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Existence and percolation results for stopped germ-grain models with unbounded velocities

David Coupier¹, David Dereudre² and Simon Lestum³

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Abstract

We investigate the existence and first percolation properties of general stopped germ-grain models. They are defined via a random set of germs generated by a homogeneous planar Poisson point process in \mathbf{R}^2 . From each germ, a grain, composed by a random number of branches, grows. This grain stops to grow whenever one of its branches hits another grain. The classical and historical example is the line segment model for which the grains are segments growing in a random direction in $[0, 2\pi)$ with random velocity. In the bilateral line segment model the segments grow in both directions. Other examples are considered here such as the Brownian model where the branches are simply given by independent Brownian motions in \mathbf{R}^2 . The existence of such dynamics for an infinite number of germs is not obvious and our first result ensures it in a very general setting. In particular the existence of the line segment model is proved as soon as the random velocity admits a moment of order 4 which extends the result by Daley et al (Theorem 4.3 in [4]) for bounded velocity. Our result covers also the Brownian dynamic model. In a second part of the paper, we show that the line segment model with random velocity admitting a super exponential moment does not percolate. This improves a recent result (Theorem 3.2 [3]) in the case of bounded velocity.

Key words: continuum percolation, geometric random graph, lilypond model, Brownian dynamic.

1 Introduction

Consider a stationary Poisson point process \mathbf{X} in \mathbf{R}^2 where each germ $\xi \in \mathbf{X}$ is marked by a uniformly random direction Θ on $[0, 2\pi]$. At time 0, each germ gives off a growing line segment at unit rate, in the associated direction. One of the ends of the growing line segment is ξ , and the other one determines the stop of the segment when it hits another line segment. The almost sure existence of a unique stopped system of non-overlapping finite line segments has been proved by Daley et al (Theorem 4.3 in [4]). The authors conjectured the absence of percolation for this model and this has been recently solved by Coupier et al. (Theorem 3.2 in [3]). Percolation means here the existence of an unbounded connected component produced by the union of stopped segments. When the random velocity \mathbf{V} takes its values in a compact set, the existence

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and absence of percolation come from slight modifications of theorems mentioned above. In the case of random unbounded velocities \mathbf{V} , the problem is much more complicated because very quick segments may destroy the locality of the dynamic. As a consequence of our main theorems, we establish the existence of the stopped line segment model as soon as $\mathbf{E}(\mathbf{V}^4) < +\infty$ and the absence of percolation if there exists $s > 1$ such that $\mathbf{E}(e^{\mathbf{V}^s}) < +\infty$. Obviously this super-exponential moment condition is quite restrictive but it covers the Gaussian random velocities case corresponding to the classical Maxwellian velocity distribution for an ideal gas in statistical mechanics.

In a general setting, a planar germ-grain dynamic is a system of growing particles defined on a Poisson point process. The set of germs is distributed by a standard Poisson point process in \mathbf{R}^2 . A typical grain is defined as a finite collection of random processes identically distributed in \mathbf{R}^2 . Each grain can be viewed as a finite collection of growing branches coming from the associated germ. Any growing grain ceases its propagation when one of its extremities hits another grain. Given an initial infinite configuration of growing grains, the existence and uniqueness of such a stopped germ-grain dynamic is not guaranteed. In Theorem 3.1 we establish the almost sure existence and uniqueness under a very general assumption involving the fourth moment of the expansion of the grain (see Assumption (3)). As mentioned above this assumption is equivalent to a finite moment of order four (i.e. $\mathbf{E}(\mathbf{V}^4) < +\infty$) in the line segment model. Our result covers also the existence of the Brownian model where each grain is a collection of independent Brownian motions in \mathbf{R}^2 .

Let us now turn to our results on the absence of percolation for some stopped germ-grain models. As mentioned above we prove such absence of percolation for the line segment model in the case $\mathbf{E}(e^{\mathbf{V}^s}) < +\infty$ for some $s > 1$. For simplicity we deal only with the line segment model although the result could be easily extended to several stopped germ-grain models having similar geometric properties. "Similar geometric properties" means that the direction of expansion of the grain is random but does change during the evolution. The randomness is completely encoded in the initial condition. For instance our results do not cover the case where the grains grow with unpredictable directions as in the Brownian model. The absence of percolation for the Brownian model is still a conjecture today.

The absence of percolation for the line segment model with unbounded velocities is an extension of a recent result in [3] with bounded velocities. This non-trivial extension required to develop new concepts and ideas which we explain briefly now. From any stopped germ-grain model, we associate an outdegree-one graph where the vertices are simply the points of the Poisson point process itself and the outgoing edges are defined by pointing out the stopping germ. With this formalism, the absence of percolation for a stopped germ-grain model is equivalent to the absence of percolation for its associated outdegree-one graph, for which a general theorem is developed in [3] under the so-called assumptions Loop and Shield. Roughly speaking, the Loop assumption means that any forward branch merges on a loop provided that the Poisson point process is augmented with a finite collection of well-chosen points. This assumption is still true in our model with unbounded velocity. In our setting, the shield assumption means that a large square box $[-n, n]^2$ has high probability (i.e. the probability tends to 1 when $n \rightarrow \infty$) to be not crossed by a segment from the left to the right, uniformly with respect to the configuration outside the box. Obviously this property does not occur for unbounded velocities since it is always possible to build a very quick segment starting outside the box and crossing any line structure inside the box before its formation. So the major issue here is to control that germs with high velocities do not pollute so much the space \mathbb{R}^2 in order to apply the general strategy

developed in [3] in the non-polluted part of the space. In particular, we need that this non-polluted part is large enough and percolates. The super exponential moment condition is then required to build shield blocks with high probability in the non-polluted domain.

The paper is organized as follows. In Section 2, we provide a precise description of stopped germ-grain models and we give the three main examples. In Section 3 we present our results on existence and absence of percolation. Sections 4 and 5 are devoted to the proofs.

2 The stopped germ-grain model

This section is devoted to the notion of stopped germ-grain model. The set of **germs** is generated by a homogeneous planar Poisson point process. From each germ, a **grain** is growing, made up with a random number of **branches** which are identically distributed but not necessarily independent. The grains are assumed to be independent from each other. Any grain ceases to grow whenever one of its branches hits another grain.

Our first example of stopped germ-grain model is the famous line segment model introduced in [4] in which each grain simply corresponds to one (unilateral or bilateral) segment growing with a constant velocity.

2.1 The germ-grain model

All models in this paper take place in the Euclidean space \mathbf{R}^2 . The associated Lebesgue measure is denoted by Leb . The intensity of the (homogeneous) Poisson point process generating the germs is λLeb with $\lambda > 0$. The numbers of branches per grain are i.i.d. positive random integers with distribution δ and whose generic r.v. is denoted by K . The state space for the collection of grains is

$$\mathbf{F} := \left(\mathcal{C}(\mathbf{R}^+, \mathbf{R}^2) \right)^{\mathbf{N}} = \left\{ (f_n)_{n \geq 0} ; \forall n \geq 0, f_n : \mathbf{R}^+ \rightarrow \mathbf{R}^2 \text{ is continuous} \right\} .$$

The product space \mathbf{F} is equipped with the cylindric σ -algebra \mathcal{S} where each set $\mathcal{C}(\mathbf{R}^+, \mathbf{R}^2)$ is equipped with the Borel σ -algebra generated by the uniform convergence on compact sets. Let us note that we consider an infinite collection of branches for each grain although we will only use a finite number of them.

Let us consider a probability measure \mathcal{L} on the measurable space $(\mathbf{F}, \mathcal{S})$ satisfying two assumptions:

- Given $Y = (Y_0, Y_1, \dots, Y_n, \dots)$ distributed according to \mathcal{L} , all marginals are identically distributed (but not necessarily independent);
- For any index i , a.s. $Y_i(0) = 0$, i.e. all the branches of a given grain start their trajectories from the corresponding germ.

The general mark space \mathcal{M} of our model is defined by $\mathcal{M} = \mathbf{N}^* \times \mathbf{F}$ and the configuration space $\mathcal{C}^{\mathcal{M}}$ on \mathbf{R}^2 with marks in \mathcal{M} is defined by

$$\mathcal{C}^{\mathcal{M}} = \left\{ \varphi \subset \mathbf{R}^2 \times \mathcal{M} ; \#(\varphi \cap (\Lambda \times \mathcal{M})) < \infty, \text{ for any bounded } \Lambda \subset \mathbf{R}^2 \right\} .$$

It is equipped with the σ -algebra \mathcal{F} generated by the counting events $P_{(A,n)} = \{ \varphi \in \mathcal{C}^{\mathcal{M}} ; \#(\varphi \cap A) \leq n \}$ for all $n \geq 0$ and A in the sigma field $\mathcal{P}(\mathbf{N}^*) \otimes \mathcal{S}$.

Definition 2.1 A $(\lambda, \delta, \mathcal{L})$ -*germ-grain model* is a Poisson point process on $\mathcal{C}^{\mathcal{M}}$ with intensity $\lambda \text{Leb} \otimes \delta \otimes \mathcal{L}$.

Let \mathbf{X} be such a germ-grain model. The associated germ process is denoted by

$$\mathbf{X}_{\text{germs}} := \{\xi : (\xi, \cdot, \cdot) \in \mathbf{X}\} \subset \mathbf{R}^2 .$$

Given a marked point $(\xi, k, Y) \in \mathbf{X}$, the integer k corresponds to the number of branches starting at the germ ξ . And, in the countable collection of random variables $(Y_i)_{i \geq 0}$, only the k first paths are used in the dynamic. For a time $t \geq 0$ and a marked point $x = (\xi, k, Y)$, we define the corresponding **grain** until time t by

$$\text{Grain}(x, t) := \{\xi + Y_i(s) : 0 \leq i \leq k - 1, s \in [0, t]\} .$$

Also, the **extremity** at time t of the previous grain is

$$\text{H}(x, t) := \{\xi + Y_i(t) : 0 \leq i \leq k - 1\}$$

whose elements are called **particles**. Let us note that a grain does not contain its extremity. This convention has been chosen in order to simplify the stopping rules in Definition 2.2.

2.2 Examples of germ-grain model

2.2.1 Unilateral line segment model (Model 1)

This model is directly inspired by recent works [3, 4] on planar line segment dynamics. First, each grain is made up with only one branch, i.e. the distribution δ is the Dirac measure on 1. To specify the probability measure \mathcal{L} , we need two independent r.v.'s. On the one hand, Θ which is uniformly distributed on $[0, 2\pi]$ gives the direction of the line segment. On the other hand, V on \mathbf{R}_+^* is the growth velocity of the line segment. Then the variable Y_1 is defined by:

$$\forall t \geq 0, Y_1(t) := tV(\cos \Theta, \sin \Theta)$$

and the probability measure \mathcal{L} is obtained as the law of the sequence $Y := (Y_1, Y_1, \dots)$. In other words, the germ-grain $(\xi, 1, Y)$ is a single and unilateral line segment growing from the germ ξ according to the direction Θ and with velocity V .

For simplicity, in several place of the paper, the unilateral line segment model is described by a Poisson point process \mathbf{X} on $\mathbf{R}^2 \times [0, 2\pi] \times \mathbf{R}_+^*$ with intensity $\lambda \text{Leb} \otimes \Xi \otimes \mu$, where Ξ is the uniform distribution on $[0, 2\pi]$ and μ is the law of V . We will then use the notation

$$\mathbf{X} = \bigcup_{\xi \in \mathbf{X}_{\text{germs}}} (\xi, \Theta_\xi, V_\xi) .$$

2.2.2 Bilateral line segment model (Model 2)

The bilateral line segment model has been introduced in [4] as well. Its definition also involves the r.v.'s Θ , V and Y_1 from the previous section. The probability measure \mathcal{L} is now the law of the sequence $Y := (Y_1, -Y_1, Y_1, Y_1, \dots)$ and the distribution δ is the Dirac measure on 2. Hence, the germ-grain $(\xi, 2, Y)$ is a bilateral line segment growing from ξ according to the two opposite directions Θ and $-\Theta$ with the same velocity V .

This example highlights the possibility to have dependent marginals in \mathcal{L} . Several other examples could be constructed similarly.

2.2.3 Brownian model (Model 3)

In this model, any given grain is made up with a random number K of branches which are driven by i.i.d. Brownian trajectories. For technical reason, we assume that $\mathbf{E}(K^2)$ is finite. Thus let us set $B := (B_i)_{i \geq 0}$ a sequence of i.i.d. Brownian motions in \mathbf{R}^2 starting from 0. The probability measure \mathcal{L} is then simply the distribution of B . Hence, a germ-grain is completely determined by the triplet (ξ, K, B) .

To illustrate the variety of possible models considering here, we could also imagine a single Brownian path starting from each germ, or exactly two paths but one reflected (w.r.t. the germ) from the other, etc.

2.3 Stopped germ-grain model

This section is devoted to the notion of **stopped germ-grain model**. Instead of a dynamical definition as described in the Introduction, we prefer here to define a stopped germ-grain model through the concept of lifetime. This point of view was already used to define the Lilypond model in [5].

Definition 2.2 *Let $\varphi \in \mathcal{C}^M$ be a configuration. An **exploration** is a function defined on φ associating a lifetime to each marked point.*

$$f : \varphi \longrightarrow (0, +\infty] .$$

An exploration f of φ is said **stopped** if it satisfies the two following conditions:

- (i) (**Hardcore property**). $\forall x \neq y \in \varphi$ then $\text{Grain}(x, f(x)) \cap \text{Grain}(y, f(y)) = \emptyset$;
- (ii) (**Uniqueness of the stopping grain**). $\forall x \in \varphi$ with $f(x) < +\infty$ then $\exists! y \in \varphi \setminus \{x\}$ such that $H(x, f(x)) \cap \text{Grain}(y, f(y)) \neq \emptyset$.

Item (ii) asserts that either a marked point $x \in \varphi$ has an infinite lifetime, i.e. $f(x) = +\infty$, and will be never stopped or it will be eventually stopped, i.e. $f(x) < +\infty$, but by only one grain.

Definition 2.3 *A $(\lambda, \delta, \mathcal{L})$ -germ-grain model \mathbf{X} is said **stopped** if it a.s. admits a unique **stopped exploration**, denoted by $f_{\mathbf{X}}$. Moreover, \mathbf{X} satisfies the **finite time property** if*

$$\mathbf{P}(\forall x \in \mathbf{X}, f_{\mathbf{X}}(x) < +\infty) = 1 . \tag{1}$$

The existence and uniqueness of a stopped exploration is not obvious in general because of the infinite number of marked points. We will prove in Corollary 3.1 and Proposition 3.1 that the three germ-grain models introduced in Section 2.2 are stopped and satisfy the finite time property.

