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# Small cycle structure for words in conjugation invariant random permutations

Mohamed Slim Kammoun\* Mylène Maïda†

October 24, 2023

## Abstract

We study the cycle structure of words in several random permutations. We assume that the permutations are independent and that their distribution is conjugation invariant, with a good control on their short cycles. If, after successive cyclic simplifications, the word  $w$  still contains at least two different letters, then we get a universal limiting joint law for short cycles for the word in these permutations. These results can be seen as an extension of our previous work [Kammoun and Maïda, 2020] from the product of permutations to any non-trivial word in the permutations and also as an extension of the results of [Nica, 1994] from uniform permutations to general conjugation invariant random permutations. In particular, we get optimal assumptions in the case of the commutator of two such random permutations.

## 1 Introduction and statement of the results

The object of this paper is to study the behavior of the cycles up to a fixed size for a word in several independent conjugation invariant random permutations.

We start by recalling that the asymptotics of the short cycle structure of one permutation chosen uniformly in the symmetric group is well known and Poisson. If we denote by  $\mathfrak{S}_n$  the symmetric group of size  $n \in \mathbb{N}^*$ , and by  $\#\ell\sigma$  the number of cycles of length  $\ell$  of the permutation  $\sigma$ , we have :

**Theorem 1.1.** [Arratia et al., 2000, Theorem 3.1] For  $\rho_n$  following the uniform distribution on  $\mathfrak{S}_n$ , for any  $d' \geq 1$ ,

$$(\#\_1\rho_n, \#\_2\rho_n, \dots, \#\_{d'}\rho_n) \xrightarrow[n \rightarrow \infty]{(d)} (\xi_1, \dots, \xi_{d'}),$$

where  $\xrightarrow[n \rightarrow \infty]{(d)}$  is the convergence in distribution and  $(\xi_1, \dots, \xi_{d'})$  are independent variables, where the distribution of  $\xi_\ell$  is Poisson with parameter  $\frac{1}{\ell}$ .

Moreover, the joint moments converge and in particular,

$$\mathbb{E}(\#\_1\rho_n) \xrightarrow[n \rightarrow \infty]{} 1.$$

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It is also known that the asymptotic short cycle structure for words in independent uniformly distributed permutations is Poisson. More precisely, let  $k > 1$  be fixed and  $F_k := \langle x_1, \dots, x_k \rangle$  be the free group on  $k$  generators  $\{x_1, \dots, x_k\}$ . In the sequel, the index  $k$  will always refer to the number of generators. An element  $w$  of  $F_k$  can be written

$$w = x_{k_r}^{\varepsilon_r} \dots x_{k_1}^{\varepsilon_1},$$

with  $\varepsilon_j = \pm 1$  and if  $k_{j+1} = k_j$  then  $\varepsilon_{j+1} = \varepsilon_j$ . Such a form is unique, we call it in the sequel the *canonical form* of the word  $w$ .

For every group  $G$ ,  $w = x_{k_r}^{\varepsilon_r} \dots x_{k_1}^{\varepsilon_1}$  induces a word map from the Cartesian product  $G^k$  into  $G$  by substitution :

$$\begin{aligned} w : \quad G^k &\rightarrow G \\ (g_1, \dots, g_k) &\mapsto g_{k_r}^{\varepsilon_r} \dots g_{k_1}^{\varepsilon_1}. \end{aligned}$$

We consider the case when  $G = \mathfrak{S}_n$ , the symmetric group of size  $n \in \mathbb{N}^*$ , equipped with its composition law. When  $g_1, \dots, g_k$  is a  $k$ -tuple of random permutations, their image by the word map  $w$  is again a random permutation and one can study the distribution of its cycle structure. The more natural case to consider is independent uniformly distributed random permutations. It has been investigated by Nica, and more recently by Puder and coauthors. They have shown that the limiting distribution depends only on the algebraic structure of the word  $w$ . For any word  $\Omega$ , and  $d \in \mathbb{N}^*$ ,  $\Omega^d$  denotes the composition of  $d$  copies of the word  $\Omega$ . We say that the word  $w$  is a *power* if there exists a word  $\Omega \neq 1$  and  $d \geq 2$  such that  $w = \Omega^d$ . Note that any word  $w \neq 1$  can be written  $w = \Omega^d$  where  $\Omega$  is not a power and  $d \geq 1$ . We have the following :

**Proposition 1.2.** [*Hanany and Puder, 2020, Theorem 1.3*] [*Nica, 1994, Theorem 1.1*] *Let  $w = \Omega^d \neq 1$  be a word in  $F_k$  such that  $d \geq 1$  and  $\Omega$  is not a power. Let  $\rho_{1,n}, \dots, \rho_{k,n}$  be independent and **uniformly distributed** on  $\mathfrak{S}_n$ . Then the random vector  $(\#_1 w(\rho_{1,n}, \dots, \rho_{k,n}), \rho_{k,n}), \dots, \#_d w(\rho_{1,n}, \dots, \rho_{k,n}))$  converges in distribution to a limit that depends only on the power  $d$  and the maximal length  $d'$  of the cycles under consideration. Moreover, the joint moments converge also to the corresponding moments of the limiting law. In particular,*

$$\mathbb{E}(\#_1 w(\rho_{1,n}, \dots, \rho_{k,n})) \xrightarrow[n \rightarrow \infty]{} \psi(d) \tag{1}$$

where  $\psi(d) := \sum_{\ell|d} 1$  is the number of divisors of the integer  $d$ .

Remarkably, the limiting law does not depend on  $\Omega$ . In particular, the limiting law is the same if we replace the word  $\Omega$  by the word  $x_1$  in just one letter.

Our goal in this paper will be to explore universality for these beautiful results. Before stating ours, let us give a few motivations to explore words in random permutations. The initial motivations of [Nica, 1994] was related to free probability theory and the notion of asymptotic freeness. Without going too much into the details, one can say that for a  $k$ -tuple of random matrices  $X_{1,N}, \dots, X_{k,N}$  with a prescribed joint law, their asymptotic joint distribution in the framework of free probability theory is determined by the limit of  $\mathbb{E}(\frac{1}{N} \text{Tr } w(X_{1,N}, \dots, X_{k,N}))$ , for any word  $w$ . If the random matrices are independent and their distribution is unitarily invariant, the behavior of the limit is well known and related to the notion of freeness - one says that such random matrices are asymptotically free. It was natural in this context to ask the same question for permutation matrices. This is the point of view of Nica. As the trace of a permutation matrix is related to the number of fixed points, one can see that the question of asymptotic freeness in this case is closely related to the study of the short cycle structure. In the same direction, one can also mention the theory of traffic distribution, developed by Male and coauthors (see e.g. [Male, 2020, Au et al., 2021]), exploring the

case when unitary invariance is replaced by permutation invariance and the notion of second order freeness introduced by Mingo and Speicher (see e.g. [Mingo and Speicher, 2006, Male, 2021]). Talking about links with random matrix theory, [Dubach, 2021] studies the total number of cycles for the commutator of two independent uniform permutations by using a connection with the moments in non-Hermitian random matrices.

In a series of papers [Puder and Parzanchevski, 2015, Hanany and Puder, 2020, Magee and Puder, 2021], Puder and coauthors use the properties of the distribution of some statistics of  $w(\rho_{1,n}, \dots, \rho_{k,n})$ , as the number of fixed points, short cycles etc. to characterize the algebraic properties of the word  $w$ , through what they call the primitivity rank of the word  $w$ . Understanding  $w(\rho_{1,n}, \dots, \rho_{k,n})$  can then give access to properties of the word map for other groups  $G$ .

In the present work, the goal is to extend Proposition 1.2 to a  $k$ -tuple  $(\sigma_{1,n}, \dots, \sigma_{k,n})$  of **independent** and **conjugation invariant** random permutations.

**Definition 1.3.** *For any  $n \geq 1$ , let  $(\sigma_{1,n}, \dots, \sigma_{k,n})$  be random permutations in  $(\mathfrak{S}_n)^k$ . We say that the (distribution of the)  $k$ -tuple  $(\sigma_{1,n}, \dots, \sigma_{k,n})$  is conjugation invariant if for any fixed permutations  $\sigma_1, \dots, \sigma_k \in \mathfrak{S}_n$ ,  $(\sigma_1^{-1} \circ \sigma_{1,n} \circ \sigma_1, \dots, \sigma_k^{-1} \circ \sigma_{k,n} \circ \sigma_k)$  has the same distribution as  $(\sigma_{1,n}, \dots, \sigma_{k,n})$ .*

One can give several natural motivations to extend the results known for uniform permutations in this direction. First, if we go back to the question of asymptotic freeness mentioned above, seeking for universality for these questions has always been a strong motivation. For example, the work [Benaych-Georges, 2010] is clearly presented as an extension of [Nica, 1994] with motivations arising from free probability theory and studies some particular conjugation invariant distributions. It is also worth mentioning a connection with random maps : in [Budzinski et al., 2019] for example, the study of random maps is strongly related to the study of the cycle structure of the product of two permutations, one with prescribed cycle structure (corresponding to the faces of the map), the other being an involution without fixed points (corresponding to the edges), both naturally inheriting conjugation invariant distributions.

Before stating our theorem, let us detail our assumptions on random permutations having few short cycles.

**Assumption 1.4.** *Let  $S_1, S_2$  be two finite subsets of  $\mathbb{N}^*$ . Let  $(\sigma_n)_{n \in \mathbb{N}^*}$  be a sequence of random permutations (with for all  $n \geq 1, \sigma_n \in \mathfrak{S}_n$ ). We say that the sequence  $(\sigma_n)_{n \in \mathbb{N}^*}$  has few short cycles with respect to  $(S_1, S_2)$*

- in the weak sense if

$$\forall i \in S_1, \frac{\#_i \sigma_n}{\sqrt{n}} \xrightarrow[n \rightarrow \infty]{\mathbb{P}} 0, \quad (2)$$

and

$$\forall i \in S_2, \frac{\#_i \sigma_n}{n} \xrightarrow[n \rightarrow \infty]{\mathbb{P}} 0, \quad (3)$$

where  $\xrightarrow[n \rightarrow \infty]{\mathbb{P}}$  stands for the convergence in probability,

- and in the strong sense if

$$\forall p \in \mathbb{N}^*, c_1, \dots, c_p \in S_1, \lim_{n \rightarrow \infty} \mathbb{E} \left( \prod_{i=1}^p \left( \frac{\#_{c_i} \sigma_n}{\sqrt{n}} \right) \right) = 0 \quad (4)$$

and

$$\forall i \in S_2, \lim_{n \rightarrow \infty} \mathbb{E} \left( \frac{\#_i \sigma_{j,n}}{n} \right) = 0. \quad (5)$$

Otherwise stated, we ask for a good probabilistic control on some cycles of length less than the maximum of  $S_1 \cup S_2$ . This control may be in probability for the weak version or on in mixed moments for the strong version.

In the assumptions of our results, we will need to choose the sets  $S_1$  and  $S_2$  according to the word  $w$  we are interested in. We detail now this choice.

For a given word  $w$  in  $F_k$ , we will alternatively use its canonical form - that we have already introduced above :  $w = x_{k_r}^{\varepsilon_r} \dots x_{k_1}^{\varepsilon_1}$ , with  $\varepsilon_j = \pm 1$  and if  $k_{j+1} = k_j$  then  $\varepsilon_{j+1} = \varepsilon_j$  - and its *reduced form*. Any word  $w \in F_k$  can be written  $w := x_{\alpha_1}^{\beta_1} \dots x_{\alpha_\ell}^{\beta_\ell}$  with  $\beta_i \in \mathbb{Z}^*$ ,  $\ell \geq 1$ ,  $\alpha_i \in \llbracket k \rrbracket$ ,  $\alpha_{i+1} \neq \alpha_i$ , where  $\llbracket k \rrbracket = \{1, \dots, k\}$ . We call this rewriting the *reduced form* of the word.

