  $M(u, x_C)$  is given by the interval  $[\underline{b}, \bar{b}]$ , where  $\underline{b}$  is determined by the larger solution of

$$p_2^2(0) = u (p_0^2 - \phi \underline{b} |\underline{b}| L_1) = p_{\max}^2 \quad \text{and} \quad p_2^2(L_2) = u p_1^2(L_1) - \phi \underline{b} |\underline{b}| L_2 = p_{\max}^2, \quad (20)$$

and  $\bar{b}$  is determined by the smaller solution of

$$p_1^2(L_1) = p_0^2 - \phi \bar{b} |\bar{b}| L_1 = p_{\min}^2 \quad \text{and} \quad p_2^2(L_2) = u p_1^2(L_1) - \phi \bar{b} |\bar{b}| L_2 = p_{\min}^2.$$

For  $\bar{u}$  sufficiently large, s.t.  $\underline{b}$  is sufficiently large (determined by (20)), s.t.

$$p_1^2(L_1) = p_0^2 - \phi \underline{b} |\underline{b}| L_1 < p_{\min}^2 \quad \text{or} \quad p_2^2(L_2) = u p_1^2(L_1) - \phi \underline{b} |\underline{b}| L_2 < p_{\min}^2,$$

the set  $M(\bar{u}, x_C)$  is empty for all  $x_C \in [0, L]$  and thus

$$\mathbb{P}(b \in M(\bar{u}, x_C)) = \int_{M(\bar{u}, x_C)} \rho_b(z) dz = 0 < \alpha.$$

So the admissible set  $\mathcal{A}$  is bounded.

Suppose that  $\mathcal{A}$  is open. Consider a convergent sequence  $(u_k, x_{C,k})_{k \in \mathcal{N}}$  in  $\mathcal{A}$  with limit  $(\hat{u}, \hat{x}_C) \notin \mathcal{A}$  and

$$\mathbb{P}(b \in M(\hat{u}, \hat{x}_C)) = \alpha - \epsilon_1,$$

for  $\epsilon_1 > 0$ . In the proof of *Lemma 9* we have shown, that for  $\epsilon_2 > 0$  sufficiently small, there exists  $\nu$  with  $|\nu|$  sufficiently small, s.t.

$$\mathbb{P}(b \in M(\hat{u} \pm \epsilon_2, \hat{x}_C)) = \alpha - \epsilon_1 + \nu < \alpha.$$

So all sequence elements in  $(\hat{u} \pm \epsilon_2, \hat{x}_C)$  cannot be in  $\mathcal{A}$  and we have a contradiction. Thus the admissible set  $\mathcal{A}$  is bounded and closed and by the *extreme value theorem*, a solution of (OPT 2) exists.  $\square$

$\square$

The statements in *Theorem 10* correspond to the statements **(i)** and **(ii)** of *Lemma 5* resp. *Corollary 7*. In the probabilistic setting the statement **(iii)** of *Lemma 5* and *Corollary 7* implies, that if the admissible set of (OPT 2) is empty, then there exists no solution. We also mention that the results of *Lemma 9* and *Theorem 10* are not restricted to Gaussian distributions. If the gas outflow is given by a random variable, that only allows positive loads (e.g., truncated Gaussian distribution or logarithmic distribution), the condition  $p_2^2(L_2) \leq p_{\max}^2$  is implied by the condition  $p_2^2(0) \leq p_{\max}^2$ , see also *Remark 4*. In the following theorem, we establish the existence of a unique solution for Gaussian distributed loads.

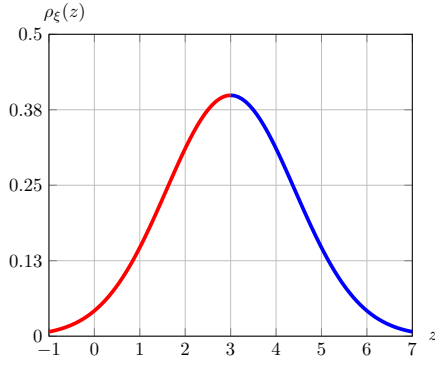
**Theorem 11.** *Let  $\alpha > \frac{1}{2}$  be given. For a Gaussian distribution, Statement (ii) in Theorem 10 guarantees the existence of a unique solution  $(u^*, x_C^*)$  of (OPT 2) with  $u^* > 1$  and  $x_C^* \in [0, L]$ .*

Note that if  $u^* = 1$ , then the assumptions in statement (i) in *Theorem 10* are satisfied and every point  $(1, x_C)$  is a solution of (OPT 2). The assumption  $\alpha > 0.5$  is necessary to use the convexity resp. concavity property of the probability distribution function of the Gaussian distribution for values smaller than resp. larger than the mean value (see *Figure 3*).

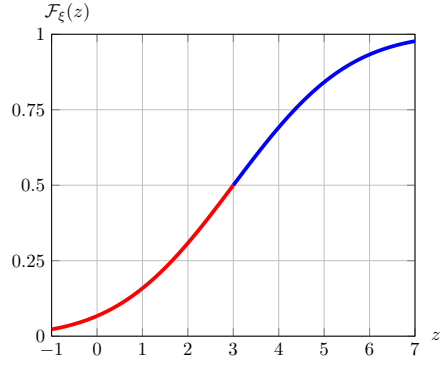
For the proof of *Theorem 11* we use a contradiction argument. We show that if the assumptions are not satisfied, then we can construct an admissible point with lower control cost. Since the following proof is quite technical we split it in four parts.

*Proof. Proof of Theorem 11.* The existence of a solution was already shown in *Theorem 10*. We only have to show the uniqueness for Gaussian distributed loads. Therefore we use a proof by contradiction. Consider two solutions  $(u^*, x_{C,1})$  and  $(u^*, x_{C,2})$  with  $u^* > 1$  and  $x_{C,1} \neq x_{C,2}$ . We define the set

$$M(u, x_C) := \left\{ b \in \mathbb{R} \left| \begin{array}{l} p_1^2(L_1) \geq p_{\min}^2 \\ p_2^2(0) \leq p_{\max}^2 \\ p_2^2(L_2) \geq p_{\min}^2 \\ p_2^2(L_2) \leq p_{\max}^2 \end{array} \right. \right\},$$



(a) Probability density function of a Gaussian distribution, the increasing part is shown in red and decreasing part is shown in blue



(b) Probability distribution function of a Gaussian distribution, the strictly convex part is shown in red and strictly concave part is shown in blue

Figure 3: Probability density function and probability distribution function of a Gaussian distribution with mean value  $\mu = 3$  and standard deviation  $\sigma = \sqrt{2}$

where the pressures depend on the control  $u$  and on the compressor location  $x_C$ . As above  $b$  denotes the gas outflow at the end of the pipe. We split the proof in four main parts: In the first part we write  $M(u, x_C)$  depending on the solutions  $(u^*, x_{C,1})$  and  $(u^*, x_{C,2})$  as intervals and determine their bounds. In the second part we construct a new pair  $(u^*, x_{C,3})$ . In the third part we determine the bounds of  $M(u^*, x_{C,3})$ . In the fourth part, we show that  $(u^*, x_{C,3})$  is admissible with probability larger than  $\alpha$ , which contradicts *Lemma 9*. Consequently we can find a smaller control for  $x_{C,3}$ , s.t. neither  $(u^*, x_{C,1})$  nor  $(u^*, x_{C,2})$  can be optimal.

*Proof of Theorem 11, Part I:* In the first part of the proof, we determine the bounds for  $M$  depending on the solutions  $(u^*, x_{C,1})$  and  $(u^*, x_{C,2})$ . Since the pressure drops monotonously before and after the compressor station, we can write  $M(u, x_C)$  as interval. Considering the bounds on the pressures

$$\begin{aligned}
 p_{\min}^2 &\leq p_1^2(L_1) = p_0^2 - \phi b |b| x_C, \\
 p_2^2(0) &= u p_0^2 - u \phi b |b| x_C, && \leq p_{\max}^2, \\
 p_{\min}^2 &\leq p_2^2(L_2) = u p_0^2 - \phi b |b| ((u-1)x_C + L) && \leq p_{\max}^2,
 \end{aligned} \tag{21}$$

one can see that the upper bound of  $M$  depends on the first and third inequality in (21), while the lower bound of  $M$  depends on the second and third inequality in (21). Let  $M(u^*, x_{C,1}) = [\underline{b}_1, \bar{b}_1]$  and  $M(u^*, x_{C,2}) = [\underline{b}_2, \bar{b}_2]$ . Then for  $i \in \{1, 2\}$  we have

$$\begin{aligned}
 p_{\min}^2 &\leq p_0^2 - \phi \bar{b}_i |\bar{b}_i| x_{C,i}, \\
 \text{and } p_{\min}^2 &\leq u^* p_0^2 - \phi \bar{b}_i |\bar{b}_i| ((u^*-1)x_{C,i} + L).
 \end{aligned} \tag{22}$$

Due to the fact that  $p_0 \geq p_{\min}^2$ , the upper bounds  $\bar{b}_i$  are non negative. Hence, for  $i = 1, 2$ , we have

$$\bar{b}_i = \min \left\{ \sqrt{\frac{p_0^2 - p_{\min}^2}{\phi x_{C,i}}}, \sqrt{\frac{u^* p_0^2 - p_{\min}^2}{\phi ((u^*-1)x_{C,i} + L)}} \right\} \tag{23}$$

Note that for  $x_{C,1} = 0$ , the first inequality in (22) is always satisfied and  $\bar{b}_1$  is determined by the second inequality. The first key result of this part of the proof is, considering (23), due  $x_{C,1} < x_{C,2}$  we have

$$\bar{b}_1 > \bar{b}_2. \tag{24}$$

Next, for the lower bounds, we have ( $i \in \{1, 2\}$ )

$$\begin{aligned}
 u^* p_0^2 - \phi \underline{b}_i |\underline{b}_i| x_{C,i} &\leq p_{\max}^2, \\
 \text{and } u^* p_0^2 - \phi \underline{b}_i |\underline{b}_i| ((u^*-1)x_{C,i} + L) &\leq p_{\max}^2.
 \end{aligned} \tag{25}$$

Since  $p_0 \leq p_{\max}$ , the lower bounds  $\underline{b}_i$  can be negative or positive, depending on  $u^*$ . Therefore we define  $\mathfrak{s} := \text{sgn}(u^* p_0^2 - p_{\max}^2)$  as indicator for the sign of the lower bounds  $\underline{b}_i$ . Thus, for  $i = 1, 2$  we have

$$\underline{b}_i = \max \left\{ \mathfrak{s} \sqrt{\frac{|u^* p_0^2 - p_{\max}^2|}{u^* \phi x_{C,i}}}, \mathfrak{s} \sqrt{\frac{|u^* p_0^2 - p_{\max}^2|}{\phi ((u^* - 1) x_{C,i} + L)}} \right\}. \quad (26)$$

As above, for  $x_{C,1} = 0$ , if the first equation in (25) is satisfied, the lower bounds are determined by the second equation in (25). If  $\mathfrak{s} = 0$  we have  $\underline{b}_1 = \underline{b}_2 = 0$ . If  $\mathfrak{s} < 0$ , for  $x_{C,1} < x_{C,2}$ , we have  $\underline{b}_1 < \underline{b}_2$ . In both cases, due to (24), this implies  $[\underline{b}_2, \bar{b}_2] \subseteq [\underline{b}_1, \bar{b}_1]$  with  $\bar{b}_1 \neq \bar{b}_2$ . Then both intervals lead to different probabilities, which is a contradiction to the optimality conditions in *Lemma 9*, so for  $\mathfrak{s} \leq 0$  the pairs  $(u^*, x_{C,1})$  and  $(u^*, x_{C,2})$  cannot both be solutions.

