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# Fully well-balanced entropy stable Godunov numerical schemes for the Shallow Water equations with the topography source term.

Ludovic Martaud\*, Christophe Berthon†

January 15, 2024

## Abstract

This work concerns the design of well-balanced entropy stable numerical schemes for the Shallow Water equations. The fully discrete entropy inequality is reached with a local condition incorporated in the scheme design. The source term is discretized to preserve the steady states and the entropy stability. The method yields explicit schemes. We illustrate the schemes relevancy with several tests cases in the wet and dry areas.

**Keywords.** Shallow Water equations, Finite Volume Schemes, Discrete Entropy Stability, Well-Balanced Schemes

**Math. classification**

## 1 Introduction

The present work is devoted to the numerical approximation of the weak solutions of the Shallow Water equations with the topography source term in one space dimension given by

$$\partial_t \begin{pmatrix} h \\ hu \end{pmatrix} + \partial_x \begin{pmatrix} hu \\ hu^2 + gh^2/2 \end{pmatrix} = \begin{pmatrix} 0 \\ -gh \end{pmatrix} \partial_x z.$$

This model governs the water height  $h \geq 0$  and the velocity  $u \in \mathbb{R}$  of a fluid which is defined as follows (see [5, 41] for the details):

$$u = \begin{cases} (hu)/h, & \text{if } h > 0, \\ 0, & \text{otherwise.} \end{cases}$$

The gravitation constant is  $g > 0$  and  $z : \mathbb{R} \rightarrow \mathbb{R}$  is a given time independent smooth topography function. The unknown state vector  $w = (h, hu)^T$  is assumed to be in the convex set  $\Omega = \{(h, hu) \in \mathbb{R}^2 \mid h \geq 0, hu \in \mathbb{R}\}$ . We consider  $w_0 : \mathbb{R} \rightarrow \Omega$  a given measurable function of  $L^1_{\text{loc}}(\mathbb{R})$  as initial condition of  $w$  at time  $t = 0$  and we study the following Cauchy problem:

$$\begin{cases} \partial_t w + \partial_x f(w) &= S(w) \partial_x z, & x \in \mathbb{R}, t > 0, \\ w(x, t = 0) &= w_0(x), & x \in \mathbb{R}, \end{cases} \quad (1.1)$$

where  $f(w) = (hu, hu^2 + gh^2/2)^T$ ,  $S(w) = (0, -gh)^T$ . For the sake of clarity, we introduce  $\hat{w} = (h, hu, z)^T$  which takes its values in the convex set  $\hat{\Omega}$  defined by

$$\hat{\Omega} = \{(h, hu, z) \in \mathbb{R}^3 \mid h \geq 0, hu \in \mathbb{R}, z \in \mathbb{R}\}.$$

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\*Laboratoire de Mathématiques Jean Leray, CNRS UMR 6629, Nantes Université, 2 rue de la Houssinière, BP 92208, 44322 Nantes, France *Email address:* ludovic.martaud@univ-nantes.fr.

†Laboratoire de Mathématiques Jean Leray, CNRS UMR 6629, Nantes Université, 2 rue de la Houssinière, BP 92208, 44322 Nantes, France *Email address:* christophe.berthon@univ-nantes.fr.

In the flat regions (*i.e.*  $\partial_x z = 0$ ), it is well-known that the homogeneous Shallow Water system is endowed with the following entropy inequality

$$\partial_t \eta(w) + \partial_x G(w) \leq 0,$$

with  $\eta(w) = hu^2/2 + gh^2/2$ ,  $G(w) = hu^3/2 + gh^2u$ . A generic  $z$  function modifies the above inequality with the term  $-ghu\partial_x z$  in its right hand side. As a consequence, the entropy inequality associated to Shallow Water system (1.1) now reads

$$\partial_t \eta(w) + \partial_x G(w) \leq -ghu\partial_x z. \quad (1.2)$$

According [12] (for instance, see also [21]) and since  $\partial_t z = 0$ , the inequality (1.2) reformulates equivalently as a conservative form given by

$$\partial_t (\eta(w) + ghz) + \partial_x (G(w) + g(hu)z) \leq 0.$$

Let us set

$$\hat{\eta}(\hat{w}) = \frac{(hu)^2}{2h} + \frac{gh^2}{2} + ghz, \quad \hat{G}(\hat{w}) = \frac{(hu)^3}{2h^2} + g(hu)(h+z), \quad \forall \hat{w} \in \hat{\Omega}, \quad (1.3)$$

so that the above entropy inequality is equivalent to

$$\partial_t \hat{\eta}(\hat{w}) + \partial_x \hat{G}(\hat{w}) \leq 0. \quad (1.4)$$

Therefore, the presence of the source term  $S(w)\partial_x z$  in the system (1.1) modifies the entropy inequality  $\partial_t \eta(w) + \partial_x G(w) \leq 0$  into an other conservative entropy equality that includes a contribution of the topography function.

In addition, the presence of the source term  $S(w)\partial_x z$  involves the existence of non-trivial stationary solutions that satisfy

$$hu = \text{cst}, \quad \frac{u^2}{2} + g(h+z) = \text{cst}. \quad (1.5)$$

Among these steady states, a special attention is paid to the lake at rest (for instance, see [11, 22, 34, 37, 38, 43, 46]) which is given by

$$u = 0, \quad h + z = \text{cst}. \quad (1.6)$$

From a numerical point of view, the solutions of the Shallow Water system (1.1) are approximated on uniform space meshes  $(x_{i+\frac{1}{2}})_{i \in \mathbb{Z}}$  in  $\mathbb{R}$  of constant size  $\Delta x > 0$ . Thus, the equality  $x_{i+\frac{1}{2}} = x_{i-\frac{1}{2}} + \Delta x$  holds for all  $i \in \mathbb{Z}$ . Uniform meshes in time  $(t^n)_{n \in \mathbb{N}}$  in  $[0, +\infty)$  of constant size  $\Delta t > 0$  are also considered and they satisfy  $t^{n+1} = t^n + \Delta t$  for all  $n$  in  $\mathbb{N}$ . At the initial datum  $t^0 = 0$ , the initial condition  $w_0$  and the given regular function  $z$  are discretized with a sequence  $((w_i^0, z_i))_{i \in \mathbb{Z}}$  in  $\hat{\Omega}$  such that

$$(w_i^0, z_i)^T = \frac{1}{\Delta x} \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} (w_0, z)^T(x) dx, \quad \forall i \in \mathbb{Z}. \quad (1.7)$$

The sequence  $((w_i^0, z_i)^T)_{i \in \mathbb{Z}}$  define a piecewise constant approximation of  $(w(\cdot, t=0), z)^T$ . As a consequence, a numerical approximation of  $w(\cdot, t^{n+1})$  is entirely defined by a numerical scheme that gives the updated sequence  $(w_i^{n+1})_{i \in \mathbb{Z}}$  from the sequence  $(w_i^n)_{i \in \mathbb{Z}}$ . However, a suitable updated sequence  $(w_i^{n+1})_{i \in \mathbb{Z}}$  has to satisfy some properties. In order to give this properties and for the sake of clarity, the notation  $\hat{w}_i^n = (w_i^n, z_i)^T$  is now considered but we emphasize  $z_i$  is a given quantity.