We define the following sets of integers : for any  $w \in F_k$  with reduced form  $x_{\alpha_1}^{\beta_1} \dots x_{\alpha_\ell}^{\beta_\ell}$  and  $1 \leq j \leq k$ , let

$$\mathcal{N}'_j(w) := \{|\beta_i|, \alpha_i = j\}, \widehat{\mathcal{N}}_j(w) := \{|d| : \exists \beta \in \mathcal{N}'_j(w), d|\beta\} \text{ and } \mathcal{N}_j(w) := \llbracket \max \widehat{\mathcal{N}}_j(w) \rrbracket \cup \{\mathcal{N}'_j(w) + \mathcal{N}'_j(w)\}.$$

Note that  $\mathcal{N}_j(w) \subset \llbracket 2 \max(\widehat{\mathcal{N}}_j(w)) \rrbracket = \llbracket 2 \max(\mathcal{N}'_j(w)) \rrbracket$ .

For example, for  $w = x_1 x_2^3 x_1^{-1} x_2^6$ ,  $\mathcal{N}'_1(w) = \widehat{\mathcal{N}}_1(w) = \{1\}$ ,  $\mathcal{N}_1(w) = \{1, 2\}$ ,  $\mathcal{N}'_2(w) = \{3, 6\}$ ,  $\widehat{\mathcal{N}}_2(w) = \{1, 2, 3, 6\}$ ,  $\mathcal{N}_2(w) = \{1, 2, 3, 4, 5, 6, 9, 12\}$ .

Our main result is the following extension of Proposition 1.2.

**Theorem 1.5.** *Let  $w$  be a word in  $F_k$  with canonical form  $w = x_{k_r}^{\varepsilon_r} \dots x_{k_1}^{\varepsilon_1}$ , where  $k_1 \neq k_r$  (there is at least two different letters). We write  $w = \Omega^d$ , with  $\Omega$  which is not a power and  $d \geq 1$ .*

*Let  $\sigma_{1,n}, \dots, \sigma_{k,n}$  be independent random permutations, with conjugation invariant distributions such that  $\forall j \in \llbracket k \rrbracket$ , the sequence  $(\sigma_{j,n})_{n \geq 1}$  has few short cycles with respect to  $(\widehat{\mathcal{N}}_j(w), \mathcal{N}_j(w))$  in the strong sense.*

*Then, for any  $d' \in \mathbb{N}^*$ ,  $(\#_1 w(\sigma_{1,n}, \dots, \sigma_{k,n}), \dots, \#_{d'} w(\sigma_{1,n}, \dots, \sigma_{k,n}))$  converges in distribution to a universal limit that depends only on the power  $d$  and the maximal length  $d'$  of the cycles under consideration. Moreover, the joint moments converge also to those of the limit.*

The convergence obtained in Theorem 1.5 is a convergence in distribution and for all the moments. By standard probabilistic arguments, that will be briefly presented at the end of Section 4, the assumptions can be weakened to get the following corollary.

**Corollary 1.6.** *Let  $w$  be a word in  $F_k$  with canonical form  $w = x_{k_r}^{\varepsilon_r} \dots x_{k_1}^{\varepsilon_1}$ , where  $k_1 \neq k_r$  (there is at least two different letters). We write  $w = \Omega^d$ , with  $\Omega$  which is not a power and  $d \geq 1$ .*

*Let  $\sigma_{1,n}, \dots, \sigma_{k,n}$  be independent random permutations, with conjugation invariant distributions such that  $\forall j \in \llbracket k \rrbracket$ , the sequence  $(\sigma_{j,n})_{n \geq 1}$  has few short cycles with respect to  $(\widehat{\mathcal{N}}_j(w), \mathcal{N}_j(w))$  in the weak sense.*

*Then, for any  $d' \in \mathbb{N}^*$ , the random vector  $(\#_1 w(\sigma_{1,n}, \dots, \sigma_{k,n}), \dots, \#_{d'} w(\sigma_{1,n}, \dots, \sigma_{k,n}))$  converges in distribution to a universal limit that depends only on the power  $d$  and the maximal length  $d'$  of the cycles under consideration.*

Let us first comment on the condition,  $k_1 \neq k_r$ . For example, if  $w = x_1 x_2 x_1^{-1}$ ,  $w(\sigma_1, \sigma_2)$  has the same cycle structure as  $\sigma_2$  so obviously there is no universality.

One can easily check the following cyclic invariance : for any  $\sigma = (\sigma_1, \dots, \sigma_k) \in \mathfrak{S}_n^k$  and  $w \in F_k$ , for any  $\ell \geq 1, i \in \llbracket k \rrbracket$ ,

$$\#_\ell(\sigma_i w(\sigma)) = \#_\ell(w(\sigma) \sigma_i).$$

Using this remark, one can check that when  $k_1 = k_r$ , there are 3 cases :

- either  $w = 1$  : in this case there is nothing to prove since  $w(\sigma) = I_n$ , that fixes all integers,
- or  $w = w_1 x_j^\alpha w_1^{-1}$  : in this case, the cycle structure of  $w(\sigma)$  is the same as that of  $\sigma_j^\alpha$  and there is no universality,
- or  $w = w_1 w_2$  with  $w_1$  and  $w_2$  two words such that the canonical form of  $w' := w_2 w_1 = x_{k'_r}^{\varepsilon'_r} \dots x_{k'_1}^{\varepsilon'_1}$  satisfies  $k'_1 \neq k'_{r'}$ . In this case, the cycle structure of  $w'$  is the same as that of  $w$ .

In other words, if, after successive cyclic simplifications, the word  $w$  still contains at least two different letters, then, by applying this word to independent permutations with conjugation invariant distributions and few short cycles, we get a random permutation with an universal limiting joint distribution for short cycles.

Similar results have already been proved in the case when the permutations are uniformly chosen among permutations with restrictions on cycle lengths in [Benaych-Georges, 2010, Theorem 3.7] (the latter being an extension of [Neagu, 2007], in which only the fixed points of the words in permutations were studied). They can also be seen as an extension of our previous work [Kammoun and Maïda, 2020] for the product of permutations to any non-trivial word in the permutations. In the case of the product, we could get optimal assumptions, requiring the permutations to have only few fixed points and cycles of size 2. These assumptions were reminiscent of the connectivity assumptions for random maps appearing in [Budzinski et al., 2019]. In the case when  $|\beta_i| = 1$ , for all  $i \leq \ell$ , we recover the exact same assumptions. For general words, we nevertheless require a more stringent control on short cycles, that may not be optimal. Optimality is discussed in Section 5, it does not hold in full generality but we can show that the assumptions we got for the product are also optimal in the case of the commutator of two permutations.

Note that in this work we focus only on short cycles but in the literature, there is also interest in the limit shape of the Young diagram associated to the cycle structure of a random permutation. We refer to [Cipriani and Zeindler, 2015] for further details and exact definitions. This kind of convergence requires also the study of long cycles, which is an interesting question beyond the scope of this study. In the framework of conjugation invariant permutations, short cycles are still very interesting since they gives access to local events i.e. if  $\sigma_n$  and  $\rho_n$  are conjugation invariant random permutations such that  $(\#_1 \sigma_n, \dots, \#_\ell \sigma_n) \stackrel{d}{=} (\#_1 \rho_n, \dots, \#_\ell \rho_n)$  for some  $\ell < \frac{n}{2}$ , then for any  $1 \leq i_1, i_2, \dots, i_\ell \leq n$  and for any  $1 \leq j_1, j_2, \dots, j_\ell \leq n$ , the equality

$$\mathbb{P}(\sigma_n(i_1) = j_1, \dots, \sigma_n(i_\ell) = j_\ell) = \mathbb{P}(\rho_n(i_1) = j_1, \dots, \rho_n(i_\ell) = j_\ell)$$

holds true. This remark is a direct consequence, for example, of [Hamaker and Rhoades, 2022, Equation (5.68)] (see also [Diaconis, 1989]). Theorem 1.5 can then be seen as a local convergence result.

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## 2 Technical lemmas

The technique of proof is inspired by [Kammoun and Maïda, 2020]. It is based on a graphical representation of the image of several points by the permutations composing the word  $w$ . The strategy of the proof

will be to identify the class of graphs giving the main contribution in the uniform case and then to show that, under our assumptions, the non vanishing contributions are the same in the conjugation invariant case.

Let us introduce some definitions. Some of them have already been used in [Kammoun and Maïda, 2020] but we will need colored versions of the graphs used therein.

- We denote by  $\mathbb{G}_\ell^n$  the set of oriented graphs with vertices  $\llbracket n \rrbracket$  having exactly  $\ell$  (oriented) edges and  $\mathbb{G}^n$  the set of oriented graphs with vertices  $\llbracket n \rrbracket$ . We allow here loops but not multiple edges.

We denote also by  $\mathbb{H}_\ell^n$  the set of oriented graphs with vertices  $\llbracket n \rrbracket$  each colored by red or white and having exactly  $\ell$  (oriented) edges and  $\mathbb{H}^n$  the set of oriented graphs with vertices  $\llbracket n \rrbracket$ , each colored by red or white. Given a (possibly colored) graph  $g$  we denote by  $E_g$  the set of its edges.

- Given  $g \in \mathbb{G}^n$  and  $h \in \mathbb{H}^n$ , we say that  $h$  is of type  $g$  and we denote it by  $\dot{h} = g$  if  $E_g = E_h$  i.e.  $g$  is the non-colored version of  $h$ .
- For any  $\sigma \in \mathfrak{S}_n$ , we denote by  $g_\sigma$  the graph with vertices  $\llbracket n \rrbracket$  and oriented edges

$$\bigcup_{\ell=1}^n \{(\ell, \sigma(\ell))\}.$$

- Given  $g \in \mathbb{G}^n$ , we denote by

$$\mathfrak{S}_{n,g} := \{\sigma \in \mathfrak{S}_n, \forall (i, j) \in E_g, \sigma(i) = j\}.$$

In other words,  $\mathfrak{S}_{n,g}$  is the set of permutations  $\sigma$  such that  $g$  is a subgraph of  $g_\sigma$ .

- A vertex  $i$  of  $g$  is called *isolated* if  $E_g$  does not contain any edge of the form  $(i, j)$  or  $(j, i)$  nor a loop  $(i, i)$ . Let  $g \in \mathbb{G}^n$ , we denote by  $\tilde{g}$  the graph obtained from  $g$  after removing isolated vertices.
- A connected component of  $g$  is called *trivial* if it is reduced to one isolated vertex.
- Two graphs (resp. colored graphs) are isomorphic if there exists a permutation of their vertices that preserves edges (resp. edges and colors). Let  $\mathcal{R}$  be the equivalence relation on colored graph such that  $h_1 \mathcal{R} h_2$  if  $\tilde{h}_1$  and  $\tilde{h}_2$  are two colored graphs that are isomorphic and  $\dot{\mathcal{R}}$  be the equivalence relation on non-colored graph such that  $g_1 \dot{\mathcal{R}} g_2$  if  $\tilde{g}_1$  and  $\tilde{g}_2$  two graphs that are isomorphic. We denote by  $\hat{\mathbb{H}}_\ell := \cup_{n \geq 1} \mathbb{H}_\ell^n / \mathcal{R}$  and  $\hat{\mathbb{G}}_\ell := \cup_{n \geq 1} \mathbb{G}_\ell^n / \dot{\mathcal{R}}$  the respective set of equivalence classes of  $\cup_{n \geq 1} \mathbb{H}_\ell^n$  and  $\cup_{n \geq 1} \mathbb{G}_\ell^n$  and we set  $\hat{\mathbb{H}} := \bigcup_{\ell \geq 1} \hat{\mathbb{H}}_\ell$  and  $\hat{\mathbb{G}} := \bigcup_{\ell \geq 1} \hat{\mathbb{G}}_\ell$ .