Hence from now on we only consider the case  $\mathfrak{s} > 0$ . For  $i \in \{1, 2\}$ , due to

$$\phi ((u^* - 1) x_{C,i} + L) = u^* \phi x_{C,i} + \underbrace{\phi (L - x_{C,i})}_{\geq 0},$$

the second terms in (26) are always smaller than or equal to the first terms. Equality is only satisfied, if  $x_{C,2} = L$ . Thus for  $\mathfrak{s} > 0$  and  $i = 1, 2$ , we have

$$\underline{b}_i = \sqrt{\frac{|u^* p_0^2 - p_{\max}^2|}{u^* \phi x_{C,i}}}. \quad (27)$$

So in addition to (24), the second key result of this part of the proof is, considering (27) and  $\mathfrak{s} > 0$ , due to  $x_{C,1} < x_{C,2}$  we have

$$\underline{b}_1 > \underline{b}_2. \quad (28)$$

This completes the first part of the proof.

*Proof of Theorem 11, Part II:* In the second part of the proof we construct a new admissible pair  $(u^*, x_{C,3})$ . Since the bounds of  $M$  depend continuously on  $x_C$ , we can choose  $x_{C,1} < x_{C,3} < x_{C,2}$ , s.t. the lower bound of  $M(u^*, x_{C,3})$  is given by  $\underline{b}_3 = \frac{\underline{b}_1 + \underline{b}_2}{2}$ . Inserting (27) this leads to

$$\underline{b}_3 = \frac{1}{2} \left( \sqrt{\frac{|u^* p_0^2 - p_{\max}^2|}{u^* \phi x_{C,1}}} + \sqrt{\frac{|u^* p_0^2 - p_{\max}^2|}{u^* \phi x_{C,2}}} \right) = \frac{1}{2} \sqrt{\frac{|u^* p_0^2 - p_{\max}^2|}{u^* \phi x_{C,1} x_{C,2}}} (\sqrt{x_{C,1}} + \sqrt{x_{C,2}}). \quad (29)$$

Further, due to (27) for  $i = 3$ , with  $\mathfrak{s} > 0$ , we have

$$\underline{b}_3 = \sqrt{\frac{|u^* p_0^2 - p_{\max}^2|}{u^* \phi x_{C,3}}},$$

and thus together with (29), it follows

$$\begin{aligned} \sqrt{\frac{|u^* p_0^2 - p_{\max}^2|}{u^* \phi x_{C,3}}} &= \frac{1}{2} \sqrt{\frac{|u^* p_0^2 - p_{\max}^2|}{u^* \phi x_{C,1} x_{C,2}}} (\sqrt{x_{C,1}} + \sqrt{x_{C,2}}) \\ \Leftrightarrow \sqrt{\frac{1}{x_{C,3}}} &= \frac{1}{2} \frac{(\sqrt{x_{C,1}} + \sqrt{x_{C,2}})}{\sqrt{x_{C,1} x_{C,2}}}. \end{aligned}$$

Then this implies

$$x_{C,3} = \frac{4 x_{C,1} x_{C,2}}{(\sqrt{x_{C,1}} + \sqrt{x_{C,2}})^2}. \quad (30)$$

Now we can compute the upper bound  $\bar{b}_3$  for  $M(u^*, x_{C,3})$ . Due to (23), we have

$$\begin{aligned} \bar{b}_3 &= \min \left\{ \sqrt{\frac{p_0^2 - p_{\min}^2}{\phi x_{C,3}}}, \sqrt{\frac{u^* p_0^2 - p_{\min}^2}{\phi ((u^* - 1) x_{C,3} + L)}} \right\} \\ &= \min \left\{ \frac{\sqrt{p_0^2 - p_{\min}^2} (\sqrt{x_{C,1}} + \sqrt{x_{C,2}})}{2 \sqrt{\phi x_{C,1} x_{C,2}}}, \frac{\sqrt{u^* p_0^2 - p_{\min}^2}}{\sqrt{\phi \left( (u^* - 1) \frac{4 x_{C,1} x_{C,2}}{(\sqrt{x_{C,1}} + \sqrt{x_{C,2}})^2} + L \right)}} \right\}. \end{aligned} \quad (31)$$

Due to (24), for  $x_{C,1} < x_{C,3} < x_{C,2}$  we have  $\bar{b}_2 < \bar{b}_3 < \bar{b}_1$ , which completes the second part of the proof.

*Proof of Theorem 11, Part III:* In this part of the proof, we now show that

$$\bar{b}_3 \geq \frac{1}{2}(\bar{b}_1 + \bar{b}_2).$$

We split this part depending on whether  $\bar{b}$  is defined by the first or second term in (31). Note that due to  $x_{C,1} < x_{C,3} < x_{C,2}$  we have  $x_{C,3} \notin \{0, L\}$  and due to  $u^* > 1$  we have  $\bar{b}_3 > 0$ .

*Proof of Theorem 11, Part III a):* Let  $\bar{b}_3$  be given by the first term in (31). Then due to  $\bar{b}_i \geq 0$  ( $i \in \{1, 2, 3\}$ ), for  $p_0^2 > p_{\min}^2$ , we have

$$\frac{p_0^2 - p_{\min}^2}{\phi x_{C,3}} \leq \frac{u^* p_0^2 - p_{\min}^2}{\phi \left( (u^* - 1) x_{C,3} + L \right)} \Leftrightarrow \frac{\phi x_{C,3}}{p_0^2 - p_{\min}^2} \geq \frac{\phi \left( (u^* - 1) x_{C,3} + L \right)}{u^* p_0^2 - p_{\min}^2}.$$

Since  $x_{C,3} < x_{C,2}$  we set  $\Delta x := x_{C,2} - x_{C,3} > 0$ . Thus we have

$$\begin{aligned} \frac{\phi (x_{C,2} - \Delta x)}{p_0^2 - p_{\min}^2} &\geq \frac{\phi \left( (u^* - 1) (x_{C,2} - \Delta x) + L \right)}{u^* p_0^2 - p_{\min}^2} \\ \Leftrightarrow \frac{\phi x_{C,2}}{p_0^2 - p_{\min}^2} &\geq \frac{\phi \left( (u^* - 1) x_{C,2} + L \right)}{u^* p_0^2 - p_{\min}^2} + \frac{\phi \Delta x}{p_0^2 - p_{\min}^2} - \frac{\phi (u^* - 1) \Delta x}{u^* p_0^2 - p_{\min}^2}. \end{aligned} \quad (32)$$

Further we have

$$\frac{\phi \Delta x}{p_0^2 - p_{\min}^2} \frac{(u^* - 1)}{(u^* - 1)} = \frac{\phi (u^* - 1) \Delta x}{u^* p_0^2 - p_{\min}^2 + \underbrace{2 p_{\min}^2 - p_0^2}_{\geq p_{\min}^2} - \underbrace{u^* p_{\min}^2}_{> p_{\min}^2}} > \frac{\phi (u^* - 1) \Delta x}{u^* p_0^2 - p_{\min}^2}. \quad (33)$$

For (32) this implies

$$\frac{\phi x_{C,2}}{p_0^2 - p_{\min}^2} > \frac{\phi \left( (u^* - 1) x_{C,2} + L \right)}{u^* p_0^2 - p_{\min}^2},$$

and thus  $\bar{b}_2$  is also defined by the first value in (23) for  $i = 2$ . For  $p_0 = p_{\min}$  the second term in (23) is positive and thus  $\bar{b}_2$  is always defined by the first value in (23). Assume that  $\bar{b}_1$  is also defined by the first value in (23) (for  $i = 1$ ). Then we have

$$\begin{aligned} \left( \frac{\bar{b}_1 + \bar{b}_2}{2} \right)^2 &= \frac{1}{4} \left( \frac{p_0^2 - p_{\min}^2}{\phi x_{C,1}} + 2 \sqrt{\frac{p_0^2 - p_{\min}^2}{\phi x_{C,1}}} \sqrt{\frac{p_0^2 - p_{\min}^2}{\phi x_{C,2}} + \frac{p_0^2 - p_{\min}^2}{\phi x_{C,2}}} \right) \\ &= \frac{1}{4} \left( \frac{(p_0^2 - p_{\min}^2) x_{C,2}}{\phi x_{C,1} x_{C,2}} + \frac{2 (p_0^2 - p_{\min}^2) \sqrt{x_{C,1} x_{C,2}}}{\phi x_{C,1} x_{C,2}} + \frac{(p_0^2 - p_{\min}^2) x_{C,1}}{\phi x_{C,1} x_{C,2}} \right) \\ &= \frac{p_0^2 - p_{\min}^2}{4 \phi x_{C,1} x_{C,2}} (\sqrt{x_{C,1}} + \sqrt{x_{C,2}})^2 \\ &= \bar{b}_3^2. \end{aligned} \quad (34)$$

If  $\bar{b}_1$  is defined by the second value in (23), then we can estimate  $\bar{b}_1$  from above by the first value in (23) and thus, if  $\bar{b}_3$  is defined by the first value in (31), then we have

$$\bar{b}_3 \geq \frac{1}{2}(\bar{b}_1 + \bar{b}_2). \quad (35)$$



*Proof of Theorem 11, Part III b):* Let  $\bar{b}_3$  be given by the second term in (31). This implies  $p_0 > p_{\min}$ . Then due to  $\bar{b}_i \geq 0$  ( $i \in \{1, 2, 3\}$ ) we have

$$\frac{p_0^2 - p_{\min}^2}{\phi x_{C,3}} \geq \frac{u^* p_0^2 - p_{\min}^2}{\phi \left( (u^* - 1) x_{C,3} + L \right)} \Leftrightarrow \frac{\phi x_{C,3}}{p_0^2 - p_{\min}^2} \leq \frac{\phi \left( (u^* - 1) x_{C,3} + L \right)}{u^* p_0^2 - p_{\min}^2}.$$

Since  $x_{C,1} < x_{C,3}$  we set  $\Delta x := x_{C,3} - x_{C,1} > 0$ . Thus we have

$$\frac{\phi \left( (u^* - 1) x_{C,1} + L \right)}{u^* p_0^2 - p_{\min}^2} \geq \frac{\phi x_{C,1}}{p_0^2 - p_{\min}^2} + \frac{\phi \Delta x}{p_0^2 - p_{\min}^2} - \frac{\phi (u^* - 1) \Delta x}{u^* p_0^2 - p_{\min}^2}.$$

Due to (33) this implies

$$\frac{\phi \left( (u^* - 1) x_{C,1} + L \right)}{u^* p_0^2 - p_{\min}^2} > \frac{\phi x_{C,1}}{p_0^2 - p_{\min}^2},$$

and thus,  $\bar{b}_1$  is also defined by the second value in (23) for  $i = 1$ . If  $\bar{b}_2$  is defined by the second value in (23), then we get (35) analogously to (34) by writing  $\bar{b}_3$  as

$$\bar{b}_3 = \frac{\sqrt{u^* p_0^2 - p_{\min}^2} (\sqrt{x_{C,1}} + \sqrt{x_{C,2}})}{\sqrt{\phi \left( (u^* - 1) 4 x_{C,1} x_{C,2} + L (\sqrt{x_{C,1}} + \sqrt{x_{C,2}})^2 \right)}}.$$

If  $\bar{b}_2$  is defined by the first value in (23) that we can estimate  $\bar{b}_2$  from above by the second value in (23) and we also get (35).