It is worth pointing out here that the finite time property is crucial in our work since it allows to interpret any stopped germ-grain model with such property as a **Poisson outdegree-one graph** (see Section 5.1). Indeed, each marked point has a finite lifetime and then admits a unique outgoing vertex corresponding to its stopping grain.

3 Results

3.1 Existence of stopped germ-grain models

In this section we state a general result (Theorem 3.1) ensuring the existence of stopped germ-grain models. Our main assumption involves the modulus of continuity for branches of a given grain. Precisely, for \mathbf{X} a $(\lambda, \delta, \mathcal{L})$ -germ-grain model where K is a δ -distributed r.v. and $Y = (Y_i)_{i \geq 0}$ is a \mathcal{L} -distributed sequence of paths, and for all $t, t' \geq 0$, we set:

$$M_{t,t'} := \max_{0 \leq k \leq K-1} \sup_{0 \leq s \leq t'} \|Y_k(t+s) - Y_k(t)\| . \quad (2)$$

Theorem 3.1 *Let \mathbf{X} be a $(\lambda, \delta, \mathcal{L})$ -germ-grain model such that $\mathbf{E}_\delta(K^2) < +\infty$ and*

$$\lim_{t' \rightarrow 0} \sup_{t \geq 0} \mathbf{E} \left(M_{t,t'}^4 \right) = 0. \quad (3)$$

Moreover, for any U_1, U_2 two independent random paths on $\mathcal{C}(\mathbf{R}^+, \mathbf{R}^2)$ distributed as the marginals of \mathcal{L} we assume that

$$\text{a.s. } \text{Leb}(\{U_1(t), t \geq 0\}) = 0 \quad \text{and} \quad \text{Leb}(\{U_1(t) - U_2(t), t \geq 0\}) = 0 . \quad (4)$$

Then \mathbf{X} is a stopped germ-grain model.

The Assumption (4) is technical and satisfied for most natural models. It actually ensures that pathological random paths do not occur. Theorem 3.1 will be proved in Section 4.

As a consequence of Theorem 3.1, we easily get that the three germ-grain models presented in Section 2.2 are stopped.

Corollary 3.1 *The unilateral and bilateral line segment models (Models 1 and 2) with $\mathbf{E}(V^4) < +\infty$ and the Brownian model (Model 3) are stopped germ-grain models.*

Let us note that D.J. Daley et al have established in [4] the existence of the bilateral line segment model with constant velocity. They proved the finite time property as well.

Proof: For both line segment models, we have $M_{t,t'} = Vt'$ and Assumption (3) is obviously satisfied. The Assumption (4) is easily checked as well.

Let us focus on the Brownian model. By stationarity of increments for Brownian paths, Assumption (3) is equivalent to

$$\lim_{t' \rightarrow 0} \mathbf{E} \left(M_{0,t'}^4 \right) = 0 . \quad (5)$$

Recall that

$$M_{0,t'} = \max_{0 \leq k \leq K-1} \sup_{0 \leq s \leq t'} \|B_k(s)\| ,$$

where $(B_k)_{0 \leq k \leq K-1}$ is a collection of K independent Brownian motions. Using the scaling property, we can write $\mathbf{E}(M_{0,t'}^4) = (t')^2 \mathbf{E}(M_{0,1}^4)$. It then remains to prove that $\mathbf{E}(M_{0,1}^4)$ is finite. This simply comes from

$$\mathbf{E} \left(M_{0,1}^4 \right) = \mathbf{E} \left(\max_{0 \leq k \leq K-1} W_k^4 \right) \leq \mathbf{E} \left(\sum_{k=0}^{K-1} W_k^4 \right) = \mathbf{E}(K) \mathbf{E}(W_0^4) < +\infty .$$

Finally, the Brownian model also satisfied Assumption (4) since the Lebesgue measure of the bi-dimensional Brownian path in \mathbb{R}^2 is a.s. equal to 0. \square

3.2 Absence of percolation

Let us focus on the graph produced by a stopped germ-grain model.

Definition 3.1 For any $(\lambda, \delta, \mathcal{L})$ germ-grain model \mathbf{X} , we set

$$\Sigma(\mathbf{X}) := \bigcup_{x \in \mathbf{X}} \text{Grain}(x, f_{\mathbf{X}}(x)) .$$

We say that \mathbf{X} percolates if $\Sigma(\mathbf{X})$ contains an unbounded connected component.

We conjecture the absence of percolation for a large class of stopped germ-grain models. This conjecture is supported by the underlying outdegree-one graph structure of stopped germ-grain models with finite time property (see Section 5.1). The absence of percolation has been conjectured in [4] and proved in [3] for the line segment model with constant velocity. Our initial ambitious was to state the absence of percolation for a large class of stopped germ-grain models but we do not succeed in this task and the conjecture is still largely open today. Nevertheless, we significantly improve the result of [3] by allowing velocities of line segments to be random and especially unbounded.

Before giving our main theorem below (Theorem 3.2), let us briefly discuss the finite time property given in Definition 2.3. The percolation question is relevant only if the studied stopped germ-grain model satisfies the finite time property. Otherwise, some of its grains have an infinite lifetime and then are (generally) unbounded. Besides, a large class of stopped germ-grain models should satisfy the finite time property (but we have not investigated such a general result in the present paper). This is the case for the three models introduced in Section 2.2.

Proposition 3.1 *The unilateral and bilateral line segment models (Models 1 and 2) with $\mathbf{E}(V^4) < +\infty$ and the Brownian model (Model 3) are stopped germ-grain models satisfying the finite time property.*

The proof of Proposition 3.1 is given at the end of this section. Here is our second main result stating the absence of percolation for the unilateral line-segment model with unbounded velocities. The proof is given in Section 5.

Theorem 3.2 *Assume that there exists $s > 1$ such that $\mathbf{E}(\exp(V^s)) < +\infty$. Then the unilateral line segment model does not percolate.*

Proof:(of Proposition 3.1). First, let us focus on the unilateral line segment model which is described by a Poisson point process \mathbf{X} on $\mathbf{R}^2 \times [0, 2\pi] \times \mathbf{R}_+^*$. Let γ be a typical point located at the origin: $\gamma = (0, \Theta, V)$ where Θ is a uniform r.v. on $[0, 2\pi]$ and $\mathbf{E}(V^4) < +\infty$. Let us proceed by absurd, assuming that with positive probability the typical point γ is never stopped: $\mathbf{P}(f_{\mathbf{X} \cup \{\gamma\}}(\gamma) = +\infty) > 0$. The Campbell-Mecke formula implies that, for any Borelian set $\Delta \subset \mathbf{R}^2$,

$$\mathbf{E} \left(\sum_{x \in \mathbf{X}_\Delta} \mathbb{1}_{\{f_{\mathbf{X}}(x) = +\infty\}} \right) = \lambda \text{Leb}(\Delta) \mathbf{P} \left(f_{\mathbf{X} \cup \{\gamma\}}(\gamma) = +\infty \right) > 0 , \quad (6)$$

where $\mathbf{X}_\Delta = \{x = (\xi, \Theta, V) \in \mathbf{X} : \xi \in \Delta\}$. For an introduction to the Palm theory, the reader may refer to [2]. The isotropy of the r.v. Θ and (6) allow us to state that for any $n \geq 1$

$$\mathbf{P} \left(\exists x = (\xi, \Theta, V) \in \mathbf{X} ; f_{\mathbf{X}}(x) = \infty, \xi \in [n, n+1) \times (0, 1), \Theta \in \left(\frac{\pi}{2}, \frac{3\pi}{4} \right) \right) > 0 .$$

Thus, the ergodicity of the Poisson point process leads to

$$\mathbf{P} \left(\exists n \geq 1, \exists x = (\xi, \Theta, V) \in \mathbf{X} ; f_{\mathbf{X}}(x) = \infty, \xi \in [n, n+1) \times (0, 1), \Theta \in \left(\frac{\pi}{2}, \frac{3\pi}{4} \right) \right) = 1 . \quad (7)$$

Besides, a similar argument gives

$$\mathbf{P} \left(\exists n \geq 1, \exists x = (\xi, \Theta, V) \in \mathbf{X} ; f_{\mathbf{X}}(x) = \infty, \xi \in [-n-1, -n) \times (0, 1), \Theta \in \left(\frac{\pi}{4}, \frac{\pi}{2} \right) \right) = 1 . \quad (8)$$

From (7) and (8), we deduce by a simple geometric argument that

$$\mathbf{P} (\exists x \neq x' \in \mathbf{X} ; f_{\mathbf{X}}(x) = f_{\mathbf{X}}(x') = \infty, \text{ and } \text{Grain}(x, \infty) \cap \text{Grain}(x', \infty) \neq \emptyset) = 1 .$$

This last statement contradicts the hardcore property in Definition 2.2. The proof for the unilateral line segment model is complete.

The proof for the bilateral line segment model and the Brownian model are simpler. In both cases it is easy to see that

$$\mathbf{P} (\forall x \neq x' \in \mathbf{X}, \text{ Grain}(x, \infty) \cap \text{Grain}(x', \infty) \neq \emptyset) = 1$$

and we conclude as above. □

4 Proof of Theorem 3.1

Let us first present the main idea of the proof. Let us assume that the states of grains, already stopped or still alive at a given time t , are known. During the time interval $[t, t + t']$, each particle still alive evolves inside a (random) disk, represented by a red circle in Fig. 1, whose radius is given by the variable $M_{t,t'}$ defined in (2). Recall that the radius $M_{t,t'}$ is the same for all the particles belonging to the same grain. The main ingredient of the proof (Proposition 4.1, proved in Section 4.2) consists in establishing the absence of percolation for the random set made up with all the disks mentioned before, for some (small but positive) time t' , which will be *uniform on t* . Henceforth, the dynamic of the germ-grain model could be rigorously constructed on the time interval $[t, t + t']$ by treating independently each (finite) cluster and the corresponding grains. The stopped germ-grain model will be first built on the time interval $[0, t']$, thus on $[t', 2t']$ and so on.

4.1 A percolation argument

Let us consider a germ-grain model \mathbf{X} satisfying the assumptions of Theorem 3.1. For any times $t, t' \geq 0$ and any marked points $x = (\xi, k, Y)$, $x' = (\xi', k', Y')$ of \mathbf{X} still alive at time t , we will

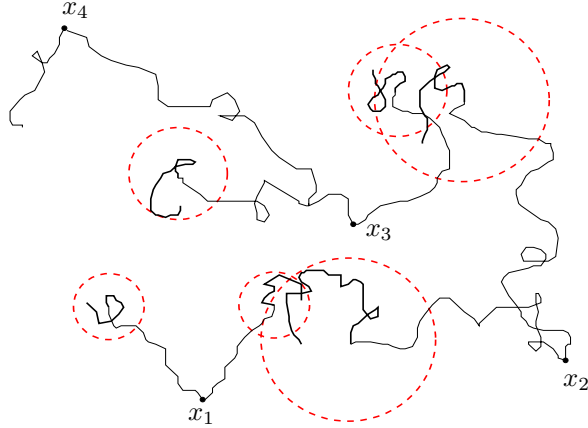


Figure 1: On this picture, the four grains (in black) associated to the marked points x_1, x_2, x_3, x_4 , each contain two branches. At time t , only the grain of x_4 is already stopped (by the one associated to x_3). For the three other marked points still alive, we draw the red circles centered at the corresponding (six) particles with radii given by the $M_{t,t'}$'s. Remark that x_1, x_2, x_3 belong to the same genealogical cluster: $x_1 \sim_{t,t'} x_2$ and $x_2 \sim_{t,t'} x_3$

write $x \sim_{t,t'} x'$ if and only if

$$\left(\bigcup_{l=0}^{k-1} B(\xi + Y_l(t), M_{t,t'}(x)) \right) \cap \left(\bigcup_{l=0}^{k'-1} B(\xi' + Y'_l(t), M_{t,t'}(x')) \right) \neq \emptyset$$

where $M_{t,t'}(x)$ is the random radius associated to the marked point $x = (\xi, k, Y)$ between times t and $t + t'$, and defined in (2). Idem for $M_{t,t'}(x')$.

Hence, let us consider the **genealogical graph** $\mathcal{G}_{t,t'}(\mathbf{X})$ as the (non oriented) random graph whose vertex set is given by the marked points of \mathbf{X} still alive at time t , and whose edge set is defined by the (non oriented) relation $\sim_{t,t'}$. The connected components of $\mathcal{G}_{t,t'}(\mathbf{X})$ are called **genealogical clusters**. For instance, in Fig. 1, the three marked points x_1, x_2, x_3 still alive at time t belong to the same genealogical cluster.

Let us consider a marked point $x \in \mathbf{X}$. The genealogical cluster of x in $\mathcal{G}_{t,t'}(\mathbf{X})$, denoted by $\mathcal{C}_{t,t'}(x)$, corresponds to the set of marked points still alive at time t on which the evolution of the grain associated to x during the time interval $[t, t + t']$ depends. The next result which is the key ingredient for the proof of Theorem 3.1 asserts that the genealogical cluster $\mathcal{C}_{t,t'}(x)$ of x is a.s. finite. Its proof is postponed to Section 4.2.

Proposition 4.1 *There exists a (small) $t' > 0$ such that for any time $t \geq 0$, the genealogical graph $\mathcal{G}_{t,t'}(\mathbf{X})$ a.s. admits only finite clusters.*

From now on, we set t' as the time given by Proposition 4.1. In order to determine the evolution of the grain associated to x during the time interval $[t, t + t']$, we also have to take into account the trajectories of all the grains produced before t and intersecting the random set

$$\mathcal{C}_{t,t'}(x) := \bigcup_{(\xi', k', Y') \in \mathcal{C}_{t,t'}(x)} \bigcup_{l=0}^{k'-1} B(\xi' + Y'_l(t), M_{t,t'}(x'))$$

which is bounded thanks to the choice of t' . Fortunately,

Lemma 4.1 *Let t' given by Proposition 4.1. For any time t and any marked point $x \in \mathbf{X}$ still alive at time t , the set of $y \in \mathbf{X}$ whose grain $\text{Grain}(y, t)$ until time t hits $C_{t, t'}(x)$ is a.s. finite.*

Hence, the combination of Proposition 4.1 and Lemma 4.1 makes it possible to define a finite algorithm determining the evolution of the grain of x during the time interval $[t, t + t']$. For details the reader may refer to [4], Section 3.

In the case where the grain of x would be stopped, the next result based on Assumption (4) allows to avoid some pathological situations and to identify with no ambiguity its stopping grain. Proofs of Lemmas 4.1 and 4.2 are given at the end of this section.