We will now introduce the graphs useful for our purpose. We start from an integer  $m$  and look at its orbit along the word  $w(\sigma_1, \dots, \sigma_k)$ . If the reduced form of the word is  $w = x_{\alpha_\ell}^{\beta_\ell} \dots x_{\alpha_1}^{\beta_1}$ , we perform from  $m$  a walk of length  $|\beta_1|$  on the graph  $g_{\sigma_{\alpha_1}}$ , then a walk of length  $|\beta_2|$  on the graph  $g_{\sigma_{\alpha_2}}$ , etc. This will provide the vertices and edges of the  $k$  graphs corresponding to each permutation and each vertex is colored red if it is the entering or exiting point of the walks (see the definitions and the proof of Lemma 2.1 below for more details). In comparison to [Kammoun and Maïda, 2020], considering graphs colored this way will allow more accurate control to show that the graphs containing loops do not contribute to the limiting distribution.



- Let  $n \in \mathbb{N}^*$  and  $\sigma = (\sigma_1, \dots, \sigma_k) \in (\mathfrak{S}_n)^k$ . Let  $m \in \llbracket n \rrbracket$  be fixed. For a word with canonical form  $w = x_{k_r}^{\varepsilon_r} \dots x_{k_1}^{\varepsilon_1}$ , we define

$$i_0^{m,w}(\sigma) := m, i_1^{m,w}(\sigma) := \sigma_{k_1}^{\varepsilon_1}(m), \dots, i_r^{m,w}(\sigma) := \sigma_{k_r}^{\varepsilon_r} \dots \sigma_{k_1}^{\varepsilon_1}(m) = w(\sigma)(m),$$

for any  $a \geq 1$  and  $0 \leq b \leq r-1$ ,

$$i_{ar+b}^{m,w}(\sigma) := w^a(\sigma)(i_b^{m,w}(\sigma)).$$

We also define the colors of the vertices as follows : the color of the vertex  $j$  with respect to the permutation  $\sigma_i$  is given by

$$c_{i,j}^{m,w}(\sigma) = \text{red, if } \begin{cases} j = m \text{ and } i = k_1 \\ \text{or } j = w(\sigma)(m) \text{ and } i = k_r \\ \text{or } \exists \ell \text{ such that } j = i_\ell^{m,w}(\sigma), i \in \{k_\ell, k_{\ell+1}\} \text{ and } k_\ell \neq k_{\ell+1} \end{cases},$$

and white otherwise.

For any  $i \in \llbracket k \rrbracket$ ,  $\mathcal{G}_i^{m,w}(\sigma) \in \mathbb{G}^n$  is the graph with vertices  $\llbracket n \rrbracket$  and edges

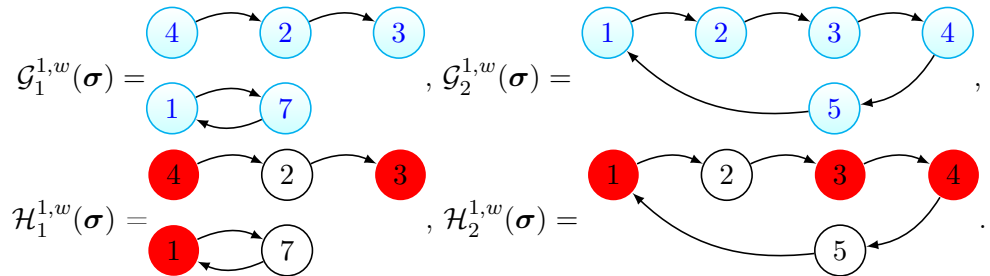
$$E_{\mathcal{G}_i^{m,w}(\sigma)} := \left( \bigcup_{\{\ell: k_\ell = i, \varepsilon_\ell = 1\}} \{(i_\ell^{m,w}(\sigma), i_{\ell+1}^{m,w}(\sigma))\} \right) \cup \left( \bigcup_{\{\ell: k_\ell = i, \varepsilon_\ell = -1\}} \{(i_{\ell+1}^{m,w}(\sigma), i_\ell^{m,w}(\sigma))\} \right),$$

$\mathcal{H}_i^{m,w}(\sigma) \in \mathbb{H}^n$  is the graph with vertices  $\llbracket n \rrbracket$ , the color of vertex  $j$  being  $c_{i,j}^{m,w}(\sigma)$ , and edges  $E_{\mathcal{G}_i^{m,w}(\sigma)}$ .

Finally, let  $\widehat{\mathcal{G}}_i^{m,w}(\sigma)$  and  $\widehat{\mathcal{H}}_i^{m,w}(\sigma)$  be respectively the equivalence class of  $\mathcal{G}_i^{m,w}(\sigma)$  and  $\mathcal{H}_i^{m,w}(\sigma)$ .

Given  $s \subset \llbracket k \rrbracket$ , let  $\mathcal{G}_i^{s,w}(\sigma)$  be the graph such that  $E_{\mathcal{G}_i^{s,w}(\sigma)} = \bigcup_{m \in s} E_{\mathcal{G}_i^{m,w}(\sigma)}$  and let  $\mathcal{H}_i^{s,w}(\sigma)$  be the graph such that  $E_{\mathcal{H}_i^{s,w}(\sigma)} = \bigcup_{m \in s} E_{\mathcal{H}_i^{m,w}(\sigma)}$  and the color of a vertex  $j$  is red in  $\mathcal{H}_i^{s,w}(\sigma)$  if and only if the color of  $j$  is red in  $\mathcal{H}_i^{m,w}(\sigma)$  for some  $m \in s$ .

For example, if  $w = (x_1^2 x_2^3)^2$ ,  $\sigma_1 = (4, 2, 3, 8)(1, 7)(5, 6)$  and  $\sigma_2 = (1, 2, 3, 4, 5)(9, 10)$ , we have

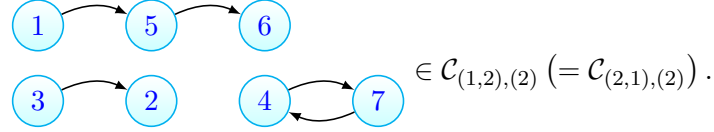


One can notice that for any  $s, i$  and  $w$ ,  $\mathcal{G}_i^{s,w}(\sigma)$  is a subgraph of  $g_{\sigma_i}$ . Fro any permutation  $\sigma$ , the graph  $g_\sigma$  and all its subgraphs lie in classes of type  $\mathcal{C}_{\gamma, \gamma'}$ , that we define now.

- Let  $\ell, \ell' \in \mathbb{N}^*$ ,  $\gamma := (\gamma_1, \dots, \gamma_\ell) \in (\mathbb{N}^*)^\ell$  and  $\gamma' := (\gamma'_1, \dots, \gamma'_{\ell'}) \in (\mathbb{N}^*)^{\ell'}$ . We denote by  $\mathcal{C}_{\gamma, \gamma'}$  the set of classes of graphs  $g_{\gamma, \gamma'}$  defined as follows : for  $n \geq \ell + \sum_{i=1}^{\ell'} \gamma_i + \sum_{i=1}^{\ell'} \gamma'_i$ ,  $g_{\gamma, \gamma'} \in \mathbb{G}^n$  has  $\ell + \ell'$  non trivial connected components, for any  $j, j' \in \llbracket \ell + \ell' \rrbracket$ , if  $j \neq j'$  then  $j$  and  $j'$  are in two



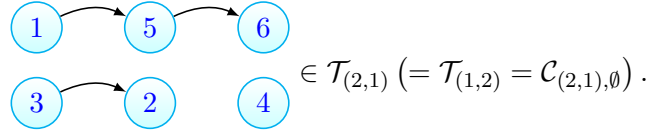
distinct connected components. For  $j \leq \ell'$ , the component containing  $j$  is a directed cycle of length  $\gamma'_j$  (the cycles of length 1 being loops). For  $\ell' + 1 \leq j \leq \ell + \ell'$ , the component containing  $j$  has  $\gamma_j$  edges and  $\gamma_j + 1$  vertices :  $j$  is a vertex of incoming degree 0 and outgoing degree 1, that we call the *head*, one vertex of incoming degree 1 and outgoing degree 0, that we call the *tail* and  $\gamma_j - 1$  vertices of incoming degree 1 and outgoing degree 1. We call such a component *straight*. For example,



Note that the multiplicity of the entries in the vectors  $\gamma$  and  $\gamma'$  is relevant, but their order is not.

- By convention, we will extend the previous definition to the case when  $\ell' = 0$ , that is the case when there is no loop nor cycle. In this case, for  $\ell \in \mathbb{N}^*$ , for  $\gamma := (\gamma_1, \dots, \gamma_\ell) \in (\mathbb{N}^*)^\ell$ , we denote by  $\mathcal{T}_\gamma$  the class  $\mathcal{C}_{\gamma, \emptyset}$  of graphs  $g_\gamma$  having  $\ell$  non trivial connected components, all of them being straight, the component containing  $j \leq \ell$  having  $\gamma_j$  edges and these components being pairwise distinct.

For example,



For a tuple  $\gamma := (\gamma_1, \dots, \gamma_\ell) \in (\mathbb{N}^*)^\ell$ , and  $h \in \mathbb{N}^*$ ,  $h \cdot \gamma \in (\mathbb{N}^*)^{h\ell}$  is the t-tuple obtained by repeating  $h$  times the t-tuple  $\gamma$ .

For example  $\mathcal{T}_{3 \cdot (1,4,5)} = \mathcal{T}_{(1,4,5,1,4,5,1,4,5)} = \mathcal{T}_{(1,1,1,4,4,4,5,5,5)}$ .

- For any  $j \in \llbracket k \rrbracket$ , we define  $\gamma^i(w) := (|\beta_j|)_{\alpha_j=i}$ , and  $\gamma(w) := (\gamma^1(w), \dots, \gamma^k(w))$ .  
For example, for  $w = x_1^4 x_2^{-3} x_3^2 x_2^5$ ,  $\gamma^1(w) = (4)$ ,  $\gamma^2(w) = (3, 5)$  and  $\gamma^3(w) = (2)$ .

The structure of the graphs and their colored versions follow the following rules :

**Lemma 2.1.** *Let  $w$  be a word in  $F_k$  with reduced form  $w = x_{\alpha_q}^{\beta_q} \dots x_{\alpha_1}^{\beta_1}$ , with  $q \geq 2$  and  $1 \leq i \leq k$ . For any  $s \subset \llbracket n \rrbracket$ ,*

1. A connected component of  $\mathcal{H}_i^{s,w}(\sigma)$  either is a cycle or is straight.
2. A connected component of  $\mathcal{H}_i^{s,w}(\sigma)$  has at least one red vertex.
3. If a connected component of  $\mathcal{H}_i^{s,w}(\sigma)$  has exactly one red vertex, then it is a cycle of length  $d$  dividing  $\beta_\ell$  for some  $\ell$  such that  $\alpha_\ell = i$ .
4. If a connected component of  $\mathcal{H}_i^{s,w}(\sigma)$  is straight, then the head and the tail are red.
5. If a connected component of  $\mathcal{H}_i^{s,w}(\sigma)$  has exactly two red points, then it is either

- a straight component of length  $|\beta_\ell|$  for some  $\ell$  such that  $\alpha_\ell = i$ ,
- a cycle of length  $|\beta_\ell| + |\beta_{\ell'}|$  for some  $\ell, \ell'$  such that  $\alpha_\ell = \alpha_{\ell'} = i$ ,
- a cycle of length  $j < |\beta_\ell|$  for some  $\ell$  such that  $\alpha_\ell = i$ .

6. Under the condition  $w(\boldsymbol{\sigma})(m) = m$ , if  $\ell$  is red in  $\mathcal{H}_i^{s,w}(\boldsymbol{\sigma})$ , then it is red in  $\mathcal{H}_j^{s,w}(\boldsymbol{\sigma})$  for some  $j \neq i$ .