Thus, if  $\bar{b}_3$  is defined by the second value in (31), then (35) also holds. Consequently, for the upper bound  $\bar{b}_3$  of  $M(u^*, x_{C,3})$ , we always have

$$\bar{b}_3 \geq \frac{\bar{b}_1 + \bar{b}_2}{2}, \quad (36)$$

which is the key result of the third part of the proof.

*Proof of Theorem 11, Part IV:* In the last part of this proof, we show, that  $(u^*, x_{C,3})$  is an admissible pair and from this we infer a contradiction. Since  $u^* > 1$  by assumption and  $0 \leq x_{C,1} < x_{C,3} < x_{C,2} \leq L$  we only have to make sure that the probabilistic constraint in (OPT 2) is satisfied. For  $(u^*, x_{C,3})$ , we have

$$\mathbb{P}(b \in M(u^*, x_{C,3})) = \mathbb{P} \left( \begin{array}{l} p_1(L_1) \geq p_{\min} \\ p_2(0) \leq p_{\max} \\ p_2(L_2) \geq p_{\min} \\ p_2(L_2) \leq p_{\max} \end{array} \right) = \int_{\bar{b}_3}^{\bar{b}_3} \rho_{\xi}(z) dz,$$

where  $\rho_{\xi}$  is the probability density function of the Gaussian distribution. Due to (36) it follows

$$\int_{\bar{b}_3}^{\bar{b}_3} \rho_{\xi}(z) dz \geq \int_{\frac{1}{2}(\bar{b}_1 + \bar{b}_2)}^{\frac{1}{2}(\bar{b}_1 + \bar{b}_2)} \rho_{\xi}(z) dz = \mathcal{F}_{\xi} \left( \frac{1}{2} \bar{b}_1 + \frac{1}{2} \bar{b}_2 \right) - \mathcal{F}_{\xi} \left( \frac{1}{2} \bar{b}_1 + \frac{1}{2} \bar{b}_2 \right),$$

where  $\mathcal{F}_{\xi}$  is the probability distribution function of the Gaussian distribution. The distribution function  $\mathcal{F}_{\xi}(z)$  is strictly convex for  $z < \mu$  and strictly concave for  $z > \mu$ , where  $\mu$  is the expected value of the Gaussian distribution. Since we have  $\alpha > \frac{1}{2}$  by assumption, the lower bounds  $\bar{b}_1$  and  $\bar{b}_2$  are smaller than the expected value  $\mu$ , and the upper bounds  $\bar{b}_1$  and  $\bar{b}_2$  are larger than the expected value  $\mu$  (cf. Figure 3). Thus we can apply the inequalities for strict convexity on the lower bounds and for strict concavity on the upper bounds here. This implies

$$\begin{aligned} \mathcal{F}_{\xi} \left( \frac{1}{2} \bar{b}_1 + \frac{1}{2} \bar{b}_2 \right) - \mathcal{F}_{\xi} \left( \frac{1}{2} \bar{b}_1 + \frac{1}{2} \bar{b}_2 \right) &> \frac{1}{2} \mathcal{F}_{\xi}(\bar{b}_1) + \frac{1}{2} \mathcal{F}_{\xi}(\bar{b}_2) - \frac{1}{2} \mathcal{F}_{\xi}(\bar{b}_1) - \frac{1}{2} \mathcal{F}_{\xi}(\bar{b}_2) \\ &= \frac{1}{2} \left( \mathcal{F}_{\xi}(\bar{b}_1) - \mathcal{F}_{\xi}(\bar{b}_1) \right) + \frac{1}{2} \left( \mathcal{F}_{\xi}(\bar{b}_2) - \mathcal{F}_{\xi}(\bar{b}_2) \right). \end{aligned}$$

Due to *Lemma 9* the solutions  $(u^*, x_{C,1})$  and  $(u^*, x_{C,2})$  satisfy the probabilistic constraint with equality, so we have

$$\left( \mathcal{F}_\xi(\bar{b}_1) - \mathcal{F}_\xi(b_1) \right) = \left( \mathcal{F}_\xi(\bar{b}_2) - \mathcal{F}_\xi(b_2) \right) = \alpha,$$

and thus for  $(u^*, x_{C,3})$  it follows

$$\mathbb{P} \left( \begin{array}{l} p_1(L_1) \geq p_{\min} \\ p_2(0) \leq p_{\max} \\ p_2(L_2) \geq p_{\min} \\ p_2(L_2) \leq p_{\max} \end{array} \right) > \alpha.$$

So  $(u^*, x_{C,3})$  is feasible with probability larger than  $\alpha$ . Following the proof of *Lemma 9*, for  $x_{C,3}$  we can find  $\bar{u}$  with  $1 < \bar{u} < u^*$ , s.t.  $(\bar{u}, x_{C,3})$  is feasible. Thus,  $(u^*, x_{C,1})$  and  $(u^*, x_{C,2})$  cannot be solutions of (OPT 2) and we have a contradiction. So under the assumptions of *Theorem 11*, a unique solution of (OPT 2) exists and the proof is complete.  $\square$

$\square$

**Remark 12.** *The properties of the Gaussian distribution were only applied in Part IV of the proof. Theorem 11 also holds for other distributions, that allow the last step in proof (e.g., logistic distribution, Laplace distribution or chi distribution with at least two degrees of freedom), by appropriate strict convexity and concavity properties and a proper lower bound for  $\alpha$ .*

### 2.3 A Numerical Example on a Single Pipe

As an illustrative example we consider the pipe with length 30km from *Figure 2* with the data given in *Table 1*. The data in this example is based on the data of methane, the main component of natural gas. Our aim is to find the optimal control and the optimal compressor location in both the deterministic and probabilistic setting. Both settings are implemented in *MATLAB<sup>®</sup> 2019a* using the optimization routine *fmincon.m* with default settings. The probability is approximated using a *kernel density estimator approach* similarly as it was done in [42, 43].

Variable	Letter	Value	Unit
inlet pressure	$p_0$	58	bar
lower pressure bound	$p_{\min}$	40	bar
upper pressure bound	$p_{\max}$	60	bar
gas outflow (=mean value)	$b (= \mu)$	180	kg/m <sup>2</sup> s
variance	$\sigma$	9	
pipe friction coefficient	$\lambda^F$	0.1	
pipe diameter	$D$	0.5	m
pipe length	$L$	30	km
specific gas constant	$R_S$	515	J/kg K
gas temperature	$T$	293	K
probability level	$\alpha$	0.9	

Table 1: Values for the example in *Section 2.3* with lower and upper pressure bound for the whole pipe

The unique solution (cf. *Figure 2*) of the deterministic optimization problem (OPT 1) with the data of *Table 1* is given by

$$\begin{pmatrix} u_{\text{det}}^* \\ x_{C,\text{det}}^* \end{pmatrix} = \begin{pmatrix} 1.4811 \\ 9545.957 \text{ km} \end{pmatrix}.$$

As it was stated in *Corollary 6* a compressor station with smaller  $x_C$  is not possible since we have  $p_2(0) = p_{\max}$  and  $p_2(L_2) = p_{\min}$  (otherwise the admissible set would be empty). A compressor station with larger  $x_C$  is possible but this would increase the compressor cost and thus not be optimal. This is illustrated in *Figure 4*.

We now consider uncertain gas outflow. We define a Gaussian distributed random variable

$$\xi \sim \mathcal{N}(180, 9),$$

and we identify the random gas outflow  $b^\omega$  with the image of the random variable  $\xi(\omega)$  for  $\omega \in \Omega$ . The optimal solution of the deterministic problem  $(u_{\text{det}}^*, x_{C,\text{det}}^*)$  leads to infeasible pressures in the probabilistic problem for every  $\alpha \in (0, 1)$ . This can be seen as follows. Due to

$$\begin{cases} p_2(0; (u_{\text{det}}^*, x_{C,\text{det}}^*)) > p_{\max} & \text{for } b^\omega < b, \\ p_2(L_2; (u_{\text{det}}^*, x_{C,\text{det}}^*)) < p_{\min} & \text{for } b^\omega > b, \end{cases}$$

for every load  $b^\omega \neq b$  the pressure bounds are not satisfied and we have

$$\mathbb{P} \left( \begin{array}{l} p_1(L_1) \geq p_{\min} \\ p_2(0) \leq p_{\max} \\ p_2(L_2) \geq p_{\min} \\ p_2(L_2) \leq p_{\max} \end{array} \right) = 0.$$

Thus the deterministic optimal solution is not acceptable in the probabilistic setting and it is necessary to consider the probabilistic constrained optimization problem (OPT 2). As mentioned before the probabilistic constraint is evaluated using a kernel density estimator approach similarly as it was done and explained in [42, 43]. We choose 1000 samples for the random gas outflow  $b^\omega$ . The unique optimal solution of the probabilistic constrained optimization problem (OPT 2) is given by

$$\begin{pmatrix} u_{\text{prob}}^* \\ x_{C,\text{prob}}^* \end{pmatrix} = \begin{pmatrix} 1.6431 \\ 12969.569 \text{ km} \end{pmatrix}. \quad (37)$$

Even if the probabilistic optimal compressor cost is slightly larger than the deterministic optimal compressor cost, this slight increase in the cost guarantees feasibility of the pressure in 90% of all scenarios, which is shown in *Figure 4*.

