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# Pre-service teachers' generalization strategies solving figurative growth patterns

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*The aim of this paper is to analyse the algebraic thinking of future elementary education teachers (3-12 years old) when they solve tasks involving figurative growth patterns. The theoretical framework related to generalization processes and strategies and also to learning styles in problem solving is taken as reference. The study involved 43 pre-service teachers and followed a qualitative, interpretative methodology. Data was mainly collected through participant observation and written productions. The results show that the participants were able to use a diversity of generalization strategies, allowing the development of representational fluency, and showed more difficulties with the transition from near to far generalization when they resorted to analytical approaches. In general, we noticed that the exploration of figurative growth patterns facilitates the integration of different learning.*

*Keywords: Algebraic thinking, figurative growth patterns, generalization strategies, learning styles.*

## Introduction

There are recommendations for the inclusion of algebra in the early-years mathematics curricula, defending the proposal of a set of meaningful mathematical experiences, such as identifying patterns, formulating conjectures and generalizations, thinking from particular to general, thinking relationally and functionally between numbers and operations, making sense of symbolism and formalizing mathematical structures (Kieran, 2011; Radford, 2008). Reinforcing the importance of algebraic thinking, we developed some studies that showed the pertinence of using tasks that involve the discovery and study of patterns in visual contexts, as an essential component of building generalization (e.g. Barbosa & Vale, 2015; Vale & Cabrita, 2008). It was identified that, traditionally, students are more analytical, but there are those who prefer visual reasoning, also, we found that visual thinking is teachable. Based on these ideas, we are interested in the development of algebraic thinking through the study of figurative growth patterns. Therefore, and considering the fundamental role of the teacher in the work developed in the classroom, this study seeks to characterize the algebraic thinking of future elementary education teachers in solving tasks involving figurative growth patterns. Based on this problem, we formulated two research questions: 1) Which strategies do the participants use in the generalization process?; 2) What difficulties are experienced by the participants during the generalization process?

## Theoretical framework

In the literature it is possible to observe the attribution of different connotations to the term algebraic thinking. In general terms, we highlight the definition of Blanton and Kaput (2011) who consider algebraic thinking as “a process in which students generalize mathematical ideas from a set of particular cases, establishing these generalizations through the discourse of argumentation,

expressing them gradually in formal, age-appropriate, ways” (p. 413). Algebraic thinking can assume different forms, among which is functional reasoning, identified as one of the main axes of this type of thinking (Kaput, 2008) and one of the central contents in research in algebra in the early years. It can be defined as a process used in the construction and generalization of patterns and relationships, using various linguistic and representational tools, exploring generalized relationships or functions, which constitute mathematical objects in themselves (Blanton & Kaput, 2011). Thinking about the approach to algebra by younger students, it can be said that the intuitive understanding of the concept of function begins to develop before the formal introduction of functions and the teacher can help students to develop functional reasoning, for example, through tasks that imply the discovery and generalization of patterns, facilitating the construction of the function concept.

The search for patterns has long been associated with generalization, considering that it may naturally lead to the expression of generality. This type of mathematical activity can be a powerful vehicle for understanding relationships between quantities underlying mathematical functions, thus contributing to the establishment of functional-type relationships (Barbosa & Vale, 2015; Blanton & Kaput, 2011). Pattern generalization tasks are often used in the classroom to engage students to the extent of finding the value of a near or far term and further articulation with the underlying functional relationship to the pattern using symbols.

Visualization plays a crucial role in mathematics, constituting an important cognitive tool in the development of mathematical concepts and processes in different domains, including problem solving (Rivera, 2011). Several authors (Krutetskii, 1976; Vale et al., 2018) underline the need to diversify the approaches used by the teacher, going beyond analytical strategies, therefore considering three types of students, depending on their preference of thinking in solving mathematical tasks: Visualizers or geometric – use visual solution strategies (figures, diagrams) or pictorial-visual schemes, even when the problems could be more easily solved with analytical tools; Verbalizers, non-visual or analytical – use logical-verbal approaches or non-visual solution strategies (algebraic, numerical, verbal representations), even when the problems could be more easily solved using a visual approach; and Harmonics, mixed or integrated – have no specific preference for logical-verbal or visual-pictorial thinking, and tend to combine analytical and visual strategies, presenting an integrated thinking style.

