Unitarity excess in Schwartzschild metric

Philippe Jacquet, Veronique Joly

Unitarity excess in Schwartzschild metric

Philippe Jacquet[1][0000−0001−7919−1206] Véronique Joly[2]

[1] Inria Saclay Ile-de-France, France
philippe.jacquet@inria.fr
[2] Onera Alumni, France
vjoly58@icloud.com

Abstract. We refer to the black hole information paradox. We look after the existence of eigenvalues with non zero imaginary part in the Gordon Klein equation with Schwarzschild metric. Such eigenvalues exist because the Schwartzschild metric is singular on the event horizon. The eigenvalues should be proportional to the inverse of black hole radius. The existence has many impacts, among other that black holes should be again eternal. However the effects of the unitary violation should not be detectable within known black holes with existing technologies.

Keywords: Unitarity · Schwartzschild metric · Gordon Klein equation.

1 Motivation

It is known that the black hole thermal radiation, described in 1976 [1], leads to the *Black Hole Information Loss Paradox*. This paradox suggests a violation of quantum unitarity. Since then, many solutions have been proposed to dissipate the paradox. In fact collecting all the solutions to the paradox proposed so far is already a challenge. To make it short, the solutions proposed since almost 50 years span over a great variety of ideas:

- (a) information could be encoded in thermal radiation, in correlation between the future and the past of the black hole [2, 3];
- (b) information stay hidden in a Planck size Black Hole remain [4];
- (c) information escape in a baby universe [5];
- (d) an hypothetical firewall prevents the information to enter the black hole or simply prevents the formation of the black hole [6, 9];
- (e) the black hole has soft hair which make the event horizon to have fluctuations which allow information to escape [8, 7];
- (f) a quantum theory of gravity where "graviton" are not massless [10].

Most of the proposed solutions struggle for keeping intact the principle of quantum unitarity. On the other hand the "anti-solutions", *i.e.* the solutions which accept the possibility of an actual violation of unitarity, are by far less numerous. Our aim is to try to recover the simplest non unitary solution. By "simple" we mean that we limit our investigations in a semi-classic Schwarzschild metric. Surprisingly we got imaginary eigenvalues which indeed hold the signature

of non unitary evolution. The reason why we get a non unitary evolution from an equation which is basically unitary, comes from the fact the Schwartzschild metric around a black-hole shows singularities at the event horizon.

However there is a real challenge in the fact that the occurrence of such imaginary eigenvalues should not bring any significant departures from what has been measured via today labs experiments or astronomical observations. To make it simple, if the consequence of non unitary quantum physics would make stones to rush out of our atmosphere toward the closest black hole, then the result of this paper would be highly questionable.

An unexpected outcome of our results (under some hypotheses), is that the average apparent lifetime of a black hole before complete evaporation is made again infinite. This is a kind of paradoxical since it is precisely the evaporation of the black hole which led to the unitary violation hypothesis. Our paper is divided in three main contributions:

1. The handling of the semi classical Klein-Gordon equation of scalar field in Schwarzschild metric for a single particle;
2. the asymptotic estimating of its main eigenvalues when the event horizon radius is large. In particular the eigenvalues with non zero imaginary part have their imaginary part asymptotically smaller than $\frac{1}{4R}$, R being the radius of the black hole, all expressed in Planck unit, the value is rather large since it leads to an unitary excess of one unit per traversal time of the event horizon at light speed;
3. The global behaviour of non unitary black-hole when the number of particles is proportional to the area of the event horizon. Under this hypothesis the non unitary black hole is eternal. But a *doomsday* analysis show that the actual *posterior* effect on quantum measurement is in fact not very important (although real) of the order of $\frac{1}{R^3}$ and might be difficult to detect through measurement.

2 Semi-classic scalar field equation

The scalar field function Ψ under the relativistic Gordon-Klein equation

$$\hbar^2 \Box \Psi + m^2 c^2 \Psi = 0. \tag{1}$$

The d'Alembertian operator $\Box\Psi$ defined as follow

$$\Box \Psi = \frac{1}{\sqrt{\det(\mathbf{g})}} \sum_{ij} \partial_i (g^{ij} \sqrt{\det(\mathbf{g})} \partial_j \Psi)$$

where $\mathbf{g}$ is the metric matrix. The relativistic version of the operator $\partial_t^2 \Psi - \Delta\Psi$, where Δ is the spatial Laplacian.