Lemma 4.2 (i) *Two different grains can not meet simultaneously: a.s. for any $x \neq y \in \mathbf{X}$ and for any time t , $H(x, t) \cap H(y, t) = \emptyset$.*

(ii) *Two branches belonging to the same grain can not hit simultaneously two different grains: a.s. for any three different marked points x, y, z of \mathbf{X} and any $t \geq 0$,*

$$H(x, t) \cap \text{Grain}(y, t) \neq \emptyset \Rightarrow H(x, t) \cap \text{Grain}(z, t) = \emptyset .$$

Now, let us explain how to combine the three previous results to check that a.s. there exists a unique stopped exploration $f_{\mathbf{X}}$.

Let us start with $t = 0$ and the genealogical graph $\mathcal{G}_{0, t'}(\mathbf{X})$. We treat all the genealogical clusters in the same way (and independently). Let \mathcal{C} be one of them: it is finite by Proposition 4.1. For this first step, Lemma 4.1 is not needed. For any $x \neq y \in \mathcal{C}$, we determine by continuity of trajectories a first hitting time between the grains of x and y if such (first) meeting occurs. In this case, Lemma 4.2 (i) allows to know the first arrived at the meeting point among x and y . Then, we obtain for the cluster \mathcal{C} a finite sequence of hitting times: all those concerning a given grain are different by Lemma 4.2 (ii). Then a well-defined algorithm (see Section 3 of [4]) identifies among these hitting times the real stopping times. Hence it determines if x is stopped or not during the time interval $[0, t']$ and, if it is, by whom.

At the end of this first step, let us set $f_{\mathbf{X}}^{(1)}(x) = t'$ if x is still alive at time t' . Otherwise $f_{\mathbf{X}}^{(1)}(x)$ is defined as its stopping time. We then obtain the set of grains already explored until time t' :

$$\mathbf{G}_1 := \bigcup_{x \in \mathbf{X}} \text{Grain}(x, f_{\mathbf{X}}^{(1)}(x)) .$$

Let us continue the proof by induction. Let k be a positive integer and let us assume explored and known all the grains until time kt' : i.e. the values $f_{\mathbf{X}}^{(k)}(x)$ for any x as well as the set

$$\mathbf{G}_k := \bigcup_{x \in \mathbf{X}} \text{Grain}(x, f_{\mathbf{X}}^{(k)}(x)) .$$

The marked points still alive at time kt' are those such that $f_{\mathbf{X}}^{(k)}(\cdot) = kt'$. For the others (those stopped before kt'), we can immediately set $f_{\mathbf{X}}^{(k+1)}(x) = f_{\mathbf{X}}^{(k)}(x)$. Thus we proceed as in the first step but taking into account this time the grains produced before time kt' (and using Lemma 4.1). A similar algorithm allows to determine the evolution of grains still alive at kt' until time

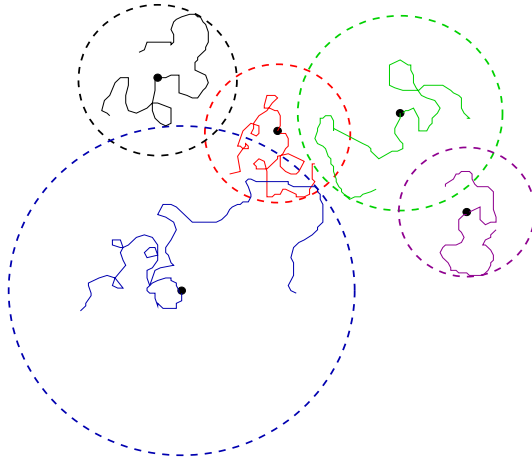


Figure 2: Here is an illustration of a genealogical cluster of $\mathcal{G}_{0,t'}(\mathbf{X})$ with 5 marked points (with different colors) having each exactly two branches which are pictured till time t' . The black, green and purple marked points will be still alive at time t' . Only the blue and red grains meet during the time interval $[0, t']$: say the blue one stops the red one. The blue marked point will then survive until time t' . The parts of both red branches created after the “red” stopping time have to be deleted.

$(k+1)t'$. If x is stopped during the time interval $[kt', (k+1)t']$, then $f_{\mathbf{X}}^{(k+1)}(x)$ is defined as its stopping time. Otherwise $f_{\mathbf{X}}^{(k+1)}(x) = (k+1)t'$. Thus we update the set of explored grains:

$$\mathbf{G}_{k+1} := \bigcup_{x \in \mathbf{X}} \text{Grain}(x, f_{\mathbf{X}}^{(k+1)}(x))$$

which contains \mathbf{G}_k .

We can then define a function $f_{\mathbf{X}}$ valued in $(0, \infty]$ as the pointwise limit of the non-decreasing sequence $(f_{\mathbf{X}}^{(k)})_{k>0}$: for any $x \in \mathbf{X}$, $f_{\mathbf{X}}(x) := \lim \nearrow f_{\mathbf{X}}^{(k)}(x)$. In particular, $f_{\mathbf{X}}(x)$ is infinite if and only if the grain associated to x is never stopped. Hence we get a unique exploration function $f_{\mathbf{X}}$ of \mathbf{X} which is stopped by construction (see Definition 2.2). Indeed, any two different grains do not overlap. Moreover, a stopped marked point x , i.e. such that $f_{\mathbf{X}}(x) < \infty$, admits only one stopping marked point by Lemma 4.2 (ii).

Proof:(of Lemma 4.1). Let t' given by Proposition 4.1. For any $t \geq 0$ and $x \in \mathbf{X}$ still alive at time t , the set $\mathbf{C}_{t,t'}(x)$ is a.s. bounded. So, it suffices to prove that for any $R > 0$, a.s. the set of $y \in \mathbf{X}$ such that $\text{Grain}(y, t) \cap B(0, R) \neq \emptyset$ is finite. We are going to prove that

$$I := \mathbf{E}(\#\{y \in \mathbf{X} ; \text{Grain}(y, t) \cap B(0, R) \neq \emptyset\}) < +\infty .$$

This holds by standard computations based on the Campbell-Mecke formula

$$\begin{aligned}
I &\leq \lambda \left(\sum_{k=1}^{+\infty} \mathbf{P}(K = k) k \right) \int_{[0,t]} \int_{\mathbf{R}^2} \int_{\mathcal{C}(\mathbf{R}_+, \mathbf{R}^2)} \mathbb{I}_{\{\xi + U(s) \in B(0,R)\}} \text{Leb}(ds) \text{Leb}(d\xi) \chi(dU) \\
&\leq \lambda \mathbf{E}(K) \int_{[0,t]} \left(\int_{\mathcal{C}(\mathbf{R}_+, \mathbf{R}^2)} \text{Leb}(B(-U(s), R)) \chi(dU) \right) \text{Leb}(ds) \\
&\leq \lambda \pi R^2 \mathbf{E}(K) t.
\end{aligned}$$

□

Proof:(of Lemma 4.2). Let us focus on Item (i). Without loss of generality, we can assume that each grain contains exactly one branch. The extremity $\mathbf{H}(\cdot, t)$ is identified to the single point that it contains. Let us show that

$$J := \mathbf{E} \left(\# \left\{ (x, y) \in \mathbf{X}^2 ; x \neq y \text{ and } \exists t \geq 0, \mathbf{H}(x, t) = \mathbf{H}(y, t) \right\} \right) = 0 .$$

By the Campbell-Mecke formula, we obtain:

$$\begin{aligned}
J &= \lambda^2 \int \mathbb{I}_{\{\exists t \geq 0 ; \xi_1 + U_1(t) = \xi_2 + U_2(t)\}} (\text{Leb} \otimes \chi)^2 (d(\xi_1, U_1), d(\xi_2, U_2)) \\
&= \lambda^2 \int \mathbb{I}_{\{\exists t \geq 0 ; U_1(t) - U_2(t) = \xi_2 - \xi_1\}} (\text{Leb} \otimes \chi)^2 (d(\xi_1, U_1), d(\xi_2, U_2))
\end{aligned}$$

where χ denotes the marginal of \mathcal{L} . Thus by substitution,

$$\begin{aligned}
J &= \lambda^2 \int \mathbb{I}_{\{\exists t \geq 0 ; (U_1 - U_2)(t) = u\}} \text{Leb}(du) \chi \otimes \chi (dU_1, dU_2) \\
&= \lambda^2 \mathbf{E} (\text{Leb} (\{U_1(t) - U_2(t) ; t \geq 0\}))
\end{aligned}$$

which is equal to 0 by Assumption (4) of Theorem 3.1.

Let $x_i := (\xi^{(i)}, k^{(i)}, Y^{(i)}) \in \mathbf{X}$ for $i = 1, 2, 3$. If the grain associated to x_1 admits at least two branches and if it is stopped by the grain of x_2 , we denote by $\tau = \tau(x_1, x_2)$ this stopping time. Otherwise, we set $\tau = 0$. To prove Item (ii), it is enough to show

$$L := \mathbf{E} \left(\# \left\{ (x_1, x_2, x_3) \in \mathbf{X}^3 ; \begin{array}{l} x_1, x_2, x_3 \text{ are all different and} \\ \exists t \leq \tau, \xi^{(1)} + Y_1^{(1)}(\tau) = \xi^{(3)} + Y_0^{(3)}(t) \end{array} \right\} \right) = 0 .$$

The Campbell-Mecke formula gives:

$$\begin{aligned}
L &= \lambda^3 \int \mathbb{I}_{\{\exists t \leq \tau ; \xi^{(1)} + Y_1^{(1)}(\tau) = \xi^{(3)} + Y_0^{(3)}(u)\}} (\text{Leb} \otimes \delta \otimes \mathcal{L})^3 (dx_1, dx_2, dx_3) \\
&= \lambda^3 \int_{(\mathbf{N}^* \times \mathcal{C}(\mathbf{R}_+, \mathbf{R}^2))^2} \int_{(\mathbf{R}^2)^3} \int_{\mathcal{C}(\mathbf{R}_+, \mathbf{R}^2)} \mathbb{I}_{\{\exists t \leq \tau ; Y_0^{(3)}(t) = \xi^{(1)} - \xi^{(3)} + Y_1^{(1)}(\tau)\}} \\
&\quad \chi(dY_0^{(3)}) d\xi^{(1)} d\xi^{(2)} d\xi^{(3)} (\delta \otimes \mathcal{L})^2 (d(k^{(1)}, Y^{(1)}), d(k^{(2)}, Y^{(2)})) .
\end{aligned}$$

Now, assume that $Y^{(1)}, k^{(1)}$ and $Y^{(2)}, k^{(2)}$ are fixed. Let us specify that τ does not depend only on $Y^{(1)}, k^{(1)}$ and $Y^{(2)}, k^{(2)}$ but also on $\xi^{(1)}$ and $\xi^{(2)}$. Hence,

$$\begin{aligned}
& \int_{(\mathbf{R}^2)^3} \left(\int_{\mathcal{C}(\mathbf{R}_+, \mathbf{R}^2)} \mathbb{I}_{\{\exists t \leq \tau ; Y_0^{(3)}(t) = \xi^{(1)} - \xi^{(3)} + Y_1^{(1)}(\tau)\}} \chi(dY_0^{(3)}) \right) d\xi^{(1)} d\xi^{(2)} d\xi^{(3)} \\
&= \int_{\mathcal{C}(\mathbf{R}_+, \mathbf{R}^2)} \left(\int_{(\mathbf{R}^2)^2} \left(\int_{\mathbf{R}^2} \mathbb{I}_{\{\exists t \geq 0 ; Y_0^{(3)}(t) = \xi^{(1)} - \xi^{(3)} + Y_1^{(1)}(\tau)\}} d\xi^{(3)} \right) d\xi^{(1)} d\xi^{(2)} \right) \chi(dY_0^{(3)}) \\
&\leq \int_{\mathcal{C}(\mathbf{R}_+, \mathbf{R}^2)} \left(\int_{(\mathbf{R}^2)^2} \text{Leb}(\{\eta \in \mathbf{R}^2 ; \exists t \geq 0 ; Y_0^{(3)}(t) = \eta\}) d\xi^{(1)} d\xi^{(2)} \right) \chi(dY_0^{(3)}) \\
&\leq \int_{(\mathbf{R}^2)^2} \left(\int_{\mathcal{C}(\mathbf{R}_+, \mathbf{R}^2)} \text{Leb}(\{\eta \in \mathbf{R}^2 ; \exists t \geq 0 ; Y_0^{(3)}(t) = \eta\}) \chi(dY_0^{(3)}) \right) d\xi^{(1)} d\xi^{(2)} \\
&\leq \int_{(\mathbf{R}^2)^2} \mathbf{E} \left(\text{Leb}(\{Y^{(3)}(t) ; t \geq 0\}) \right) d\xi^{(1)} d\xi^{(2)}
\end{aligned}$$

which is null by Assumption (4) in Theorem 3.1. This implies $L = 0$. \square

4.2 Proof of Proposition 4.1

Our proof is inspired by Theorem 2 in [7]. Given a $(\lambda, \delta, \mathcal{L})$ -germ grain model \mathbf{X} and times $t \geq 0$, $t' > 0$, we define a Poisson point process \mathbf{Y} on $\mathbf{R}^2 \times \mathbf{N}^* \times (\mathbf{R}^2 \times \mathbf{R}_+^*)^{\mathbf{N}}$ as follows: any marked point $(\cdot, \cdot, (Y_i(t) : t \geq 0)_{i \geq 0})$ is replaced with $(\cdot, \cdot, (Z_i^{(1)}, Z_i^{(2)})_{i \geq 0})$ where

$$Z_i^{(1)} = Y_i(t) \quad \text{and} \quad Z_i^{(2)} = \sup_{0 \leq s \leq t'} \|Y_i(t+s) - Y_i(t)\|.$$

This transformation will simplify the notations in the following. The distribution of the sequence $(Z_i^{(1)}, Z_i^{(2)})_{i \geq 0}$ is denoted by \mathscr{W} and its marginal is denoted by Γ . Moreover we consider the random variable

$$\mathbf{M} = \max_{0 \leq i \leq K-1} Z_i^{(2)} \tag{9}$$

in place of $M_{t,t'}$. Thus, to the Poisson process \mathbf{Y} , we associate the Boolean model

$$\text{Bool}(\mathbf{Y}) := \bigcup_{(\xi, k, Y) \in \mathbf{X}} \bigcup_{l=0}^{k-1} B \left(\xi + Y_l(t), \max_{0 \leq l \leq k-1} \sup_{0 \leq s \leq t'} \|Y_l(t+s) - Y_l(t)\| \right) \tag{10}$$

and the non oriented graph $\mathcal{G}(\mathbf{Y})$ whose vertex set is given by \mathbf{Y} and two vertices $(\xi, k, (Z_i^{(1)}, Z_i^{(2)})_{i \geq 0})$ and $(\bar{\xi}, \bar{k}, (\bar{Z}_i^{(1)}, \bar{Z}_i^{(2)})_{i \geq 0})$ are connected by an edge whenever

$$\left(\bigcup_{i=0}^{k-1} B(\xi + Z_i^{(1)}, \max_{0 \leq i \leq k-1} Z_i^{(2)}) \right) \cap \left(\bigcup_{i=0}^{\bar{k}-1} B(\bar{\xi} + \bar{Z}_i^{(1)}, \max_{0 \leq i \leq \bar{k}-1} \bar{Z}_i^{(1)}) \right) \neq \emptyset.$$

Lemma 4.3 *If $\lambda \mathbf{E}(\mathbf{M}^4) \mathbf{E}_\delta(K^2)$ is smaller than $1/16\pi$. Then $\mathcal{G}(\mathbf{Y})$ a.s. does not percolate.*

Proposition 4.1 is a direct consequence of Lemma 4.3. Indeed, by hypotheses on \mathbf{X} , we can choose t' small enough so that $16\pi\lambda\mathbf{E}(\mathbf{M}^4)\mathbf{E}_\delta(K^2)$ is smaller than 1, uniformly on t thanks to (3). Thus Lemma 4.3 applies: $\mathcal{G}(\mathbf{Y})$ and also $\mathcal{G}_{t,t'}(\mathbf{X})$ does not percolate with probability 1.

