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# Disclosing parametric functions in the transition from pen-and-paper to GeoGebra

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*Students and teachers struggle with calculus topics. In recent decades, various technologies have been developed for learning and teaching calculus. In this paper, we report an empirical study on the exploration of parametric functions by 11th-grade students, in the transition from pen-and-paper to GeoGebra. In particular, we use the phenomenological perspective outlined by Rota to study the meaning-making processes of students engaged in the study of functions, paying attention to the role of GeoGebra in their disclosure processes. The analysis of two case studies revealed that students disclosed four layers of meaning-making. These layers testify an increased mastery of the topic of parametric functions.*

*Keywords: GeoGebra, meaning-making, disclosure, parametric functions.*

## Introduction

Research has often highlighted how many students struggle to learn calculus and, similarly, teachers struggle to teach calculus in a meaningful way (Baker et al., 2000). The research community is thus engaged in the search for new ways and tools to improve the teaching and learning of calculus and pre-calculus on the one hand, and to support the creation of meaning in this subject on the other hand (Bressoud et al., 2016; Nardi, 2008;). As Bos et al. (2022) point out, various technologies for learning and teaching calculus have been developed in recent decades. Besides traditional tools that are still widely used (e.g., the scientific calculator), new technologies include, for example, augmented reality, virtual reality, GeoGebra, or MATLAB software. They are all used to teach pre-calculus concepts in secondary school and more advanced calculus concepts at university level.

In this paper, we report on an empirical study concerning the exploration of parametric functions by two 11-grade students, first with the use of pen-and-paper and then with the dynamic geometry software GeoGebra, what is called a *transition beyond* (Geraniou et al., 2023). For the purpose of our investigation, the most essential feature of such a digital tool is its ability to initiate the “process of producing something” (Monaghan et al., 2016, p. 8) - graphs, equations, illustrations, movements, etc. - as well as the process of meaning-making. A number of authors have explored the teaching and learning of specific calculus content with GeoGebra: limits, derivatives and definite integrals (Caligaris et al., 2015); how GeoGebra can help students to visualize functions and to determine slopes (Garber & Picking, 2010); to visualize the effects of varying parameters on the graph of functions by using sliders (Wassie & Zergaw, 2018).

In this study, we use the phenomenological perspective set out by Rota (1991) to study students’ meaning-making processes when engaging in the study of parametric functions, paying attention to the role of GeoGebra in their disclosure processes. Our aim is to contribute to the literature by analyzing how GeoGebra can support students’ disclosure process when learning parametric

functions. Our study is part of ongoing research. What we report here are the preliminary analysis results that have emerged.

## **Meaning-making as disclosure**

Research in Mathematics Education has emphasized the notion of ‘meaning-making’ for concepts in calculus (Thompson, 2013). However, this notion is not uniquely defined in the literature: among all the possibilities, here we will adopt the phenomenological perspective as elaborated by Rota (1991). Starting from the fundamental assumption that there is no unique way of seeing things, the phenomenological perspective conceives learning as a process by which people make sense of things in the world in the contexts to which they are exposed. This process of meaning-making is named *disclosure*, and it happens when someone is able to grasp an object’s functionality in a specific learning situation. Different people’s backgrounds and different contexts can generate different layers of meaning in time and over time. According to Rota, these layers of meaning should not be intended as isolated or hierarchical but layered upon one another. Disclosure happens first through perceptual senses, but people should be trained in it, and the mediation of some tools can be extremely relevant. For instance, “appropriate tools can play mediating roles in how the students make sense of novel concepts. Tools can not only trigger the disclosure process, they can also sustain it through different layers of meaning” (Swidan et al., 2020, p. 85). Indeed, Swidan and colleagues (2020) explored how students gave meaning to the function-antiderivative relationship when working with dynamic software showing the two draggable functions’ graphs in two related Cartesian systems. Three layers of meaning were identified in students’ cognitive processes: disclosing objects, disclosing relationships and disclosing functional relationships. Such layers of meaning were disclosed due to the features of the digital software and the interactions with their peers. Indeed, one of the main findings of the study was that “the layers of meaning indicate how making sense of mathematical concepts emerges from the specific interpretative processes employed by the students because of their interactions with the different situations they produce and observe on the two screens [displayed by the digital tool]” (Swidan et al., 2020, p. 97).