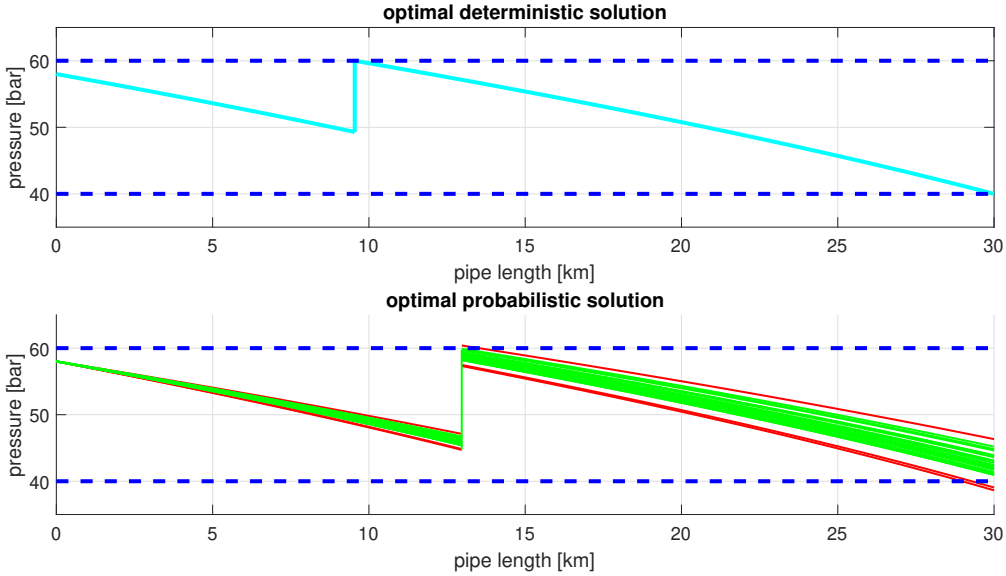


Figure 4: Pressure profile for the unique optimal solution of the deterministic problem (OPT 1) (upper picture) and pressure profiles of 25 scenarios for unique optimal solution of the probabilistic problem (OPT 2) (lower picture). In the lower picture the feasible pressure profiles are shown in green, the infeasible pressure profiles are shown in red.

### 3 Optimal Compressor Location on a Network of Pipes

In this section we consider the problem of finding optimal compressor locations on graphs under uncertain gas demand. A binary variable will indicate whether a compressor station is located at a certain pipe in the network. This leads to mixed integer non-convex optimal compressor location problems with probabilistic constraints.

#### 3.1 The Optimal Location of a Finite Number of Compressor Stations on a Graph

Consider a connected, directed graph  $\mathcal{G} = (\mathcal{V}, \mathcal{E})$  with the vertex set  $\mathcal{V} = \{v_1, \dots, v_n\}$  and the set of edges  $\mathcal{E} = \{e_1, \dots, e_m\} \subseteq \mathcal{V} \times \mathcal{V}$ . For  $i = 1, \dots, m$  let binary variables  $\delta_i \in \{0, 1\}$  be given. The binary variables  $\delta_i$  state if a compressor station is located on edge  $e_i$ . For the pressure on every edge  $e_i$  we have

$$p_{i,1}^2(x) = p_{i,1}^2(0) - \phi q_i |q_i| x \quad x \in [0, \delta_i x_{C,i}], \quad (38)$$

and

$$p_{i,2}^2(x) = \left(1 - \delta_i + \delta_i u_i\right) p_{i,1}^2(L_{i,1}) - \phi q_i |q_i| x \quad x \in [0, L - \delta_i x_{C,i}], \quad (39)$$

where  $q_i$  is the (constant) flow and  $u_i \geq 1$  is the control on edge  $e_i$ . For the convenience of the reader we assume that  $\phi$  is equal on every edge. For  $\delta_i = 0$  this implies

$$p_{i,2}^2(x) = p_{i,1}^2(0) - \phi q_i |q_i| x \quad x \in [0, L],$$

and for  $\delta_i = 1$  this implies

$$p_{i,1}^2(L_{i,1}) = p_{i,1}^2(0) - \phi q_i |q_i| x_{C,i},$$

$$p_{i,2}^2(0) = u_i p_{i,1}^2(L_{i,1}),$$

$$p_{i,2}^2(L_{i,2}) = u_i p_{i,1}^2(L_{i,1}) - \phi q_i |q_i| L_{i,2},$$

which coincides with (3) and (4). So an edge  $e_i$  without compressor station is modeled as an edge with compressor station at  $x_{C,i} = 0$ , on which the control does not influence the change in pressure. Due to this modelling, optimization later will always lead to  $u_i = 1$  (i.e., the compressor at  $x_{C,i} = 0$  is switched off), if a compressor station is not needed on edge  $e_i$ .

For every node  $v \in \mathcal{V}$  let  $\mathcal{E}_-(v)$  be the index set of all ingoing edges (i.e., edges that end in  $v$ ) and let  $\mathcal{E}_+(v)$  be the index set of all outgoing edges (i.e., edges that start in  $v$ ). Since we consider gas transport on networks in this section we need to define suitable coupling conditions at the nodes. Consider a load vector  $b \in \mathbb{R}^n$ , where  $b_i$  is the amount of gas that enters or leaves the network at node  $v_i$ . We have  $b_i < 0$  if gas enters the network and  $b_i \geq 0$  if gas leaves the network. We assume conservation of mass, i.e. for every node  $v_i \in \mathcal{V}$  ( $i = 1, \dots, n$ ) we assume

$$\sum_{j \in \mathcal{E}_-(v_i)} q_j = b_i + \sum_{j \in \mathcal{E}_+(v_i)} q_j. \quad (40)$$

Further we assume continuity in the pressures for the graph, i.e., for every node  $v_i \in \mathcal{V}$  ( $i = 1, \dots, n$ ) we assume

$$p_{j,2}(L_{j,2}) = p_{k,1}(0) \quad \forall j \in \mathcal{E}_-(v_i), k \in \mathcal{E}_+(v_i). \quad (41)$$

Every node  $v_i$  is either an inflow node, where gas enters the network (i.e.,  $b_i < 0$ ), or an outflow node, where gas leaves the network (i.e.,  $b_i \geq 0$ ). Let  $\mathcal{V}_{\text{in}} \subseteq \mathcal{V}$  be the set of all inflow nodes. For every inflow node  $v_i \in \mathcal{V}_{\text{in}}$  an inlet pressure  $p_{j,0} \in [p_{\min}, p_{\max}]$  is given, i.e. we have

$$p_{j,1}^2(0) = p_{j,0}^2 \quad \forall j \in \mathcal{E}_+(v_i). \quad (42)$$

Let  $\mathcal{V}_{\text{out}}$  be the set of outflow nodes. For every outflow node  $v_i \in \mathcal{V}_{\text{out}}$  an outflow  $b_i \geq 0$  (gas demand) is given. A similar model without the binary variables was also considered in [29] and [42]. For given numbers  $\delta_i$  ( $i = 1, \dots, m$ ) the existence of a unique stationary state follows from [25].

Let pressure bounds  $0 < p_{\min} < p_{\max}$  be given. For convenience we assume that the pressure bounds are equal on every edge, but every result in this section can easily be adapted to different pressure bounds on every edge. For a graph with  $n_C \in \{1, \dots, m\}$  compressor stations we consider the deterministic optimization problem

$$(OPT\ 3) \quad \left\{ \begin{array}{l} \min_{u, x_C, \delta} \quad \|u\|_2^2 = \sum_{i=1}^m u_i^2, \\ \text{s.t.} \quad \text{for all } i = 1, \dots, m, \text{ we have} \\ \quad p_{i,1}(L_{i,1}) \geq p_{\min}, \\ \quad p_{i,2}(0) \leq p_{\max}, \\ \quad p_{i,2}(L_{i,2}) \geq p_{\min}, \\ \quad u_i \geq 1, \\ \quad x_{C,i} \in [0, L_i], \\ \quad \delta_i \in \{0, 1\}, \\ \quad \sum_{j=1}^m \delta_j = n_C, \end{array} \right.$$

where the pressures at the nodes  $p_{i,k}$  ( $i = 1, \dots, m, k = 1, 2$ ) are given by the solution of the static model (38), (39), (40), (41) and (42).

**Lemma 13.** *For all  $v_i \in \mathcal{V}_{in}$  let  $p_{i,0} \in [p_{\min}, p_{\max}]$  be given and for all  $v_i \in \mathcal{V}_{out}$  let  $b_i \geq 0$  be given. Further let a number  $n_C \in \{0, \dots, m\}$  be given.*

- (i) *If a triple  $(u, x_C, \delta) \in \mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R}^m$  with  $u_j = 1$  for all  $j = 1, \dots, m$  satisfies the constraints of (OPT 3), every triple  $(\mathbb{1}_m, x_C, \delta)$  with  $x_{C,j} \in [0, L_j]$ ,  $\delta_j \in \{0, 1\}$  and  $\sum_{j=1}^m \delta_j = n_C$  is a solution of (OPT 3).*
- (ii) *If there exists a triple  $(u, x_C, \delta) \in \mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R}^m$  that satisfies the constraints in (OPT 3), and if  $(\mathbb{1}_m, x_C, \delta)$  is infeasible for at least one pair  $(x_C, \delta)$  with  $x_{C,j} \in [0, L_j]$  and  $\delta_j \in \{0, 1\}$ , the optimization problem (OPT 3) has at least one solution.*

*Proof. Proof of Lemma 13.*

(i) The first statement of the proof is similar to the first statement of *Theorem 10*. It means that compressor stations are not needed and thus their location can be arbitrary.

(ii) In the second statement we show that if the admissible set is nonempty, then there exists at least one solution of (OPT 3). Let

$$\mathcal{D} := \left\{ \delta \in \{0, 1\}^m \mid \sum_{j=1}^m \delta_j = n_C \right\}, \quad (43)$$

with  $|\mathcal{D}| = n_D$  be the set of all possible distributions of  $n_C$  compressor stations on the graph. We define the admissible set  $\mathcal{A}_k$  ( $k = 1, \dots, n_D$ ) for a fixed distribution of compressor stations on the graph  $\delta \in \mathcal{D}$  as

$$\mathcal{A}_k := \left\{ (u, x_C) \mid \text{The constraints of (OPT 3) are satisfied} \right\} \quad k = 1, \dots, n_D,$$

and we define the admissible set for the whole optimization problem (OPT 3) as

$$\mathcal{A} := \bigcup_{k=1}^{n_D} \mathcal{A}_k.$$

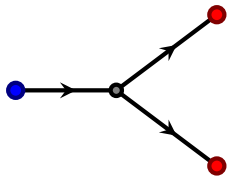
The admissible set  $\mathcal{A}$  is nonempty by assumption, i.e.,  $\mathcal{A}_k$  is nonempty for at least one  $k \in \{1, \dots, n_D\}$ .