At first, the sequence  $(\hat{w}_i^{n+1})_{i \in \mathbb{Z}}$  has to be well-balanced that means it exactly preserves stationary solutions. In the one hand, the well-balanced property for the lake at rest (1.6) writes

$$u_i^n = 0, \quad h_i^n + z_i = \text{cst}, \quad \text{then,} \quad w_i^{n+1} = w_i^n, \quad \forall i \in \mathbb{Z}. \quad (1.8)$$

Several schemes satisfying this property have been proposed during the two last decays (for instance, see [11, 22, 34, 37, 38, 43, 46]). In the other hand, the well-balanced property for the moving equilibria (1.5) is given by

$$(hu)_i^n = \text{cst}, \quad \frac{(u_i^n)^2}{2} + g(h_i^n + z_i) = \text{cst}, \quad \text{then,} \quad w_i^{n+1} = w_i^n, \quad \forall i \in \mathbb{Z}, \quad (1.9)$$

and a such property is satisfied by the schemes described in [16, 25, 35, 41] for instance.

In addition to a well-balanced property, the sequence  $(\hat{w}_i^{n+1})_{i \in \mathbb{Z}}$  has to verify a discrete entropy inequality. Denoting  $\hat{\mathcal{G}}_{i+1/2}$  a consistent approximation of  $\hat{G}$  defining in (1.3), such an inequality writes

$$\frac{\hat{\eta}(\hat{w}_i^{n+1}) - \hat{\eta}(\hat{w}_i^n)}{\Delta t} + \frac{\hat{\mathcal{G}}_{i+1/2} - \hat{\mathcal{G}}_{i-1/2}}{\Delta x} \leq 0, \quad \forall i \in \mathbb{Z}. \quad (1.10)$$

In the flat regions (*i.e.*  $\partial_x z = 0$ ), several schemes verifying the discrete entropy inequality (1.10) have already been introduced [23, 24, 28, 29, 30, 31, 33]. In the present work, we focus on Godunov type scheme based on approximate Riemann solver made of two intermediate constant states (see [28] for details). Considering the well-known Euler equations, the fully discrete entropy inequality (1.10) can be established for the HLLC scheme [45] or equivalently, for the Suliciu relaxation schemes [7, 12, 18]. Let us underline Section b.ii in [28] that presents an alternative to derive an entropy satisfying Godunov type scheme with two intermediate constant states.

For a generic function  $z$  (*i.e.*  $\partial_x z \neq 0$ ), the design of a well-balanced scheme that satisfies a discrete entropy inequality (1.10) turn out to be very challenging. Both properties are satisfied by some Godunov schemes [1, 2, 6, 17, 27, 36, 47] or a relaxation well-balanced for the lake at rest scheme [12], but all of these schemes need to solve a set of non linear equation for each cells of the mesh and for each time iterations. These non linear equations are not used to define the lake at rest well-balanced schemes (1.8) in [4, 5] nor to define the moving equilibria well-balanced schemes (1.9) in [8, 9, 10]. However, the fully discrete entropy estimations proposed in these works contain an error term of the form  $\mathcal{O}(\Delta x^2)$  such that

$$\frac{\hat{\eta}(\hat{w}_i^{n+1}) - \hat{\eta}(\hat{w}_i^n)}{\Delta t} + \frac{\hat{\mathcal{G}}_{i+1/2} - \hat{\mathcal{G}}_{i-1/2}}{\Delta x} \leq \mathcal{O}(\Delta x^2).$$

The error term  $\mathcal{O}(\Delta x^2)$  does not occur in the works of [20, 32] but the proposed well-balanced schemes satisfied a global version of the entropy inequality (1.10) that writes  $\sum_{i \in \mathbb{Z}} \hat{\eta}(\hat{w}_i^N) \Delta x \leq \hat{\eta}(\hat{w}_i^0) \Delta x$  where  $t^N, t^0$  respectively denote the final and the initial time of the simulation.

In this work, we propose to design numerical schemes approximating the weak solutions of the system (1.1) and that ensure the weak consistency of the scheme and the discrete entropy inequality (1.10). These schemes also satisfy the well-balanced property. More precisely, we consider schemes written under the following form

$$w_i^{n+1} = w_i^n - \frac{\Delta t}{\Delta x} (\hat{\mathcal{F}}(\hat{w}_i^n, \hat{w}_{i+1}^n) - \hat{\mathcal{F}}(\hat{w}_{i-1}^n, \hat{w}_i^n)) + \frac{\Delta t}{2} (\hat{S}_{i+1/2}^n + \hat{S}_{i-1/2}^n), \quad \forall i \in \mathbb{Z}. \quad (1.11)$$

where  $\hat{\mathcal{F}} : (\hat{\Omega})^2 \rightarrow \mathbb{R}^2$  stands for the numerical flux and  $\hat{S}_{i+1/2}^n$  denotes a consistent approximation of the source term  $(0, -ghu)^T \partial_x z$ .

The paper is organized as follows. In Section 2, we reformulate the numerical scheme (1.11) under the Godunov type schemes form then we give sufficient conditions to obtain the weak consistency, the discrete entropy inequality (1.10) and the well-balanced property. The two first sufficient conditions are used in Section 3 to define an approximate Riemann solver made of two intermediate constant states. This approximate Riemann solver is governed by under-determined equations that ensure the consistency and the discrete entropy stability (1.10). These equations are completed in Section 4 to obtain consistent, entropic schemes in the case of  $z = \text{cste}$  then Sections 5 and 6 propose other closures in order to define consistent, entropic, well-balanced schemes for the lac at rest (1.6) then for the moving equilibria (1.5). In Section 7, numerical tests are carried out to illustrate our numerical schemes.

## 2 Consistent, well-balanced, entropy stable Godunov type scheme for the Shallow Water equations

In this section, we establish an equivalent reformulation of the numerical scheme (1.11) under the form of a Godunov type scheme. Namely, considering a given approximate Riemann solver  $\tilde{w}$  :

$\mathbb{R} \times (\hat{\Omega})^2 \rightarrow \Omega$ , a Godunov type scheme writes

$$w_i^{n+1} = \frac{1}{\Delta x} \int_0^{\frac{\Delta x}{2}} \tilde{w}(x/\Delta t, \hat{w}_{i-1}^n, \hat{w}_i^n) dx + \frac{1}{\Delta x} \int_{-\frac{\Delta x}{2}}^0 \tilde{w}(x/\Delta t, \hat{w}_i^n, \hat{w}_{i+1}^n) dx, \quad \forall i \in \mathbb{Z}. \quad (2.1)$$

According to the above formulation, the sequence  $(w_i^{n+1})_{i \in \mathbb{Z}}$  is given by the  $L^2$  projection of the sequence  $(\tilde{w}(\cdot, \hat{w}_i^n, \hat{w}_{i+1}^n))_{i \in \mathbb{Z}}$  that is a set of approximate juxtaposed Riemann solvers without interaction, This non interaction is ensured by a restriction on the time step  $\Delta t > 0$  also called the non interaction CFL condition. In the sequel, such a CFL condition is assumed satisfied.