2.1 Schwarzschild metric

Assuming an isolated black hole of event horizon radius R and a point with polar coordinates (r, θ, ϕ) taken from black hole center, r is the distance to the center, θ is the polar angle, and ϕ the azimuthal angle, the metric matrix is:

$$\mathbf{g} = [g_{ij}] = \mathrm{diag}(-(1-\frac{R}{r})^{-1}, -r^2, -r^2\sin^2\phi, (1-\frac{R}{r}))$$
$$\mathbf{g}^{-1} = [g^{ij}] = \mathrm{diag}((-(1-\frac{R}{r}), r^{-2}, -r^{-2}\sin^{-2}\phi, (1-\frac{R}{r})^{-1}).$$

where diag(.) is the diagonal matrix. The term in $(1-\frac{R}{r})^{-1}$ confirms that the metric is singular on the event horizon (the points at distance R from the center, and on the center itself. Under a spherical symmetry hypothesis which makes $\Psi = \Psi(r,t)$ to depend only on parameter r and time t, the Gordon-Klein equation becomes:

$$\frac{1}{1-R/r}\frac{1}{c^2}\partial_t^2\Psi - \frac{1}{r^2}\partial_r(r^2(1-R/r)\partial_r\Psi) + m^2\beta^2\Psi = 0 \tag{2}$$

with $\beta^2 = c^2/\hbar^2$.

3 Eigenvectors of Gordon-Klein equation

In this section we will assume R large compared to Compton wavelength $\lambda_C = \frac{\hbar}{mc}$. The case where R is of the same order than the Compton wavelength will be investigated in a further section.

We investigate the case where $\Psi(r,t) = e^{-i\omega t}\Psi(r)$ thus ω is an eigenvalue and function $\Psi(r)$ is the spatial part at time $t = 0$. Quantity ω is a complex number; in an unitary setting have the imaginary part $\Im(\omega)$ should always be null. In any case the function $\Psi(r)$ satisfies the equation:

$$(1-R/r)\left(\frac{1}{r^2}\partial_r(r^2(1-R/r)\partial_r\Psi) - m^2\beta^2\Psi\right) = -\frac{\omega^2}{c^2}\Psi.$$

thus it comes that $\Psi(r,t) = e^{\pm i\omega t}\Psi(r)$ since $\partial_t^2\Psi(r,t) = -\omega^2\Psi(r,t)$.

We consider as area of interest the horizon area where $r \approx R$. The central area with $r = 0$ can be investigated by analytical continuation but does not not give interesting insight.

3.1 Analysis in the horizon area

In the horizon area with $r = R + x$, we have the first order approximation

$$\frac{1}{R^2}x\partial_x(x\partial_x\Psi(r)) - \frac{m^2\beta^2}{R}x\Psi(r) = -\frac{\omega^2}{c^2}\Psi(r).$$

We introduce the auxiliary function f_a which satisfies for some $a \in \mathbb{C}$,

$$x\partial_x(x\partial_x f_a(x)) - xf_a(x) = a^2 f_a(x).$$

In other words we are looking for the eigenvectors of the operator $x\partial_x(x\partial_x f(x)) - xf(x)$, for the eigenvalue a^2. When $R \to \infty$ we have $\Psi(r) \approx f_a(Ax)$ with $A = Rm^2\beta^2$ and $a = \pm i\frac{R}{c}\omega$. Indeed the above equation can be rewritten in

$$x\partial_x(x\partial_x\Psi(r)) - Ax\Psi(r) = -\frac{R^2}{c^2}\omega^2\Psi(r).$$

where $f_{iR\omega/c}(Ax)$ is clearly the solution. Let $f_a^*(s)$ denotes the Mellin transform of $f_a(x)$, it must satisfy:

$$s^2 f_a^*(s) - f_a^*(s+1) = a^2 f_a^*(s)$$

or $(s-a)(s+a)f_a^*(s) = f_a^*(s+1)$ which has solution $f_a^*(s) = \Gamma(s-a)\Gamma(s+a)$, where $\Gamma(.)$ denotes the Euler function.