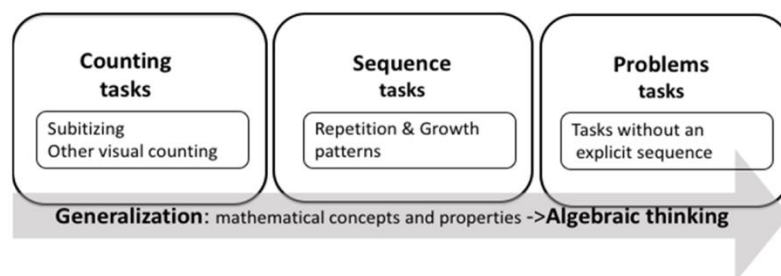
Within the scope of Algebra, pattern exploration tasks can be proposed in a variety of contexts (e.g. numerical, geometric, figurative), however the use of a visual support can lead to the application of strategies of different nature in an attempt to reach generalization, whether visual or analytical (Barbosa & Vale, 2015; Radford, 2008; Rivera, 2011; Vale et al., 2018). In visual strategies, the figures play a key role in discovering the invariant, while in analytical ones, the work developed focuses on the numbers that emerge from the figures (Barbosa & Vale, 2015). The literature shows that students opt for a recursive strategy when generalization tasks provide a clear connection with incremental change, whether proposed in visual or non-visual contexts. However, although this strategy is useful for quickly determining the next terms, it is not efficient to immediately calculate any term whose order is a large value or when the pattern is presented non-successively. Furthermore, it also does not promote the ability to analyse the functional relationship between each term and its position (Kaput, 2008; Radford, 2008). The successful use of visual strategies implies that the

properties of the figures are identified in a discursive way (Duval, 2014). It means that the spatial arrangement of the elements that make up the figure is interpreted as a configuration of objects that are related through an invariant, attribute or property. In the words of Duval (2014), this visual process is intrinsically mathematical and involves discerning and recognizing figurative units, that is, elements that are mathematically relevant in a given visual representation. Students who choose this type of approach can do so in two different ways: identify disjoint sets of elements that are combined in order to obtain the figure as a whole, constructive generalization; detect the existence of overlapping sets, counting certain elements more than once, which are subsequently subtracted, deconstructive generalization (Rivera, 2011).

It can be said that patterns in figurative contexts make an important contribution towards generalization, even allowing the generation of different rules, helping to: reinforce the connections between arithmetic and spatial relationships; assign meaning to formulated rules; realize the need to formulate and validate conjectures. Generally, there are different ways of expressing the relationship between two variables in tasks of this nature, making them privileged contexts to discuss the application of multiple strategies, as well as to explore the formulation of equivalent expressions (Barbosa & Vale, 2015; Vale & Cabrita, 2008).

## Methods

This study seeks to characterize the algebraic thinking of future elementary school teachers (3-12 years old) in solving tasks involving figurative growth patterns. In particular, we aim to answer the following research questions: 1) Which strategies do the participants use in the generalization process?; 2) What difficulties are experienced by the participants during the generalization process? Based on the outlined objectives, a qualitative methodology was chosen. Forty-three students of the 3<sup>rd</sup> year of an Undergraduate Course in Elementary Education (future teachers of students aged 3 to 12 years) participated in this study. The data were collected in the context of a Didactics of Mathematics curricular unit taught by the researchers, which, among other topics, included a module dedicated to Algebra. In this module, that lasted 9 hours (3x3h), we followed a didactical proposal (Vale et al., 2011), whose sequence is shown in Figure 1. The authors define a learning trajectory, consisting of three fundamental steps, for the development of algebraic thinking: Counting; Sequences; and Problems. This didactical sequence assigns a fundamental role to visualization, favouring figurative contexts.



**Figure 1: Didactical sequence**


In this curricular unit, we started by exploring basic visual counting tasks (subitizing), evolving to visual counting in different contexts, to promote pattern recognition with several arrangements. In

the next step, tasks involving the discovery of patterns and the construction of generalizations were explored, with repetition and growth patterns. However, we chose to work particularly on growth patterns as they more naturally lead to the discovery of relationships between two variable quantities (Rivera, 2011). It should also be added that tasks centred on the exploration of figurative growth patterns have been gaining relevance, with the analysis of different approaches and representations to promote algebraic thinking and the understanding of the structure. In the last step, problems were proposed involving the construction of sequences and the establishment of generalizations.