The following lemma precises the definition of an approximate Riemann solver  $\tilde{w} : \mathbb{R} \times (\hat{\Omega})^2 \rightarrow \Omega$  then shows the equivalence between the numerical scheme (1.11) and the Godunov type scheme (2.1).

**Lemma 2.1** (Equivalent reformulation of the numerical scheme (1.11)). *Let consider  $\hat{w} = (w, z)^T$  in  $\hat{\Omega}$ , a function  $\hat{s} : (\hat{\Omega})^2 \rightarrow \mathbb{R}$  such that*

$$\hat{s}(\hat{w}, \hat{w}) = -gh, \quad \forall \hat{w} \in \hat{\Omega},$$

and  $(\hat{S}_{i+1/2}^n)_{i \in \mathbb{Z}}$  a sequence of  $\mathbb{R}^2$  that writes

$$\hat{S}_{i+1/2}^n = \begin{pmatrix} 0 \\ \hat{s}(\hat{w}_i^n, \hat{w}_{i+1}^n) \end{pmatrix} \frac{z_{i+1} - z_i}{\Delta x}, \quad \forall i \in \mathbb{Z}, \quad (2.2)$$

where  $(z_i)_{i \in \mathbb{Z}}$  is defined in (1.7). Let also consider a Godunov type scheme (2.1) defined by an approximate Riemann solver  $\tilde{w} : \mathbb{R} \times (\hat{\Omega})^2 \rightarrow \Omega$  that satisfies  $\tilde{w}(\cdot, \hat{w}, \hat{w}) = w$  and that satisfies in addition the following consistency integral relation:

$$\frac{1}{\Delta x} \int_{-\frac{\Delta x}{2}}^{\frac{\Delta x}{2}} \tilde{w}(x/\Delta t, \hat{w}_i^n, \hat{w}_{i+1}^n) dx = \frac{w_i^n + w_{i+1}^n}{2} - \frac{\Delta t}{\Delta x} (f(w_{i+1}^n) - f(w_i^n)) + \Delta t \hat{S}_{i+1/2}^n. \quad (2.3)$$

The Godunov type scheme (2.1) is equivalent to the numerical scheme (1.11) in which the numerical flux function  $\hat{\mathcal{F}} : (\hat{\Omega})^2 \rightarrow \mathbb{R}^2$  is given by

$$\begin{aligned} \hat{\mathcal{F}}(\hat{w}_i^n, \hat{w}_{i+1}^n) &= \frac{f(w_{i+1}^n) + f(w_i^n)}{2} - \frac{\Delta x}{4\Delta t} (w_{i+1}^n - w_i^n) + \frac{1}{2\Delta t} \int_0^{\frac{\Delta x}{2}} \tilde{w}(x/\Delta t, \hat{w}_i^n, \hat{w}_{i+1}^n) dx \\ &- \frac{1}{2\Delta t} \int_{-\frac{\Delta x}{2}}^0 \tilde{w}(x/\Delta t, \hat{w}_i^n, \hat{w}_{i+1}^n) dx. \end{aligned} \quad (2.4)$$

*Proof.* Since a CFL condition holds, the integral formulation of a Godunov type scheme (2.1) can be rewritten as follows:

$$\begin{aligned} w_i^{n+1} &= \frac{1}{2\Delta x} \int_{-\frac{\Delta x}{2}}^0 \tilde{w}(x/\Delta t, w_i^n, w_{i+1}^n) dx + \frac{1}{2\Delta x} \int_0^{\frac{\Delta x}{2}} \tilde{w}(x/\Delta t, w_{i-1}^n, w_i^n) dx \\ &+ \frac{1}{2\Delta x} \int_{-\frac{\Delta x}{2}}^{\frac{\Delta x}{2}} \tilde{w}(x/\Delta t, w_{i-1}^n, w_i^n) dx + \frac{1}{2\Delta x} \int_{-\frac{\Delta x}{2}}^{\frac{\Delta x}{2}} \tilde{w}(x/\Delta t, w_i^n, w_{i+1}^n) dx \\ &- \frac{1}{2\Delta x} \int_{-\frac{\Delta x}{2}}^0 \tilde{w}(x/\Delta t, w_{i-1}^n, w_i^n) dx - \frac{1}{2\Delta x} \int_0^{\frac{\Delta x}{2}} \tilde{w}(x/\Delta t, w_i^n, w_{i+1}^n) dx. \end{aligned}$$

Using the integral consistency relation (2.3), the above equality re-writes

$$\begin{aligned} w_i^{n+1} &= \frac{1}{2\Delta x} \int_{-\frac{\Delta x}{2}}^0 \tilde{w}(x/\Delta t, w_i^n, w_{i+1}^n) dx + \frac{1}{2\Delta x} \int_0^{\frac{\Delta x}{2}} \tilde{w}(x/\Delta t, w_{i-1}^n, w_i^n) dx \\ &+ \frac{1}{2} \left( \frac{w_{i-1}^n + w_i^n}{2} - \frac{\Delta t}{\Delta x} (f(w_i^n) - f(w_{i-1}^n)) + \Delta t \hat{S}_{i-1/2}^n \right) \\ &+ \frac{1}{2} \left( \frac{w_i^n + w_{i+1}^n}{2} - \frac{\Delta t}{\Delta x} (f(w_{i+1}^n) - f(w_i^n)) + \Delta t \hat{S}_{i+1/2}^n \right) \\ &- \frac{1}{2\Delta x} \int_{-\frac{\Delta x}{2}}^0 \tilde{w}(x/\Delta t, w_{i-1}^n, w_i^n) dx - \frac{1}{2\Delta x} \int_0^{\frac{\Delta x}{2}} \tilde{w}(x/\Delta t, w_i^n, w_{i+1}^n) dx. \end{aligned}$$

Considering the formulation of the numerical flux function  $\hat{\mathcal{F}}$  given by (2.4) in the above equality, we deduce the formulation of the numerical scheme given by (1.11). Since the converse is direct, that concludes the proof.  $\square$

The literature [9, 12, 37] devoted to the numerical approximation of the weak solutions of the Shallow Water system (1.1) gives several definitions of the weak consistency for the discrete source term  $(\hat{S}_{i+1/2}^n)_{i \in \mathbb{Z}}$ . The definition (2.2) is proposed in this work because it is a sufficient condition to the definition suggested in [12, Eq 4.17] and it degenerates toward the usual definition [28] if the sequence  $(z_i)_{i \in \mathbb{Z}}$  is locally constant.

The sequence  $(\hat{S}_{i+1/2}^n)_{i \in \mathbb{Z}}$  is used in the consistency integral relation (2.3) that defines a suitable approximate Riemann solver  $\tilde{w} : \mathbb{R} \times (\hat{\Omega})^2 \rightarrow \Omega$ . Therefore, a Godunov type scheme (2.1) is entirely defined by the given of a such approximate Riemann solver. As both numerical schemes (2.1) and (1.11) are equivalent, only the integral form given by (2.1) will be used in the rest of this work. This integral form is very useful to show the well-balanced property that is now defined for the sake of completeness.