Lemma 1. *When $a \neq 0$ we have $f_a(x)$ which behave like $x^{-|\Re(a)|}$ when $x \to 0$, where $\Re(a)$ denotes the real part of a. The quantity $f_0(x)$ behaves like $\log x$ when $x \to 0$. In both case we have $f_a(x)$ which exponentially decays when $x \to +\infty$.*

Proof. For convenience, we assume that $\Re(a) \geq 0$. The Mellin transform f_a^* is defined for all s such that $\Re(s) > \Re(a)$. The inverse Mellin transform tells that $f_a = \frac{1}{2i\pi}\int_{c-i\infty}^{c+\infty} f_a^*(s)x^{-s}dx$ for c in the definition domain of $f_a^*(s)$. When moving the integration line toward the left one meets a first pole on $s = a$ with residue $x^{-\lambda}\Gamma(2a)$ which gives the leading term. When $a = 0$, the pole becomes a double pole at $s = 0$ which has residue $\log(x)$. See figure 1 for a plot of function $f_0(x)$.

Theorem 1. *The eigenvalue ω of the wave function is $\frac{ic}{R}a$ with $\Re(a) \in]-\frac{1}{4}, \frac{1}{4}[$ and the eigenvector is close to $f_a(Ax)$ with $A = Rm^2\beta^2$, m being the mass of the particle.*

Proof. We notice that the local integrability of $|\Psi(r,t)|^2$ on a neighbourhood of the event horizon implies that $|\Re(a)| < \frac{1}{4}$. Indeed in Schwartzschild metric the integral $\int |\Psi(r,t)|^2\sqrt{\det(\mathbf{g}_t)}drd\theta d\phi \approx \pi^2\int |f_a(Ax)|R^{5/2}x^{-1/2}dx$ ($\mathbf{g}_t$ being $\mathbf{g}$ restricted on its spatial components $\det(\mathbf{g}_t) \approx \frac{R^5}{x}\sin\phi$.

We have $\frac{R}{c}\Im(\omega) \in]-\frac{1}{4}, \frac{1}{4}[$ which allows for the possibility of non zero imaginary part. A more involved analysis shows that when $R \to \infty$ the admissible values of $i\frac{R}{c}\omega$ forms a countable set which ends to be dense in the strip $\{s, \Re(s) \in]-\frac{1}{4}, \frac{1}{4}[\}$. What is remarkable in the above analysis is that the asymptotic eigenvalues do not depend on the mass m.

3.2 Preponderent solution, unitarity excess rate

From the above we get that $\Psi(r,t) = \Psi(r)e^{\pm i\omega t}$. The quantity $|\Psi(r,t)|^2$ integrated on a whole spatial slice at time t would be in $e^{\pm 2\Im(\omega)t}$. The solutions with positive or negative $\Im(\omega)$ have same spatial expression. However the preponderant solutions are those with the negative imaginary part of factor ω. Indeed if $\Im(\omega) < 0$, then a combination of $e^{-i\Im(\omega)t}$ and $e^{i\Im(\omega)t}$ will have its components other than $e^{-2\Im(\omega)t}$ vanishing exponentially before $e^{-2\Im(\omega)t}$. Therefore the solutions with negative $\Im(\omega)$ are preponderant when $t \to +\infty$. Let's define the *unitarity excess* at time T as the logarithm of the integral of $|\Psi(r,t)|^2$ on a space-time slice defined by $t = T$, and the *unitarity excess rate* the unitarity excess derived by T. When $\Psi(r,t) = e^{-i\omega t}\Psi(r)$ the unitarity excess should be $2\Im(\omega)T + \log \int |\Psi(r)|^2$ and the unitarity excess rate should be $2\Im(\omega)$.

One should notice that since the largest value of $2\Im(\omega)$ is $\frac{c}{2R}$, the unitarity excess rate correspond to one unit per time taken by light to travel over a distance equivalent to the horizon diameter, which is rather considerable. The traversal time of the horizon of a black hole of one solar mass would be 10^{-5} seconds, for the massive black hole in the center of our galaxy it would be 3 minutes.

The results we have so far, *i.e.* that the imaginary part of the main eigenvalue is asymptotically equal to $\frac{c}{4R}$, are valid as long as the Compton wavelength $\frac{\hbar}{mc}$ of the particle is negligible, compared to horizon radius R. When $m\frac{c}{\hbar}$ is of order $1/R$ we are no longer in this situation. But a careful look leads to the fact that in the case where the Compton wavelength is proportional to $1/R$, the eigenvalues of the Klein-Gordon equation are also proportional to $1/R$ for all radius R. However this non asymptotic case makes the analysis difficult, since the Klein-Gordon equation looks much more complicated to solve. Anyhow by continuity we assume that the main eigenvalue is $\frac{c}{4R}$ as with the small Compton wavelength, although this is not a proof.