We now show that  $\mathcal{A}$  is compact. For every  $k = 1, \dots, n_D$  due to the constraint  $u_j \geq 1$  ( $j = 1, \dots, m$ ) and due to the pressure bounds, for the controls we have

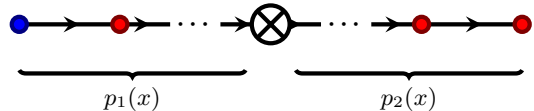
$$u_j \in \left[ 1, \frac{p_{\max}^2}{p_{\min}^2} \right].$$

Further the compressor locations are bounded by 0 and by the lengths of the pipes, i.e.,  $x_{C,j} \in [0, L_j]$  ( $j = 1, \dots, m$ ). So the sets  $\mathcal{A}_k$  are bounded for every  $k \in \{1, \dots, n_D\}$  and thus,  $\mathcal{A}$  is bounded. Analogously to *Theorem 10*, for a fixed distribution  $\delta \in \mathcal{D}$  of compressor stations on the graph, the sets  $\mathcal{A}_k$  are closed. Consequently, the union  $\mathcal{A}$  is closed as well and the existence of a solution follows by the *extreme value theorem* due to the continuity of the objective function.  $\square$

The uniqueness of a solution mainly depends on the topology of the graph. For the graph shown in *Figure 5a* with  $n_C = 2$  one can easily construct a scenario in which a single compressor station on the first edge is needed to guarantee the pressure bounds, but due to  $n_C = 2$  the second compressor station is switched off and can be located anywhere on the second or third edge, so we do not have a unique solution. On the other hand if a point  $(u, x_C, \delta)$  with  $u_j = 1$  for some  $j \in \{1, \dots, m\}$  (but not all, cf. *Lemma 13*) is feasible, nevertheless, a unique solution might exist, depending on the graph and the scenario. We refer to [27], where the authors analyze the properties of the admissible set depending on the graph topology. For linear graphs with a single compressor station we can formulate a statement for the uniqueness of a solution. By considering the linear graph as a single edge we can consider the subgraph in front of the compressor as  $p_1(x)$  and the subgraph behind the compressor as  $p_2(x)$  as it is shown in *Figure 5b* and thus we can apply the results of *Lemma 5* and *Corollary 7*.



(a) Scheme of a symmetric graph with one source and two sinks



(b) Scheme of a linear graph with one source and  $n$  sinks

Figure 5: A scheme of a symmetric and a linear graph, source nodes (gas supply) are drawn in blue and sink nodes (gas demand) are drawn in red

Another interesting question is to find the optimal number of compressor stations and their locations, s.t. the control cost is minimal. This implies, that  $n_C$  is also an optimization variable, i.e., we consider the optimization problem

$$\text{(OPT 4)} \quad \left\{ \begin{array}{l} \min_{u, x_C, \delta, n_C} \quad \|u\|_2^2 = \sum_{i=1}^m u_i^2, \\ \text{s.t.} \quad \text{for all } i = 1, \dots, m, \text{ we have} \\ \quad p_{i,1}(L_{i,1}) \geq p_{\min}, \\ \quad p_{i,2}(0) \leq p_{\max}, \\ \quad p_{i,2}(L_{i,2}) \geq p_{\min}, \\ \quad u_i \geq 1, \\ \quad x_{C,i} \in [0, L_i], \\ \quad \delta_i \in \{0, 1\}, \\ \quad \sum_{j=1}^m \delta_j = n_C. \end{array} \right.$$

The existence of a solution for the optimization problem (OPT 4) directly follows from the statements of *Lemma 13* by defining the set  $\mathcal{D}$  (stated in (43)) as  $\{0, 1\}^m$ . Thus we can state the following result:

**Corollary 14.** For all  $v_i \in \mathcal{V}_{in}$  let  $p_{i,0} \in [p_{\min}, p_{\max}]$  be given and for all  $v_i \in \mathcal{V}_{out}$  let  $b_i \geq 0$  be given.

- (i) If a quadruple  $(u, x_C, \delta, n_C) \in \mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R}^m \times \{0, \dots, m\}$  with  $u_j = 1$  for all  $j = 1, \dots, m$  satisfies the constraints of (OPT 4), every quadruple  $(\mathbb{1}_m, x_C, \delta, n_C)$  with  $x_{C,j} \in [0, L_j]$ ,  $\delta_j \in \{0, 1\}$ ,  $n_C \in \{0, \dots, m\}$  and  $\sum_{j=1}^m \delta_j = n_C$  is a solution of (OPT 4).

(ii) If there exists a quadruple  $(u, x_C, \delta, n_C) \in \mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R}^m \times \{0, \dots, m\}$ , that satisfies the constraints in (OPT 4), and if  $(\mathbb{1}_m, x_C, \delta, n_C)$  is infeasible for at least one triple  $(x_C, \delta, n_C)$  with  $x_{C,j} \in [0, L_j]$ ,  $\delta_j \in \{0, 1\}$  and  $n_C \in \{0, \dots, m\}$ , the optimization problem (OPT 4) has at least one solution.

**Remark 15.** The assumption in statement (i) of Corollary 14 is equivalent to the assumption that there exists a quadruple  $(u, x_C, \delta, n_C)$  with  $n_C = 0$ , that satisfies the constraints of (OPT 4). This also leads to the fact, that the bounds for the pressure are satisfied without compressor stations.

As before the uniqueness of a solution cannot be guaranteed since it mainly depends on the topology of the graph (cf. Figure 5a and Figure 5b). The objective to find the minimal number of compressor stations on a graph without considering the control cost leads to the optimization problem

$$(OPT 5) \quad \left\{ \begin{array}{l} \min_{u, x_C, \delta, n_C} \quad n_C, \\ \text{s.t.} \quad \text{for all } i = 1, \dots, m, \text{ we have} \\ p_{i,1}(L_{i,1}) \geq p_{\min}, \\ p_{i,2}(0) \leq p_{\max}, \\ p_{i,2}(L_{i,2}) \geq p_{\min}, \\ u_i \geq 1, \\ x_{C,i} \in [0, L_i], \\ \delta_i \in \{0, 1\}, \\ \sum_{j=1}^m \delta_j = n_C. \end{array} \right.$$

The following lemma guarantees the existence of a solution of (OPT 5).

**Lemma 16.** For all  $v_i \in \mathcal{V}_{in}$  let  $p_{i,0} \in [p_{\min}, p_{\max}]$  be given and for all  $v_i \in \mathcal{V}_{out}$  let  $b_i \geq 0$  be given. If there exists a quadruple  $(u, x_C, \delta, n_C) \in \mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R}^m \times \{0, \dots, m\}$ , that satisfies the constraints in (OPT 5), the optimization problem (OPT 5) has a solution.

*Proof.* Proof of Lemma 16. Similar as in the proof of the second statement of Lemma 13 we define the set

$$\mathcal{D}_j := \{ \delta \in \{0, 1\}^m \mid \sum_{i=1}^m \delta_i = j \} \quad j = 1, \dots, m,$$

with  $|\mathcal{D}_j| = n_{D,j}$  as the set of all possible distributions of  $j$  compressor stations on the graph. We define the admissible sets  $\mathcal{A}_{j,k}$  for a fix distribution of compressor stations on the graph as

$$\mathcal{A}_{j,k} := \{ (u, x_C) \mid \text{The constraints of (OPT 5) are satisfied} \} \quad k = 1, \dots, n_{D,j}.$$

Then the admissible set

$$\mathcal{A} := \bigcup_{j=0}^m \bigcup_{k=1}^{n_{D,j}} \mathcal{A}_{j,k},$$

for the optimization problem (OPT 5) is nonempty by assumption. Thus

$$n_C^* := \min \{ j = 0, \dots, m \mid \mathcal{A}_{j,k} \neq \emptyset \text{ for at least one } k \in \{1, \dots, n_{D,j}\} \}$$

is a solution of (OPT 5). □

□

**Remark 17.** If statement (i) in Corollary 14 is satisfied, a solution of (OPT 5) is given by  $n_C = 0$ .

**Remark 18.** In general, a solution  $n_C^*$  of (OPT 5) is not unique, since every quadruple  $(u, x_C, \delta, n_C) \in \mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R}^m \times \{0, \dots, m\}$  with  $n_C = n_C^*$ , that satisfies the constraints of (OPT 5), is a solution of (OPT 5) (cf. Figure 4).

In the next part of this section we consider random gas outflow at every node  $v_i \in \mathcal{V}_{\text{out}}$ . Let  $n_{\text{out}}$  be the number of exit nodes, i.e.,  $|\mathcal{V}_{\text{out}}| = n_{\text{out}}$ . As in *Section 2* we consider gas outflow that is Gaussian distributed. For mean value  $\mu \in \mathbb{R}_+^{n_{\text{out}}}$  and covariance matrix  $\Sigma \in \mathbb{R}^{n_{\text{out}} \times n_{\text{out}}}$ , consider the  $n_{\text{out}}$ -dimensional Gaussian random variable

$$\xi \sim \mathcal{N}(\mu, \Sigma),$$

on an appropriate probability space  $(\Omega, \mathcal{A}, \mathbb{P})$ . We identify the random gas outflow  $b^\omega \in \mathbb{R}^{n_{\text{out}}}$  with the image  $\xi(\omega)$  for  $\omega \in \Omega$  on this probability space. Thus also the pressures are random. We mention again that we write  $p$  and  $b$  instead of  $p^\omega$  and  $b^\omega$  for the random pressures and random loads. Let  $\rho_\xi$  be the probability density function of  $\xi$ . In contrast to *Lemma 13*, *Corollary 14* and *Lemma 16*, due to the Gaussian distribution, the gas outflow can be negative. This might lead to negative gas flow in a pipe, which increases the gas pressure. So we additionally have to guarantee, that the pressure at the end of every pipe is smaller than or equal to the upper pressure bound (cf. (17) and *Remark 8*). For  $\alpha \in (0, 1)$ , the probabilistic constrained optimization problem corresponding to the deterministic problem (OPT 3) is given by

$$\text{(OPT 6)} \quad \left\{ \begin{array}{l} \min_{u, x_C, \delta} \quad \|u\|_2^2 = \sum_{i=1}^m u_i^2, \\ \text{s.t.} \quad \mathbb{P} \left( \begin{array}{l} p_{k,1}(L_{k,1}) \geq p_{\min} \\ p_{k,2}(0) \leq p_{\max} \\ p_{k,2}(L_{k,2}) \geq p_{\min} \\ p_{k,2}(L_{k,2}) \leq p_{\max} \end{array} \quad \forall k = 1, \dots, m \right) \geq \alpha, \\ \text{and for all } i = 1, \dots, m, \text{ we have} \\ u_i \geq 1, \\ x_{C,i} \in [0, L_i], \\ \delta_i \in \{0, 1\}, \\ \sum_{j=1}^m \delta_j = n_C, \end{array} \right.$$

where the pressures  $p_{k,1}, p_{k,2}$  ( $k = 1, \dots, m$ ) are given by the solution of the model (38), (39), (40), (41) and (42) with respect to the random outflow  $b$ . Even if the probabilistic constrained optimization problems (OPT 6) contains binary variables, it has similar properties as (OPT 2).