*Proof.* Let  $w = x_{\alpha_q}^{\beta_q} \dots x_{\alpha_1}^{\beta_1}$  and  $\boldsymbol{\sigma} = (\sigma_1, \dots, \sigma_k)$ . For each  $m$ , we can describe the sequence  $i_0^{m,w}(\boldsymbol{\sigma}), \dots, i_r^{m,w}(\boldsymbol{\sigma})$  as follows : starting from  $m$ , we perform a walk of length  $|\beta_1|$  on the graph  $g_{\sigma_{\alpha_1}}$ , following the oriented edges if  $\beta_1 > 0$  or going backward if  $\beta_1 < 0$ , then, after  $|\beta_1|$  steps, we walk the same way on  $g_{\sigma_{\alpha_2}}$  during  $|\beta_2|$  steps etc. The colored graph  $\mathcal{H}_i^{m,w}(\boldsymbol{\sigma})$  is then the part of  $g_{\sigma_i}$  explored by this process where each vertex is colored by red if and only if it is the entering or exiting point for one of the walks. Given this description, the proof of the lemma is straightforward.  $\square$

We gather hereafter several technical bounds that we will need in the proofs of our main results. For  $\sigma \in \mathfrak{S}_n$  and  $j \in \llbracket n \rrbracket$ ,  $c_j(\sigma)$  is the length of the cycle of the permutation  $\sigma$  containing  $j$ .

For a graph  $g$  of class  $\mathcal{T}_\gamma$ , the probability of the event  $\{\sigma_n \in \mathfrak{S}_{n,g}\}$  is controlled by the length of the cycle of a given point, say 1. This is stated in the following lemma that we recall from [Kammoun and Maïda, 2020]:

**Lemma 2.2.** *Let  $\boldsymbol{\gamma} := (\gamma_1, \dots, \gamma_\ell) \in (\mathbb{N}^*)^\ell$  and  $n \geq v := \ell + \sum_{i=1}^\ell \gamma_i$ . For any graph  $g \in \mathbb{G}^n$  of class  $\mathcal{T}_\gamma$  and  $\sigma_n$  a random permutation with conjugation invariant distribution, we have*

$$\left( 1 - \sum_{i=1}^\ell \mathbb{P}(c_1(\sigma_n) \leq \gamma_i) - \frac{\ell-1}{n-1} \sum_{i=1}^\ell \gamma_i \right) \leq \frac{\mathbb{P}(\sigma_n \in \mathfrak{S}_{n,g})(n-\ell)!}{(n-\ell - \sum_{i=1}^\ell \gamma_i)!} \leq 1.$$

*Proof.* This is exactly Lemma 2.9 in [Kammoun and Maïda, 2020].  $\square$

We will also need a control on  $\mathbb{P}(\sigma_n \in \mathfrak{S}_{n,g})$  when  $g$  is a graph of class  $\mathcal{C}_{\boldsymbol{\gamma}, \boldsymbol{\gamma}'}$ , with  $\ell' > 0$ . This will require to look at the length of the cycles of  $\ell'$  given points, for example  $1, \dots, \ell'$ , provided they lie in pairwise distinct cycles. More precisely, we have:

**Lemma 2.3.** *Let  $\boldsymbol{\gamma} := (\gamma_1, \dots, \gamma_\ell) \in (\mathbb{N}^*)^\ell$ ,  $\boldsymbol{\gamma}' := (\gamma'_1, \dots, \gamma'_{\ell'}) \in (\mathbb{N}^*)^{\ell'}$  and  $n \geq v := \ell + \sum_{i=1}^\ell \gamma_i + \sum_{i=1}^{\ell'} \gamma'_i$ . For  $\ell' > 0$ , we define*

$$A^{\boldsymbol{\gamma}'} = \{ \sigma \in \mathfrak{S}_n : \forall 1 \leq j \leq \ell' : c_j(\sigma) = \gamma'_j \text{ and } , \forall 1 \leq j \neq j' \leq \ell', j \text{ and } j' \text{ are not in the same cycle} \}.$$

1. For  $\ell' > 0$ , if  $g \in \mathbb{G}^n$  is a graph of class  $\mathcal{C}_{\boldsymbol{\gamma}, \boldsymbol{\gamma}'}$ , for any random permutation  $\sigma_n$  with conjugation invariant distribution, we have

$$\frac{\mathbb{P}(\sigma_n \in \mathfrak{S}_{n,g})(n-\ell-\ell')!}{(n-\ell - \sum_{i=1}^\ell \gamma_i - \sum_{i=1}^{\ell'} \gamma'_i)!} \leq \mathbb{P}(\sigma_n \in A^{\boldsymbol{\gamma}'}).$$

Moreover,

$$\mathbb{P}(\sigma_n \in A^{\boldsymbol{\gamma}'}) \leq \mathbb{E} \left( \prod_{i=1}^{\ell'} (\#_{\gamma'_i} \sigma_n) \right) \frac{(n-\ell')!}{n!} \prod_{i=1}^{\ell'} \gamma'_i. \quad (6)$$

2. If  $\rho_n$  is distributed according to the uniform law then,

$$\mathbb{P}(\rho_n \in A^{\boldsymbol{\gamma}'}) \left( 1 - \frac{\ell \sum_{i=1}^{\ell'} (\gamma'_i - 1)}{n - \ell'} \right) \left( 1 - \frac{\ell \sum_{i=1}^\ell \gamma_i}{n - \sum \gamma_i} \right) \leq \frac{\mathbb{P}(\rho_n \in \mathfrak{S}_{n,g})(n-\ell-\ell')!}{(n-\ell - \sum_{i=1}^\ell \gamma_i - \sum_{i=1}^{\ell'} \gamma'_i)!}.$$

*Proof.* Let  $n \geq v := \ell + \sum_{i=1}^{\ell} \gamma_i + \sum_{i=1}^{\ell'} \gamma'_i$  and  $g \in \mathbb{G}^n$  be a graph of class  $\mathcal{C}_{\gamma, \gamma'}$ . By conjugation invariance, one can assume without loss of generality that  $g = g_{\gamma, \gamma'}$  as in the definition of  $\mathcal{C}_{\gamma, \gamma'}$  above and that the non-isolated vertices are  $\llbracket v \rrbracket$ . We denote by

$$F := \{y = (y_i)_{\ell + \ell' + 1 \leq i \leq v}, y_i \in \{\ell + \ell' + 1, \dots, n\}, \text{ pairwise distincts}\}$$

and for any  $y \in F$ , we denote by  $g_y$  the graph isomorphic to  $g$  obtained from  $g$  by fixing the vertices  $1, \dots, \ell + \ell'$  and, for any  $i \in \{\ell + \ell' + 1, \dots, v\}$ , replacing  $i$  by  $y_i$ .

It is easy to check that, for  $y, y' \in F$ , if  $y \neq y'$ , we have  $\mathfrak{S}_{n, g_y} \cap \mathfrak{S}_{n, g_{y'}} = \emptyset$ . From there, we easily get the first bound of point 1.:

$$\mathbb{P}(\sigma_n \in \mathfrak{S}_{n, g}) = \frac{\mathbb{P}(\sigma_n \in \bigcup_{y \in F} \mathfrak{S}_{n, g_y})}{\text{card}F} \leq \frac{\mathbb{P}(\sigma_n \in A^{\gamma'})}{\frac{(n - \ell - \ell')!}{(n - v)!}}.$$

For the second inequality of point 1., we write

$$\begin{aligned} \mathbb{E} \left( \prod_{i=1}^{\ell'} \gamma'_i (\#\gamma'_i \sigma_n) \right) &= \sum_{(i_1, \dots, i_{\ell'}) \in \llbracket n \rrbracket^{\ell'}} \mathbb{E} \left( \prod_{j=1}^{\ell'} 1_{\{c_{i_j}(\sigma_n) = \gamma'_j\}} \right) \geq \sum_{(i_1, \dots, i_{\ell'}) \in \llbracket n \rrbracket^{\ell'} \text{ pairwise distinct}} \mathbb{E} \left( \prod_{j=1}^{\ell'} 1_{\{c_{i_j}(\sigma_n) = \gamma'_j\}} \right) \\ &= \frac{n!}{(n - \ell')!} \mathbb{E} \left( \prod_{j=1}^{\ell'} 1_{\{c_j(\sigma_n) = \gamma'_j\}} \right) \\ &\geq \frac{n!}{(n - \ell')!} \mathbb{P}(\sigma_n \in A^{\gamma'}). \end{aligned}$$

We now go to the bound in the uniform case and define :

$$B^{\gamma, \gamma'} := A^{\gamma'} \cap \left( \bigcap_{i=1}^{\ell} \bigcap_{j=1}^{\gamma_i} \{\sigma \in \mathfrak{S}_n, \sigma^j(i + \ell') > \ell + \ell'\} \right)$$

$$C^{\gamma, \gamma'} := A^{\gamma'} \cap \left( \bigcap_{i=1}^{\ell} \bigcap_{j=1}^{\gamma_i} \{\sigma \in \mathfrak{S}_n, \sigma^j(i + \ell') > \ell'\} \right) = A^{\gamma'} \cap \left( \bigcap_{i=1}^{\ell'} \bigcap_{j=1}^{\gamma'_i - 1} \{\sigma \in \mathfrak{S}_n, \sigma^j(i) > \ell + \ell'\} \right).$$

By observing that for  $g = g_{\gamma, \gamma'}$ ,

$$\mathbb{P}(\sigma_n \in \mathfrak{S}_{n, g} | \sigma_n \in \mathfrak{S}_n \setminus B^{\gamma, \gamma'}) = 0,$$

and

$$\mathbb{P}(\sigma_n \in \mathfrak{S}_{n, g} | \sigma_n \in B^{\gamma, \gamma'}) = \frac{(n - v)!}{(n - \ell - \ell')!},$$

we obtain that

$$\mathbb{P}(\sigma_n \in \mathfrak{S}_{n, g}) = \mathbb{P}(\sigma_n \in B^{\gamma, \gamma'}) \frac{(n - v)!}{(n - \ell - \ell')!}.$$

Moreover, by definition,  $B^{\gamma, \gamma'} \subset C^{\gamma, \gamma'} \subset A^{\gamma'}$  and then

$$\mathbb{P}(\sigma_n \in B^{\gamma, \gamma'}) = \mathbb{P}(\sigma_n \in B^{\gamma, \gamma'} | \sigma_n \in C^{\gamma, \gamma'}) \mathbb{P}(\sigma_n \in C^{\gamma, \gamma'} | \sigma_n \in A^{\gamma'}) \mathbb{P}(\sigma_n \in A^{\gamma'}).$$

For any conjugation invariant permutation,

$$\mathbb{P}(\sigma_n \in C^{\gamma, \gamma'} | \sigma_n \in A^{\gamma'}) \geq 1 - \sum_{i=1}^{\ell'} \sum_{j=1}^{\gamma'_i - 1} \sum_{p=1}^{\ell} \mathbb{P}(\sigma_n^j(i) = p + \ell' | \sigma_n \in A^{\gamma'}) = 1 - \sum_{i=1}^{\ell'} \frac{\ell(\gamma'_i - 1)}{n - \ell'}.$$

When  $\rho_n$  is a uniform random permutation, then conditionally to  $\{\rho_n \in C^{\gamma, \gamma'}\}$ , the restriction of  $\rho_n$  to  $\llbracket n \rrbracket \setminus \{\sigma^j(i), 1 \leq i \leq \ell, 1 \leq j \leq \gamma'_i\}$  is uniform. Therefore,

$$\mathbb{P}(\rho_n \in B^{\gamma, \gamma'} | \rho_n \in C^{\gamma, \gamma'}) \geq 1 - \frac{\ell \sum_{i=1}^{\ell} \gamma_i}{n - \sum \gamma'_i}.$$