4 Impact of unitarity excess in quantum measurement

From now we switch to *Planck units* to simplify the presentation (for which $c = 1$, $h = 1$, $G = 1$, *etc*). In Planck unit, black hole mass M and black hole radius R are just linked by the relation $R = 2M$. In this section we assume that the average number N of particles, or independent constituent at the event horizon, depends on the radius R. The classic hypothesis is that this number is proportional to the area of the event horizon: $N = N(R) \approx \gamma R^2$ for some constant γ. There are several possibilities for γ but they are mostly of the same order. The largest estimate would be $\gamma = \frac{\pi}{\log 2}$, as if an event horizon particle would represents the minimal mass for carrying one bit of the entropy of the black hole. This would be in accordance with the conjectured holographic principle [11] based on the remark that the black hole entropy is proportional to the event horizon area. The smallest value would be $\gamma = \frac{1}{\alpha}$ where $\frac{\alpha}{R}$ is the average mass of one evaporated photon when the black hole is of radius R. (see next section for the estimate of α). Whatever the estimate of γ, the consequence of the hypothesis $N = O(R^2)$

makes any black hole to be again eternal with an infinite apparent average delay between each evaporated particle. This will be detailed in a next subsection.

4.1 Posterior impact of unitarity excess on binary events

The unitary excess rate of the whole black hole system of radius R is $U(R) = \frac{N(R)}{2R}$, since it is made of $N(R)$ particles, each having the same eigenvalue imaginary part $\frac{1}{4R}$. We define the *final unitarity excess* as the integral of the average unitary excess rate computed at the end of the black hole lifetime, and let's denote it $\Omega(R)$ where R is the black-hole initial radius R. Notice that the integral may not be trivial since the radius R and N may randomly vary with the time.

Let $\Psi_R(r,t)$ be the wave function at time t of a particle of the black hole of radius R, we have $\partial_t \Psi_R(r,t) = -i\omega_R \Psi(r,t)$ with $\Im(\omega(R)) = \frac{1}{4R}$, thus

$$\Psi_R(r,T) = \exp\left(-i\int_0^T \omega(R(t))dt\right)\Psi_R(r,0),$$

where $R(t)$ is the black hole radius at time t. The integral of the squared wave function $\int |\Psi_R(r,t)|^2$ on a space slice $t = T$ is equal to

$$\int |\Psi_R(r,0)|^2 \exp\left(\int_0^T 2\Im(\omega R(t))\right)dt).$$

The density operator $\rho(R)$ of the black hole, made of $N(R)$ particles (*a priori* in mixed state for a non zero entropy, but it does not matter here) satisfies

$$\mathrm{trace}(\rho(R(T))) = \exp\left(\int_0^T N(R)2\Im(\omega(R))dt\right)\left(\int |\Psi_R(r,0)|^2\right)^{N(R)}.$$

This quantity is interpreted as a probability, and after evaporation this quantity is equal to $e^{\Omega(R)}\left(\int |\Psi(r,0)|^2\right)^{N(R)}$ when $T \to \infty$, since $\Im(\omega_R) = \frac{1}{4R}$. By normalization (no unitarity excess at time 0) we assume that $\int |\Psi_R(r,0)|^2 = 1$

Let assume two black holes of respective initial radii R_1 and R_2. We consider the following binary event (*e.g.* a spin measurement over a particle made on Earth with outcome $(-1,+1)$ with fair probability $(\frac{1}{2},\frac{1}{2})$:

- if the measurement is $+1$ then a mass m is thrown in black hole 1;
- if the measurement is -1, then the mass is thrown in black hole 2.