Data were collected in a holistic, descriptive and interpretive way and included classroom observations and written productions of the proposed tasks. All written records produced by the participants were analysed and triangulated with field notes referring to the classes observed. The field notes were elaborated during each class, and discussed after these sessions, assuming the form of free-flowing notes, focusing on the participant's reactions, interactions, discussions and interpretations, having the problem in mind, and included a record of the facts but also the researchers' commentaries about those facts. The analysis of these data was guided by two broad categories: type of strategy (visual/analytical); difficulties experienced.

## Results and discussion

In this article, we chose to present the results of the implementation of one of the tasks proposed throughout the study (Figure 2). This task was selected because it allows to illustrate the most representative results, referring to the strategies used and the difficulties felt by the participants in situations of near and far generalization.

<p>Observe the three figures in the sequence. Each figure has more squares than the previous one:</p> <ol style="list-style-type: none"> <li>1. If these figures are the first three terms of a sequence, describe the following figure and determine the number of squares.</li> <li>2. Keeping the pattern, determine the number of squares in the 10th figure. Explain how you thought.</li> <li>3. How many squares does the 100th figure have? Explain how you thought.</li> <li>4. How many squares will figure <math>n</math> have? Explain how you thought.</li> </ol>	 <div style="display: flex; justify-content: space-around; margin-top: 5px;"> <span>Fig. 1</span> <span>Fig. 2</span> <span>Fig. 3</span> </div>
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**Figure 2: Task – Pattern with squares**

It is a growth pattern, presented in a figurative context, however future teachers could freely resort to strategies of visual or analytical nature in an attempt to generalize. Multiple approaches emerged in solving this task, because the context allowed it, but also because some participants found it easier to recognize regularities spatially, while others showed a preference for analytical recognition (Krutetskii, 1976; Vale et al., 2018). The strategies used varied between visual and analytical, with the prevalence of visual approaches, possibly due to the influence of the didactical proposal implemented in the curricular unit's classes.

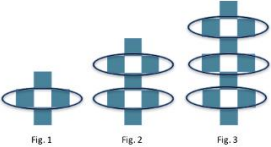
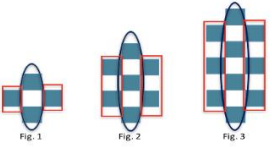
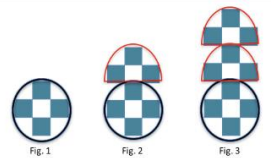
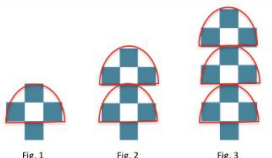
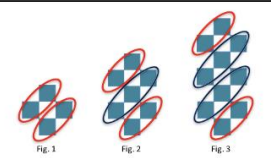
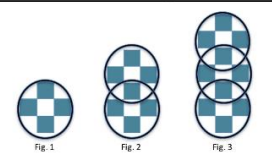
Among the analytical strategies, recursive reasoning, in near generalization, and trial and error and functional reasoning, in far generalization, stood out. In these cases, the participants converted the figures into numbers and identified the variation from one term to the next (+3) until they reached the 10th figure. Situations of transition to distant generalization were observed using trial and error, trying to test numerical expressions for the first terms that could show some structure: “To discover the number of squares in figure 100, I used trial and error until I found a formula. This sequence varies from 3 to 3 so I multiplied 3 by the order of the figure adding 1, so I have  $3 \times 100 + 1$ , 301 squares”. Alternatively, some of the participants recognized numerical relationships in the values that led to a general expression, managing to establish a functional-type relationship. Figure 3 represents an example of a solution of this nature presented by one of the participants.

N° fig	N° cuadrados	Total
1	4	
2	7 $\downarrow +3$	$2 \times 2 + 3 = 7$
3	10 $\downarrow +3$	$3 \times 2 + 4 = 10$
4	13 $\downarrow +3$	$4 \times 2 + 5 = 13$
5	16 $\downarrow +3$	$5 \times 2 + 6 = 16$
6	19 $\downarrow +3$	$6 \times 2 + 7 = 19$
7	22 $\downarrow +3$	$7 \times 2 + 8 = 22$
8	25 $\downarrow +3$	$8 \times 2 + 9 = 25$
9	28 $\downarrow +3$	$9 \times 2 + 10 = 28$
10	31	$10 \times 2 + 11 = 31$
(...)	(...)	(...)
$100 \times 2 + 101 = 301$		
A figura 100 terá 301 quadrados.		