This concludes the proof of the lower bound in the uniform case.  $\square$

### 3 Asymptotics of fixed points

For readability reasons, before proving the general statements, we will prove the convergence of the number of fixed points under weaker assumptions. The statement is the following :

**Theorem 3.1.** *Let  $w$  be a word in  $F_k$  with canonical form  $w = x_{k_r}^{\varepsilon_r} \dots x_{k_1}^{\varepsilon_1}$ , where  $k_1 \neq k_r$  (there is at least two different letters). For  $j \leq k$ , we denote by  $r_j := \text{card}\{i, k_i = j\}$ . We write  $w = \Omega^d$ , with  $\Omega$  which is not a power and  $d \geq 1$ .*

*Let  $\sigma_{1,n}, \dots, \sigma_{k,n}$  be independent random permutations, with conjugation invariant distributions satisfying the following assumptions:*

$$\forall j \in \llbracket k \rrbracket, \forall p \in \mathbb{N}^*, c_1, \dots, c_p \in \widehat{\mathcal{N}}_j(w) \text{ such that } \sum_{i=1}^p c_i \leq r_j, \lim_{n \rightarrow \infty} \mathbb{E} \left( \prod_{i=1}^p \left( \frac{\#_{c_i} \sigma_{j,n}}{\sqrt{n}} \right) \right) = 0 \quad (7)$$

and

$$\forall j \in \llbracket k \rrbracket, \forall i \in \mathcal{N}_j(w), \lim_{n \rightarrow \infty} \mathbb{E} \left( \frac{\#_i \sigma_{j,n}}{n} \right) = 0. \quad (8)$$

Then we have that

$$\mathbb{E}(\#_1 w(\sigma_{1,n}, \dots, \sigma_{k,n})) \xrightarrow{n \rightarrow \infty} \psi(d),$$

where we recall that  $\psi(d)$  is the numbers of divisors of the integer  $d$ .

In particular, in the case when  $d = 1$ , Theorem 3.1 reads :

**Corollary 3.2.** *Let  $\Omega \in F_k$  with canonical form  $\Omega = x_{k_r}^{\varepsilon_r} \dots x_{k_1}^{\varepsilon_1} \neq 1$  with  $k_1 \neq k_r$ , and which is not a power.*

*Under Assumptions (7) and (8), we have that*

$$\mathbb{E}(\#_1 \Omega(\sigma_{1,n}, \dots, \sigma_{k,n})) \xrightarrow{n \rightarrow \infty} 1.$$

We start with the identification of the graphs that are contributing to the limit in the uniform case :

**Proposition 3.3.** *Let  $w = \Omega^d$  be a word such that  $\Omega$  is not a power. We write the reduced form of  $\Omega$  as  $x_{\alpha_\ell}^{\beta_\ell} \dots x_{\alpha_1}^{\beta_1}$ , with  $\alpha_\ell \neq \alpha_1$ . Assume that  $\rho_{1,n}, \dots, \rho_{k,n}$  are independent and **uniformly distributed** on  $\mathfrak{S}_n$ . Then, for any  $\hat{g}_1, \hat{g}_2, \dots, \hat{g}_k \in \widehat{\mathbb{G}}$ ,*

$$\lim_{n \rightarrow \infty} n\mathbb{P} \left( \forall i \in \llbracket k \rrbracket, \widehat{\mathcal{G}}_i^{1,w}(\rho_{1,n}, \dots, \rho_{k,n}) = \hat{g}_i, c_1(w(\rho_{1,n}, \dots, \rho_{k,n})) = 1 \right) = \sum_{h|d} \prod_{i=1}^k \mathbb{1}_{\{\hat{g}_i = \mathcal{T}_{h \cdot \gamma^i(\Omega)}\}},$$

where the sum runs over all divisors  $h$  of  $d$  and  $\forall i \in \llbracket k \rrbracket$ ,  $\gamma^i(\Omega) = (|\beta_j|)_{\alpha_j=i}$ .

*Proof.* We first recall that for any random permutation  $\tau_n$ ,

$$\mathbb{E}(\#\mathbb{1}(\tau_n)) = \mathbb{E} \left( \sum_{i=1}^n \mathbb{1}_{\{c_i(\tau_n)=1\}} \right) = \sum_{i=1}^n \mathbb{E}(\mathbb{1}_{\{c_i(\tau_n)=1\}}) = \sum_{i=1}^n \mathbb{P}(c_i(\tau_n) = 1)$$

and if  $\tau_n$  is conjugation invariant then  $\mathbb{P}(c_i(\tau_n) = 1) = \mathbb{P}(c_1(\tau_n) = 1)$  and

$$\mathbb{E}(\#\mathbb{1}(\tau_n)) = n\mathbb{P}(c_1(\tau_n) = 1).$$

According to Proposition 1.2, for  $\rho_{1,n}, \dots, \rho_{k,n}$  independent and uniformly distributed,

$$\lim_{n \rightarrow \infty} \mathbb{E}(\#\mathbb{1}w(\rho_{1,n}, \dots, \rho_{k,n})) = \psi(d).$$

This is equivalent to the following:

$$\lim_{n \rightarrow \infty} \sum_{(\hat{g}_i)_{1 \leq i \leq k}} n\mathbb{P} \left( \forall i \in \llbracket k \rrbracket, \widehat{\mathcal{G}}_i^{1,w}(\rho_{1,n}, \dots, \rho_{k,n}) = \hat{g}_i, c_1(w(\rho_{1,n}, \dots, \rho_{k,n})) = 1 \right) = \psi(d).$$

Our candidates for being the main contributions are the  $k$ -tuples of classes  $(\mathcal{T}_{h \cdot \gamma^1(\Omega)}, \dots, \mathcal{T}_{h \cdot \gamma^k(\Omega)})$  with  $\gamma^i(\Omega) = (|\beta_j|)_{\alpha_j=i}$  and  $h|d$ . As there are  $\psi(d)$  such  $k$ -tuples, one only needs to show that

$$\lim_{n \rightarrow \infty} n\mathbb{P} \left( \forall i \in \llbracket k \rrbracket, \widehat{\mathcal{G}}_i^{1,w}(\rho_{1,n}, \dots, \rho_{k,n}) = \mathcal{T}_{h \cdot \gamma^i(\Omega)}, c_1(w(\rho_{1,n}, \dots, \rho_{k,n})) = 1 \right) \geq 1.$$

We denote by  $t := \sum_{i=1}^\ell |\beta_i|$  the total length of the word  $\Omega$  and  $r = td$  the length of  $w$ . Let  $h|d$  and, for  $n$  large enough, let  $j_1, j_2, \dots, j_{ht-1}$  be  $ht-1$  indices in  $\{2, \dots, n\}$  which are two by two distinct.

Let  $\sigma := (\sigma_1, \dots, \sigma_k)$  be such that

$$i_0^{1,w}(\sigma) = 1, i_1^{1,w}(\sigma) = j_1, \dots, i_{ht-1}^{1,w}(\sigma) = j_{ht-1}, i_{ht}^{1,w}(\sigma) = 1.$$

By construction,  $\forall 1 \leq i \leq k$ ,  $\widehat{\mathcal{G}}_i^{1,w}(\sigma) = \mathcal{T}_{h \cdot \gamma^i(\Omega)}$ . To simplify the notations, for any  $i \in \llbracket k \rrbracket$ , we denote by  $g_i := \mathcal{G}_i^{1,w}(\sigma)$ . By Lemma 2.2 and independence, we have the following lower bound :

$$\begin{aligned} & \mathbb{P}(\forall i \in \llbracket k \rrbracket, \rho_{i,n} \in \mathfrak{S}_{n, g_i}) \\ & \geq \prod_{i=1}^k \frac{(n - h\ell_i - h \sum_{j=1}^{\ell_i} \gamma_j^i(\Omega))!}{(n - h\ell_i)!} \left( 1 - h \sum_{j=1}^{\ell_i} \mathbb{P}(c_1(\rho_{n,i}) \leq \gamma_j^i(\Omega)) - \frac{h\ell_i - 1}{n-1} h \sum_{j=1}^{\ell_i} \gamma_j^i(\Omega) \right), \end{aligned}$$

where  $\ell_i$  is the number of non-trivial connected components of  $\gamma^i(\Omega)$ .

From there, we get that

$$\begin{aligned} & \mathbb{P} \left( \forall i \in \llbracket k \rrbracket, \widehat{\mathcal{G}}_i^{1,w}(\rho_{1,n}, \dots, \rho_{k,n}) = \mathcal{T}_{h \cdot \gamma^i(\Omega)}, c_1(w(\rho_{1,n}, \dots, \rho_{k,n})) = 1 \right) \\ & \geq \prod_{i=1}^k \frac{(n - h\ell_i - h \sum_{j=1}^{\ell_i} \gamma_j^i(\Omega))!}{(n - h\ell_i)!} \left( 1 - \sum_{j=1}^{\ell_i} \mathbb{P}(c_1(\rho_{n,i}) \leq \gamma_j^i(\Omega)) - \frac{h\ell_i - 1}{n-1} h \sum_{j=1}^{\ell_i} \gamma_j^i(\Omega) \right) \frac{(n-1)!}{(n-h\ell_i)!} \end{aligned}$$

Since, for any  $i \in \llbracket k \rrbracket$ ,  $\sum_{j=1}^{\ell_i} \mathbb{P}(c_1(\rho_{n,i}) \leq \gamma_j^i(\Omega)) + \frac{h\ell_i - 1}{n-1} h \sum_{j=1}^{\ell_i} \gamma_j^i(\Omega) \rightarrow 0$ , and we get that

$$\begin{aligned} \liminf_{n \rightarrow \infty} n \mathbb{P} \left( \forall i \in \llbracket k \rrbracket, \widehat{\mathcal{G}}_i^{1,w}(\rho_{1,n}, \dots, \rho_{k,n}) = \mathcal{T}_{h \cdot \gamma^i(\Omega)}, c_1(w(\rho_{1,n}, \dots, \rho_{k,n})) = 1 \right) \\ \geq \liminf_{n \rightarrow \infty} n \frac{1}{n^{h \sum_{i=1}^k \sum_{j=1}^{\ell_i} \gamma_j^i(\Omega)}} n^{ht-1} \geq 1, \end{aligned}$$

where the last inequality comes from the fact that  $\sum_{i=1}^k \sum_{j=1}^{\ell_i} \gamma_j^i(\Omega) = t$ . This concludes the proof.  $\square$

We are now ready for the proof of Theorem 3.1.