In the case where $\delta = \delta_1$ the Dirac measure on 1, Lemma 4.3 is exactly Theorem 2 in [7]. Our proof are based on the same arguments but we take account that the number of branches per germ is random.

Proof:(of Lemma 4.3). We denote by C_0 the genealogical cluster of 0 in $\mathcal{G}(\mathbf{Y} \cup \{(0, K, Z)\})$. We will show that $\#C_0$ is almost surely finite where $\#C_0$ denotes the number of vertices in C_0 . Without loss of generality, let us replace \mathbf{M} by $\lfloor \mathbf{M} \rfloor + 1$ (in other words, we consider that \mathbf{M} is distributed on \mathbf{N}^*). Therefore we assume that the radii in $\text{Bool}(\mathbf{Y})$ have positive integer values. In the following a disc is said of type $i \in \mathbf{N}^*$ if its radius is equal to i . By the same way, a given germ ξ is of type i if its associated discs have the type i . The probability $\mathbf{P}(\mathbf{M} = i)$ is denoted by p_i .

Following the strategy in [7]), we construct a multi-type branching process such that the number of individuals dominates stochastically the number of discs in C_0 . And we show that the expected number of individuals, given that 0 is of type i , is bounded for any $i \geq 1$. See Athreya and Ney ([1] page 184) for the relevant theory of multi-type branching processes. The individuals in the branching process are discs. The individuals in the 0th generation are the K discs related to the marked point $(0, K, Z)$. Given the $N^{(n)}$ individuals $\{B_l^{(n)}\}_{1 \leq l \leq N^{(n)}}$ in the n^{th} generation, we define the $(n+1)^{\text{th}}$ generation as follows. Let $\mathbf{Y}^{(n+1)}$ be a Poisson process in $\mathbf{R}^2 \times \mathbf{N}^* \times (\mathbf{R}^2 \times \mathbf{R}_+^*)^{\mathbf{N}}$ with intensity $\lambda \text{Leb} \otimes \delta \otimes \mathscr{W}$, independent with the previous history of the process. The individuals in the $(n+1)^{\text{th}}$ generation are the discs of $\text{Bool}(\mathbf{Y}^{(n+1)})$ which have at least one associated disc overlapping the boundary of one disc in the n^{th} generation.

Let us introduce some useful random variables related to the branching process. For two integers $n \in \mathbf{N}$ and $i \in \mathbf{N}^*$, we define the random variable $N_i^{(n)}$ as the number of discs of type i in the n^{th} generation, then $N^{(n)} = \sum_i N_i^{(n)}$. The expected value of $N_i^{(n)}$ will be denoted by $v_i^{(n)}$. Then, we define:

$$\begin{aligned} v^{(n)} &= (v_1^{(n)}, v_2^{(n)}, \dots, v_i^{(n)}, \dots), \\ \#v^{(n)} &= \sum_{j=1}^{+\infty} v_j^{(n)}. \end{aligned}$$

In other words, $\#v^{(n)}$ is simply the expected total number of individuals in the n^{th} generation. Then, we have the following relation

$$N_i^{(n+1)} = \sum_{j=1}^{+\infty} \sum_{l=1}^{N_j^{(n)}} \mathbf{U}_{(j,i)}^{(l)}, \quad (11)$$

where

- For $j \in \mathbf{N}^*$, the $N_j^{(n)}$ discs of type j are ranked using the lexicographic order of their centres $(B_{(j,l)}^{(n)})_{1 \leq l \leq N_j^{(n)}}$.

- For $i, j, l \in \mathbf{N}^*$,

$$\mathbf{U}_{(j,i)}^{(l)} = \sum_{(\xi, k, Z) \in \mathbf{Y}^{(n+1)}} k \mathbb{1}_{\{\xi \text{ is type } i \text{ and one of the } k \text{ discs starting from } \xi \text{ overlap } \partial B_{(j,l)}^{(n)}\}}.$$

The probability measure of $\mathbf{U}_{(j,i)}^{(l)}$ does not depend on $N_j^{(n)}$. Hence, for any $1 \leq l \leq N_j^{(n)}$

$$\mathbf{E} \left(\mathbf{U}_{(j,i)}^{(l)} \right) = \mathbf{E} \left(\sum_{(\xi, k, Z) \in \mathbf{Y}} k \mathbb{1}_{\{\xi \text{ is type } i \text{ and one of the } k \text{ discs starting from } \xi \text{ overlaps } \partial B(0, j)\}} \right).$$

In the sequel $\mathbf{E} \left(\mathbf{U}_{(j,i)}^{(l)} \right)$ is denoted by $\mu_{(i,j)}$. Then,

$$\mu_{(i,j)} \leq \mathbf{E} \left(\sum_{(\xi, k, Z) \in \mathbf{Y}} k^2 \mathbb{1}_{\{\xi \text{ is type } i \text{ and the first disc starting from } \xi \text{ overlaps } \partial B(0, j)\}} \right),$$

where the first ball of a given marked point (ξ, k, Z) is the one with centre $\xi + Z_1^{(1)}$.

For a given marked point $y = (\xi, k, Z)$, let us define the genre of y as the couple of integers (i, k) such that i is the radius of any disc starting from ξ , and k is the number of discs starting from ξ . It follows that:

$$\begin{aligned} \mu_{(j,i)} &\leq \sum_{k=1}^{+\infty} k^2 \mathbf{E} \left(\# \left\{ y = (\xi, k, Z) \in \mathbf{Y} \text{ of genre } (i, k) \text{ such that } B(\xi + Z_0^{(1)}, i) \cap B(0, j) \neq \emptyset \right\} \right), \\ &\leq \sum_{k=1}^{+\infty} k^2 \mathbf{E} \left(\# \left\{ y = (\xi, k, Z) \in \mathbf{Y} \text{ of genre } (i, k), \text{ such that } \xi + Z_0^{(1)} \in B(0, i + j) \right\} \right), \end{aligned}$$

We have $\text{Leb}(B(0, i+j)) = 4\pi(i+j)^2$. We define $\mathcal{Q} := \mathbf{E} \left(\# \{ y \in \mathbf{Y} \text{ de genre } (i, k) ; \xi + Z_0^{(1)} \in R_j^i \} \right)$ and Υ_i the law of $Z_0^{(1)}$ given the event $\{\mathbf{M} = i\}$. Using the Slivnyak-Mecke formula,

$$\begin{aligned} \mathcal{Q} &= \lambda \delta(\{k\}) \int_{\mathbf{R}^2} \left(\int_{(\mathbf{R}^2 \times \mathbf{R}_+^*)^{\mathbf{N}}} \mathbb{1}_{\{(\xi, k, Z) \text{ est de type } i\}} \mathbb{1}_{\{\xi + Z_0^{(1)} \in B(0, i+j)\}} \text{Leb}(d\xi) \right) \mathscr{W}(dZ) \\ &= \lambda \delta(\{k\}) p_i \int_{\mathbf{R}^2} \left(\int_{\mathbf{R}^2} \mathbb{1}_{\{\xi \in \tau_{-Z_0^{(1)}}(B(0, i+j))\}} \text{Leb}(d\xi) \right) \Upsilon_i(dZ_0^{(1)}), \\ &= \lambda \delta(\{k\}) p_i \int_{\mathbf{R}^2} \text{Leb}(B(0, i+j)) \Upsilon_i(dZ_0^{(1)}), \\ &= \lambda \delta(\{k\}) p_i \text{Leb}(B(0, i+j)), \\ &= 4\pi \lambda \delta(\{k\}) p_i (i+j)^2. \end{aligned}$$

Then

$$\begin{aligned} \mu_{(j,i)} &\leq \sum_{k=1}^{+\infty} 4k^2 \pi \delta(\{k\}) \lambda p_i (i+j)^2 \\ &\leq 4\mathbf{E}(K^2) \pi \lambda p_i (i+j)^2. \end{aligned} \tag{12}$$

Using the independence of processes $(\mathbf{Y}^{(n)})_n$, Equation (11) gives us the bound

$$v_i^{(n+1)} \leq \sum_{j=1}^{+\infty} v_j^{(n)} \mu_{(j,i)}. \quad (13)$$

The vectors $v^{(n)}$ and $v^{(n+1)}$ are related by the following matrix product $v^{(n+1)} \leq v^{(n)} \mathbf{A}$, where $\mathbf{A} = (\mu_{(k,l)})$ is a matrix with infinite number of rows (indexed by k) and columns (indexed by l). By iteration, we obtain that $v^{(n)} \leq v^{(0)} \mathbf{A}^n$, for all $n \geq 0$. Let us assume that the initial individuals have the type $i \in \mathbf{N}^*$ (in this case, $v^{(0)} = (0, \dots, \mathbf{E}(K), \dots)$ the row vector whose i^{th} element is $\mathbf{E}(K)$ and has all others elements zero), and μ_i is the expected total number of individuals in all generations, given that the initial individuals were of type i . Then, we obtain an upper bound for μ_i

$$\mu_i \leq \mathbf{E}(K) + \mathbf{E}(K) \sum_{n=1}^{+\infty} \sum_{j=1}^{+\infty} a_{(i,j)}^{(n)}, \quad (14)$$

where $a_{(i,j)}^{(n)}$ is the $(i,j)^{\text{th}}$ coefficient of \mathbf{A}^n . The branching process is defined such that

$$\mathbf{E}(\#C_0 \mid 0 \text{ is of type } i) \leq \mu_i. \quad (15)$$

This general point of view is described in [9] for example. The rest of the proof consists to obtain a good upper bound for μ_i . Using (12), we obtain upper bounds for $a_{(i,j)}^{(n)}$. For $i, j \in \mathbf{N}^*$, we have:

$$\begin{aligned} a_{(i,j)}^{(2)} &= \sum_{l=1}^{+\infty} a_{(i,l)}^{(1)} a_{(l,j)}^{(1)} = \sum_{l=1}^{+\infty} \mu_{(i,l)} \mu_{(l,j)}, \\ &\leq \left(4\mathbf{E}(K^2)\pi\lambda\right)^2 p_j \sum_{l=1}^{+\infty} p_l (j+l)^2 (i+l)^2, \\ &\leq \left(4\mathbf{E}(K^2)\pi\lambda\right)^2 p_j \sum_{l=1}^{+\infty} 16p_l (ijl)^4, \\ &\leq \left(16\mathbf{E}(K^2)\pi\mathbf{E}(\mathbf{M}^4)\lambda\right)^2 p_j j^4 i^4. \end{aligned}$$

If, for all i, j , we have $a_{(i,j)}^{(n-1)} \leq (16\mathbf{E}(K^2)\pi\mathbf{E}(\mathbf{M}^4)\lambda)^{n-1} p_j j^4 i^4$, it follows easily that $a_{(i,j)}^{(n)} \leq (16\mathbf{E}(K^2)\pi\mathbf{E}(\mathbf{M}^4)\lambda)^n p_j j^4 i^4$, and so the latter formula must be true for all i, j and n , using mathematical induction. Substituting this estimate into (14) we see that

$$\mu_i \leq \mathbf{E}(K) + i^4 \mathbf{E}(K) \mathbf{E}(\mathbf{M}^4) \sum_{n=1}^{+\infty} \left(16\mathbf{E}(K^2)\pi\mathbf{E}(\mathbf{M}^4)\lambda\right)^n.$$

which is finite if $16\mathbf{E}(K^2)\pi\mathbf{E}(\mathbf{M}^4)\lambda < 1$. In this case, the expected total number of individuals in all generations (given that the initial individuals were of type i) is finite, and consequently $\mathbf{E}(\#C_0 \mid 0 \text{ is type } i) < +\infty$. \square

5 Proof of Theorem 3.2

This section aims to prove that the unilateral line segment model (Model 1) with the following moment condition on the velocity \mathbf{V} , i.e.

$$\exists s > 1, \mathbf{E}\left(\exp(\mathbf{V}^s)\right) < +\infty, \quad (16)$$

does not percolate. In Section 5.1, we introduce the concept of *Poisson outdegree-one graph* (POG) which is our main structural tool to deeply understand the percolation properties of stopped germ-grain models having the finite time property. Thus, in Section 5.2, we describe our strategy to prove Theorem 3.2. Although it is largely inspired by the proof of Theorem 3.1 in [3] for bounded velocities, the adaptation to the unbounded velocities case requires some consequent work. See Sections 5.3 and 5.4.

5.1 Interpretation in terms of Poisson outdegree-one graphs

Recall that under (16), the unilateral line segment model \mathbf{X} is a stopped germ-grain model satisfying the finite time property: see Corollary 3.1 and Proposition 3.1. Hence, it is natural to associate to it an oriented (Poisson) outdegree-one graph since each marked point $x \in \mathbf{X}$ points out to its **unique** stopping marked point.