**Definition 2.1** (Local equilibrium and well-balanced scheme for the Shallow Water equations (1.1)). *Consider the Shallow Water equations (1.1) endowed with the notation*

$$\hat{B}(\hat{w}) = \frac{u^2}{2} + g(h + z), \quad \forall \hat{w} \in \hat{\Omega}.$$

At the datum  $t^n$ , let also consider  $\hat{w}_i^n$  and  $\hat{w}_{i+1}^n$  two constants states in  $\hat{\Omega}$  respectively being on the left and on the right of the interface  $x_{i+1/2}$ .

i) The two states  $\hat{w}_i^n$  and  $\hat{w}_{i+1}^n$  define a local moving equilibrium if

$$h_i^n u_i^n = h_{i+1}^n u_{i+1}^n, \quad \hat{B}(\hat{w}_i^n) = \hat{B}(\hat{w}_{i+1}^n). \quad (2.5)$$

ii) The two states  $\hat{w}_i^n$  and  $\hat{w}_{i+1}^n$  define a local lake at rest equilibrium if

$$u_i^n = u_{i+1}^n = 0, \quad h_i^n + z_i = h_{i+1}^n + z_{i+1}. \quad (2.6)$$

iii) A numerical scheme is well-balanced for the moving equilibrium (1.5) if it exactly preserves a sequence  $(\hat{w}_i^n)_{i \in \mathbb{Z}}$  that satisfies, at each interfaces of the mesh, a local moving equilibrium (2.5). This property writes

$$\text{if } \forall i \in \mathbb{Z} \left\{ \begin{array}{l} h_i^n u_i^n = h_{i+1}^n u_{i+1}^n, \\ \hat{B}(\hat{w}_i^n) = \hat{B}(\hat{w}_{i+1}^n), \end{array} \right. \text{ then } w_i^{n+1} = w_i^n, \quad \forall i \in \mathbb{Z}.$$

A numerical scheme is well-balanced for the lake at rest (1.6) if it exactly preserves a sequence  $(\hat{w}_i^n)_{i \in \mathbb{Z}}$  that satisfies, at each interfaces of the mesh, a local lake at rest equilibrium (2.6). This property writes

$$\text{if } \forall i \in \mathbb{Z} \left\{ \begin{array}{l} u_i^n = u_{i+1}^n = 0, \\ h_i^n + z_i = h_{i+1}^n + z_{i+1}, \end{array} \right. \text{ then } w_i^{n+1} = w_i^n, \quad \forall i \in \mathbb{Z}.$$

According to the above definitions, if a local equilibrium (2.5) or (2.6) holds then the  $hu$  continuity is ensured for all  $(z_i, z_{i+1})$  in  $(\mathbb{R})^2$ . In the case  $z_i = z_{i+1}$ , the continuity of  $h$  is also ensured for the discrete lake at rest (2.6). In this work, in the case  $z_i = z_{i+1}$ , the continuity of  $h$  in the local moving equilibrium (2.5) is also assumed. This assumption writes

$$\text{if } \left\{ \begin{array}{l} z_i = z_{i+1} \\ h_i^n u_i^n = h_{i+1}^n u_{i+1}^n, \\ \hat{B}(\hat{w}_i^n) = \hat{B}(\hat{w}_{i+1}^n), \end{array} \right. \text{ then } h_i^n = h_{i+1}^n. \quad (2.7)$$

Lemma A.1-ii) given in the Appendix A shows that the smooth stationary solutions of the Shallow Water system (1.1) satisfy a continuous analogous of the equalities (2.7). These equalities formally mean if the given function  $z$  is locally constant then the smooth stationary solutions of the Shallow Water system (1.1) are constant.

Using Definition 2.1, a sufficient condition to the well-balanced property is reachable. This condition is used in [9, 40, 41] for instance and we detail it in this work for the sake of completeness. The well-balanced property is also completed in the following lemma with a sufficient condition to the discrete entropy inequality (1.10) that is the main originality of this work.

**Lemma 2.2** (Consistent, well-balanced, entropy stable Godunov type scheme for the Shallow Water equations (1.1)). *Consider the functions  $\eta, G : \Omega \rightarrow \mathbb{R}$  respectively defined by*

$$\eta(w) = \frac{hu^2}{2} + g\frac{h^2}{2}, \quad G(w) = \frac{hu^3}{2} + gh^2u. \quad (2.8)$$

Let also consider  $(\hat{S}_{i+1/2}^n)_{i \in \mathbb{Z}}$  a sequence of  $\mathbb{R}^2$  and an approximate Riemann solver  $\tilde{w} : \mathbb{R} \times (\hat{\Omega})^2 \rightarrow \Omega$  receptively satisfying the consistency definitions (2.2) and the consistency integral relation (2.3). Let denote  $\mathcal{F}_{i+1/2}^h$  the numerical flux of the variable  $h$  given by the first component of  $\hat{\mathcal{F}}(\hat{w}_i^n, \hat{w}_{i+1}^n)$  defined in (2.4). Assume a CFL condition of non interaction holds.

i) If the approximate Riemann solver satisfies the consistency integral relation

$$\frac{1}{\Delta x} \int_{-\frac{\Delta x}{2}}^{\frac{\Delta x}{2}} \tilde{w}(x/\Delta t, \hat{w}_i^n, \hat{w}_{i+1}^n) dx = \frac{w_i^n + w_{i+1}^n}{2} - \frac{\Delta t}{\Delta x} (f(w_{i+1}^n) - f(w_i^n)) + \Delta t \hat{S}_{i+1/2}^n, \quad \forall i \in \mathbb{Z}, \quad (2.9)$$

then the Godunov type scheme (2.1) is consistent with the Shallow Water system (1.1).

ii) If the approximate Riemann solver satisfies the inequality

$$\begin{aligned} \frac{1}{\Delta x} \int_{-\frac{\Delta x}{2}}^{\frac{\Delta x}{2}} \eta(\tilde{w}(x/\Delta t, \hat{w}_i^n, \hat{w}_{i+1}^n)) dx \leq \\ \frac{\eta(w_i^n) + \eta(w_{i+1}^n)}{2} - \frac{\Delta t}{\Delta x} (G(w_{i+1}^n) - G(w_i^n)) - g \frac{\Delta t}{\Delta x} (z_{i+1} - z_i) \mathcal{F}_{i+1/2}^h, \quad \forall i \in \mathbb{Z}, \end{aligned} \quad (2.10)$$

then the Godunov type scheme (2.1) verifies a discrete entropy for the couple  $(\hat{\eta}, \hat{G})$  defined in (1.3). This inequality writes

$$\frac{\hat{\eta}(\hat{w}_i^{n+1}) - \hat{\eta}(\hat{w}_i^n)}{\Delta t} + \frac{\hat{G}_{i+1/2} - \hat{G}_{i-1/2}}{\Delta x} \leq 0, \quad \forall i \in \mathbb{Z}, \quad (2.11)$$