One should notice that the natural fate of any mass in the galaxy is to be eventually absorbed either by the central black hole of our galaxy or of another galaxy, if the mass is expelled with the liberation speed. The surprising effect of unitary excess is that it may have a posterior impact on the outcome probability of the measurement made on Earth. The time line of the outcome $+1$ is affected with an unitarity excess of $\exp(\Omega(R_1+2m))\exp(\Omega(R_2))$, since the first black hole radius rises to $R_1 + 2m$ and the second black hole radius stays at R_2. In

Planck unit the radius of a black hole of radius R which absorbs a mass m is exactly $R+2m$. The timeline of the outcome -1 is affected by a final unitarity excess of $\exp(\Omega(R_1))\exp(\Omega(R_2+2m))$. Considering the unitarity excess as a posterior impact on the initial probabilities at time $t=0$ (the simplest way to interpret unitarity excess), the final probabilities of event -1 and $+1$ are:

$$\begin{aligned} P(-1) &= \frac{\exp(\Omega(R_2+2m))\exp(\Omega(R_1))}{\exp(\Omega(R_2+2m))\exp(\Omega(R_1))+\exp(\Omega(R_2))\exp(\Omega(R_1+2m))} \\ &= \frac{1}{1+\exp\left((\Omega(R_2)-\Omega(R_2+2m)+\Omega(R_1+2m)-\Omega(R_1)\right)} \end{aligned} \tag{3}$$

If this quantity differs of $\frac{1}{2}$, it characterizes a *posterior* effect. However we will show that all unitarity excesses are infinite, and this complicates the analysis.

4.2 The evaporation effect impacted by unitarity excess

Let $\beta=\frac{\zeta(3)}{1024\pi^2}$ and $\alpha=\frac{\pi^4}{16\zeta(3)}$ where $\zeta(.)$ is the Riemann *zeta* function. The evaporation is considered like a classical black body thermal radiation. Without taking into account the loss of unitarity we have a rate $\frac{\beta}{R}$ of particle evaporation, mostly a photon with a wavelength $\frac{\alpha}{R}$ in Planck unit (thus equivalent to the subsequent black hole loss of mass). The temperature of a black hole of radius R is $\frac{1}{8\pi R}$ in Planck unit.

Theorem 2. *Let a black hole with large radius, then $\Omega(R)=\infty$ and the black hole has an infinite average lifetime.*

Proof. Let $R(t)$ be the radius of the black hole at time t. In the transition from time t to time $t+dt$ the estimate of the quantity $\Omega(R(t))$ increases of $U(R(t))dt$. Meanwhile, with probability $\frac{\beta}{R}dt$, the black hole loses a mass $\frac{\alpha}{R}$. This translates into the following functional equation where $R=R(t)$

$$\exp(\Omega(R))=e^{U(R)dt}\left((1-\frac{\beta}{R}dt)\exp(\Omega(R))+\frac{\beta}{R}dt\exp(\Omega(R-\alpha/R))\right) \tag{4}$$

which when $dt\to 0$ resolves into

$$0=U(R)-\frac{\beta}{R}\left(1-\exp\left(\Omega\left(R-\frac{\alpha}{R}\right)-\Omega(R)\right)\right)$$

Taking $N(R)\approx\gamma R^2$, thus $U(R)\approx\gamma R/2$, the final identity

$$\exp\left(\Omega\left(R-\frac{\alpha}{R}\right)-\Omega(R)\right)=1-\frac{\gamma}{2\beta}R^2$$

is clearly not solvable for large enough values of $R(t)$, in theory larger than $\sqrt{2\beta/\gamma}$, sufficient to make the right hand side non positive, but this corresponds to black holes with tiny masses for which the evaporation theory is not yet established.

Notice that in this analysis we assume that the black hole is isolated and does not absorb any other mass (this would not change the nature of the result as long as the quantity of absorbed mass remain the same on the two timelines). The proof of the infinite lifetime is given next with the "Doomsday" analysis.

4.3 Posterior effect analysis in "Doomsday" hypothesis

In this section, we use the "Doomsday" argument to get around the problem of infinite excess unitarity. The trick is similar to the calculation of the Casimir effect where two infinite terms subtracted produce a finite result. We assume that a fictitious term T applies to any object subject to a non-zero unitary excess and we let T grow to infinity. Let $\Omega(T,R)$ be the excess unitarity accumulated by a black hole of radius R during a time interval of length T. This leads to a new expression of the equation (4) :

$$\exp(\Omega(T,R)) = e^{U(R)dt} \times \left((1-\frac{\beta}{R}dt)\exp(\Omega(T-dt,R)) + \frac{\beta}{R}dt\exp(\Omega(T-dt,R-\frac{\alpha}{R}))\right). \tag{5}$$