In a similar stream of thought, Bagossi et al. (2022) analyzed students’ meaning-making of mathematical concepts by considering the role of the interactions with the digital tools at stake, specifically a GeoGebra applet and augmented reality. The analysis of two learning episodes focused on the mathematical conceptualization of real phenomena revealed that students disclosed three layers of meaning: relating the real phenomenon to the mathematical model; variation of a specific quantity, and covariational relationships between quantities. Such layers witness an increase in mastery of the mathematical concepts: they are “neither hierarchical nor ordered. They are intertwined throughout the episodes and may be disclosed by the students at different moments” (Bagossi et al., 2022, p. 377).

Given our theoretical assumptions and the purpose of our investigation, the research questions which we will try to answer with this contribution can be formulated as follows:

- Which layers of meaning emerge when students learn parametric functions in the transition from a pen-and-paper to a GeoGebra task?
- How do students use GeoGebra to disclose such layers of meaning?

## Research context

Our research is situated within a laboratorial and interdisciplinary project that involved an 11-grade class of 21 students in a scientific-oriented secondary school in the province of Turin, Italy, during the school year 2020/2021. Within this project, the students received various prompts to reflect on and understand the role of variables and parameters after engaging in a learning experiment on the conceptualization of a real phenomenon (Bagossi, 2022). Here we focus on one of the intermediate moments of the project when the students were tested to assess their knowledge and skills about function study, the mathematical topics they were dealing with at that time. In particular, we focus on one of the tasks of the test shown in Figure 1.

**Task**

Consider the relationship  $y = \frac{3ax}{x^2-x}$ .

- Represent the probable graphs of the function  $y = f(x)$  and  $a = f(x)$  and identify domain, zeros, sign and asymptotes of the functions.
- Explain the role of the parameter in analytical and graphical representations.
- Draw the probable graphs of  $y = f(a)$ , comment on them and compare them with the previous ones.

**Figure 1: Task instruction**

Note that the task required the students to draw the probable graphs of the functions, namely their graphs according to the mathematical knowledge they possessed until that moment: the domain and range, zeros, sign, intersection points, and vertical asymptotes. For this reason, the teacher introduced the term ‘probable graph’ in the class terminology. Indeed, it is not usual for Italian students to engage in function study in the early years of high school, but in this case, the teacher initiated them to such study from the earliest years through the sketching of probable graphs and using finite differences as a pre-calculus tool.

The task was designed taking into consideration three intertwined aspects, which should jointly concur to produce students’ solution processes:

- The *semiotic* one, which refers to the presence of different registers (the symbolic and the graphical);
- The *conceptual* one, which requires students to understand the meaning the relation takes on before and after the algebraic manipulation. In fact, the students had to grasp that the letters used as independent/dependent variables or parameters (in our case  $x$ ,  $y$ ,  $a$ ) are not to be understood as rigid designators (Arzarello et al., 1994), but rather than their role may vary depending on how they are considered. For instance, when considering  $a=f(x)$  [ $a = \frac{y(x^2-x)}{3x}$ ]  $a$  should be intended as the dependent variable,  $x$  as the independent and  $y$  as the parameter; when studying  $y=f(a)$  [ $y = \frac{3a}{x-1}$ ],  $a$  should be intended as the independent variable,  $y$  as the dependent one and  $x$  as the parameter;
- The *procedural* one consisting of treatments (in the sense of Duval, 2017) within the same register and specifically the symbolic one:  $y=f(x)$  is a relation to be converted into different functions ( $a=f(x)$ ;  $y=f(a)$ ), and this requires manipulating the algebraic expression by doing

factorization and shortening. Then conversions (Duval, 2017) are required to move from the symbolic register to the graphical one.