with,

$$\begin{aligned} \hat{G}_{i+1/2} &= \frac{G(w_{i+1}^n) + G(w_i^n)}{2} + g \mathcal{F}_{i+1/2}^h \frac{z_{i+1} + z_i}{2} - \frac{\Delta x}{4\Delta t} (\eta(w_{i+1}^n) - \eta(w_i^n)) \\ &+ \frac{1}{2\Delta t} \int_0^{\frac{\Delta x}{2}} \eta(\tilde{w}(x/\Delta t, \hat{w}_i^n, \hat{w}_{i+1}^n)) dx - \frac{1}{2\Delta t} \int_{-\frac{\Delta x}{2}}^0 \eta(\tilde{w}(x/\Delta t, \hat{w}_i^n, \hat{w}_{i+1}^n)) dx. \end{aligned} \quad (2.12)$$

iii) If the approximate Riemann solver satisfies

$$\tilde{w}(x/\Delta t, \hat{w}_i^n, \hat{w}_{i+1}^n) = \begin{cases} w_i^n & \text{if } x < 0, \\ w_{i+1}^n & \text{otherwise,} \end{cases} \quad \forall i \in \mathbb{Z}, \quad (2.13)$$

as soon as the sequence  $(\hat{w}_i^n)_{i \in \mathbb{Z}}$  verifies, at each interface of the mesh, a local moving equilibrium (2.5) (resp. a local lake at rest equilibrium (2.6)) then the Godunov type scheme (2.1) is well-balanced for the moving equilibrium (resp. for the lake at rest).

*Proof.* For the statement *i*), according to the definition of  $\hat{S}_{i+1/2}^n$  given by (2.2) and since the function  $z$  is regular,  $\hat{S}_{i+1/2}^n$  est consistent with  $(0, -gh\partial_x z)^T$ . In addition, we have

$$\hat{S}(\hat{w}, \hat{w}) = 0, \quad \forall \hat{w} \in \hat{\Omega}. \quad (2.14)$$

As a consequence, an approximate Riemann solver that verifies the consistency integral relation (2.9) defines a consistent scheme.

For the statement *ii*), let consider the function  $w \mapsto \eta(w)$  defined by (2.8). This function is convex and it satisfies the conditions described in [42]. Hence, the Jensen inequality applied to the Godunov type scheme (2.1) gives

$$\begin{aligned} \eta(w_i^{n+1}) &\leq \\ &\frac{1}{2\Delta x} \int_{-\frac{\Delta x}{2}}^0 \eta(\tilde{w}(x/\Delta t, \hat{w}_i^n, \hat{w}_{i+1}^n)) dx + \frac{1}{2\Delta x} \int_0^{\frac{\Delta x}{2}} \eta(\tilde{w}(x/\Delta t, \hat{w}_{i-1}^n, \hat{w}_i^n)) dx \\ &+ \frac{1}{2\Delta x} \int_{-\frac{\Delta x}{2}}^{\frac{\Delta x}{2}} \eta(\tilde{w}(x/\Delta t, \hat{w}_{i-1}^n, \hat{w}_i^n)) dx + \frac{1}{2\Delta x} \int_{-\frac{\Delta x}{2}}^{\frac{\Delta x}{2}} \eta(\tilde{w}(x/\Delta t, \hat{w}_i^n, \hat{w}_{i+1}^n)) dx \\ &- \frac{1}{2\Delta x} \int_{-\frac{\Delta x}{2}}^0 \eta(\tilde{w}(x/\Delta t, \hat{w}_{i-1}^n, \hat{w}_i^n)) dx - \frac{1}{2\Delta x} \int_0^{\frac{\Delta x}{2}} \eta(\tilde{w}(x/\Delta t, \hat{w}_i^n, \hat{w}_{i+1}^n)) dx. \end{aligned}$$

Using the inequality (2.10) in the above inequality, we have

$$\begin{aligned} \eta(w_i^{n+1}) &\leq \\ &\frac{1}{2\Delta x} \int_{-\frac{\Delta x}{2}}^0 \eta(\tilde{w}(x/\Delta t, \hat{w}_i^n, \hat{w}_{i+1}^n)) dx + \frac{1}{2\Delta x} \int_0^{\frac{\Delta x}{2}} \eta(\tilde{w}(x/\Delta t, \hat{w}_{i-1}^n, \hat{w}_i^n)) dx \\ &+ \frac{1}{2} \left( \frac{\eta(w_{i-1}^n) + \eta(w_i^n)}{2} - \frac{\Delta t}{\Delta x} (G(w_i^n) - G(w_{i-1}^n)) - \frac{\Delta t}{\Delta x} g(z_i - z_{i-1}) \mathcal{F}_{i-\frac{1}{2}}^h \right) \\ &+ \frac{1}{2} \left( \frac{\eta(w_i^n) + \eta(w_{i+1}^n)}{2} - \frac{\Delta t}{\Delta x} (G(w_{i+1}^n) - G(w_i^n)) - \frac{\Delta t}{\Delta x} g(z_{i+1} - z_i) \mathcal{F}_{i+\frac{1}{2}}^h \right) \\ &- \frac{1}{2\Delta x} \int_{-\frac{\Delta x}{2}}^0 \eta(\tilde{w}(x/\Delta t, \hat{w}_{i-1}^n, \hat{w}_i^n)) dx - \frac{1}{2\Delta x} \int_0^{\frac{\Delta x}{2}} \eta(\tilde{w}(x/\Delta t, \hat{w}_i^n, \hat{w}_{i+1}^n)) dx. \end{aligned} \quad (2.15)$$

Since there is no contribution of the source term  $S(w)$  in the first equation of the Shallow Water system (1.1), the numerical scheme for the variable  $h$  writes

$$h_i^{n+1} = h_i^n - \frac{\Delta t}{\Delta x} (\mathcal{F}_{i+\frac{1}{2}}^h - \mathcal{F}_{i-\frac{1}{2}}^h), \quad \forall i \in \mathbb{Z}. \quad (2.16)$$

Multiplying the equation (2.16) by  $gz_i$  then adding the result to the inequality (2.15), we obtain an inequality of the form

$$\eta(w_i^{n+1}) + gh_i^{n+1} z_i \leq \eta(w_i^n) + gh_i^n z_i - \frac{\Delta t}{\Delta x} (\hat{\mathcal{G}}_{i+\frac{1}{2}} - \hat{\mathcal{G}}_{i-\frac{1}{2}}), \quad \forall i \in \mathbb{Z},$$

with  $\hat{\mathcal{G}}_{i+\frac{1}{2}}$  given by (2.12). Since the equality  $\hat{\eta}(\hat{w}) = \eta(w) + ghz$  is obvious and since the function  $z$  does not depend on the time, this last inequality rewrites under the form of (2.11). Before to conclude the proof of the statement *ii*), we have to show that the quantity  $\hat{\mathcal{G}}_{i+\frac{1}{2}}$  given by (2.12) is consistent with  $\hat{G}$  defined in (2.8).