*Proof.* It is enough to show that, under the assumptions (7) and (8), for any  $\hat{g}_1, \hat{g}_2, \dots, \hat{g}_k \in \widehat{\mathcal{G}}$ ,

$$\lim_{n \rightarrow \infty} n \mathbb{P} \left( \forall i \in \llbracket k \rrbracket, \widehat{\mathcal{G}}_i^{1,w}(\sigma_{1,n}, \dots, \sigma_{k,n}) = \hat{g}_i, c_1(w(\sigma_{1,n}, \dots, \sigma_{k,n})) = 1 \right) = \sum_{h|d} \prod_{i=1}^k \mathbb{1}_{\{\hat{g}_i = \mathcal{T}_{h \cdot \gamma^i(\Omega)}\}}.$$

Let  $g$  be of class  $\mathcal{T}_\gamma$ , with  $\gamma = (\gamma_1, \gamma_2, \dots, \gamma_\ell)$ . Then, by Lemma 2.2, for any random permutation  $\sigma_n$  with conjugation invariant distribution on  $\mathfrak{S}_n$  and  $\rho_n$  uniformly distributed we have

$$\mathbb{P}(\sigma_n \in \mathfrak{S}_{n,g}) \leq \frac{1}{1 - \frac{1}{n} \sum_{i=1}^{\ell} \gamma_i - \frac{\ell-1}{n-1} \sum_{i=1}^{\ell} \gamma_i} \mathbb{P}(\rho_n \in \mathfrak{S}_{n,g}). \quad (9)$$

Therefore, for any  $\varepsilon > 0$ , there exists  $n_0$  such that for any  $n \geq n_0$ ,

$$\mathbb{P}(\sigma_n \in \mathfrak{S}_{n,g}) \leq (1 + \varepsilon) \mathbb{P}(\rho_n \in \mathfrak{S}_{n,g}). \quad (10)$$

Similarly, for any random permutation  $\sigma_n$  with conjugation invariant distribution on  $\mathfrak{S}_n$  satisfying Assumption (8), we have

$$\mathbb{P}(\sigma_n \in \mathfrak{S}_{n,g}) \geq \left( 1 - \sum_{i=1}^{\ell} \mathbb{P}(c_1(\sigma_n) = i) - \frac{\ell-1}{n-1} \sum_{i=1}^{\ell} \gamma_i \right) \mathbb{P}(\rho_n \in \mathfrak{S}_{n,g}) \geq (1 - \varepsilon) \mathbb{P}(\rho_n \in \mathfrak{S}_{n,g}). \quad (11)$$

Therefore, if for any  $i \in \llbracket k \rrbracket$ , there exists  $\gamma^i$  such that  $\hat{g}_i = \mathcal{T}_{\gamma^i}$ , from Proposition 3.3, we get that

$$\lim_{n \rightarrow \infty} n \mathbb{P} \left( \forall i \in \llbracket k \rrbracket, \widehat{\mathcal{G}}_i^{1,w}(\sigma_{1,n}, \dots, \sigma_{k,n}) = \hat{g}_i, c_1(w(\sigma_{1,n}, \dots, \sigma_{k,n})) = 1 \right) = \sum_{h|d} \prod_{i=1}^k \mathbb{1}_{\{\hat{g}_i = \mathcal{T}_{h \cdot \gamma^i(w)}\}}.$$

We now want to show that graphs containing loops do not contribute. Let  $g$  be a fixed (non-colored) graph. If  $h$  is a colored graph such that  $\hat{h} = g$ , then  $\hat{h}$  belongs to a finite set of classes of colored graphs,

with cardinal independent of  $n$ . Therefore, if we fix some classes of colored graphs  $(\hat{h}_i)_{1 \leq i \leq k}$  such that one of the components is not straight, it is enough to show that

$$\lim_{n \rightarrow \infty} n\mathbb{P} \left( \forall i \in \llbracket k \rrbracket, \widehat{\mathcal{H}}_i^{1,w}(\sigma_{1,n}, \dots, \sigma_{k,n}) = \hat{h}_i, c_1(w(\sigma_{1,n}, \dots, \sigma_{k,n})) = 1 \right) = 0. \quad (12)$$

One can assume that there exists  $(\sigma_1, \dots, \sigma_k)$  such that  $c_1(w(\sigma_{1,n}, \dots, \sigma_{k,n})) = 1$  and  $\forall i \in \llbracket k \rrbracket$ ,  $\widehat{\mathcal{H}}_i^{1,w}(\sigma_1, \dots, \sigma_k) = \hat{h}_i$ , otherwise (12) is trivial. To simplify the notations, we denote in the sequel  $h_i := \widehat{\mathcal{H}}_i^{1,w}(\sigma_1, \dots, \sigma_k)$ , for any  $i \in \llbracket k \rrbracket$ .

Let  $\tilde{h}_i$  be the same graph as  $h_i$  after removing one edge from each cycle having at least two red vertices. In particular,  $\tilde{h}_i$  and  $h_i$  have the same set of non-trivial vertices. We denote by  $\mathcal{C}^{\gamma^i, \gamma'^i}$  the class of  $\tilde{h}_i$ ,  $\ell_i$  and  $\ell'_i$  being the number of non-zero components of  $\gamma^i$  and  $\gamma'^i$  respectively. Let  $\mathcal{V}_i$  be the set of non-trivial vertices of  $h_i$  and  $r_i$  is the number of red vertices of  $h_i$ .

A direct consequence of Lemma 2.1 is that

$$\text{card} \left( \bigcup_{i=1}^k \mathcal{V}_i \right) \leq \sum_{i=1}^k (\text{card}(\mathcal{V}_i) - r_i/2) \leq \sum_{i=1}^k \sum_{j=1}^{\ell_i} \gamma_j^i + \sum_{j=1}^{\ell'_i} (\gamma_j'^i - 1/2) =: \text{deg}.$$

There is at most  $O(n^{\text{deg}-1})$  possible  $k$ -tuples of graphs  $(h'_1, \dots, h'_k)$  such that  $\forall i \in \llbracket k \rrbracket$ ,  $\hat{h}'_i = \hat{h}_i$  and there exists  $(\sigma_1, \dots, \sigma_k)$  satisfying  $\forall i \in \llbracket k \rrbracket$ ,  $\widehat{\mathcal{H}}_i^{1,w}(\sigma_1, \dots, \sigma_k) = \hat{h}'_i$  and  $c_1(w(\sigma_1, \dots, \sigma_k)) = 1$ . Indeed, one need to choose at most  $\text{deg} - 1$  non-trivial vertices in  $\llbracket 2, n \rrbracket$  (1 is a non-trivial vertex), and then choose their positions (not all choices are admissible).

We have

$$\begin{aligned} \mathbb{P} \left( \forall i \in \llbracket k \rrbracket, \widehat{\mathcal{H}}_i^{1,w}(\sigma_{1,n}, \dots, \sigma_{k,n}) = \hat{h}_i, c_1(w(\sigma_{1,n}, \dots, \sigma_{k,n})) = 1 \right) \\ \leq O(n^{\text{deg}-1}) \mathbb{P} \left( \forall i \in \llbracket k \rrbracket, \mathcal{H}_i^{1,w}(\sigma_{1,n}, \dots, \sigma_{k,n}) = h_i \right) \\ \leq O(n^{\text{deg}-1}) \mathbb{P} \left( \forall i \in \llbracket k \rrbracket, \sigma_{i,n} \in \mathfrak{S}_{n, h_i} \right) \\ \leq O(n^{\text{deg}-1}) \prod_{i=1}^k \mathbb{P} \left( \sigma_{i,n} \in \mathfrak{S}_{n, h_i} \right) \end{aligned}$$

By Lemma 2.3, we have

$$\mathbb{P} \left( \sigma_{i,n} \in \mathfrak{S}_{n, h_i} \right) \leq \mathbb{P} \left( \sigma_{i,n} \in \mathfrak{S}_{n, \tilde{h}_i} \right) = O \left( n^{-\sum_{j=1}^{\ell_i} \gamma_j^i - \sum_{j=1}^{\ell'_i} \gamma_j'^i + \ell'_i} \right) \mathbb{P}(\sigma_{1,n} \in A^{\gamma'^i}). \quad (13)$$

Moreover, when  $\ell'_i \neq 0$ , by Lemma 2.1, for any  $j \leq \ell'_i$ ,  $\gamma_j'^i | \beta_\ell$  for some  $\ell$  such that  $\alpha_\ell = i$ . Consequently, using (6) and Assumption (7),

$$\mathbb{P}(\sigma_{i,n} \in A^{\gamma'^i}) \leq o(n^{\frac{\ell'_i}{2}}) O(n^{-\ell'_i}) = o(n^{-\frac{\ell'_i}{2}})$$

Therefore, if there exists  $i \leq k$  such that  $\ell'_i \neq 0$ , then (12) holds.



We now consider the case when  $\forall i \leq k, \ell'_i = 0$ . It means that every cycle of the  $h_i$ 's has at least two red vertices. In this case, by Lemma 2.1, we have

$$\text{card} \left( \bigcup_{i=1}^k \mathcal{V}_i \right) \leq \sum_{i=1}^k (\text{card}(\mathcal{V}_i) - r_i/2) \leq \sum_{i=1}^k \sum_{j=1}^{\ell_i} \gamma_j^i = \text{deg}.$$

The second inequality is strict as soon as one component has at least three red vertices. In this case, we trivially bound  $\mathbb{P}(\sigma_{i,n} \in A^{\gamma^i})$  by 1 in (13) to conclude the proof.

If all the components have at most two red vertices, then, by Lemma 2.1, the components are straight or a cycle of length  $|\beta_\ell| + |\beta_{\ell'}|$  for some  $\ell$  and  $\ell'$  or a cycle of length  $j < |\beta_\ell|$  for some  $\ell$  such that  $\alpha_\ell = i$ , for some  $1 \leq i \leq k$ . In this case, one can conclude similarly to the preceding case, using Assumption (7) to conclude. □

## 4 Proof of Theorem 1.5 and Corollary 1.6

Let us first introduce some more notations. Let  $\lambda = (\lambda_1, \dots, \lambda_m)$  be a Young diagram, that is a sequence of nonnegative integers in decreasing order  $\lambda_1 \geq \dots \geq \lambda_m \geq 0$ . It is customary to represent them with  $\lambda_i$  empty boxes on row number  $i$ . For two Young diagrams  $\lambda, \mu$ , we say that  $\mu \subset \lambda$  if  $\mu$  can be obtained from  $\lambda$  by removing some rows of  $\lambda$ . We denote by  $\ell(\lambda)$  the number of non-empty rows of  $\lambda$  and  $|\lambda| = \sum_{i=1}^{\ell(\lambda)} \lambda_i$  the number of boxes of the diagram.

For example,  $\mu = (3, 1) = \begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \square & & \\ \hline \end{array} \subset \begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \square & \square & \square \\ \hline \square & & \\ \hline \end{array} = (3, 3, 1) = \lambda$

For any pair  $(\lambda, \mu)$  of Young diagrams such that  $\mu \subset \lambda$ , for any  $n \geq \ell(\lambda)$ , an admissible filling of type  $(\lambda, \mu, n)$  is a filling of  $\mu$  such that

- all entries are in  $\llbracket n \rrbracket$  and are pairwise distinct,
- the entries in the first column are increasing,
- in every row, the first entry is the smallest entry,
- $\forall i \leq \ell(\lambda)$ ,  $i$  is in a row of  $\mu$  of length  $\lambda_i$ .

Let  $K_{\lambda, \mu, n}$  be the number of admissible fillings of type  $(\lambda, \mu, n)$ . For example,

- $K_{(3,1),(1),n} = 0$ , because 1 should be in a row of  $\mu$  of length 3.
- $K_{(2,2,1),(2,1),n} = 0$ , because 1, 2 and 3 should go in the first row of  $\mu$ , which has only two boxes.
- $K_{(3,3,1),(3,1),n} = 2(n-3)$  because the only admissible fillings in this case are of the form  $\begin{array}{|c|c|c|} \hline 1 & 2 & j \\ \hline \square & \square & \square \\ \hline \square & & \\ \hline \end{array}$  and  $\begin{array}{|c|c|c|} \hline 1 & j & 2 \\ \hline \square & \square & \square \\ \hline \square & & \\ \hline \end{array}$  for some  $4 \leq j \leq n$ .