First, the test was carried out using pen and paper; in a second moment, the teacher asked the students to do the task again as homework. This time, however, they could freely use GeoGebra and then submit the new task resolution, either as a scan of their paper worksheet or on a digital document such as a .doc file.

## Data collection and data analysis

The collected data consists of the final productions of both the pen-and-paper task (PPT) and the task solved with the aid of GeoGebra (GGT). We have chosen to consider the productions of two students, Marta and Saro. They are two students of medium-high level and, compared to their peers' protocols, their productions contain the development of all three points (a, b, c) of the task. In our analysis, of qualitative nature, we compared their resolutions of the three points of the task, highlighting which disclosures were generated by the transition to the software: we identified disclosures when the two students revised the analysis elaborated during the PPT in light of the new information provided by the use of GeoGebra and elaborated on it by enriching their analysis or revising their previous conjectures. We coded the layers of meaning to which such disclosures refer by focusing on the mathematical concepts that students elaborated on (e.g., degree of the function, points not belonging to the domain, variables and parameters). We performed the descriptive coding of the disclosures and related layers of meaning independently and then revised it until we reached an agreement.

## Results and discussion

In the following, we will present four layers of meaning that we identified by analyzing the two students' productions. The presentation of the layers does not follow their order of appearance when solving the task, but it is consistent with the exposition of the disclosed meanings.

*Layer 1:* The first layer of meaning concerns the degree of the function, specifically through the conversion between the symbolic and the graphical register. Answering item (c) of the task, Marta elaborates on the comparison between  $y=f(a)$  and the other two functions.

Marta: This latter function  $[y=f(a)]$  is of first degree, and therefore its graphs are distinguished from those of the first one  $[y=f(x)]$ , which is instead of second degree (also the second one  $[a=f(x)]$  is actually of first degree because I could have simplified it to  $a = y(x - 1)/3$ ). [...] I chose to keep the factors of the initial function unchanged when I obtained  $a=f(x)$ , and this resulted in the fact that it remained in fractional form (and so interrupted in  $x=0$ ).

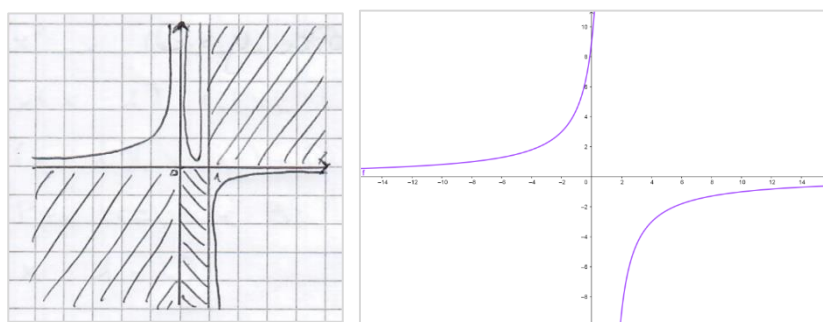
From Marta's answer, it seems that the use of GeoGebra helped her to connect between the graphical and symbolic registers. Indeed, the graphical representation in the software of the function  $a=f(x)$  made her realize that actually the function is a first-degree (linear) one and that its analytical expression can be simplified. Anyway, she maintains all the factors and observes the fractional form of the function. Saro, too, in switching from the PPT to the GGT proceeds by simplifying the functions  $a=f(x)$  and  $y=f(a)$ . However, he does not transcribe anything to justify this new way of writing functions. Therefore, we cannot claim that the software has generated any layer of further disclosure in him about this aspect.