Where $U(R)$ is the unitary excess rate of the black hole system. In fact to avoid trivially not acceptable solutions we should consider a black hole system made by a single mass 2β plus γR^2 particles so that $U(R) = \gamma\frac{R}{2} + \frac{\beta}{R}$. When $dt \to 0$ we get the equation:

$$0 = U(R) - \partial_T\Omega(T,R) - \frac{\beta}{R}\left(1-\exp(\Omega(T,R-\frac{\alpha}{R}) - \Omega(T,R))\right) \tag{6}$$

Theorem 3. *Under the doomsday analysis we have the posterior impact when R_1 and R_2 are much larger than unity and m is much smaller than R_1 and R_2:*

$$P(-1) = \frac{1}{1+\exp\left(-\frac{4\beta^2}{\alpha\gamma^2}\left(\frac{1}{(R_1+2m)^2} - \frac{1}{R_1^2} + \frac{1}{R_2^2} - \frac{1}{(R_2+2m)^2}\right) + O(\frac{1}{R_1^4}+\frac{1}{R_2^4})\right)} \tag{7}$$

$$\approx \frac{1}{2} + \frac{m\beta^2}{\alpha\gamma^2}\left(R_2^{-3} - R_1^{-3}\right) \tag{8}$$

Thus the posterior effect is small but not null, however paving the way to a theoretical ability to transmit information backward in time, but not in a practical way with existing technologies.

Proof. To simplify the discussion we approximate $\Omega(T,R-\frac{\alpha}{R}) - \Omega(T,R)$ by $-\frac{\alpha}{R}\partial_R\Omega(T,R)$. Notice that both expressions are wrong for small values of R because the evaporation theory does not apply well to small radii. Thus our analysis will only be valid for large values of radius R. Taking the second expression, equation (6) becomes

$$0 = \frac{\gamma R}{2} - \partial_T\Omega(T,R) + \frac{\beta}{R}\exp(-\frac{\alpha}{R}\partial_R\Omega(T,R)). \tag{9}$$

Thus if we integrate, keeping the fact that $\Omega(0,R) = 0$:

$$\Omega(T,R) = \frac{\gamma R}{2}T + \frac{\beta}{R}\int_0^T \exp(-\frac{\alpha}{R}\partial_R\Omega(t,R))dt.$$

Let $C(R) = \frac{\beta}{R}\int_0^\infty \exp(-\frac{\alpha}{R}\partial_R \Omega(t,R))dt$, we have

$$\Omega(T,R) = \frac{\gamma R}{2}T + C(R) - \int_T^\infty \exp(-\frac{\alpha}{R}\partial_R\Omega(t,R))dt.$$

The last term is an exponentially decreasing function of T. It turns out that indeed $\Omega(T,R) \to \infty$ when $T \to \infty$. The consequence is that black holes are now back to be eternal as it was assumed before Hawking.

Under the assumption of an absolute black hole lifetime limit set at T, the quantity $\frac{e^{\Omega(t,R)}}{e^{\Omega(T,R)}}$, for any $t < T$, is interpreted as the probability that the apparent lifetime of the black hole is less than t. The reason is that if the black hole evaporates completely before t, then the system returns to simple unitary mode, which means that between t and T the imaginary parts of its eigenvalues are all zero. But since $\lim_T \Omega(T,R) = \infty$, it follows that the probability that the lifetime of the black hole is less than t is in fact 0 whatever t since for any fixed t $\frac{e^{\Omega(t,R)}}{e^{\Omega(T,R)}} \to 0$ when $T \to \infty$. Moreover, the black hole simply "abstains" from radiating. This surprising effect is not detectable because the thermal radiations of known black holes are already far too weak (in R^{-2}) to be observed, even compared to the cosmic background radiation.

Coming back to the posterior impact analysis, the terms in $\gamma RT/2$ cancel in (3):

$$\begin{aligned} P(-1) &= \lim_{T\to\infty} \frac{1}{1+\exp\left((\Omega(T,R_2) - \Omega(T,R_2+2m) + \Omega(T,R_1+2m) - \Omega(T,R_1)\right)} \\ &= \frac{1}{1+\exp\left(C(R_1+2m) - C(R_1) + C(R_2) - C(R_2+2m)\right)} \end{aligned}$$

We notice that the expression of $P(-1)$ does not trivially tend to 0 or 1 when $T \to \infty$ because we choose $U(R) = \gamma\frac{R}{2} + \frac{\beta}{R}$. We could have chosen $U(R) = CR + D + \frac{\beta}{R}$, with arbitrary constants C and D. It is particularly important to have these cancellations because a trivial convergence of $P(-1)$ would systematically make all masses in the universe to rush toward a single black hole.