Definition 5.1 *Let \mathbf{X} be the unilateral line segment model defined in Section 2.2.1. A.s. the unique stopping marked point of any $x \in \mathbf{X}$ is denoted by $h(\mathbf{X}, x)$. Hence, we associate to \mathbf{X} a Poisson outdegree-one graph (POG) $\mathcal{G}(\mathbf{X})$ which is an oriented graph whose vertex set is \mathbf{X} and edge set is $\{(x, h(\mathbf{X}, x)) : x \in \mathbf{X}\}$. Moreover, we denote by $h_g(\mathbf{X}, x)$ the single point contained in $H(x, f_{\mathbf{X}}(x))$. Geometrically, $h_g(\mathbf{X}, x)$ represents the impact point of the line segment x over the line segment $h(\mathbf{X}, x)$.*

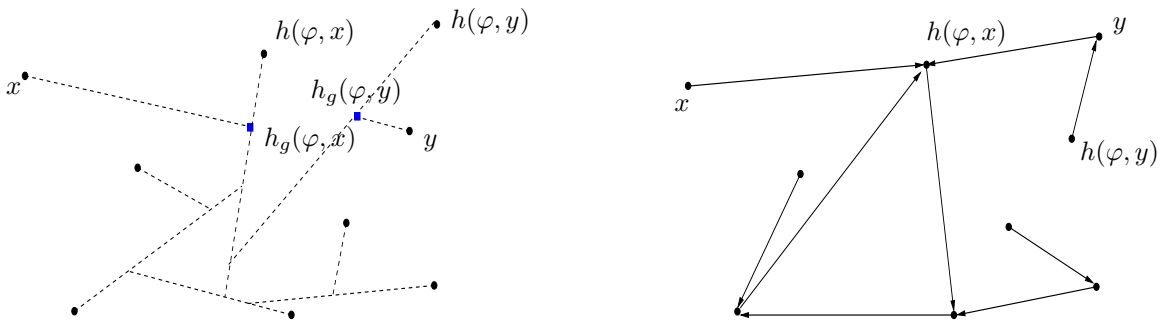


Figure 3: Here is a finite configuration of the unilateral line segment model (to the left) viewed as a POG (to the right). The blue squares represent the impact points $h_g(\mathbf{X}, x)$ and $h_g(\mathbf{X}, y)$ where $x, y \in \mathbf{X}$ are marked points. On this picture, $h(\mathbf{X}, x)$ is the outgoing vertex of $h(\mathbf{X}, y)$ what is written by $h(\mathbf{X}, x) = h(\mathbf{X}, h(\mathbf{X}, y))$. Remark also that the Forward sets of x and y contain a loop (the same one) of size 3.

This formalism corresponds exactly to the concept of POGs developed in [3] whose we recall the main notions right now. With probability 1, for any marked point $x \in \mathbf{X}$, the *Forward set*

$\text{For}(\mathbf{X}, x)$ of x in \mathbf{X} is defined as the sequence of outgoing vertices starting at x :

$$\text{For}(\mathbf{X}, x) := \{x, h(\mathbf{X}, x), h(\mathbf{X}, h(\mathbf{X}, x)), \dots\} .$$

By construction, the forward set $\text{For}(\mathbf{X}, x)$ is a branch of the POG possibly infinite. The *Backward set* $\text{Back}(\mathbf{X}, x)$ of x in \mathbf{X} is made up with all the marked points y having x in their Forward set, i.e.

$$\text{Back}(\mathbf{X}, x) := \{y \in \mathbf{X} : x \in \text{For}(\mathbf{X}, y)\} .$$

The Backward set $\text{Back}(\mathbf{X}, x)$ then admits a tree structure whose x is the root. The Forward and Backward sets of x may overlap; they (at least) contain x . Their union $C(\mathbf{X}, x)$ forms the *Cluster* of x in \mathbf{X} . Although the Cluster $C(\mathbf{X}, x)$ is a subset of the connected component of x in the POG $\mathcal{G}(\mathbf{X})$, the absence of infinite clusters clearly implies the one of infinite connected components.

Also the outdegree-one property of $\mathcal{G}(\mathbf{X})$ forces each cluster to contain at most one *loop*, i.e. a finite subset $\{y_1, \dots, y_l\} \subset \text{For}(\mathbf{X}, x)$ such that for any $1 \leq i \leq l$, $h(\mathbf{X}, y_i) = y_{i+1}$ (where the index $i + 1$ is taken modulo l). The integer l is called the *size* of the loop. It is not difficult to observe that the Forward set $\text{For}(\mathbf{X}, x)$ is finite if and only if it contains a loop. Hence, a finite cluster is made up with some finite trees merging on the same loop as illustrated in Figure 3. The notion of loop will be central in our study.

Let us end this section with recalling a general percolation result for POGs which is a direct consequence of the mass transport principle. See Proposition 4.1 of [3] for details.

Proposition 5.1 *Let \mathbf{X} be a stopped germ-grain model satisfying the finite time property. Then,*

$$\mathbf{P}(\forall x \in \mathbf{X}, \#C(\mathbf{X}, x) < +\infty) = 1 \iff \mathbf{P}(\forall x \in \mathbf{X}, \#\text{For}(\mathbf{X}, x) < +\infty) = 1 .$$

Henceforth, our goal is to show that a.s. each Forward set in the unilateral line segment model contains a loop. The main result of [3] (Theorem 3.1) provides two sufficient assumptions, namely the Loop assumption and the Shield assumption, ensuring the absence of forward percolation for POGs. This result applies to the line segment model with *bounded velocities*: see Theorem 3.2 in [3]. However, the Shield assumption which expresses a certain stabilizing property of the model, is no longer true in the context of *unbounded velocities*: roughly speaking the reader may think about a very distant– but very quick –marked point whose line segment would destroy any structure in a given domain.

We then have to adapt the proofs of [3] to unbounded velocities.

5.2 Strategy of the Proof of Theorem 3.2

We globally follow the strategy established in the proof of Theorem 3.1 in [3]. To help the reader we briefly recall the main steps (with precise references to [3]) and thus we insist on the novelties due to unbounded velocities.

Looping points. The marked point $x = (\xi, \Theta, V) \in \mathbf{X}$ is said *looping inside* a given subset Γ of \mathbf{R}^2 if its Forward set $\text{For}(\mathbf{X}, x)$ contains a loop $\{y_1, \dots, y_l\}$ for some $l \geq 1$ such that the center of mass of the set $\{y_1, \dots, y_l\}$ belongs to Γ . A *looping point* is merely a marked point whose Forward set admits a loop whose center of mass is localized. In the sequel, we will use

the notation $B(x, r)$ (instead of $B(\xi, r)$) for the open Euclidean ball with center ξ and radius $r > 0$, and \mathbf{X}_Γ for the elements of \mathbf{X} whose first coordinate is in Γ .

Definition 5.2 *Let $r < R$ be some positive real numbers and K be a positive integer. A marked point $x \in \mathbf{X}$ is said a (r, R, K) -looping point of \mathbf{X} if $\#\mathbf{X}_{B(x, R)} \leq K$ and x is looping inside the ball $B(x, r)$.*

Let $\gamma := (0, \Theta, V)$ be a typical marked point at the origin with direction Θ (uniformly distributed on $[0, 2\pi]$) and velocity V . Let us set $\mathbf{X}_\gamma := \mathbf{X} \cup \{\gamma\}$. Then, the mass transport principle leads to the next result.

Proposition 5.2 *(Proposition 4.5 of [3]) Any triplet (r, R, K) satisfies*

$$\mathbb{E}\left(\#\text{Back}(\mathbf{X}_\gamma, \gamma)\mathbb{1}_{\{\gamma \text{ is a } (r, R, K)\text{-looping point of } \mathbf{X}_\gamma\}}\right) < \infty . \quad (17)$$

Almost looping points. Let $v \in \mathbf{R}^2$. The translation operator τ_v acts on \mathbf{R}^2 and $\mathbf{R}^2 \times [0, 2\pi] \times \mathbf{R}_+^*$ as follows: for any $w \in \mathbf{R}^2$, $x = (\xi, \Theta, V) \in \mathbf{R}^2 \times [0, 2\pi] \times \mathbf{R}_+^*$ and $A \subset (\mathbf{R}^2 \times [0, 2\pi] \times \mathbf{R}_+^*)$, we set $\tau_v(w) = v + w$, $\tau_v(x) = (\xi + v, \Theta, V)$ and $\tau_v(A) = \bigcup_{x \in A} \tau_v(x)$. An *almost looping point* whose definition below is directly inspired by Definition 4.2 of [3], is set to become a looping point by adding some suitable marked points. In other words, the Forward set of an almost looping point can be stopped (by a loop) provided that the configuration is augmented with a finite number of well chosen points.

Definition 5.3 *Let us consider real numbers $0 < r < R$, a positive integer $K \in \mathbf{N}^*$, a maximal velocity $W > 0$ and an open ball $A \subset (B(0, r) \times [0, 2\pi] \times [0, W])^3$. A marked point $x \in \mathbf{X}$ is said a (r, R, W, K, A) -almost looping point of \mathbf{X} if $\#\mathbf{X}_{B(x, R)} \leq K$ and for any triplet $(x_1, x_2, x_3) \in A_x$, we have:*

- (i) *For $(\mathbf{X} \cup \{x_1, x_2, x_3\}, x) = \{x, x_1, x_2, x_3\}$;*
- (ii) *$\#\text{Back}(\mathbf{X} \cup \{x_1, x_2, x_3\}, x) \geq \#\text{Back}(\mathbf{X}, x)$;*

where $A_x = \tau_\xi(A)$ with $x = (\xi, \cdot)$.

Definition 5.3 says that a (r, R, W, K, A) -almost looping point x of \mathbf{X} becomes a $(r, R, K+3)$ -looping point of the augmented configuration $\mathbf{X} \cup \{x_1, x_2, x_3\}$ whenever the marked points x_1, x_2, x_3 are added in A_x ; these three marked points produce a loop of size 3 which stops the line segment of x . Hence, A_x can be understood as a suitable region to break the Forward set of x .

Let us specify also that the surprising Item (ii) in Definition 5.3 is crucial to get the next implication which compares mean sizes of Backward sets.

Proposition 5.3 *(Proposition 4.4 of [3]) If there exist parameters r, R, W, K, A such that*

$$\mathbb{E}\left(\#\text{Back}(\mathbf{X}_\gamma, \gamma)\mathbb{1}_{\{\gamma \text{ is a } (r, R, W, K, A)\text{-almost looping point for } \mathbf{X}_\gamma\}}\right) = \infty$$

then

$$\mathbb{E}\left(\#\text{Back}(\mathbf{X}_\gamma, \gamma)\mathbb{1}_{\{\gamma \text{ is a } (r, R, K+3)\text{-looping point for } \mathbf{X}_\gamma\}}\right) = \infty .$$

The following result states that the mean size of the Backward set of a typical almost looping point is infinite as soon as the Forward set of a typical marked point contains an infinite number of almost looping points with positive probability.

Proposition 5.4 (*Proposition 4.3 of [3]*) *If there exist parameters r, R, W, K, A such that*

$$\mathbb{P}\left(\#\{y \in \text{For}(\mathbf{X}_\gamma, \gamma); y \text{ is a } (r, R, W, K, A)\text{-almost looping point of } \mathbf{X}_\gamma\} = \infty\right) > 0 \quad (18)$$

then

$$\mathbb{E}\left(\#\text{Back}(\mathbf{X}_\gamma, \gamma) \mathbb{1}_{\{\gamma \text{ is a } (r, R, W, K, A)\text{-almost looping point of } \mathbf{X}_\gamma\}}\right) = \infty .$$

Conclusion. According to the Palm theory, the statement $\mathbf{P}(\#\text{For}(\mathbf{X}_\gamma, \gamma) < +\infty) = 1$ is enough to prove that a.s. the Forward set of any marked point is finite, and then to get Theorem 3.2. Hence, we proceed by contradiction, assuming that the typical point γ admits an infinite forward branch with positive probability:

$$\mathbf{P}(\#\text{For}(\mathbf{X}_\gamma, \gamma) = +\infty) > 0 . \quad (19)$$

Combining Propositions 5.2, 5.3 and 5.4, we have to prove the implication “(19) \Rightarrow (18)” to get a contradiction. This task was already present in [3] but in the bounded velocities context. Proving this implication for unbounded velocities is our main contribution here.

To do it, the idea consists in identifying (and isolating) the region of \mathbf{R}^2 perturbed (or *polluted*) by the too quick line segments as a subcritical percolation set, and in adapting the strategy of [3] outside this polluted set.

5.3 Pollution by quick line segments

Let m be a positive integer devoted to tend to infinity. A marked point (or a line segment) $x = (\xi_x, \Theta_x, V_x)$ is said *quick* whenever the growth velocity V_x of its line segment is larger than some *critical velocity* $V_c(m)$ (which will be specified below). Thus, let us partition the space \mathbf{R}^2 into blocks $mz \oplus \Lambda_m$ of size m , with $z \in \mathbf{Z}^2$ and $\Lambda_m := [-m/2, m/2]^2$. For any $z \in \mathbf{Z}^2$, let us define $V_m^{\max}(z)$ as the highest velocity inside the block $mz \oplus \Lambda_m$:

$$V_m^{\max}(z) := \max\{V_x : x \in \mathbf{X}_{mz \oplus \Lambda_m}\} .$$

First remark that at time 1, the extremity $H(\cdot, 1)$ of any line segment coming from the block $mz \oplus \Lambda_m$ is necessarily included in the square $mz \oplus [-V_m^{\max}(z) - \frac{m}{2}, V_m^{\max}(z) + \frac{m}{2}]^2$. Moreover, the inequality $V_m^{\max}(z) \geq V_c(m)$ means that the block $mz \oplus \Lambda_m$ contains at least one quick line segment.

For the sequel, we set

$$V_c(m) := \left(\log(m^3)\right)^{1/s} ,$$

where $s > 1$ is given by (16). The choice of the critical velocity $V_c(m)$ comes from the following compromise. On the one hand, $V_c(m)$ has to tend to infinity so that the probability for a given block $mz \oplus \Lambda_m$ to contain at least one quick line segment tends to 0– see Lemma 5.1. On the other hand, our construction of shield events– see Lemma 5.3 in Section 5.4 –requires that $V_c(m)$ increases very slowly. Satisfying both conditions needs a strong moment hypothesis on the speed distribution, namely (16).

Lemma 5.1 For any vertex $z \in \mathbf{Z}^2$,

$$\lim_{m \rightarrow \infty} \mathbf{P}(V_m^{\max}(z) \geq V_c(m)) = 0 .$$

Proof: It is enough to write, using stationarity of the model and the Markov inequality:

$$\begin{aligned} \mathbf{P}(V_m^{\max}(z) \geq V_c(m)) &= 1 - \exp\left(-\lambda m^2 \mathbf{P}(\mathbf{V} \geq V_c(m))\right) \\ &\leq \lambda m^2 \mathbf{P}(\mathbf{V} \geq V_c(m)) \\ &\leq \lambda \frac{\mathbf{E}(\exp(\mathbf{V}^s))}{m} . \end{aligned}$$

□

Let us introduce the *polluted set* (by quick line segments). Recall that $\lfloor \cdot \rfloor$ denotes the floor function and $B_\infty(z, r) = \{y \in \mathbf{Z}^2 : \|z - y\|_\infty < r\}$.