Considering the consistency equality  $\tilde{w}(\cdot, \hat{w}, \hat{w}) = w$  and the consistency of the numerical flux  $\mathcal{F}_{i+\frac{1}{2}}^h$ , we have

$$\hat{G}(\hat{w}, \hat{w}) = \frac{G(w) + G(w)}{2} + g\mathcal{F}^h(w, w)z - \frac{\Delta x}{4\Delta t} (\eta(w) - \eta(w))$$



$$\begin{aligned}
& + \frac{1}{2\Delta t} \int_0^{\frac{\Delta x}{2}} \eta(\tilde{w}(x/\Delta t, \hat{w}, \hat{w})) \, dx - \frac{1}{2\Delta t} \int_{-\frac{\Delta x}{2}}^0 \eta(\tilde{w}(x/\Delta t, \hat{w}, \hat{w})) \, dx, \\
& = G(w) + g(hu)z + \frac{1}{2\Delta t} \int_0^{\frac{\Delta x}{2}} \eta(w) \, dx - \frac{1}{2\Delta t} \int_{-\frac{\Delta x}{2}}^0 \eta(w) \, dx, \\
& = G(w) + g(hu)z.
\end{aligned}$$

As a direct computing gives  $\hat{G}(\hat{w}) = G(w) + g(hu)z$ , the above equality achieves to show the statement *ii*).

Finally, for the statement *iii*), let assume the sequence  $(\hat{w}_i^n)_{i \in \mathbb{Z}}$  is such that a local equilibrium is satisfied at each interface of the mesh. In this case, using the condition (2.13) in a Godunov type scheme (2.1), we have

$$\begin{aligned}
w_i^{n+1} & = \frac{1}{\Delta x} \int_0^{\frac{\Delta x}{2}} \tilde{w}(x/\Delta t, \hat{w}_{i-1}^n, \hat{w}_i^n) \, dx + \frac{1}{\Delta x} \int_{-\frac{\Delta x}{2}}^0 \tilde{w}(x/\Delta t, \hat{w}_i^n, \hat{w}_{i+1}^n) \, dx, \\
& = \frac{1}{\Delta x} \int_0^{\frac{\Delta x}{2}} w_i^n \, dx + \frac{1}{\Delta x} \int_{-\frac{\Delta x}{2}}^0 w_i^n \, dx, \\
& = w_i^n.
\end{aligned}$$

According to Definition 2.1-*iii*), this last equality gives the well-balanced property that concludes the proof.  $\square$

Since the entropy  $\hat{w} \mapsto \hat{\eta}(\hat{w})$  defined in (1.3) is not convex, the proof of the discrete entropy inequality (2.11) uses the sufficient condition (2.10) and the equality (2.16) that arises from the numerical scheme. This proof is quite similar to a computation described in [12, Section 4.4] but the originality of this work is the definition of the inequality (2.10). This inequality ensures the discrete entropy stability (2.11) and it is now associated to the consistency integral relation (2.9) to define an approximate Riemann solver made of two intermediate symmetric constant states.

### 3 Consistent, entropy stable two states approximate Riemann solver for the Shallow Water equations

For the sake of clarity and in order to define an approximate Riemann solver made of two intermediate symmetric constant states, the following notations are considered:

$$(w_i^n, z_i) = (w_L, z_L) = \hat{w}_L, \quad (w_{i+1}^n, z_{i+1}) = (w_R, z_R) = \hat{w}_R, \quad \hat{s}_{i+\frac{1}{2}} = \hat{s}_{LR}. \quad (3.1)$$

Then, with a real  $\lambda > 0$ , a CFL condition is given by the inequalities

$$\lambda \geq \max_{\alpha \in \{L, R\}} |u_\alpha \pm \sqrt{gh_\alpha}| \quad \text{and} \quad \frac{\lambda \Delta t}{\Delta x} \leq \frac{1}{2}. \quad (3.2)$$

In the sequel, the above inequalities are assumed satisfied. As a consequence, an approximate Riemann solver denoted  $\tilde{w}(\cdot, \hat{w}_L, \hat{w}_R) : \mathbb{R} \rightarrow \mathbb{R}^2$  and made of two intermediate symmetric constant states writes

$$\tilde{w}(x/\Delta t, \hat{w}_L, \hat{w}_R) = \begin{cases} w_L & \text{if } \frac{x}{\Delta t} \leq -\lambda, \\ w_L^* & \text{if } -\lambda < \frac{x}{\Delta t} \leq 0, \\ w_R^* & \text{if } 0 < \frac{x}{\Delta t} \leq \lambda, \\ w_R & \text{if } \lambda < \frac{x}{\Delta t}. \end{cases} \quad (3.3)$$

The intermediate states  $w_L^*$  and  $w_R^*$  that define the approximate Riemann solver (3.3) have to be determined and they write

$$w_\alpha^* = (h_\alpha^*, h_\alpha^* u_\alpha^*)^\top, \quad \forall \alpha \in \{L, R\}.$$

An approximate Riemann solver of the form (3.3) is now defined with the consistency integral relation (2.9) and an additional relation obtained with the equality case in the inequality (2.10). The following lemma gives the equations satisfied by  $w_L^*$  and  $w_R^*$ .

**Lemma 3.1.** *Consider  $\hat{w}_L, \hat{w}_R$  two constant states being in the interior of  $\hat{\Omega}$ ,  $\hat{s}_{LR}$  a consistent discretization of  $\frac{1}{\Delta x} \int_{-\Delta x/2}^{\Delta x/2} (-gh \partial_x z) dx$  in the sens of Definition (2.2) and  $\tilde{w}(\cdot, \hat{w}_L, \hat{w}_R) : \mathbb{R} \rightarrow \mathbb{R}^2$  an approximate Riemann solver of the form (3.3). Let also consider the couples  $(\hat{\eta}, \hat{G})$  and  $(\eta, G)$  respectively given by (1.3), (2.8). Assume the CFL condition (3.2) holds and consider*

$$w^{\text{HLL}} = (h^{\text{HLL}}, (hu)^{\text{HLL}})^\top = \frac{w_R + w_L}{2} - \frac{f(w_R) - f(w_L)}{2\lambda} \in \mathbb{R}^2, \quad (3.4a)$$

$$h^{\text{HLL}} \hat{u}^{\text{HLL}} = (hu)^{\text{HLL}} + \frac{\Delta x \hat{s}_{LR}}{2\lambda} \in \mathbb{R}, \quad (3.4b)$$

$$\hat{w}^{\text{HLL}} = (h^{\text{HLL}}, h^{\text{HLL}} \hat{u}^{\text{HLL}}, \frac{z_L + z_R}{2})^\top \in \mathbb{R}^3, \quad (3.4c)$$

$$\eta^{\text{HLL}} = \frac{\eta(w_R) + \eta(w_L)}{2} - \frac{G(w_R) - G(w_L)}{2\lambda} \in \mathbb{R}, \quad (3.4d)$$

$$\hat{\eta}^{\text{HLL}} = \frac{\hat{\eta}(\hat{w}_R) + \hat{\eta}(\hat{w}_L)}{2} - \frac{\hat{G}(\hat{w}_R) - \hat{G}(\hat{w}_L)}{2\lambda} \in \mathbb{R}. \quad (3.4e)$$