An important remark is that, for any Young diagram  $\lambda$ , we obviously have  $\lambda \subset \lambda$  and, for any  $n \geq |\lambda|$ ,

$$K_{\lambda, \lambda, n} = \frac{(n - \ell(\lambda))!}{(n - |\lambda|)!},$$

because  $1, 2, \dots, \ell(\lambda)$  should go in the first position of each row of  $\lambda$  and we then complete freely the  $|\lambda| - \ell(\lambda)$  remaining boxes with distinct numbers between  $\ell(\lambda) + 1$  and  $n$ . By a similar argument, for any  $\mu \subset \lambda$ , there exists  $C_{\lambda, \mu}$  such that for any  $n \geq |\lambda|$ ,

$$K_{\lambda, \mu, n} = C_{\lambda, \mu} \frac{(n - \ell(\lambda))!}{(n - |\mu|)!} = C_{\lambda, \mu} n^{|\mu| - \ell(\lambda)} (1 + o(1)). \quad (14)$$

Let us now go to the proof of Theorem 1.5 itself. The scheme is similar to the proof of Theorem 3.1, except we have to follow simultaneously the trajectory of several starting points  $1, 2, \dots, m$ . As we will see, a crucial point is that it is enough to consider the case when they lie in different cycles. We introduce the following event : for a Young diagram  $\lambda$ ,  $j \leq \ell(\lambda)$ ,  $k_1, \dots, k_j$  distinct integers in  $[[\ell(\lambda)]]$ , we denote by

$$A_{k_1, \dots, k_j}^{\lambda, \Omega} := \left\{ \sigma \in (\mathfrak{S}_n)^k, \forall i \leq j, c_{k_i}(\Omega(\sigma)) = \lambda_{k_i} \right. \\ \left. \text{and } \forall i \neq j, k_i \text{ and } k_j \text{ lie in pairwise distinct cycles of } \Omega(\sigma) \right\}.$$

It is an extension of the event  $A^{\gamma'}$  defined in Lemma 2.3, in the sense that if  $\lambda$  is a Young diagram with rows  $\gamma'_1, \dots, \gamma'_\ell$  in decreasing order then  $A_{1, \dots, \ell(\lambda)}^{\lambda, x_1} = A^{\gamma'}$ .

The first key step is the following lemma :

**Lemma 4.1.** *Let  $\sigma_n$  and  $\rho_n$  be two conjugation invariant random permutations in  $\mathfrak{S}_n$ . Then the following two properties are equivalent :*

1. *For any Young diagram  $\mu$ ,  $\lim_{n \rightarrow \infty} n^{\ell(\mu)} \mathbb{P} \left( \sigma_n \in A_{1, \dots, \ell(\mu)}^{\mu, x_1} \right) = \lim_{n \rightarrow \infty} n^{\ell(\mu)} \mathbb{P} \left( \rho_n \in A_{1, \dots, \ell(\mu)}^{\mu, x_1} \right)$ .*
2. *For any  $P \in \mathbb{R}[x_1, x_2, \dots, x_d]$ , monomial,*

$$\lim_{n \rightarrow \infty} \mathbb{E} (P(\#_1 \sigma_n, \dots, \#_d \sigma_n)) = \lim_{n \rightarrow \infty} \mathbb{E} (P(\#_1 \rho_n, \dots, \#_d \rho_n)).$$

*Proof.* For any  $m \in \mathbb{N}^*$ ,  $p_1, \dots, p_m \in \mathbb{N}$ ,

$$a_{p_1, \dots, p_m}^n := \mathbb{E} \left( \left( \sum_{i=1}^n \mathbf{1}_{\{c_i(\sigma_n)=1\}} \right)^{p_1} \cdots \left( \sum_{i=1}^n \mathbf{1}_{\{c_i(\sigma_n)=m\}} \right)^{p_m} \right) \\ = \sum_{\substack{1 \leq i_1^1, \dots, i_{p_1}^1 \\ i_1^m, \dots, i_{p_m}^m \leq n}} \mathbb{P} \left( \forall j \leq m, \forall s \leq p_j, c_{i_s^j}(\sigma_n) = j \right).$$

We denote by  $\lambda$  the Young diagram  $(m, \dots, m, \dots, 1, \dots, 1)$  where  $j$  appears  $p_j$  times. For any  $j \leq m$ , let us denote by  $t_j := \text{card}\{i_1^j, \dots, i_{p_j}^j\} \leq p_j$  the number of distinct indices among  $i_1^j, \dots, i_{p_j}^j$  and  $\pi$  the Young tableau  $(m, \dots, m, \dots, 1, \dots, 1)$  where  $j$  appears  $t_j$  times so that  $\ell(\pi) = t_1 + \dots + t_m$ . Note that  $\pi \subset \lambda$ .

Then, by conjugation invariance of the distribution, we have,

$$\mathbb{P} \left( \forall j \leq m, \forall s \leq p_j, c_{i_s^j}(\sigma_n) = j \right) = \mathbb{P} \left( \forall j \leq m, \forall s \text{ such that } 1 + \sum_{\ell=1}^{j-1} t_\ell \leq s \leq \sum_{\ell=1}^j t_\ell, c_s(\sigma_n) = j \right),$$

so that

$$\begin{aligned} a_{p_1, \dots, p_m}^n &= \sum_{\pi \subset \lambda} n^{t_1 + \dots + t_m} (1 + o(1)) \left( \prod_{j=1}^m \mathcal{P}(p_j, t_j) \right) \mathbb{P}(\forall i \leq t_1 + \dots + t_m, c_i(\sigma_n) = \pi_i) \\ &= \sum_{\pi \subset \lambda} n^{\ell(\pi)} (1 + o(1)) \left( \prod_{j=1}^m \mathcal{P}(p_j, t_j) \right) \mathbb{P}(\forall i \leq \ell(\pi), c_i(\sigma_n) = \pi_i), \end{aligned}$$

where  $\mathcal{P}(p, t)$  is the number of partitions of  $p$  with  $t$  parts.

We now denote by  $B_\pi = \{\sigma \in \mathfrak{S}_n, \forall i \leq \ell(\pi), c_i(\sigma) = \pi_i\}$ .

$\forall \sigma \in B_\pi$ , we define  $f_\pi(\sigma)$  as follows : let  $\pi_\sigma$  be the Young diagram following the cycle structure of  $\sigma$  restricted to the cycles having at least one element in  $[[\ell(\pi)]]$ . Then  $f_\pi(\sigma)$  is the filling of  $\pi_\sigma$  with the elements of the cycles of  $\sigma$  starting with the smallest element in each cycle and so that the first column is increasing.

For example, for  $\sigma = (1, 7, 8)(9, 3, 2)(4, 6)(10, 13, 6)(12, 11)$  and  $\pi = (3, 3, 3, 2)$ ,  $f_\pi(\sigma) = \begin{array}{|c|c|c|} \hline 1 & 7 & 8 \\ \hline 2 & 9 & 3 \\ \hline 4 & 6 & \\ \hline \end{array}$ .

Note that for  $n$  large enough,  $f_\pi(\sigma)$  is necessarily an admissible filling of type  $(\pi, \pi_\sigma, n)$ . We have then,

$$\mathbb{P}(\sigma_n \in B_\pi) = \sum_{\mu \subset \pi} \sum_{f \text{ admissible filling of type } (\pi, \mu, n)} \mathbb{P}(f_\pi(\sigma_n) = f).$$

By conjugation invariance, for any admissible filling of type  $(\pi, \mu, n)$ ,

$$\mathbb{P}(f_\pi(\sigma_n) = f) \frac{(n - \ell(\mu))!}{(n - |\mu|)!} = \mathbb{P}(\sigma_n \in A_{1, \dots, \ell(\mu)}^{\mu, x_1})$$

so that

$$\mathbb{P}(\forall i \leq \ell(\pi), c_i(\sigma_n) = \pi_i) = \sum_{\mu \subset \pi} \frac{K_{\pi, \mu, n}}{K_{\mu, \mu, n}} \mathbb{P}(\sigma_n \in A_{1, \dots, \ell(\mu)}^{\mu, x_1}).$$

$$\begin{aligned} a_{p_1, \dots, p_m}^n &= (1 + o(1)) \left( n^{\ell(\lambda)} \mathbb{P}(\sigma_n \in A_{1, \dots, \ell(\lambda)}^{\lambda, x_1}) + \sum_{\mu \subset \lambda, \lambda \neq \mu} C_{\lambda, \mu} \frac{n^{\ell(\lambda)} K_{\lambda, \mu, n}}{K_{\mu, \mu, n}} \mathbb{P}(\sigma_n \in A_{1, \dots, \ell(\mu)}^{\mu, x_1}) \right) \\ &= (1 + o(1)) \left( n^{\ell(\lambda)} \mathbb{P}(\sigma_n \in A_{1, \dots, \ell(\lambda)}^{\lambda, x_1}) + \sum_{\mu \subset \lambda, \lambda \neq \mu} C'_{\lambda, \mu} n^{\ell(\mu)} \mathbb{P}(\sigma_n \in A_{1, \dots, \ell(\mu)}^{\mu, x_1}) \right), \end{aligned}$$

where in the last equality, we have used (14). By replacing  $\sigma_n$  by  $\rho_n$  in the previous calculus, the first implication is direct and the other implication can be obtained by iteration on  $\ell(\lambda)$ . □

We give now the counterpart of Proposition 3.3.

**Proposition 4.2.** *Let  $\Omega$  a word which is not a power. Suppose that its reduced form  $x_{\alpha_\ell}^{\beta_\ell} \dots x_{\alpha_1}^{\beta_1}$ , satisfies  $\alpha_\ell \neq \alpha_1$ . Assume that  $\rho_{1,n}, \dots, \rho_{k,n}$  are independent and **uniformly distributed** on  $\mathfrak{S}_n$ . Then, for any  $\hat{g}_1, \hat{g}_2, \dots, \hat{g}_k \in \widehat{\mathfrak{G}}$ , for any Young diagram  $\mu$ ,*

$$\begin{aligned} \lim_{n \rightarrow \infty} n^{\ell(\mu)} \mathbb{P} \left( \forall i \in \llbracket k \rrbracket, \widehat{\mathcal{G}}_i^{\llbracket \ell(\mu) \rrbracket, \Omega}(\rho_{1,n}, \dots, \rho_{k,n}) = \hat{g}_i \text{ and } (\rho_{1,n}, \dots, \rho_{k,n}) \in A_{1, \dots, \ell(\mu)}^{\mu, \Omega} \right) \\ = \prod_{i=1}^k \mathbb{1}_{\{\hat{g}_i = \mathcal{T}_{|\mu| \cdot \gamma^i(\Omega)}\}}, \end{aligned} \quad (15)$$

where we recall that  $|\mu| = \sum_{j=1}^{\ell(\mu)} \mu_j$ . Consequently, if  $\sigma_{1,n}, \dots, \sigma_{k,n}$  are independent, with conjugation invariant distribution satisfying Assumptions (4) and (5), then, for any Young diagram  $\mu$ , we have that

$$\mathbb{P} \left( (\sigma_{1,n}, \dots, \sigma_{k,n}) \in A_{1, \dots, \ell(\mu)}^{\mu, \Omega} \right) = \mathbb{P} \left( (\rho_{1,n}, \dots, \rho_{k,n}) \in A_{1, \dots, \ell(\mu)}^{\mu, \Omega} \right) (1 + o(1)). \quad (16)$$

*Proof.* A direct consequence of Proposition 1.2 and Lemma 4.1 is that for any  $\Omega$  which is not a power,

$$\mathbb{P} \left( (\rho_{1,n}, \dots, \rho_{k,n}) \in A_{1, \dots, \ell(\mu)}^{\mu, \Omega} \right) = \mathbb{P} \left( \rho_{1,n} \in A_{1, \dots, \ell(\mu)}^{\mu, x_1} \right) (1 + o(1)).$$

Since, when  $n > |\mu|$ ,

$$\mathbb{P} \left( \rho_{1,n} \in A_{1, \dots, \ell(\mu)}^{\mu, x_1} \right) = \frac{(n - \ell(\mu))!}{n!},$$

we have,

$$\lim_{n \rightarrow \infty} n^{\ell(\mu)} \mathbb{P} \left( (\rho_{1,n}, \dots, \rho_{k,n}) \in A_{1, \dots, \ell(\mu)}^{\mu, \Omega} \right) = 1.$$