**Figure 3: Example of functional reasoning in an analytical context**

In this case, a recursive type of reasoning was used, however, in the face of a situation of far generalization, the strategy was changed, since the recursive relation would not be useful for terms of greater magnitude. Having observed the numbers in the second column of the table, the student teacher was able to find numerical relationships, described in the third column, which indicated the discovery of a rule.

Six different cases of visual strategies were identified, used by several participants, which are shown in Figure 4, grouped into three situations.

A	 $n \times 2 + (n+1)$	 $n \times 2 + (n-1)$
B	 $4 + (n-1) \times 3$	 $1 + n \times 3$
C	 $2 + (n-1) \times 3 + 2$	 $n \times 4 - (n-1)$

**Figure 4: Visual strategies**

The strategies presented in A illustrate the observation of the elements of the figures by rows and columns. In B, we present the strategies that emerged most frequently from the observation of groups of three elements in each of the figures. The distinction between the two lies in the consideration of the 1<sup>st</sup> figure as an integral part of the following figures or in the identification of subsets from the 1<sup>st</sup> figure onwards. The cases illustrated in C were used only once and are therefore the most original. After analysing the situations described, it should be noted that most of the participants who used a visual strategy were able to adapt it to questions of near and far generalization. These strategies allowed not only to reach the expression of generality in symbolic terms, establishing a parallelism between the rule and the way of seeing the figure, but also to generate equivalent expressions. It was also found that future teachers who presented a solution of this nature favoured constructive generalizations, with only one case representing a deconstructive generalization (2<sup>nd</sup> situation in C).

Contrary to what happened in near generalization, not all were successful in distant generalization. Difficulties translated into the application of wrong strategies or the inability to present an answer were observed. It should be noted that these situations emerged only when using approaches of analytical nature. In addition to the unsuccessful use of trial and error to formulate the algebraic expression, one of the errors that we must highlight, which appeared in only two solutions, concerns the use of direct proportionality when the pattern does not show this relationship. It is a powerful mathematical tool whose application can be influenced by the use of multiples, as happens in the transition from the 10th to the 100th term of the sequence, or by working in strictly numerical and meaningless contexts (Barbosa & Vale, 2015).

The transition from near generalization to far generalization provoked a conflict in the reasoning of some of these future teachers, with a clear change in the choice of strategy. In these situations, they started by using recursive reasoning to determine the number of squares in the 10th figure and, in the far generalization, they resorted to trial and error or the identification of a functional relationship through the recognition of generalizable numerical relationships. From the analysis of the

participants' work, the lack of representational fluency was still notorious. In the cases where they failed to arrive to the solution with an analytical strategy, they were unable to use visual representations in the construction of the generalization. In our opinion these specific results are related to the formative path of these future teachers in which symbolic representations and analytical approaches were privileged, thus justifying their difficulties in the transition to other thinking styles, particularly involving visualization.

## Concluding remarks

(Future) teachers should be aware of their students' way of thinking and use different approaches that allow exploring the abilities of each one, following this procedure with each of the tasks they use. We believe that visual contexts are an asset in understanding mathematics in global terms and, in particular, in developing algebraic thinking. This type of tasks facilitates the use of multiple generalization strategies, whether of a visual or analytical nature (Radford, 2008; Rivera, 2011; Vale et al., 2018). The presentation and establishment of a parallelism between the different approaches and representations can make an important contribution to increase the students' repertoire of strategies, as well as to develop representational fluency (Barbosa & Vale, 2015). Reflecting on the strategies also makes it possible to perceive that, despite the multiplicity of approaches that can emerge from this type of task, some may present limitations when moving from near to far generalization. We believe that the didactic proposal (Vale et al., 2011) used, with its strong visual nature, has a positive impact on students with regard to the development of algebraic thinking. Understanding the meaning of numerical and algebraic expressions, thinking flexibility in the use of different generalization strategies, the formulation of equivalent expressions are some examples. Through its different steps, the learning trajectory outlined in the didactical proposal facilitates the understanding of powerful mathematical concepts.

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