The following equalities

$$\frac{1}{\Delta x} \int_{-\frac{\Delta x}{2}}^{\frac{\Delta x}{2}} \tilde{w}(x/\Delta t, \hat{w}_L, \hat{w}_R) dx = \frac{w_L + w_R}{2} - \frac{\Delta t}{\Delta x} (f(w_R) - f(w_L)) + \Delta t (0, \hat{s}_{LR})^\top, \quad (3.5a)$$

$$\frac{1}{\Delta x} \int_{-\frac{\Delta x}{2}}^{\frac{\Delta x}{2}} \eta(\tilde{w}(x/\Delta t, \hat{w}_L, \hat{w}_R)) dx = \quad (3.5b)$$

$$\frac{\eta(w_L) + \eta(w_R)}{2} - \frac{\Delta t}{\Delta x} (G(w_R) - G(w_L)) - g \frac{\Delta t}{\Delta x} (z_R - z_L) \mathcal{F}_{LR}^h,$$

are equivalent to

$$\frac{h_L^* + h_R^*}{2} = h^{\text{HLL}}, \quad (3.6a)$$

$$\frac{h_L^* u_L^* + h_R^* u_R^*}{2} = h^{\text{HLL}} \hat{u}^{\text{HLL}}, \quad (3.6b)$$

$$\frac{h_L^* h_R^*}{8h^{\text{HLL}}} (u_R^* - u_L^*)^2 + \frac{g}{8} (h_R^* - h_L^* + z_R - z_L)^2 = \hat{\eta}^{\text{HLL}} - \hat{\eta}(\hat{w}^{\text{HLL}}) + \frac{g}{8} (z_R - z_L)^2. \quad (3.6c)$$

Before to prove the above lemma, we empathize the quantity  $h^{\text{HLL}} \hat{u}^{\text{HLL}}$  given by (3.4b) depends on the free parameter  $\hat{s}_{LR}$ . This parameter has to satisfy the consistency condition described in (2.2) and some admissible choices will be given in the sequel sections. Lemma 3.1 is now proved.

*Proof of Lemma 3.1.* At first, we show that the system (3.5) implies the system (3.6). As a CFL condition holds and noting  $\nu = \lambda \Delta t / \Delta x$ , the left hand side of the equation (3.5a) is equivalent to

$$\begin{aligned} & \frac{1}{\Delta x} \int_{-\frac{\Delta x}{2}}^{\frac{\Delta x}{2}} \tilde{w}(x/\Delta t, \hat{w}_L, \hat{w}_R) dx \\ &= \frac{1}{\Delta x} \int_{-\frac{\Delta x}{2}}^{-\lambda \Delta t} w_L dx + \frac{1}{\Delta x} \int_{-\lambda \Delta t}^0 w_L^* dx + \frac{1}{\Delta x} \int_0^{\lambda \Delta t} w_R^* dx + \frac{1}{\Delta x} \int_{\lambda \Delta t}^{\frac{\Delta x}{2}} w_R dx, \\ &= (-\nu + 1/2)w_L + \nu w_L^* + \nu w_R^* + (1/2 - \nu)w_R. \end{aligned}$$

As a consequence the equation (3.5a) equivalently rewrites

$$(-\nu + 1/2)w_L + \nu w_L^* + \nu w_R^* + (1/2 - \nu)w_R = \frac{w_L + w_R}{2} - \nu \frac{f(w_R) - f(w_L)}{\lambda} + \nu (0, \Delta x \hat{s}_{LR}/\lambda, 0)^T.$$

Writing the above equation component by component then dividing both sides by  $2\nu$ , we deduce the equations (3.6a)-(3.6b).

Now, we have to rewrite the equation (3.5b). Developing the integral of the left hand side of (3.5b) and using once again the notation  $\nu = \lambda \Delta t / \Delta x$ , we have

$$\begin{aligned} & \frac{1}{\Delta x} \int_{-\frac{\Delta x}{2}}^{\frac{\Delta x}{2}} \eta(\tilde{w}(x/\Delta t, \hat{w}_L, \hat{w}_R)) \, dx \\ &= \frac{1}{\Delta x} \int_{-\frac{\Delta x}{2}}^{-\lambda \Delta t} \eta(w_L) \, dx + \frac{1}{\Delta x} \int_{-\lambda \Delta t}^0 \eta(w_L^*) \, dx + \frac{1}{\Delta x} \int_0^{\lambda \Delta t} \eta(w_R^*) \, dx + \frac{1}{\Delta x} \int_{\lambda \Delta t}^{\frac{\Delta x}{2}} \eta(w_R) \, dx, \\ &= (-\nu + 1/2)\eta(w_L) + \nu\eta(w_L^*) + \nu\eta(w_R^*) + (1/2 - \nu)\eta(w_R). \end{aligned}$$

With the above equality, the equation (3.5b) is equivalent to

$$\begin{aligned} (-\nu + 1/2)\eta(w_L) + \nu\eta(w_L^*) + \nu\eta(w_R^*) + (1/2 - \nu)\eta(w_R) = \\ \frac{\eta(w_L) + \eta(w_R)}{2} - \nu \frac{G(w_R) - G(w_L)}{\lambda} - g\nu \frac{z_R - z_L}{\lambda} \mathcal{F}_{LR}^h. \end{aligned}$$

Multiplying this last equality by  $1/(2\nu) > 0$  and using the quantity  $\eta^{\text{HLL}}$  defined in (3.4d), we obtain a new equivalent form of the equation (3.5b) which is given by

$$\frac{\eta(w_L^*) + \eta(w_R^*)}{2} = \eta^{\text{HLL}} - g \frac{z_R - z_L}{2\lambda} \mathcal{F}_{LR}^h. \quad (3.7)$$

In order to rewrite the numerical flux  $\mathcal{F}_{LR}^h$ , the notation  $[\cdot] = \cdot_R - \cdot_L$  is now considered. According to the approximate Riemann solver (3.3), this quantity writes

$$\mathcal{F}_{LR}^h = \frac{(hu)_R + (hu)_L}{2} - \frac{\lambda}{2}[h] + \frac{\lambda}{2}[h^*].$$

As a consequence, and using the definition of  $h^{\text{HLL}}$  given by (3.4a), we have

$$\begin{aligned} -\frac{[z]}{2\lambda} \mathcal{F}_{LR}^h &= -\frac{[z][h^*]}{4} + \frac{[z]}{4} \left( [h] - \frac{(hu)_L + (hu)_R}{\lambda} \right), \\ &= -\frac{[z][h^*]}{4} + \frac{z_R h_R + z_L h_L}{2} - \frac{z_L + z_R}{2} h^{\text{HLL}} - \frac{[huz]}{2\lambda}. \end{aligned}$$

Now, considering the above equality and the quantity  $\hat{\eta}^{\text{HLL}}$  given by (3.4e) in the equation (3.7), then developing  $\eta(w_L^*)$  and  $\eta(w_R^*)$ , we deduce

$$\frac{h_R^*(u_R^*)^2 + h_L^*(u_L^*)^2}{4} + g \frac{(h_L^*)^2 + (h_R^*)^2}{4} + \frac{g}{4}[z][h^*] = \hat{\eta}^{\text{HLL}} - g \frac{z_L + z_R}{2} h^{\text{HLL}}. \quad (3.8)$$