If we make the approximation $\partial_R\Omega(t,R) = \gamma t/2 + \partial_R C(R)$ (ignoring the exponentially decreasing part), and plug it in a Taylor expansion in $\frac{1}{R}$:

$$C(R) = \frac{\beta}{R}\int_0^\infty \exp\left(-\frac{\alpha}{2R}\gamma t - \partial_R C(R)\right)dt = \frac{2\beta}{\gamma\alpha} - \frac{4\beta^2}{\gamma^2\alpha R^2} + O(\frac{1}{R^4}).$$

The first term cancels in the expression of $P(-1)$ and ignoring the $1/R^4$ terms:

$$P(-1) \approx \frac{1}{1+\exp\left(-\frac{4\beta^2}{\alpha\gamma^2}\left(\frac{1}{(R_1+2m)^2} - \frac{1}{R_1^2} + \frac{1}{R_2^2} - \frac{1}{(R_2+2m)^2}\right)\right)}.$$

5 Conclusion

We have investigated the possibility of imaginary eigenvalues for the Gordon Klein equation with the Schwarzschild metric of a black hole. We have found that

indeed imaginary eigenvalues exist and have order $\frac{ic}{4R}$, leading to a considerable unitary excess rate, being the signature of an unitary violation which comes along with the black hole information paradox.

However, under appropriate hypotheses, this unitary violation would lead to physical effects ranging from radiation suspension to posterior impact, all effects of order R^{-2} which should not be detectable on known black holes with existing technologies.

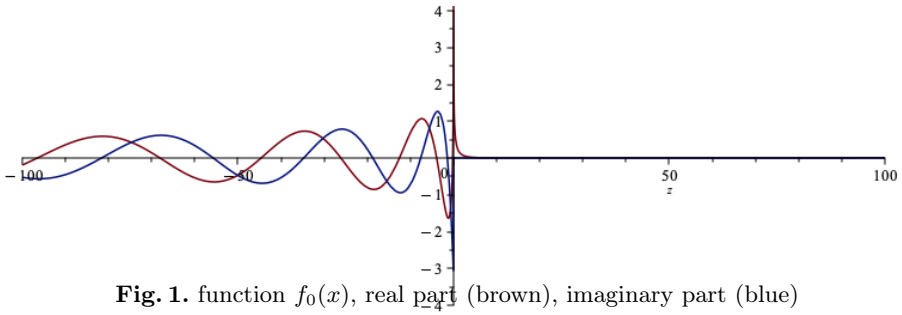

Fig. 1. function $f_0(x)$, real part (brown), imaginary part (blue)

References

1. Hawking, S. W. (1976). "Breakdown of predictability in gravitational collapse". Physical Review D. 14 (10): 2460–2473.
2. Raju, Suvrat (January 2022). "Lessons from the information paradox". Physics Reports. 943: 1–80.
3. Hartle, James B. (1998). "Generalized Quantum Theory in Evaporating Black Hole Spacetimes". Black Holes and Relativistic Stars: 195.
4. Giddings, Steven (1992). "Black Holes and Massive Remnants". Physical Review D. 46 (4): 1347–1352.
5. Preskill, John (1992). Do Black Holes Destroy Information?. International Symposium on Black Holes, Membranes, Wormholes, and Superstrings.
6. Mathur, Samir D (21 November 2009). "The information paradox: a pedagogical introduction". Classical and Quantum Gravity. 26 (22): 224001.
7. Mathur, S.D. (15 July 2005). "The fuzzball proposal for black holes: an elementary review". Fortschritte der Physik. 53 (7–8): 793–827.
8. Hawking, Stephen W.; Perry, Malcolm J.; Strominger, Andrew (2016-01-05). "Soft Hair on Black Holes". Physical Review Letters. 116 (23): 231301.
9. Raju, Suvrat (January 2022). "Lessons from the information paradox". Physics Reports. 943: 1–80.
10. Geng, Hao; Karch, Andreas (September 2020). "Massive islands". Journal of High Energy Physics. 2020 (9): 121.
11. Leonard Susskind, « The World as a Hologram », Journal of Mathematical Physics (J. Math. Phys.), vol. 36, 1995, p. 6377-6396 (DOI 10.1063/1.531249, arXiv hep-th/9409089).