*Layer 2:* The second layer of meaning refers to the nature of the points in which the function is not defined. Answering item (a) of the task, on the PPT, Marta writes that (Figure 2, on the left):

Marta: The asymptotes are 0 and 1, but if  $a$  is equal to 0, then the function is a horizontal line with  $y=0$ . The only possible zero is 0, but it is an asymptote.

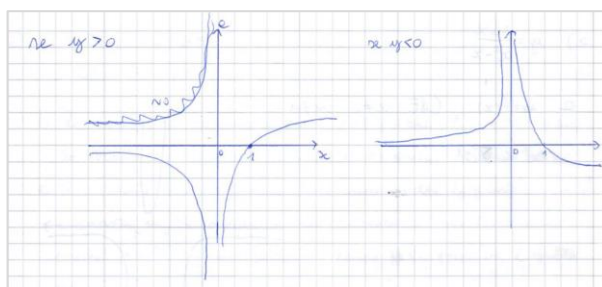
After, when solving the GGT, she elaborates on a different solution (Figure 2, on the right):

Marta: The only asymptote of the function, contrary to what I said in the [PPT] task, is at  $x=1$ , because if the variable assumed the value 0 both numerator and denominator would be 0 and so, even though it might not be seen in the graph, at  $x=0$  the function is interrupted. I understand this because on GeoGebra, if I enter the command “point of intersection” and look for the one [intersection] of the function with the  $y$ -axis, what it returns me is “undefined”.



**Figure 2: Marta’s graphical representation of the function  $y=f(x)$  when  $a > 0$  on the PPT and the GGT**

Initially, when studying  $y=f(x)$ , Marta interprets  $x=0$  as an asymptote; later, the support of the software helps her to disclose that such a point is just an interruption in the domain of the function, namely a point of discontinuity. Marta herself acknowledges the role of the software that, through the command “point of intersection” and the related output “undefined”, helps her elaborate on the nature of that point. Saro, instead, in the PPT identifies  $x=0$  and  $x=1$  as points not belonging to the domain of the function, and that could be interpreted as vertical asymptotes. When moving to the GGT, as we have pointed out before, Saro collects an  $x$  at the numerator. Hence, he simplifies the function and works entirely on the shortened version ( $y = \frac{3a}{x-1}$ ) precisizing that only  $x=1$  is not a point in the domain.



**Figure 3: Saro’s probable graph in performing point (a) of the PPT**

Even on the software he inserts the shortened function. Therefore, unlike Marta, he does not exploit the software support to locate the intersections with the axes, and does not have access to a disclosure on the real nature of these points. Saro’s disclosure here is another: while in the PPT, his probable graph was closer to that of branches of hyperbolas (Figure 3), it is by exploiting the software that the

graph he obtains is that of a straight line. Therefore, in addition to reporting the zeros of the function, he also adds specifics about its sign as follows:

Saro: If  $y=0$  the function will always equal zero. If  $y<0$  the function will be a decreasing line passing through the point  $(1,0)$ . If  $y>0$  the function will be an increasing line passing through the point  $(1,0)$ .

*Layer 3:* The third layer of meaning deals with the meaning of the parameter especially with respect to other variables. In answering item (b) of the task, the change from the PPT to the GGT allowed a different understanding of the role that the parameter plays in the analytical and graphical representation of the relation  $y = \frac{3ax}{x^2-x}$ . Saro writes in his PPT that

Saro: The parameter in the analytical representation can correspond to any number, so in the graphical representation it can change the sign of the function.

Marta, on the other hand, again working on the PPT, observes that

Marta: In the graphical representation, the parameter has the ‘ability’ to create symmetry with respect to the  $x$ -axis when its sign varies. In the analytical representation, on the other hand, it causes the sign of the entire function to vary if it is negative. In both cases, if the parameter is 0, everything cancels, and we have the expression  $y=0$  and a horizontal line at  $y=0$ .