**Lemma 19.** *If  $(u^*, x_C^*, \delta)$  is a solution of (OPT 6) such that  $u_i > 1$  for at least one  $i \in \{1, \dots, n_C\}$ , then we have*

$$\mathbb{P} \left( \begin{array}{l} p_{k,1}(L_{k,1}) \geq p_{\min} \\ p_{k,2}(0) \leq p_{\max} \\ p_{k,2}(L_{k,2}) \geq p_{\min} \\ p_{k,2}(L_{k,2}) \leq p_{\max} \end{array} \quad \forall k = 1, \dots, m \right) = \alpha,$$

that is the probabilistic inequality constraint is active.

*Proof.* *Proof of Lemma 19.* The proof is analogous to the proof of *Lemma 9*. We define the set

$$M(u^*, x_C^*) := \left\{ b \in \mathbb{R}^{|\mathcal{V}_{\text{out}}|} \left| \begin{array}{l} p_{k,1}(L_{k,1}) \geq p_{\min} \\ p_{k,2}(0) \leq p_{\max} \\ p_{k,2}(L_{k,2}) \geq p_{\min} \\ p_{k,2}(L_{k,2}) \leq p_{\max} \end{array} \quad \forall k = 1, \dots, m \right. \right\}, \quad (44)$$

and the equations (18) hold on every edge. Then the proof by contradiction of *Lemma 9* equivalently holds here.  $\square$

The next theorem guarantees, that if the admissible set of (OPT 6) is nonempty, a solution of (OPT 6) exists. Note that  $\mathbb{1}_m$  is the vector of all ones with length  $m$ .

**Theorem 20.** *Consider the probabilistic constrained optimization problem (OPT 6) and let  $p_{i,0} \in [p_{\min}, p_{\max}]$  be given for every node  $v_i \in \mathcal{V}_{\text{in}}$ . Further let a number  $n_C \in \{0, \dots, m\}$  be given.*



- (i) If there exists a triple  $(u, x_C, \delta) \in \mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R}^m$  with  $u = \mathbb{1}_m$ , that satisfies the constraints of (OPT 6), then every triple  $(\mathbb{1}_m, x_C, \delta)$  with  $x_{C,j} \in [0, L_j]$ ,  $\delta_j \in \{0, 1\}$  and  $\sum_{j=1}^m \delta_j = n_C$  is a solution of (OPT 6).
- (ii) If there exists a triple  $(u, x_C, \delta) \in \mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R}^m$ , that satisfies the constraints of (OPT 6), and if  $(\mathbb{1}_m, x_C, \delta)$  is infeasible for at least one pair  $(x_C, \delta)$  with  $x_{C,j} \in [0, L_j]$  and  $\delta \in \{0, 1\}$ , then (OPT 6) has at least one solution.

*Proof. Proof of Theorem 20.*

(i) The first statement of the proof coincides with the first statements in *Theorem 10* and *Lemma 13*. It means that compressor stations are not needed and thus their location can be arbitrary.

(ii) We define the set

$$\mathcal{D} := \{ \delta \in \{0, 1\}^m \mid \sum_{j=1}^m \delta_j = n_C \},$$

with  $|\mathcal{D}| = n_D$ . We define the (local) admissible set  $\mathcal{A}_k$  ( $k = 1, \dots, n_D$ ) for a fixed distribution of compressor stations on the graph  $\delta \in \mathcal{D}$  as

$$\mathcal{A}_k := \{ (u, x_C) \mid \text{The constraints of (OPT 6) are satisfied} \},$$

where the probability in (OPT 6) can equivalently be computed as

$$\mathbb{P}(b \in M_k(u, x_C)),$$

and  $M_k(u, x_C)$  is defined as in (44) for every compressor distribution  $k = 1, \dots, n_D$ . Then we can show the compactness of the  $\mathcal{A}_k$  equivalently to the proof of *Theorem 10*. Thus the (global) admissible set

$$\mathcal{A} := \bigcup_{k=1}^{n_D} \mathcal{A}_k,$$

as union of compact intervals is also compact and consequently, by the *extreme value theorem*, there exists a solution of (OPT 6). □

As discussed before, the uniqueness of a solution mainly depends on the graph topology and on the probability distribution. Even if the uniqueness result in *Theorem 11* holds for a Gaussian distribution on a single edge, the infinite variety in graph topology makes it impossible to state a general uniqueness result for the solution of (OPT 6). If we restrict the graph topology to linear graphs (see *Figure 5b*), then we can apply *Theorem 11*, but not allowing junctions in a graph is a rather strict assumption.

For the sake of completeness we also consider the probabilistic constrained optimization problems corresponding to (OPT 4) and (OPT 5) where the number of compressor stations appears as a decision variable, even if the following information does not provide any new insights on the topic. For  $\alpha \in (0, 1)$ , the probabilistic constrained optimization problem corresponding to the optimization problem (OPT 4) is given by

$$(OPT 7) \quad \left\{ \begin{array}{l} \min_{u, x_C, \delta, n_C} \quad \|u\|_2^2 = \sum_{i=1}^m u_i^2, \\ \text{s.t.} \quad \mathbb{P} \left( \begin{array}{l} p_{k,1}(L_{k,1}) \geq p_{\min} \\ p_{k,2}(0) \leq p_{\max} \\ p_{k,2}(L_{k,2}) \geq p_{\min} \\ p_{k,2}(L_{k,2}) \leq p_{\max} \end{array} \quad \forall k = 1, \dots, m \right) \geq \alpha, \\ \text{and for all } i = 1, \dots, m, \text{ we have} \\ u_i \geq 1, \\ x_{C,i} \in [0, L_i], \\ \delta_i \in \{0, 1\}, \\ \sum_{j=1}^m \delta_j = n_C. \end{array} \right.$$

The existence of a solution can be shown equivalently to *Theorem 20* by defining the set  $\mathcal{D}$  as  $\{0, 1\}^m$ . Thus we can state the following result.

**Corollary 21.** *Consider the probabilistic constrained optimization problem (OPT 7) and let  $p_{i,0} \in [p_{\min}, p_{\max}]$  be given for every node  $v_i \in \mathcal{V}_{in}$ .*

- (i) *If a quadruple  $(u, x_C, \delta, n_C) \in \mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R}^m \times \{0, \dots, m\}$  with  $u_j = 1$  for all  $j = 1, \dots, m$  satisfies the constraints of (OPT 7), every quadruple  $(\mathbb{1}_m, x_C, \delta, n_C)$  with  $x_{C,j} \in [0, L_j]$ ,  $\delta_j \in \{0, 1\}$ ,  $n_C \in \{0, \dots, m\}$  and  $\sum_{j=1}^m \delta_j = n_C$  is a solution of (OPT 7).*
- (ii) *If there exists a quadruple  $(u, x_C, \delta, n_C) \in \mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R}^m \times \{0, \dots, m\}$ , that satisfies the constraints in (OPT 7), and if  $(\mathbb{1}_m, x_C, \delta, n_C)$  is infeasible for at least one triple  $(x_C, \delta, n_C)$  with  $x_{C,j} \in [0, L_j]$ ,  $\delta_j \in \{0, 1\}$  and  $n_C \in \{0, \dots, m\}$ , the optimization problem (OPT 7) has at least one solution.*

To complete this section, for  $\alpha \in (0, 1)$  we consider the probabilistic constrained optimization problem corresponding to the optimization problem (OPT 5), which is given by

$$(OPT\ 8) \quad \left\{ \begin{array}{l} \min_{u, x_C, \delta, n_C} \quad n_C, \\ \text{s.t.} \quad \mathbb{P} \left( \begin{array}{l} p_{k,1}(L_{k,1}) \geq p_{\min} \\ p_{k,2}(0) \leq p_{\max} \\ p_{k,2}(L_{k,2}) \geq p_{\min} \\ p_{k,2}(L_{k,2}) \leq p_{\max} \end{array} \quad \forall k = 1, \dots, m \right) \geq \alpha, \\ \text{and for all } i = 1, \dots, m, \text{ we have} \\ u_i \geq 1, \\ x_{C,i} \in [0, L_i], \\ \delta_i \in \{0, 1\}, \\ \sum_{j=1}^m \delta_j = n_C. \end{array} \right.$$

For this problem, equivalent to *Lemma 16*, we can find solution, that in general is non unique (cf. *Remark 18*):

**Corollary 22.** *For all  $v_i \in \mathcal{V}_{in}$  let  $p_{i,0} \in [p_{\min}, p_{\max}]$  be given. If there exists a quadruple  $(u, x_C, \delta, n_C) \in \mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R}^m \times \{0, \dots, m\}$ , that satisfies the constraints in (34), the optimization problem (34) has a solution.*

### 3.2 Numerics and Implementation

In this section we give some reformulations of the problem to make it convenient for implementation.

Consider a vector of pressures at the nodes  $p \in \mathbb{R}^n$ , a vector of (constant) flows in the edges  $q \in \mathbb{R}^m$ , a vector of compressor locations  $x_C \in \mathbb{R}^m$ , a control  $u \in \mathbb{R}^m$ , a vector of gas inflows and outflows  $b \in \mathbb{R}^n$  (load vector) and a vector of binary variables  $\delta \in \{0, 1\}^m$ . Let  $A \in \mathbb{R}^{n \times m}$  with

$$A_{i,j} = \begin{cases} -1 & j \in \mathcal{E}_+(v_i) \\ 1 & j \in \mathcal{E}_-(v_i) \\ 0 & \text{else} \end{cases} \quad (45)$$

be the incidence matrix of the graph  $G$ . Then the conservation of mass (40) can equivalently be written as

$$A q = b,$$

where  $q \in \mathbb{R}^m$  is the vector of (constant) flows on every edge and  $b \in \mathbb{R}^n$  is the vector of loads at the nodes, i.e., the amount of gas that enters or leaves the network at the nodes.