Definition 5.4 Let m, α be some positive integers. Let us define the pollution radius $R_{\alpha, m}(z)$ of $z \in \mathbf{Z}^2$ as

$$R_{\alpha, m}(z) := \mathbb{1}_{\{V_m^{\max}(z) \geq V_c(m)\}} \left(\left\lfloor \frac{V_m^{\max}(z)}{m} + \frac{1 + \alpha}{2} \right\rfloor + 1 \right) .$$

Thus, the pollution set $\Sigma_{\alpha, m} \subset \mathbf{Z}^2$ is defined as the following discrete Boolean model:

$$\Sigma_{\alpha, m} := \bigcup_{z \in \mathbf{Z}^2} B_\infty(z, R_{\alpha, m}(z)) . \quad (20)$$

Finally, any vertex z in $\Sigma_{\alpha, m}$ is said polluted.

It is important to note that the block $mz \oplus \Lambda_m$ corresponding to a polluted vertex $z \in \mathbf{Z}^2$, does not necessarily contain a quick line segment but is likely to be touched by one of them before time 1; this is the meaning of “polluted”. Conversely, for any vertex $z \notin \Sigma_{\alpha, m}$, the set $mz \oplus [-\frac{\alpha m}{2}, \frac{\alpha m}{2}]^2$ is not polluted in the sense that it will not be touched by a quick line segment until time 1. We will take advantage of this time interval to build shield structures outside the pollution set. Precisely, on the blocks $mz \oplus \Lambda_m$ with $z \notin \Sigma_{\alpha, m}$, we exhibit a sequence $(\mathcal{E}_m)_{m \geq 1}$ of local events whose probability tends to 1 with m and acting as “elementary shields”. The next central result holds for $\alpha = 16$.

Proposition 5.5 There exists a sequence $(\mathcal{E}_m)_{m \geq 1}$ of events such that:

- (a) **Localization.** For any m , the event \mathcal{E}_m is observable in $[-8m, 8m]^2$ as soon as $0 \notin \Sigma_{16, m}$;
- (b) **High probability.** $\mathbb{P}(\mathcal{E}_m \mid 0 \notin \Sigma_{16, m}) \rightarrow 1$ as $m \rightarrow \infty$;
- (c) **Shield property.** Let $\mathcal{W} \subset \mathbf{Z}^2 \setminus \Sigma_{16, m}$ such that $\mathbf{Z}^2 \setminus \mathcal{W}$ contains at least two connected components A_1 et A_2 (w.r.t the l_1 -norm) satisfying for i in $\{1, 2\}$,

$$A_i := \left(A_i \oplus \left[-\frac{1}{2}, \frac{1}{2} \right]^2 \right) \setminus (\mathcal{W} \oplus [-8, 8]^2) \neq \emptyset .$$

Assume that for any $z \in \mathcal{W}$, $\tau_{-mz}(\mathbf{X}) \in \mathcal{E}_m$. Then, for any $x \in \mathbf{X}_{mA_1}$ such that $\text{For}(\mathbf{X}, x) \cap \mathbf{X}_{mA_2} \neq \emptyset$, there exists $y \in \text{For}(\mathbf{X}, x) \cap \mathbf{X}_{m\mathcal{W} \oplus [-8m, 8m]^2}$ which is an almost looping point w.r.t. parameters $(5m, 6m, V_c(m), K, A_y)$ for suitable K and A_y .

The key Shield property (c) roughly ensures the presence of an almost looping point y in a Forward set $\text{For}(\mathbf{X}, x)$ when this one crosses a set of blocks on which the event \mathcal{E}_m occurs.

It is worth pointing out here that the radius of the ball A_y is observable inside $[-8m, 8m]^2$ but not its location. More details are available in Section 5.4.

Among the non polluted blocks $mz \oplus \Lambda_m$ with $z \notin \Sigma_{16,m}$, those on which the event \mathcal{E}_m (or its translated) occurs, are said *good*. The other ones are said *bad*. We extend this terminology to the corresponding vertices $z \notin \Sigma_{16,m}$. Let us consider the set S_m of bad vertices (among the non polluted ones):

$$S_m := \{z \in \mathbf{Z}^2 \setminus \Sigma_{16,m} : \tau_{-mz}(\mathbf{X}) \notin \mathcal{E}_m\} . \quad (21)$$

Proposition 5.6 *There exists an integer m_0 from which the set $\Sigma_{16,m} \cup S_m$ a.s. does not percolate (w.r.t the l_1 -norm).*

The rest of the proof is organized as follows. Propositions 5.5 and 5.6 are the last two ingredients to get Theorem 3.2: this is explained in Section 5.5. The proof of Proposition 5.5 is long and then devoted in Section 5.4, while the one of Proposition 5.6 is given just below.

Proof:(of Proposition 5.6). Let us realize the Poisson point process \mathbf{X} as the union

$$\mathbf{X} = \mathbf{X}_{\text{quick}}^{(m)} \sqcup \mathbf{X}_{\text{slow}}^{(m)}$$

of two independent Poisson point processes $\mathbf{X}_{\text{quick}}^{(m)}$ and $\mathbf{X}_{\text{slow}}^{(m)}$ with respective intensities $\lambda \mathbf{P}(V \geq V_c(m)) \otimes \Xi \otimes \mathcal{L}(V|V \geq V_c(m))$ and $\lambda \mathbf{P}(V < V_c(m)) \otimes \Xi \otimes \mathcal{L}(V|V < V_c(m))$. For the notations λ, Ξ, V the reader may refer to Section 2.2.1.

Let us consider the random field $\zeta := \{\zeta_z^{(m)}, z \in \mathbf{Z}^2\}$ where

$$\zeta_z^{(m)} := \mathbb{I}_{\{\tau_{-mz}(\mathbf{X}_{\text{slow}}^{(m)}) \notin \mathcal{E}_m\}} .$$

This is a (stationary) site percolation model which is 16-dependent (i.e. the r.v.'s $\zeta_z^{(m)}$ and $\zeta_{z'}^{(m)}$ are independent whenever $\|z - z'\|_\infty > 16$) with parameter $p_m = \mathbf{P}(\mathbf{X}_{\text{slow}}^{(m)} \notin \mathcal{E}_m)$. Since $\mathbf{X} \cap (mz \oplus \Lambda_m)$ and $\mathbf{X}_{\text{slow}}^{(m)} \cap (mz \oplus \Lambda_m)$ are equal as soon as z is not polluted, we get

$$p_m = \mathbf{P}(\mathbf{X}_{\text{slow}}^{(m)} \notin \mathcal{E}_m) = \mathbf{P}(\mathbf{X} \notin \mathcal{E}_m \mid 0 \notin \Sigma_{16,m}) \rightarrow 0$$

as $m \rightarrow \infty$ by Proposition 5.5, Item (b). Then, a classical stochastic domination result due to Liggett et al [8] allows to stochastically dominate the (dependent) field ζ by an independent site percolation model $\xi := \{\xi_z^{(m)}, z \in \mathbf{Z}^2\}$ with parameter $f(p_m)$ tending to 0 with m .

On the one hand, the introduction of fields ζ and ξ allows to control the set S_m of bad vertices among the non polluted ones, defined in (21), since a.s.

$$S_m \subset \{z \in \mathbf{Z}^2 : \zeta_z^{(m)} = 1\} \subset \{z \in \mathbf{Z}^2 : \xi_z^{(m)} = 1\} .$$

So, all the polluted or bad vertices are included in $\Sigma_{16,m} \cup \{z : \xi_z^{(m)} = 1\}$. It is then sufficient to prove that this set does not percolate for m large enough to obtain Proposition 5.6.

On the other hand, it is useful to remark that $\Sigma_{16,m} \cup \{z : \xi_z^{(m)} = 1\}$ can be viewed as the following discrete Boolean model

$$\Sigma_{16,m} \cup \{z : \xi_z^{(m)} = 1\} = \bigcup_{z \in \mathbf{Z}^2} B_\infty \left(z, R'_{16,m}(z) \right), \quad (22)$$

where $R'_{16,m}(z) := \max(R_{16,m}(z), \xi_z^{(m)})$, for any vertex z . Indeed, the collections $\{R_{16,m}(z), z \in \mathbf{Z}^2\}$ and $\{\xi_z^{(m)}, z \in \mathbf{Z}^2\}$ are each i.i.d. families of r.v.'s and they are also independent from each other since the $R_{16,m}(z)$'s (and the polluted set $\Sigma_{16,m}$) only depends on $\mathbf{X}_{\text{quick}}^{(m)}$ whereas the fields ζ (and then ξ) only depends on $\mathbf{X}_{\text{slow}}^{(m)}$.

A discrete version of Theorem 2.1 of J.-B. Guéré [6] asserts that the (discrete) Boolean model defined in (22) is subcritical provided the mean volume of a ball is small enough. This is the reason why we are going to prove that:

$$\lim_{m \rightarrow +\infty} \mathbf{E} \left(R'_{16,m}(0)^2 \right) = 0. \quad (23)$$

Since $f(p_m) \rightarrow 0$, it is enough to prove that $\mathbf{E}(R_{16,m}(0)^2)$ tends to 0 with m . This immediately follows from (24) and (25) below. First, by definition of the radius $R_{16,m}(0)$ and using the Poisson distribution of \mathbf{X} , we can write:

$$\begin{aligned} \mathbf{P}(R_{16,m}(0) = n) &\leq \mathbf{P}(V_m^{\max}(0) \geq m(n - 19/2)) \\ &\leq 1 - \exp \left(- \lambda m^2 \mathbf{P}(\mathbf{V} \geq m(n - 19/2)) \right) \\ &\leq \frac{\lambda \mathbf{E}(\mathbf{V}^4)}{m^2 (n - \frac{19}{2})^4}, \end{aligned} \quad (24)$$

for any $n \geq 10$. Thus, for the small values of n , we proceed like this;

$$\mathbf{P}(R_{16,m}(0) = n) \leq \mathbf{P}(V_m^{\max}(z) \geq V_c(m)) \rightarrow 0 \quad (25)$$

as m tends to infinity by Lemma 5.1. \square

5.4 Proof of Proposition 5.5

Definitions 5.5 and 5.6 below provide a candidate to be the event \mathcal{E}_m of Proposition 5.5 satisfying Items (a), (b) and (c). The first item, called **Localization**, is obtained by construction while the two other ones, resp. called **High probability** and **Shield property**, are checked resp. in Lemmas 5.3 and 5.4. The construction of \mathcal{E}_m follows the strategy established in Section 5 of [3].

Let us first introduce the triangular lattice and some related notations. Let Π be the triangular lattice:

$$\Pi := \left\{ a \vec{i} + b \vec{j} : a, b \in \mathbf{Z} \right\}$$

where $\vec{i} := (\sqrt{3}, 0)$ and $\vec{j} := (\sqrt{3} \cos(\frac{\pi}{3}), \sqrt{3} \sin(\frac{\pi}{3}))$. The usual graph distance on Π is denoted by d_Π . We also denote by $B^n(z)$ and $S^n(z)$ the (closed) ball and sphere with center z

and radius n w.r.t. d_{Π} . For any $z \in \Pi$, let $\text{Hex}(z)$ be the Voronoi cell of z w.r.t. the vertex set Π :

$$\text{Hex}(z) := \left\{ y \in \mathbf{R}^2, \|y - z\|_2 \leq \inf_{w \in \Pi \setminus \{z\}} \|y - w\|_2 \right\}.$$

The set $\text{Hex}(z)$ is a regular hexagon centred at z . For any integer $n \geq 0$, let us introduce the hexagonal complex of size n centred in z as

$$\text{Hex}^n(z) = \bigcup_{y \in B^n(z)} \text{Hex}(y).$$

For $\xi \in \mathbf{R}^2$, we also set $\text{Hex}^n(\xi) = \text{Hex}^n(0) + \xi$. Finally, for any integer $n \geq 1$, we define the hexagonal ring $C_n(0) = \text{Hex}^n(0) \setminus \text{Hex}^{n-1}(0)$ (with $\text{Hex}^0(0) = \text{Hex}(0)$).

In the sequel, we consider a positive integer m and assume that $0 \notin \Sigma_{16,m}$, i.e. 0 is not polluted. Since $\text{Hex}^{4m}(0) \subset [-8m, 8m]^2$, this assumption first means that all the line segments starting from $\text{Hex}^{4m}(0)$ have a growth velocity bounded by $V_c(m)$. Moreover, given $z \in B^{4m}(0)$, we can also assert that no quick line segment (i.e. having a growth velocity larger than $V_c(m)$) may hit $\text{Hex}(z)$ before time 1, since $\text{Hex}(z) \subset [-8m, 8m]^2$. Hence, for any $x = (\xi, \Theta, V) \in \mathbf{X}_{\text{Hex}(z)}$, there exists a time $\tau_x = \tau_x(m) \in (0, 1]$ such that the realization of the event $\{\|\xi - h_g(\mathbf{X}, x)\|_2 \leq V\tau_x\}$ only depends on the configuration $\mathbf{X}_{\text{Hex}(z)}$. To do it, we set τ_x small enough so that the Euclidean ball with center $H(x, \tau_x)$ and radius $\tau_x V_c(m)$ is included in $\text{Hex}(z)$:

$$\tau_x := \sup \{0 \leq t \leq 1 : B(H(x, t), tV_c(m)) \subset \text{Hex}(z)\}.$$

See Figure 4 for an illustration.

Since then two situations may occur while observing only $\mathbf{X}_{\text{Hex}(z)}$. Either the whole exploration of the line segment of x is observed until its stop before time τ_x . In this case, we set $f_m(x) := f_{\mathbf{X}}(x) < \tau_x$. Or we can only assert that the lifetime of x will be longer than τ_x and we set $f_m(x) := \tau_x < f_{\mathbf{X}}(x)$. In both cases, we have observed a subset $\text{Grain}(x, f_m(x))$ of the entire (or real) line segment $\text{Grain}(x, f_{\mathbf{X}}(x))$. Thus, we set

$$\text{Graph}_m(z) := \bigcup_{x \in \mathbf{X}_{\text{Hex}(z)}} \text{Grain}(x, f_m(x)). \quad (26)$$

The crucial point is that the random set $\text{Graph}_m(z)$ only depends on $\mathbf{X}_{\text{Hex}(z)}$ —this is explained on Figure 4—which allows us to use later the independence property of the Poisson point process \mathbf{X} .

Let us now introduce the notion of shield hexagons.