Thus, both impute to the parameter the possibility of a change of sign for the function. Moving on to the GGT, both students benefit from a disclosure. Saro writes:

Saro: The parameter in the analytical representation represents any number, while in the graphical one, it can dilate the function if it multiplies the independent variable, it can translate it if it is added to the independent variable, but it cannot change the structure of the function.

Marta makes similar, but more detailed observations:

Marta: The parameter  $a$  in the function  $y=f(x)$  creates, from a graphical point of view, a straight line  $y=0$  in the case where it is equal to 0 [...]. As its absolute value increases, a dilation takes place, which appears to occur both horizontally and vertically. From the analytical point of view, however, the parameter cancels the result of the equation when it is equal to 0, whereas it could be said that  $f(x)=a \cdot g(x)$ , and thus that in the graph one should expect dilation in the vertical direction. The parameter  $y$  in the function  $a=f(x)$  creates, from a graphical point of view, a straight line [...] when it is equal to 0, while as its value increases, the line rotates, pivoting on the zero  $x=1$ , in a counter clockwise direction and vice versa.

For both Saro and Marta we can identify a level of meaning supported by the software that makes them visually grasp, by using the slider command, how as the parameter varies we can see dilations - in the case  $y=f(x)$  - or translations - in the cases  $a=f(x)$  and  $y=f(a)$  - of the function, although it continues to maintain its structure (as a hyperbola or straight line). Thus, both students capture the role of the parameter on an analytical and graphical level.

*Layer 4:* The fourth layer of meaning refers to the different roles of variables and parameters in a symbolic representation. Both students, who previously on the PPT had no particular difficulty in graphing the probable graphs of the three functions ( $y=f(x)$ ,  $a=f(x)$ , and  $y=f(a)$ ), by switching to GGT they become aware of a limitation of the software in naming the variables. In fact, on GeoGebra, the parameter can be realised using the slider command, but the software does not allow the letters  $x$  and

y as a label for the slider, because they are only used to identify dependent and independent variables. It is therefore essential not to intend the three letters used in the symbolic representation as rigid designators (Arzarello et al., 1994), but to be aware of the three entities considered in the relation (independent/dependent variable, and parameter) and of the role attributed to each letter ( $x$ ,  $y$ ,  $a$ ) in a specific symbolic representation. This implies, for students, not only being clear about which entity is to be considered as a parameter in the three functions but also familiarity with renaming parameters and variables correctly to pass them on to the software. This layer of meaning is not supported by the software, but such a limitation of the software forced the students to manipulate the algebraic writing.

## **Final remarks**

This study contributes to that strand of research, not so widespread, that adopts a phenomenological perspective on meaning-making. This theoretical choice was motivated by the wish to investigate how such a learning process happens within the specific context to which students are exposed. Whereas other studies have focused on the role of digital tools (Bagossi et al., 2022; Swidan et al., 2020), our research focuses on how the transition from pen-and-paper to a digital tool may lead to a disclosure of new meanings. Indeed, the preliminary results of this contribution shed light on how GeoGebra highlighted conflicts of which the students were not exactly aware (e.g., probable graphs obtained on paper did not match the ‘real’ graph of the function under investigation; points, where asymptotes had been identified, were actually points of discontinuity), thus enabling the students to initiate a process of disclosure. In fact, different layers of meaning emerged precisely in the transition from PPT to GGT resolution. These layers do not follow a hierarchical order but are intertwined, and this intertwining is inherent to the nature of the task itself. Indeed, to really benefit from a disclosure, students must have done some thinking before even using GeoGebra. A limitation of the study certainly remains having directly analyzed students’ final productions without having the opportunity to delve into the moments of disclosure emerging during the working phase with the software: this is a purpose for further research. Furthermore, our future research intentions are to focus on a more fine-grained analysis by taking into account Duval’s conversion/treatment processes (2017) in order to investigate more in-depth the meaning-making processes enacted by the students in conceptualising variables and parameters during their learning path.

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