The set of edges  $\mathcal{E}$  can be represented by an index matrix  $E \in \mathbb{N}^{2 \times m}$ , where edge  $e_k$  ( $k = 1, \dots, m$ ) connects the nodes  $v_{E_{1,k}}$  and  $v_{E_{2,k}}$ , starting in  $v_{E_{1,k}}$ . For an implementation, the state constraints for  $p_{k,1}(L_{k,1})$  and  $p_{k,2}(0)$  in (OPT 3), (OPT 4) and (OPT 5) with pressures given in (38) can be written as

$$\begin{aligned} p_{E_{1,k}}^2 - \phi q_k |q_k| \delta_k x_{C,k} &\geq p_{\min}^2, \\ \text{and } u_k \left[ p_{E_{1,k}}^2 - \phi q_k |q_k| \delta_k x_{C,k} \right] &\leq p_{\max}^2. \end{aligned}$$

Using the *Hadamard product*  $\circ$  (element-wise product), we have

$$\begin{aligned} p_{E_{1,:}}^2 - \phi q \circ |q| \circ \delta \circ x_C &\geq p_{\min}^2, \\ \text{and } u \circ \left[ p_{E_{1,:}}^2 - \phi q \circ |q| \circ \delta \circ x_C \right] &\leq p_{\max}^2, \end{aligned}$$

where  $p \in \mathbb{R}^n$  is the vector of pressures at the nodes. The inequalities have to be understood component-by-component. For the pressure  $p_{k,2}(L_{k,2})$  in (OPT 3), (OPT 4) and (OPT 5), by inserting (38) in (39), we get

$$\begin{aligned} p_{k,2}^2(L_{k,2}) &= \left(1 - \delta_k + \delta_k u_k\right) \left[ p_{k,1}^2(0) - \phi q_k |q_k| \delta_k x_{C,k} \right] - \phi q_k |q_k| (L_k - \delta_k x_{C,k}) \\ &= \left(1 - \delta_k + \delta_k u_k\right) p_{k,1}^2 - \phi q_k |q_k| \left[ L_k - \delta_k^2 x_{C,k} + \delta_k^2 u_k x_{C,k} \right]. \end{aligned}$$

Due to  $\delta_k^2 = \delta_k$  we have

$$p_{k,2}^2(L_{k,2}) = \left(1 - \delta_k + \delta_k u_k\right) p_{k,1}^2 - \phi q_k |q_k| \left[ \delta_k (u_k - 1) x_{C,k} + L_k \right], \quad (46)$$

which is equivalent to (2) for  $\delta_k = 0$  and to (4) for  $\delta_k = 1$ . For the implementation, using the index matrix  $E$ , this leads to

$$p_{E_{2,k}}^2 = \left(1 - \delta_k + \delta_k u_k\right) p_{E_{1,k}}^2 - \phi q_k |q_k| \left[ \delta_k (u_k - 1) x_{C,k} + L_k \right],$$

resp. using the *Hadamard product*, this leads to

$$p_{E_{2,:}}^2 = \left(1 - \delta + \delta \circ u\right) \circ p_{E_{1,:}}^2 - \phi q_k \circ |q_k| \circ \left[ \delta \circ (u - 1) \circ x_C + L_k \right]. \quad (47)$$

With (47) the vector of pressures at the nodes  $p \in \mathbb{R}^n$  can be computed. Since the inlet pressures satisfy the pressure bounds by assumption, the state constraint for  $p_{k,2}(L_{k,2})$  in (OPT 3), (OPT 4) and (OPT 5) can be written as

$$p \geq p_{\min}.$$

The last constraint in (OPT 3), (OPT 4) and (OPT 5) determines the number of compressor stations on the graph. It can be written as

$$\mathbb{1}_m^\top \delta = n_C.$$

To complete this section, for the computation of the probability in (OPT 6), (OPT 7) and (OPT 8), we refer again to [42, 43], where the authors use a kernel density estimator approach to estimate the probability distribution of the pressures at the nodes. The advantages of the kernel density estimator approach (e.g., compared to a Monte-Carlo approach) are specifically discussed in [42].

### 3.3 An Example on a Diamond Graph

In this section we consider a detailed example on a diamond graph (see *Figure 6*). A diamond graph is often considered as a meaningful example in the context of gas networks since it contains intertwined cycles even if the number of edges is small (see e.g., [7, 17, 25, 28]).

The data for this example is given in *Table 2*. Note that pipe friction, pipe diameter and the gas temperature are equal on every edge. The pipe length is 10km for every pipe except the vertical pipe, which has length  $\sqrt{2} \cdot 10$ km. The three values for the gas outflow are given for the upper source (firstly),

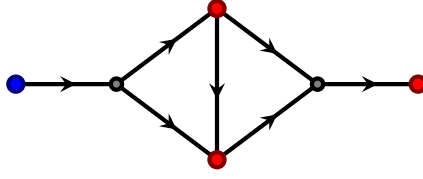


Figure 6: Scheme of the diamond graph with source (blue) and sinks (red)

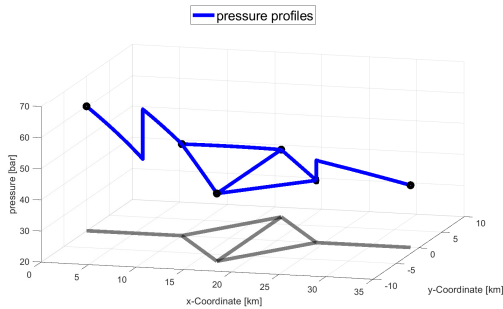
Variable	Letter	Value	Unit
inlet pressure	$p_0$	60	bar
lower pressure bound	$p_{\min}$	40	bar
upper pressure bound	$p_{\max}$	60	bar
gas outflow (=mean value)	$b (= \mu)$	[90, 60, 120]	kg/m <sup>2</sup> s
covariance matrix	$\Sigma$	diag(2.25, 2.25, 2.25)	
speed of sound in the gas	$c$	343	m/s
pipe friction coefficient	$\lambda^F$	0.1	
pipe diameter	$D$	0.5	m
specific gas constant	$R_S$	515	J/kg K
gas temperature	$T$	293	K
probability level	$\alpha$	0.8	

Table 2: Values for the example on the diamond graph

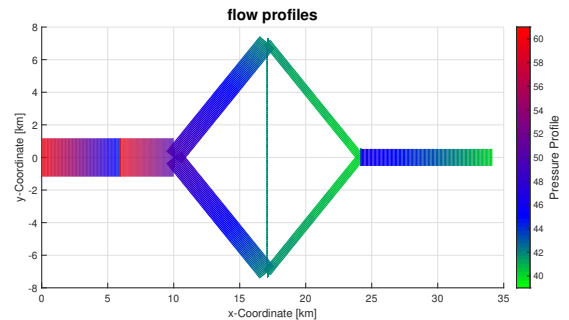
for the lower source (secondly) and for right source (thirdly).

In the deterministic case we first solve the optimization problem (OPT 3) for  $n_C = 2$ . The problem is solved using the *AMPL* software with the *Gurobi* solver. The results (visualized with *MATLAB*<sup>®</sup> 2019a) are shown in *Figure 7*. The locations of the pressure jumps indicate that the optimal location for the compressors is on the pipes outside the cycle. The solution and the corresponding locations are given by

$$u_{\det}^* = \begin{pmatrix} 1.8620 \\ 1.3484 \end{pmatrix} \quad \text{and} \quad x_{C,\det}^* = \begin{pmatrix} 5906 \\ 0 \end{pmatrix}.$$



(a) Pressure profile on the diamond graph



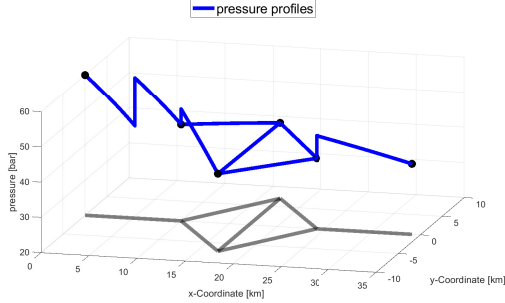
(b) Flow profile on the diamond graph

Figure 7: Solution of the deterministic problem (OPT 3) with  $n_C = 2$

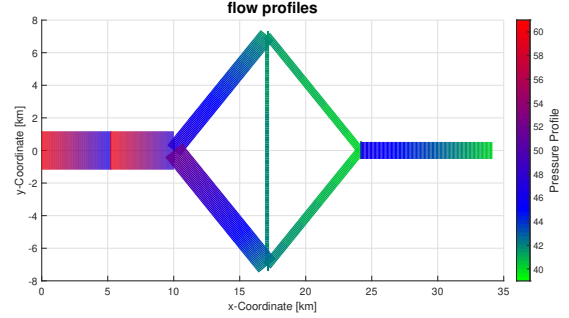
We can see similarities to the analysis of a single pipe in *Section 2*. Both compressors are located as early as possible on their edges. This means, that the first compressor steers the pressure up to the upper bound of 60bar. The second compressor is located directly at the beginning of the edge, increasing the pressure as much as necessary, such that the pressure at the end of the pipe meets the lower bound of 40bar.

The next result shows the solution of (OPT 3) for  $n_C = 3$  (see *Figure 8*), which is also the solution of (OPT 4). The solution is given by

$$u_{\text{det}}^* = \begin{pmatrix} 1.6855 \\ 1.1933 \\ 1.3484 \end{pmatrix} \quad \text{and} \quad x_{C,\text{det}}^* = \begin{pmatrix} 5189 \\ 0 \\ 0 \end{pmatrix}.$$



(a) Pressure profile on the diamond graph



(b) Flow profile on the diamond graph

Figure 8: Solution of the deterministic problem (OPT 4)

Here a third compressor station operates within the cycle of the graph. This leads to the fact that the first compressor operates weaker and earlier than in the previous example, still as early as possible. The second and the third compressor stations are located at the beginning of the pipe, respectively, and they increase the pressure sufficiently much, such that the lower pressure bound is met exactly at the end of the graph.

In the probabilistic setting, the existence of a solution strongly depends on the probability level  $\alpha$  and on the covariance matrix  $\Sigma$ . For small  $\alpha$  and small covariance matrices we can find a solution of (OPT 6) for  $n_C = 2$ , but for  $\alpha = 0.8$  the optimization problem (OPT 6) has no solution. This coincides with the result of *Theorem 20*, since in this case, the admissible set is empty. For  $\alpha = 0.8$ , for the covariance matrix given in *Table 2* and for  $n_C = 3$  the solution of (OPT 6) is given by

$$u_{\text{prob}}^* = \begin{pmatrix} 1.8498 \\ 1.3993 \\ 1.3385 \end{pmatrix} \quad \text{and} \quad x_{C,\text{prob}}^* = \begin{pmatrix} 6119 \\ 0 \\ 0 \end{pmatrix},$$

which is also the solution for (OPT 7). The probability is evaluated using a kernel density estimator approach with 1000 samples (see [42, 43]). The corresponding pressure profiles for the first 25 samples are shown in *Figure 9*. The result looks similar to the previous result from the deterministic case, but all compressors have to operate more strongly in order to prevent infeasibilities caused by the fluctuations in the uncertain gas outflow. Thus the first compressor is located later on the first edge than in the deterministic case. Besides, due to the probabilistic setting, no pressure bound is met exactly. If we further increase  $\alpha$ , then also  $n_C = 3$  is no solution of (OPT 7) and (OPT 8) anymore, which is the case e.g., for  $\alpha = 0.9$ . In this case, at least four compressors are needed.