Definition 5.5 *Let $\epsilon \in (0, 1)$ and $z \in B^{4m}(0)$. The hexagon $\text{Hex}(z)$ is said (ϵ, m) -**shield** for \mathbf{X} if for all $a, b \in \mathbf{R}^2$ such that $a \notin \text{Hex}(z)$ and $b \in \epsilon \text{Hex}(z)$, we have*

$$(a, b) \cap \text{Graph}_m(z) \neq \emptyset.$$

*Moreover, for any integer $n > 0$ and $\{z_i\}_{1 \leq i \leq n} \subset B^{4m}(0)$, the collection $\{\text{Hex}(z_i)\}_{1 \leq i \leq n}$ is said (ϵ, m) -**shield** for \mathbf{X} if each $\text{Hex}(z_i)$ is (ϵ, m) -shield for \mathbf{X} .*

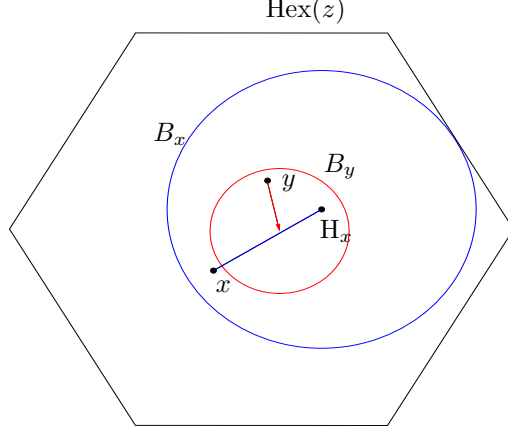


Figure 4: The assumption $0 \notin \Sigma_{16,m}$ allows to “locally” determine the line segment dynamic. Let $H_x := H(x, \tau_x)$ and $B_x := B(H_x, V_c(m)\tau_x)$ (with a blue circle). Let y be a marked point which could stop x before time τ_x . Since y is not quick, it has to belong to \mathbf{X}_{B_x} . Now let t (smaller than τ_x) be the time at which the line segment y would hit the one of x if it was not stopped before. Then, to check the survival of the line segment $\text{Grain}(y, \cdot)$ until t , we only need to observe the process \mathbf{X}_{B_y} where $B_y := B(H(y, t), tV_c(m))$ (with a red circle). Then, the triangle inequality ensures that $B_y \subset B_x$. By induction, the realization of the event $\{\|\xi - h_g(\mathbf{X}, x)\|_2 \leq V\tau_x\}$ only depends on \mathbf{X}_{B_x} .

In other words, $\text{Hex}(z)$ is (ϵ, m) -shield whenever its local exploration $\text{Graph}_m(z)$ creates a barrier (effective at time 1 in the ring $\text{Hex}(z) \setminus \epsilon\text{Hex}(z)$). Under the assumption $0 \notin \Sigma_{16,m}$, it is sufficient to observe \mathbf{X} inside $\text{Hex}(z)$ to determine if $\text{Hex}(z)$ is (ϵ, m) -shield for \mathbf{X} or not. Hence, two hexagons $\text{Hex}(z)$ and $\text{Hex}(z')$, with $z \neq z' \in B^{4m}(0)$, are independently (ϵ, m) -shield.

Furthermore, it is not difficult to convince oneself (using many small segments encircling the ring $\text{Hex}(z) \setminus \epsilon\text{Hex}(z)$, see Figure 5) that for any $\epsilon \in (0, 1)$,

$$p_{\epsilon,m} := \mathbf{P}(\text{Hex}(z) \text{ is } (\epsilon, m)\text{-shield} \mid 0 \notin \Sigma_{16,m}) > 0. \quad (27)$$

Definition 5.6 Let $\beta \in \{1, 2\}$. The set $\text{Hex}^{2\beta m}(0)$ is said (ϵ, m) -shielded for \mathbf{X} if, for all $y = (\xi', \Theta', V') \in \mathbf{X}$, the segment $[\xi', h_g(\mathbf{X}, y)]$ cannot overlap simultaneously the sets $\text{Hex}^{\beta m}(0)$ and $\mathbf{R}^2 \setminus \text{Hex}^{2\beta m}(0)$.

In other words, the set $\text{Hex}^{2\beta m}(0)$ is (ϵ, m) -shielded provided no line segment crosses completely the ring $\text{Hex}^{2\beta m}(0) \setminus \text{Hex}^{\beta m}(0)$, from inside or from outside.

In the sequel, for $\beta \in \{1, 2\}$, we establish in Lemma 5.2 the existence of an event $E_m^{(\beta)}$ on which $\text{Hex}^{2\beta m}(0)$ is (ϵ, m) -shielded. Thus, it will be proved in Lemma 5.3 that the probability of $E_m^{(\beta)}$, conditionally to $0 \notin \Sigma_{16,m}$, tends to 1 with m — this is Item (b) of Proposition 5.5. Finally, we will state in Lemma 5.4 that the sequence $(E_m^{(1)} \cap E_m^{(2)})_{m \geq 1}$ is a suitable candidate for Proposition 5.5.

The geometric construction of the event $E_m^{(\beta)}$ is directly inspired by Section 5 in [3]. We need some extra notations. Let $\eta \in \partial\text{Hex}^{\beta m}(0)$ where $\partial\Lambda$ denotes the topological boundary of

$\Lambda \subset \mathbf{R}^2$. For any $v \in [0, 1]$, we define the (semi-infinite) ray starting from η in the direction $\vec{v} := (\cos(2\pi v), \sin(2\pi v))$ by $l(\eta, \vec{v}) := \{\eta + t\vec{v}, t \geq 0\}$. Thus, we denote by \mathcal{L}^m the set of rays $l(\eta, \vec{v})$ coming from $\partial\text{Hex}^{\beta m}(0)$ which do not overlap the topological interior of $\text{Hex}^{\beta m}(0)$:

$$\mathcal{L}^m := \left\{ l(\eta, \vec{v}) : (\eta, v) \in \partial\text{Hex}^{\beta m}(0) \times [0, 1] \text{ and } l(\eta, \vec{v}) \cap \text{Int}(\text{Hex}^{\beta m}(0)) = \emptyset \right\} .$$

For any ray $l \in \mathcal{L}^m$, the set of hexagons included in $\text{Hex}^{2\beta m}(0) \setminus \text{Hex}^{\beta m}(0)$ and crossed by l is

$$\text{Cross}(l) := \{ \text{Hex}(z), \beta m + 1 \leq d_{\Pi}(0, z) \leq 2\beta m \text{ and } l \cap \text{Hex}(z) \neq \emptyset \} .$$

This set can be partitioned into different floors; $\text{Cross}_i(l)$ denotes the set of hexagons of $\text{Cross}(l)$ included in $C_i(0)$ for any $\beta m + 1 \leq i \leq 2\beta m$. Besides, let us remark that, for any ray $l \in \mathcal{L}^m$, there exists an index $\beta m + 1 \leq i(l) \leq 2\beta m$ such that, for all $i(l) \leq i \leq 2\beta m$, $\text{Cross}_i(l)$ contains at most three hexagons (when the ray l is almost parallel to the side of $\text{Hex}^{\beta m}(0)$ from which it starts, it may cross a large number of hexagons included in the same hexagonal ring $C_i(0)$ for i close to $\beta m + 1$).

The set $\text{Cross}(l)$ is said (ϵ, m) -**uncrossable** for \mathbf{X} whenever one can find two consecutive floors $\text{Cross}_i(l)$ and $\text{Cross}_{i+1}(l)$, for some index $i(l) \leq i \leq 2\beta m - 1$, which are both (ϵ, m) -shield for \mathbf{X} . Now, we can set the event $E_m^{(\beta)}(\epsilon)$ as

$$E_m^{(\beta)}(\epsilon) := \bigcap_{l \in \mathcal{L}^m} \{ \text{Cross}(l) \text{ is } (\epsilon, m)\text{-uncrossable for } \mathbf{X} \} . \quad (28)$$

Lemma 5.2 *There exists $\epsilon \in (0; 1)$ (close to 1) such that, for any $\beta \in \{1, 2\}$, $\text{Hex}^{2\beta m}(0)$ is a.s. (ϵ, m) -shielded for \mathbf{X} on the event $E_m^{(\beta)}(\epsilon) \cap \{0 \notin \Sigma_{16, m}\}$.*

Proof: Assume first that $0 \notin \Sigma_{16, m}$. There is no quick line segment in $\text{Hex}^{2\beta m}(0)$ and the quick line segments from the outside $\text{Hex}^{2\beta m}(0)$ are too far to hit $\text{Hex}^{2\beta m}(0)$ before time 1. Hence the local explorations $\text{Graph}_m(\cdot)$ defined in (26) and involved by the event $E_m^{(\beta)}(\epsilon)$ are realized without being disturbed by the quick line segments.

Let $\beta \in \{1, 2\}$. The proof of Proposition 5.2 of [3] shows that there exists ϵ_β sufficiently close to 1 such that a.s. on the event $E_m^{(\beta)}(\epsilon_\beta)$, any ray l in \mathcal{L}^m is obstructed, i.e. l necessarily hits a local exploration $\text{Graph}_m(z)$ for some z in $\text{Hex}^{2\beta m}(0) \setminus \text{Hex}^{\beta m}(0)$. Hence, the same holds for any line segments. The proof is concluded considering $\epsilon = \max\{\epsilon_1, \epsilon_2\}$. \square

From now on, we merely write $E_m^{(\beta)}$ instead of $E_m^{(\beta)}(\epsilon)$ and p_m instead of $p_{\epsilon, m}$, where ϵ is given by Lemma 5.2.

Lemma 5.3 *Given $\beta \in \{1, 2\}$,*

$$\lim_{m \rightarrow +\infty} \mathbf{P}\left(E_m^{(\beta)} \mid 0 \notin \Sigma_{16, m}\right) = 1 .$$

Proof: The proof of Proposition 5.3 in [3] gives

$$\mathbf{P}\left(\mathbf{X} \notin E_m^{(\beta)} \mid 0 \notin \Sigma_{16, m}\right) \leq Cm^3 \left(1 - p_m^6\right)^{\frac{m}{10}} , \quad (29)$$

where $C > 0$ is a constant. Let us specify that the power 6 on the probability p_m (defined in (27)) comes from the fact that we ask the hexagons belonging to two consecutive floors $\text{Cross}_i(l)$ and $\text{Cross}_{i+1}(l)$ for some ray l and some index $i \geq i(l)$ (at most 3 hexagons per floor) to be (ϵ, m) -shield for \mathbf{X} . Besides, a geometric construction (see Figure 5) leads to a lower bound for p_m : for any m ,

$$p_m \geq \left(\frac{C_1}{V_c(m)^2} \right)^{\frac{V_c(m)}{C_2}}, \quad (30)$$

where $C_1, C_2 > 0$ are constants.

Before proving (30), let us show how to conclude. Combining (29) and (30), we obtain

$$\mathbf{P} \left(\mathbf{X} \notin E_m^{(\beta)} \mid 0 \notin \Sigma_{16,m} \right) \leq C m^3 \exp \left(- \frac{m}{10} \left(\frac{C_1}{V_c(m)^2} \right)^{\frac{6V_c(m)}{C_2}} \right). \quad (31)$$

Thus, for some $C_3 > 0$,

$$\left(\frac{C_1}{V_c(m)^2} \right)^{\frac{6V_c(m)}{C_2}} \geq \exp \left(- C_3 V_c(m) \log V_c(m) \right)$$

which is larger than $m^{-1/2}$, for m large enough, using $V_c(m) = (\log m^3)^{1/s}$ with $s > 1$. The expected result then follows.

It remains to check inequality (30). To this end, we proceed geometrically by building a loop with many microscopic line segments inside $\text{Hex}(0) \setminus \epsilon \text{Hex}(0)$ and encircling $\epsilon \text{Hex}(0)$. To be sure that each microscopic line segment will contribute effectively to the loop before being stopped by another line segment coming from outside the ring $\text{Hex}(0) \setminus \epsilon \text{Hex}(0)$, it is necessary to introduce a number of order $V_c(m)$ of microscopic line segments. Each of them has a starting point located in a small domain with area of order $1/V_c(m)^2$. These constraints lead to the lower bound (30). See Figure 5 for details. \square

We are now able to prove Proposition 5.5.

Lemma 5.4 *Proposition 5.5 holds with $\mathcal{E}_m := E_m^{(1)} \cap E_m^{(2)}$.*

Proof: Let m be a positive integer. We have to check the three items of Proposition 5.5. Assuming that $0 \notin \Sigma_{16,m}$, the construction developed above ensures that the event \mathcal{E}_m only involves local explorations $\text{Graph}_m(z)$ with $z \in B^{4m}(0)$, i.e. with $\text{Hex}(z) \subset \text{Hex}^{4m}(0)$. So Item (a) **Localization** follows from the inclusion $\text{Hex}^{4m}(0) \subset [-8m, 8m]^2$. Item (b) **High probability** is given by Lemma 5.3. So it only remains to check the third one, Item (c) **Shield property**.

For this purpose, let us consider three disjoint subsets $\mathcal{W}, \mathcal{A}_1, \mathcal{A}_2$ of \mathbf{Z}^2 and $\mathcal{A}_1, \mathcal{A}_2$ as in Proposition 5.5. We assume that $\tau_{-mz}(\mathbf{X}) \in \mathcal{E}_m$ for any $z \in \mathcal{W}$. Thus, let us consider $x = (\xi, \Theta, V) \in \mathbf{X}_{m\mathcal{A}_1}$ such that $\text{For}(\mathbf{X}, x) \cap \mathbf{X}_{m\mathcal{A}_2} \neq \emptyset$. Our goal is to identify an almost looping point $y \in \text{For}(\mathbf{X}, x) \cap \mathbf{X}_{m\mathcal{W} \oplus [-8m, 8m]^2}$ (with suitable parameters).