If  $r$  is the total length of the word  $\Omega$  written under its canonical form and  $t = r \cdot |\mu| - \ell(\mu)$ , let  $j_1, \dots, j_t$  be  $t$  indices in  $\{\ell(\mu) + 1, \dots, n\}$  which are two by two distinct. Let  $(g_1, \dots, g_k)$  be equal to  $(\mathcal{G}_1^{\llbracket \ell(\mu) \rrbracket, \Omega}(\sigma), \dots, \mathcal{G}_k^{\llbracket \ell(\mu) \rrbracket, \Omega}(\sigma))$  when

$$\begin{aligned} i_0^{1, \Omega^{\mu_1}}(\sigma) = 1, i_1^{1, \Omega^{\mu_1}} = j_1, \dots, i_{\mu_1 r}^{1, \Omega^{\mu_1}}(\sigma) = 1, \\ i_0^{2, \Omega^{\mu_2}}(\sigma) = 2, i_1^{2, \Omega^{\mu_2}} = j_{\mu_1 r}, \dots, i_{\mu_2 r}^{2, \Omega^{\mu_2}}(\sigma) = 2, \\ \dots \\ i_0^{\ell(\mu), \Omega^{\mu_{\ell(\mu)}}}(\sigma) = \ell(\mu), i_1^{\ell(\mu), \Omega^{\mu_{\ell(\mu)}}}(\sigma) = j_{1 + \sum_{i=1}^{\ell(\mu)-1} (r \cdot \mu_i - 1)}, \dots, i_{\mu_{\ell(\mu)} r}^{1, \Omega^{\mu_{\ell(\mu)}}}(\sigma) = \ell(\mu). \end{aligned}$$

For such a  $k$ -tuple of graphs, by Lemma 2.3 and independence, we have the following lower bound :

$$\begin{aligned} \mathbb{P}(\forall i \in \llbracket k \rrbracket, \rho_{i,n} \in \mathfrak{S}_{n, g_i}) \\ \geq \prod_{i=1}^k \frac{(n - |\mu| \ell_i - |\mu| \sum_{j=1}^{\ell_i} \gamma_j^i(\Omega))!}{(n - |\mu| \ell_i)!} \left( 1 - |\mu| \sum_{j=1}^{\ell_i} \mathbb{P}(c_1(\rho_{n,i}) \leq \gamma_j^i(\Omega)) - \frac{|\mu| \ell_i - 1}{n - 1} |\mu| \sum_{j=1}^{\ell_i} \gamma_j^i(\Omega) \right), \end{aligned}$$

where  $\ell_i$  is the number of non-trivial connected components of  $\gamma^i(\Omega)$ .

From there, as in proof of proposition 3.3, we get that

$$\liminf_{n \rightarrow \infty} n^{\ell(\mu)} \mathbb{P} \left( \forall i \in \llbracket k \rrbracket, \widehat{\mathcal{G}}_i^{\llbracket \ell(\mu) \rrbracket, \Omega}(\rho_{1,n}, \dots, \rho_{k,n}) = \mathcal{T}_{|\mu| \cdot \gamma^i(\Omega)} \text{ and } (\rho_{1,n}, \dots, \rho_{k,n}) \in A_{1, \dots, \ell(\mu)}^{\mu, \Omega} \right) \geq 1.$$

This concludes the proof of (15). Using similar arguments as in the proof of Theorem 3.1, we get that if  $\sigma_n$  satisfies Assumptions (4) and (5), then (16) holds.  $\square$

We can now conclude the proof of Theorem 1.5 itself.

*Proof.* We assume that  $w = \Omega^d$ , with  $d \geq 1$  and  $\Omega$  is not a power. Let  $d' \in \mathbb{N}^*$  be fixed. If  $\sigma_n := (\sigma_{1,n}, \dots, \sigma_{k,n})$  satisfy (4) and  $\rho_n := (\rho_{1,n}, \dots, \rho_{k,n})$  are independent and uniformly distributed, we want to show that (16) implies that for any monomial  $P$ ,

$$\lim_{n \rightarrow \infty} \mathbb{E}(P(\#_1 w(\sigma_n), \dots, \#_{d'} w(\sigma_n))) = \lim_{n \rightarrow \infty} \mathbb{E}(P(\#_1 w(\rho_n), \dots, \#_{d'} w(\rho_n))). \quad (17)$$

Indeed, it was proved in [Nica, 1994] that the right handside depends only on  $d$  and  $P$ . The first remark is that, for any monomial  $P$ , there exists a polynomial  $Q$ , such that for any fixed  $\sigma := (\sigma_1, \dots, \sigma_k)$ ,

$$P(\#_1 w(\sigma), \dots, \#_{d'} w(\sigma)) = Q(\#_1 \Omega(\sigma), \dots, \#_{dd'} \Omega(\sigma)).$$

Indeed, for any  $j \in \mathbb{N}^*$ ,

$$\begin{aligned} \#_1 w(\sigma)^j &= \sum_{i=1}^n \mathbf{1}_{w(\sigma)^j(i)=i} = \sum_{i=1}^n \mathbf{1}_{c_i(w(\sigma))|j} \\ &= j \#_j w(\sigma) + \sum_{r|j, r \neq j} r \#_r w(\sigma). \end{aligned}$$

On the other hand,

$$\#_1 w(\sigma)^j = \#_1 \Omega(\sigma)^{dj} = \sum_{r|dj} r \#_r \Omega(\sigma).$$

Therefore, by induction, for any  $j \in \mathbb{N}^*$ ,  $\#_j w(\sigma)$  can be expressed as a linear combination of  $\{\#_r \Omega(\sigma)\}_{r|dj}$  and it is enough to show Theorem 1.5 in the particular case when  $d = 1$ . As a consequence of Lemma 4.1, we have that, if for any Young diagram  $\mu$ ,

$$\lim_{n \rightarrow \infty} n^{\ell(\mu)} \mathbb{P}\left(\Omega(\sigma_n) \in A_{1, \dots, \ell(\mu)}^{\mu, x_1}\right) = \lim_{n \rightarrow \infty} n^{\ell(\mu)} \mathbb{P}\left(\Omega(\rho_n) \in A_{1, \dots, \ell(\mu)}^{\mu, x_1}\right),$$

then for any  $P \in \mathbb{R}[x_1, x_2, \dots, x_{d'}]$ , monomial,

$$\lim_{n \rightarrow \infty} \mathbb{E}(P(\#_1 \Omega(\sigma_n), \dots, \#_{d'} \Omega(\sigma_n))) = \lim_{n \rightarrow \infty} \mathbb{E}(P(\#_1 \Omega(\rho_n), \dots, \#_{d'} \Omega(\rho_n))).$$

The convergence of joint moments is therefore a direct consequence of (16) and the convergence in distribution follows.  $\square$

We now go to the proof of Corollary 1.6. Assume that  $(\sigma_{1,n}, \dots, \sigma_{k,n})$  satisfies the assumptions of Corollary 1.6. For every  $j \leq k$ , let  $\tau_{j,n}$  be a random permutation, independent of  $(\sigma_{1,n}, \dots, \sigma_{k,n})$ , with Ewens $_n(0)$  distribution (that is the uniform law on the subset of  $\mathfrak{S}_n$  of permutations having exactly one cycle). We now define, for every  $j \leq k$ ,

$$\tilde{\sigma}_{j,n} := \begin{cases} \sigma_{j,n}, & \text{if } \forall i \in \widehat{\mathcal{N}}_j(w), \frac{\#_i \sigma_{j,n}}{\sqrt{n}} \leq 1, \\ \tau_{j,n}, & \text{otherwise.} \end{cases}$$

Then  $(\tilde{\sigma}_{1,n}, \dots, \tilde{\sigma}_{k,n})$  satisfies the assumptions of Theorem 1.5 and  $(\#_1 w(\sigma_n), \dots, \#_{d'} w(\sigma_n))$  and  $(\#_1 w(\tilde{\sigma}_n), \dots, \#_{d'} w(\tilde{\sigma}_n))$  have asymptotically the same distribution.

## 5 Discussion about optimality

In this last section, we make a few remarks on the optimality of our conditions (4), (5), (7) and (8) on short cycles. We hereafter only consider the case when the permutations are independent and have conjugation invariant distributions. In several cases, in particular the commutator, We can claim that these conditions are sharp. In [Kammoun and Maïda, 2020] we already discussed the case of the product.

### 5.1 Optimality for the commutator

- Assumption (4) is optimal in the sense that if, for some  $\ell \geq 1$ , we have

$$\liminf_{n \rightarrow \infty} n^{-\frac{\ell}{2}} \min(\mathbb{E}((\#_1 \sigma_n)^\ell), \mathbb{E}((\#_1 \rho_n)^\ell)) = \varepsilon_\ell > 0,$$

$$\text{then } \liminf_{n \rightarrow \infty} \mathbb{E}((\#_1([\sigma_n, \rho_n])^\ell) \geq \mathbb{E}(\xi_1^\ell) + \varepsilon_\ell^2.$$

Indeed, one can see that if  $g$  is the class of the graph with adjacency matrix  $I_\ell$ , the event  $\{(\widehat{\mathcal{G}}_1^{[\ell], [x_1, x_2]}(\sigma_n, \rho_n), \widehat{\mathcal{G}}_2^{[\ell], [x_1, x_2]}(\sigma_n, \rho_n)) = (\hat{g}, \hat{g})\}$  will contribute to the limit, leading to the term  $\varepsilon_\ell^2$  in the limit.

- Similarly, Assumption (5) is optimal in the sense that if

$$\liminf_{n \rightarrow \infty} \left( \frac{\min(\mathbb{E}(\#_2 \sigma_n), \mathbb{E}(\#_2 \rho_n))}{n} \right) = \varepsilon' > 0, \text{ then } \liminf_{n \rightarrow \infty} \mathbb{E} \left( (\#_1([\sigma_n, \rho_n]))^2 \right) \geq 2 + \varepsilon'^4.$$

Indeed, as above, if  $\hat{g}'$  is the class of the graph with adjacency matrix  $\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$ , the event  $\{(\widehat{\mathcal{G}}_1^{\{1,2\}, [x_1, x_2]}(\sigma_n, \rho_n), \widehat{\mathcal{G}}_2^{\{1,2\}, [x_1, x_2]}(\sigma_n, \rho_n)) = (\hat{g}', \hat{g}')\}$  will contribute to the limit.

### 5.2 Non-optimality in the general framework

On the other hand, one can find words for which the conditions are not optimal. We mention hereafter a few examples where, by easy considerations, sometimes using our previous results on the product, one could improve the assumptions on short cycles.

- Take for example the case  $w = x_1 x_2^3$ . Using our conditions in the case of the product, one could give conditions on fixed points and two cycles of  $\sigma_2^3$ , that is conditions on cycles of length 1, 3, 6 on  $\sigma_2$ . Our theorem gives conditions on cycles of lengths, 1, 2, 3, 6 therefore being suboptimal.
- More generally, if  $w = w_1 w_2$  where  $w_1$  and  $w_2$  have disjoint supports, one can try to apply the product theorem to get weaker conditions. For example, for  $w = (x_1 x_2) x_3$  asking that the number of fixed points of  $x_1 x_2$  are less than  $\varepsilon \sqrt{n}$  and the number of two-cycles of  $x_1 x_2$  is less than  $\varepsilon n$  can be obtained with only conditions on fixed points of  $\sigma_{1,n}$  and  $\sigma_{2,n}$ .
- In a similar spirit, if for some  $j \leq k$ ,  $\gcd \mathcal{N}'_j = d \neq 1$  for some  $j$ , one can obtain in many cases better conditions by considering  $x_j^d$  as a new subword  $y$  in  $w$ .

The table below summarizes some cases where we checked whether our conditions are optimal or not.

$w$	Optimality of our conditions
$x_1x_2$	Yes
$x_1x_2^2$	Yes
$[x_1, x_2]$	Yes
$x_1x_2^3$	No
$x_1x_2x_3$	No
$x_1x_2x_1x_2$	Yes
$x_1^2x_2^{-2}$	Yes
$x_3x_1x_3^{-1}x_2$	No

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