## 4 Discussion of Robustness for Transient Scenarios

In this section we discuss the probabilistic robustness of the steady state solution in a dynamic setting. The probabilistic robustness is given by the probability, that the pressure bounds are satisfied for all times  $t$  in an interval  $[0, T]$ . This type of 'proburst' constraints is discussed in [1]. Consider the isothermal

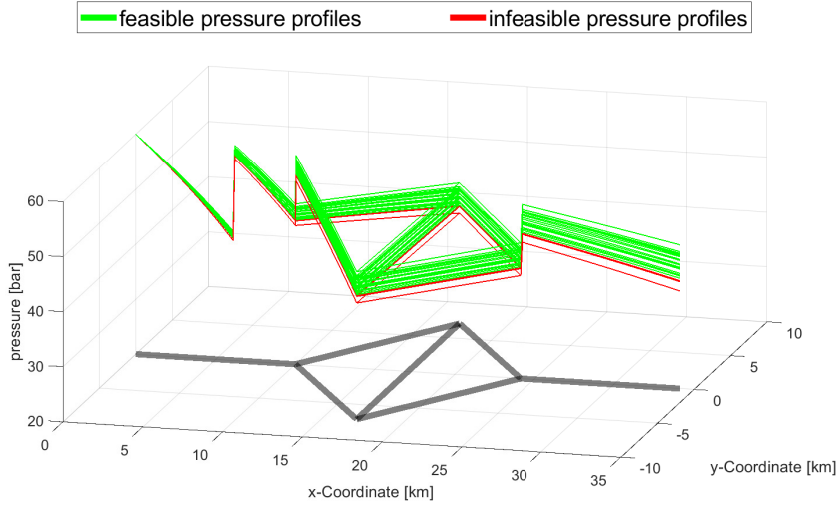


Figure 9: Pressure profiles for 25 random scenarios on the diamond graph, feasible scenarios are shown in green, infeasible scenarios are shown in red

Euler equations (see e.g., [6, 31, 32, 40]) with initial condition, inlet density and random gas outflow

$$\begin{aligned}
 \rho(0, x) &= \rho_{\text{ini}}(x), \quad q(0, x) = q_{\text{ini}}(x), \\
 \rho_t + q_x &= 0, \\
 q_t + \left( p + \frac{q^2}{\rho} \right)_x &= -\frac{\lambda}{2D} \frac{q|q|}{\rho}, \\
 \rho(t, 0) &= \rho_0(t), \quad q(t, L) = b^\omega(t),
 \end{aligned}$$

where  $\rho = \rho(t, x)$  is the gas density,  $p = p(t, x)$  is the gas pressure,  $q = q(t, x)$  is the gas flow,  $\lambda > 0$  is the pipe friction coefficient and  $D > 0$  is the pipe diameter. We consider the example from *Section 2.3* with the data given in *Table 1*. The random gas outflow is modeled by a constant flow with perturbations. Consider the (constant) outflow  $b$  from *Table 1* with a perturbation, that is given by a *Wiener process*  $(W_t)_{t \geq 0}$  (see e.g., [36, 41]). We have

$$b^\omega(t) = b + W_t.$$

For independent standard normal distributed random numbers  $\xi_1, \xi_2, \dots$  the Wiener process  $W_t$  is given by the random Fourier series

$$W_t = \sqrt{2T} \sum_{k=1}^{\infty} \xi_k \frac{\sin\left(\left(k - \frac{1}{2}\right)\pi \frac{t}{T}\right)}{\left(k - \frac{1}{2}\right)},$$

which implies  $E[W_t] = 0$  and  $\text{Var}[W_t] = t$ . An excellent overview about probability theory and stochastic processes can be found in [44]. A sample of 100 outflow scenarios is shown in *Figure 10*, in which the Fourier series was cut after ten terms.

We simulate the isothermal Euler equations on a time horizon of  $T = 12$  hours for the 100 random boundary scenarios given in *Figure 10*, where the compressor control and the compressor location is given by the optimal solution (37) of the stationary probabilistic optimization problem (OPT 2) with probability level  $\alpha = 0.9$ . The simulation results are shown in *Figure 11* in the *Appendix*. The probabilistic robustness is computed by a kernel density estimator approach similar as it is done in [42, 43]. Here this leads to

$$\mathbb{P}(p \in [p_{\min}, p_{\max}] \quad \forall t \in [0, T]) = 0.8574,$$

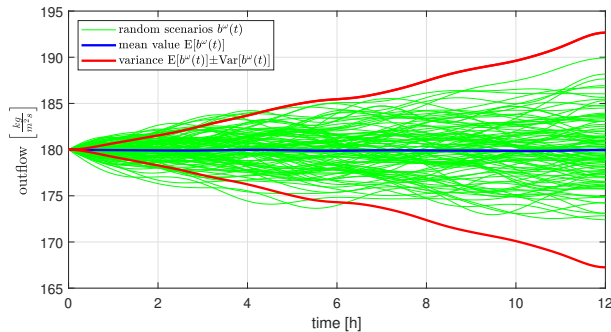


Figure 10: A sample of 100 random gas outflow scenarios

which means that 85.74% of all scenarios satisfy the pressure bounds for all times. This stands to reason that the stationary solution might be beneficial for transient scenarios with small perturbations. However, as soon as the expected gas outflow is far from being constant, the probability might decrease rapidly.

## 5 Conclusion and Open Problems

We have analyzed the problem of finding the optimal location of compressor stations on graphs, s.t. the control cost is minimal and pressure bounds are satisfied, in a deterministic as well as in a probabilistic context. For a single pipe, we have given sufficient conditions for the existence and the uniqueness for both settings. For gas networks, we have presented existence results for several optimization problems. For example, in some cases the number of compressor stations is also a decision variable. The uniqueness of solutions on networks mainly depends on the graph structure. To be precise, cycles can cause non-uniqueness.

The theory and the numerics both show, that a compressor station is better in terms of its cost, the sooner it is located on a pipe. In the compressor model chosen here, the compressor cost is independent from the flow rate. Considering a model, where the compressor cost increases with the flow rate, compressors might be more efficient if they are located at pipes with low flow rate. Other compressor models have bounded operation ranges, see e.g. [14, 34, 37]. This would lead to a more involved analysis and different results of the problems.

We have discussed the probabilistic robustness of the stationary optimal compressor locations in a transient context, in which stationary solutions are quite efficient as long as the perturbation stays small. For gas demand with high perturbation, a natural and interesting question is to consider gas flow dynamics as it was done e.g., in [6, 16, 26, 42, 43]. Then the hyperbolic structure of the gas dynamics has to be considered, which implies that e.g., a change over time in the gas inflow does change the outflow at another point in time. Also since the solution of the gas dynamics is not given explicit, a new approach for the existence (and uniqueness) of solutions is necessary. As already mentioned in the *Introduction* the problem of the optimal location of compressor stations in a gas pipeline network shows similarities to Weber problems, so results of this area might be beneficial in the transient setting.

## 6 Acknowledgements

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# Appendix.Simulation Results of the Transient Random Scenarios

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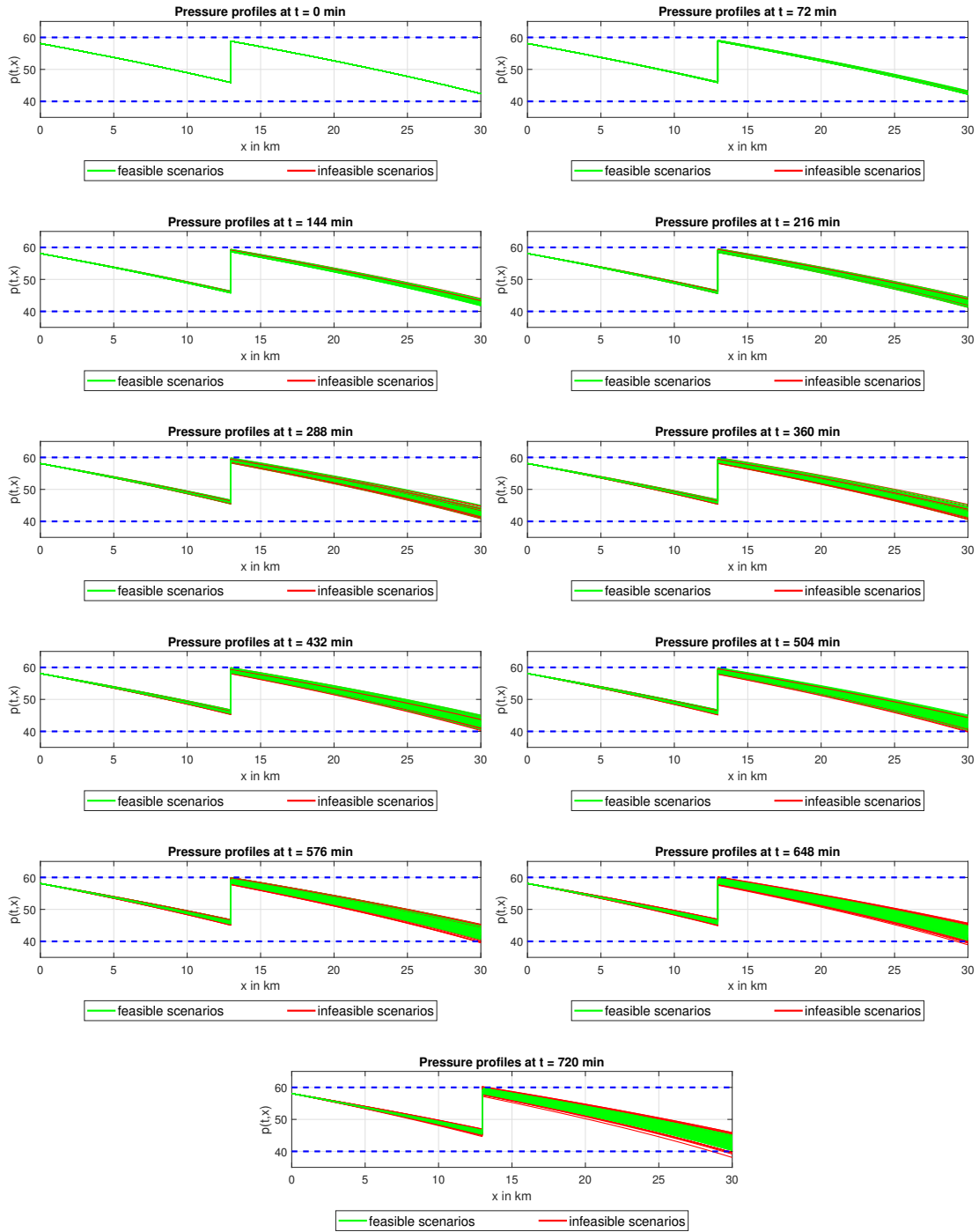


Figure 11: Simulation results of *Section 4* for every 72 minutes, feasible scenarios are shown in green, infeasible scenarios are shown in red

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