Let us write the Forward set $\text{For}(\mathbf{X}, x)$ as a sequence $(x_i)_{i \geq 0}$ with $x_0 = x$ and for all index i , $x_{i+1} = h(\mathbf{X}, x_i)$. We also set $x_i := (\xi_i, \Theta_i, V_i)$ for any i . By assumption, there exists an index $n \geq 1$ such that $x_n \in \mathbf{X}_{m\mathcal{A}_2}$ and also an index $1 \leq i \leq n$ and a vertex $u \in \mathcal{W}$ such

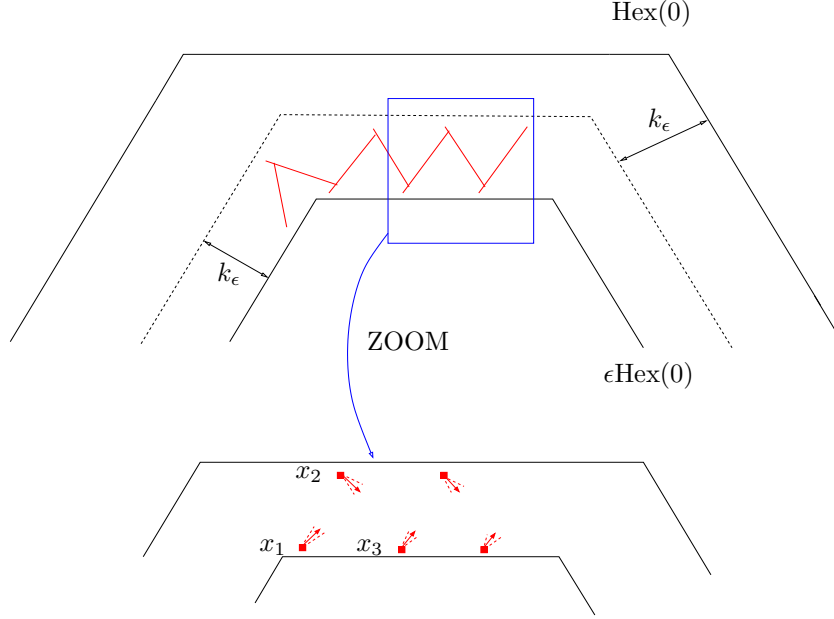


Figure 5: This picture illustrates the building of a loop $\{x_1, \dots, x_n\}$ inside on a strip of size k_ϵ . Each segment x_i of the loop can grow during a time in the order of $\frac{k_\epsilon}{V_c(m)}$ without perturbation caused by the segments starting from the outside of $\text{Hex}(0)$. The building is organized in order to allow each segment x_i to grow (with a velocity larger than a constant \mathcal{V} independent of m) during this time without be stopped. To do this, we impose that the discs centred on the growing extremities with radii $\frac{k_\epsilon \mathcal{V}}{V_c(m)}$ do not contain any other point that the associated germ. Moreover, the growth direction of any segment is chosen such that x_i is stopped by x_{i+1} (we can fix the angle of propagation in a deterministic interval). Then, there exist $C_2 > 0$ such that the number of segments contained in the loop constructed is smaller than $\frac{V_c(m)}{C_2}$. The independence property of the Poisson point process and the description above guarantee that the probability that $\epsilon \text{Hex}(0)$ is encircled in a loop is bigger than $\left(\frac{C_1}{V_c(m)^2}\right)^{\frac{V_c(m)}{C_2}}$, where C_1 is a positive constant independent of m .

that $[\xi_i, h_g(\mathbf{X}, x_i)]$ overlaps $\text{Hex}^m(mu)$. The hypothesis $\tau_{-mu}(\mathbf{X}) \in E_m^{(1)}$ actually implies that $[\xi_i, h_g(\mathbf{X}, x_i)]$ is completely included in $\text{Hex}^{2m}(mu)$ (with Lemma 5.2 and Definition 5.6).

Now, we are going to prove that the vertex x_i is an almost looping point of \mathbf{X} . Let z_i be the element of $mz \oplus B^{2m}(0)$ such that $\xi_i \in \text{Hex}(z_i)$. Recall that $\text{Grain}(x_i, f_m(x_i))$ is the subset (that we can only observe through $\mathbf{X}_{\text{Hex}(z_i)}$, see Figure 4) of the true line segment $\text{Grain}(x_i, f_{\mathbf{X}}(x_i))$. Moreover, we define $\text{Grain}(x_i, F_m(x_i))$ as the longest grain from x_i remaining inside $\text{Hex}^{2m}(mu)$ (without interaction with other marked points, see Figure 6). The inclusion $[\xi_i, h_g(\mathbf{X}, x_i)] \subset \text{Hex}^{2m}(mu)$ forces:

$$f_m(x_i) \leq f_{\mathbf{X}}(x_i) \leq F_m(x_i)$$

which means that $h_g(\mathbf{X}, x_i)$ lies somewhere on the segment $[\text{H}(x_i, f_m(x_i)), \text{H}(x_i, F_m(x_i))]$. Thus we have to identify a small ball A_{x_i} corresponding to Definition 5.3, i.e. a suitable region to break the Forward set of x_i without reducing its Backward set. So, this small ball A_{x_i}

has to be located close to (or just before) $h_g(\mathbf{X}, x_i)$. A difficulty appears at this stage: the observation of \mathbf{X} through $mu \oplus [-8m, 8m]^2$ does not guarantee to determine the stopping vertex $x_{i+1} = h(\mathbf{X}, x_i)$ and then the location of $h_g(\mathbf{X}, x_i) \in \text{Hex}^{2m}(mu)$. However, thanks to the hypothesis $\tau_{-mu}(\mathbf{X}) \in E_m^{(2)}$ we know that x_{i+1} is in $\text{Hex}^{4m}(mu)$ and consequently we only have a finite number of candidates for it.

This is the reason why we need to consider

$$\text{Ray}(u, m, x_i) := \bigcup_{x' \in \mathbf{X}_{\text{Hex}^{4m}(mz)} \setminus \{x_i\}} \text{Grain}(x', \infty) .$$

A marked point $x' \in \mathbf{X}_{\text{Hex}^{4m}(mz)} \setminus \{x_i\}$ is a potential stopping marked point for x_i if and only if $\text{Grain}(x', \infty)$ overlaps $[\mathbf{H}(x_i, f_m(x_i)), \mathbf{H}(x_i, F_m(x_i))]$. Then, there exists a random integer $l > 0$ and times $0 < t_1 < \dots < t_l \leq F_m(x_i)$ such that $\mathbf{H}(x_i, t_1), \dots, \mathbf{H}(x_i, t_l)$ are the only possible locations for $h_g(\mathbf{X}, x_i)$ which are created by $y_1, \dots, y_l \in \mathbf{X}_{\text{Hex}^{4m}(mu)} \setminus \{x_i\}$ so that for any $1 \leq j \leq l$, $\mathbf{H}(x_i, t_j) \in \text{Grain}(y_j, \infty)$. See Figure 6.

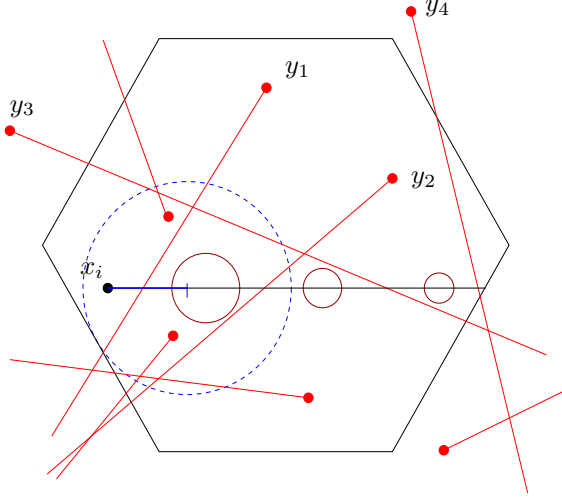


Figure 6: On this picture, $l = 4$ and $j = 2$. The three balls $B(w_k, r_k)$ (for $2 \leq k \leq 4$) are drawn in brown.

The stopping marked point of x_i in \mathbf{X} belongs to $\{y_1, \dots, y_l\}$. Moreover, if $t_k < f_m(x_i)$, the marked point y_k is not able to stop the line segment starting from x_i . Then, we set

$$j := \min \{k \in \llbracket 1, l \rrbracket ; t_k \geq f_m(x_i)\} .$$

The integer j is well defined because we know that $h(\mathbf{X}, x_i) \in \text{Hex}^{4m}(mu)$, so there exists at least one candidate. So, $h(\mathbf{X}, x_i) \in \{y_j, \dots, y_l\}$. To check which point is the real stopping point, we may have to control the state of \mathbf{X} outside of $mu \oplus [-8m, 8m]^2$. As we do not want to do such exploration in order to preserve the locality of the event \mathcal{E}_m , we identify a finite number of candidates for the "looping region" in Definition 5.3.

Let us fix $k \in \llbracket j, l \rrbracket$, let us denote by w_k the middle of the segment $[\mathbf{H}(x_i, t_{k-1}), \mathbf{H}(x_i, t_k)]$ (with $t_0 = 0$ and $\mathbf{H}(x_i, 0) = \xi_i$). There exists $r_k > 0$ such that

$$B(w_k, r_k) \cap \text{Ray}(u, m, x_i) = \emptyset \tag{32}$$

and

$$r_k < \|\xi_i - w_k\|_2 \quad (33)$$

In the cas where x_{i+1} is y_k , the obtained ball $B(w_k, r_k)$ is then a suitable region in which we could create an obstacle for the growing segment x_i without reducing its Backward set. Precisely, we add a triplet of marked points in $B(w_k, r_k)$ which shapes a triangle of stopped line segments. This triangle will be active before the arrival of the grain from x_i (33) and do not break any other line (32). Consider the set $\mathcal{A}_{x_i}^{(k)} \subset (B(w_k, r_k) \times [0, 2\pi] \times [V_i, V_c(m)])^3$, where V_i denotes the growth velocity of x_i , such that for all triplets $(a_0, a_1, a_2) \in \mathcal{A}_{x_i}^{(k)}$:

- (i) $h(\mathbf{X} \cup \{a_0, a_1, a_2\}, a_l) = a_{l+1}$ for $l = 0, 1, 2$ (where the index $l + 1$ is taken modulo 3).
- (ii) The triangle defined by the vertices $h_g(\mathbf{X} \cup \{a_0, a_1, a_2\}, a_l)$, $l = 0, 1, 2$, is included in $B(w_k, r_k)$ and contains w_k .

It is not difficult to see that $\mathcal{A}_{x_i}^{(k)}$ contains a non-empty open ball $A_{x_i}^{(k)} \subset (B(w_k, r_k) \times [0, 2\pi] \times [V_i, V_c(m)])^3$. In the cas where x_{i+1} is y_k , by (33), (i) and (ii), any add of triplet $(a_0, a_1, a_2) \in A_{x_i}^{(k)}$ forces the growing segment from x_i to hit the loop created by a_0, a_1, a_2 :

$$\text{For}(\mathbf{X} \cup \{a_0, a_1, a_2\}, x_i) = \{x_i, a_0, a_1, a_2\} .$$

Moreover, condition (33) in addition with (i) and (ii) imply that no growing segment except x_i is changing by the added marked points $\{a_0, a_1, a_2\}$:

$$\text{Back}(\mathbf{X} \cup \{a_0, a_1, a_2\}, x_i) = \text{Back}(\mathbf{X}, x_i) .$$

Then, we have proved that x_i is an almost looping point with "looping ball" $A_{x_i}^{(k)}$ in the case where x_{i+1} is y_k . More precisely, it is not difficult to check that x_i is a $(5m, 6m, V_c(m), K, \cdot)$ -almost looping point for K large enough.

Remark that, even if the "true looping ball" A_{x_i} is not precisely located, the candidates $A_{x_i}^{(k)}$, $j \leq k \leq l$ only depend on the process \mathbf{X} inside $\text{Hex}^{4m}(mu) \subset [-8m, 8m]^2$. The same holds for their radii. \square

5.5 Conclusion

Let us end with the proof of Theorem 3.2 from Propositions 5.5 and 5.6.

According to the strategy developed in Section 5.2, we have assumed by absurd that the Forward set $\text{For}(\mathbf{X}_\gamma, \gamma)$ is infinite with positive probability, where $\gamma = (0, \Theta, V)$ denotes the typical marked point at the origin. To get a contradiction, we have to prove (18), i.e. the existence of deterministic parameters r, R, W, K, A such that

$$\mathbf{P}\left(\#\{x \in \text{For}(\mathbf{X}_\gamma, \gamma) : x \text{ is a } (r, R, W, K, A)\text{-almost looping point of } \mathbf{X}_\gamma\} = +\infty\right) > 0 .$$

Using notations introduced in the previous section, for any m , we strengthen the event \mathcal{E}_m into a new event \mathcal{E}_m^l with assuming two extra conditions:

1. Any almost looping point $x \in \mathbf{X}_{\text{Hex}^{2m}(0)}$ satisfies $\text{radius}(A_x) > \delta_m$;

2. $\#\mathbf{X}_{\text{Hex}^{4m}(0)} \leq K_m$.

Remark that Proposition 5.5 still holds when \mathcal{E}_m is replaced with \mathcal{E}'_m . Indeed, its probability tends to 1 with m provided that $\delta_m \rightarrow 0$ and $K_m \rightarrow \infty$ fast enough. As \mathcal{E}_m , the event \mathcal{E}'_m is also measurable w.r.t. $\mathbf{X}_{[-8m,8m]^2}$ as soon as $0 \notin \mathbf{B}_{(16,m)}$. Of course, the **Shield property** remains true on $\mathcal{E}'_m \subset \mathcal{E}_m$.

Moreover, the subcritical result (Proposition 5.6) also holds with \mathcal{E}'_m instead of \mathcal{E}_m . So we choose m large enough so that the random set $\mathbf{B}_{(16,m)} \cup \mathbf{S}_m$ does not percolate with probability 1 (where \mathbf{S}_m is defined with \mathcal{E}'_m instead of \mathcal{E}_m). Hence, even if the trajectory $\text{For}(\mathbf{X}_\gamma, \gamma)$ visits infinitely many connected components of $\mathbf{B}_{(16,m)} \cup \mathbf{S}_m$ (which are all finite), to go from one of them to the next one, $\text{For}(\mathbf{X}_\gamma, \gamma)$ has to cross a set of blocks on which \mathcal{E}'_m occurs and then admits inside an almost looping point by Proposition 5.5. Then, with positive probability, $\text{For}(\mathbf{X}_\gamma, \gamma)$ contains infinitely many almost looping points (of \mathbf{X}_γ). It just remains to exhibit a common and deterministic set of parameters for a infinite number of those almost looping points: this is the role of the event \mathcal{E}'_m .

Let $y \in \text{For}(\mathbf{X}_\gamma, \gamma)$ be such a visited almost looping point, there exists $z_y \in \mathbf{Z}^2$ such that $\tau_{-mz_y}(\mathbf{X}) \in \mathcal{E}'_m$ and $[y, h_y(\mathbf{X}_\gamma, y)] \subset \text{Hex}^{2m}(mz_y)$. As previously mentioned, y is a $(5m, 6m, V_c(m), K_m, A_y)$ -almost looping point where the random ball A_y , with a radius larger than δ_m , is included in $(B(\eta, 5m) \times [0, 2\pi] \times [0, V_c(m)])^3$. Let us consider a finite covering of $(B(0, 5m) \times [0, 2\pi] \times [0, V_c(m)])^3$ by open euclidean balls $\{\mathcal{K}_j, 1 \leq j \leq j(m)\}$ with radius $\frac{\delta_m}{2}$. Then, the pigeonhole principle asserts that there exists a deterministic ball \mathcal{K}_{j_0} with some deterministic $1 \leq j_0 \leq j(m)$ such that, with positive probability, among the $(5m, 6m, V_c(m), K_m, A_y)$ -almost looping points visited by $y \in \text{For}(\mathbf{X}_\gamma, \gamma)$, infinitely many of them satisfy $\tau_y(\mathcal{K}_{j_0}) \subset A_y$.

We then conclude that with positive probability, $\text{For}(\mathbf{X}_\gamma, \gamma)$ contains infinitely many almost looping points of \mathbf{X}_γ w.r.t. $(5m, 6m, V_c(m), K_m, \mathcal{K}_{j_0})$.

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