Moreover, applying the square function on both sides of the equation (3.6a) then multiplying the result by  $g/2$ , we have

$$\frac{g}{8}(h_L^* + h_R^*)^2 = \frac{g}{2}(h^{\text{HLL}})^2.$$

Therefore, subtracting the above equation to the equation (3.8), we deduce

$$\frac{h_R^*(u_R^*)^2 + h_L^*(u_L^*)^2}{4} + \frac{g}{8}[h^*]^2 + \frac{g}{4}[z][h^*] = \hat{\eta}^{\text{HLL}} - \frac{g}{2}(h^{\text{HLL}})^2 - g \frac{z_L + z_R}{2} h^{\text{HLL}}. \quad (3.9)$$

The next step consists to rewrite the quantity  $h_L^*(u_L^*)^2 + h_R^*(u_R^*)^2$  in order to show that the equation (3.9) is equivalent to

$$\frac{h_R^* h_L^* [u^*]^2}{8h^{\text{HLL}}} + \frac{g}{8}[h^*]^2 + \frac{g}{4}[h^*][z] = \hat{\eta}^{\text{HLL}} - \hat{\eta}(\hat{w}^{\text{HLL}}). \quad (3.10)$$

According to the definition of  $w^{\text{HLL}}$  given by (3.4a), a direct computation shows that  $h^{\text{HLL}}$  writes

$$h^{\text{HLL}} = \frac{h_R}{2}(1 - u_{R/\lambda}) + \frac{h_L}{2}(1 + u_{L/\lambda}). \quad (3.11)$$

Since the CFL condition (3.2) holds and since  $\hat{w}_L$  and  $\hat{w}_R$  are in the interior of  $\hat{\Omega}$ , the above equality leads to  $h^{\text{HLL}} > 0$ . As a consequence, and using the equations (3.6a)-(3.6b), the following computation holds

$$\begin{aligned} \frac{h_L^*(u_L^*)^2 + h_R^*(u_R^*)^2}{4} &= \frac{h_L^*(u_L^*)^2 + h_R^*(u_R^*)^2}{4} \frac{h_L^* + h_R^*}{2h^{\text{HLL}}}, \\ &= \frac{(h_L^* u_L^*)^2 + 2h_L^* u_L^* h_R^* u_R^* + (h_R^* u_R^*)^2 + h_R^* h_L^* [u^*]^2}{8h^{\text{HLL}}}, \\ &= \frac{1}{2h^{\text{HLL}}} \left( \frac{h_L^* u_L^* + h_R^* u_R^*}{2} \right)^2 + \frac{h_L^* h_R^* [u^*]^2}{8h^{\text{HLL}}}, \\ &= \frac{(h^{\text{HLL}} \hat{u}^{\text{HLL}})^2}{2h^{\text{HLL}}} + \frac{h_L^* h_R^* [u^*]^2}{8h^{\text{HLL}}}. \end{aligned} \quad (3.12)$$

Considering the above result in the equation (3.9), we eventually deduce that the equation (3.5b) is equivalent to the equation (3.10). Adding  $g[z]^2/8$  on both sides of the equation (3.10), we deduce the equation (3.6c) that shows that the system (3.5) implies the system (3.6). The converse is a direct consequence of the above computations.  $\square$

The results of the above lemma are given with  $\hat{w}_L$  and  $\hat{w}_R$  in the interior of  $\hat{\Omega}$  that formally means  $h_L \gg 0$  and  $h_R \gg 0$ . The cases  $h_L$  or  $h_R$  closed to zero concern the transitions dry-wet and they need a specific treatment [5, 39, 41]. Indeed, the quantity  $\hat{u}^{\text{HLL}}$  given by (3.4b) is defined up to a multiplication by  $h^{\text{HLL}}$ . Since  $h^{\text{HLL}}$  is null if and only if  $h_L = 0$  and  $h_R = 0$  (see the equation (3.11)), the admitted convention in [5, 39, 41] requires to define  $\hat{u}^{\text{HLL}}$  in  $\mathbb{R}$  as follows:

$$\hat{u}^{\text{HLL}} = \begin{cases} 0, & \text{if } h_L = 0 \text{ and } h_R = 0, \\ \frac{(hu)^{\text{HLL}} + \Delta x \hat{s}_{LR}/2\lambda}{h^{\text{HLL}}}, & \text{otherwise.} \end{cases} \quad (3.13)$$

Form a general point of view, the dry-wet transitions will be treated but they are not the main interest of this work. As a consequence, and for the clarity, the intermediate results are given with  $\hat{w}_L$  and  $\hat{w}_R$  in the interior of  $\hat{\Omega}$ . In this case, both definitions (3.4b) and (3.13) are equivalent.

The equations (3.5) are chosen in this work because, according to Lemma 2.2-i)-ii), they are sufficient conditions to the consistency and to the discrete entropy stability of the Godunov type scheme. Starting with the equations (3.5), Lemma 3.1 gives a system of three equations (3.6) composed of four unknown  $h_L^*$ ,  $h_R^*$ ,  $u_L^*$ ,  $u_R^*$  and one parameter  $\hat{s}_{LR}$ . As a consequence, the system (3.6) that defines  $w_L^*$  and  $w_R^*$  is under-determined.

In particular, the parameter  $\hat{s}_{LR}$  is used in the quantity  $\hat{\eta}^{\text{HLL}} - \hat{\eta}(\hat{w}^{\text{HLL}}) + g(z_R - z_L)^2/8$  that defines the right hand side of the quadratic equation (3.6c). It is clear that the well-posed property of this quadratic equation requires the inequality

$$\hat{\eta}^{\text{HLL}} - \hat{\eta}(\hat{w}^{\text{HLL}}) + \frac{g(z_R - z_L)^2}{8} \geq 0. \quad (3.14)$$

At this point, the above inequality can not be proved yet but some properties are now given.

**Lemma 3.2** (Properties of  $\hat{\eta}^{\text{HLL}} - \hat{\eta}(\hat{w}^{\text{HLL}}) + g(z_R - z_L)^2/8$ ). *Consider  $\hat{w}_L$ ,  $\hat{w}_R$  two constant states being in the interior of  $\hat{\Omega}$ . Let also consider the couples  $(\hat{\eta}, \hat{G})$  and  $(\eta, G)$  respectively given by (1.3), (2.8) and the quantities  $(\hat{w}^{\text{HLL}}, \hat{\eta}^{\text{HLL}})$ ,  $(w^{\text{HLL}}, \eta^{\text{HLL}})$  defined in (3.4). Assume the CFL condition (3.2) holds. The following statements are verified.*

i) For all given quantity  $\hat{s}_{LR}$ , the following estimate is satisfied:

$$\begin{aligned} \hat{\eta}^{\text{HLL}} - \hat{\eta}(\hat{w}^{\text{HLL}}) + \frac{g}{8}(z_R - z_L)^2 = & \\ & \frac{h_R h_L (u_R - u_L)^2}{4(h_L + h_R)} + \frac{g}{8}(h_R - h_L + z_R - z_L)^2 \\ & - \frac{(u_R - u_L)^2 h_R h_L (h_L u_R - h_R u_L)}{4\lambda(h_L + h_R)^2} - \frac{h_L u_L + h_R u_R}{4\lambda} \left( g(z_R - z_L) + \frac{\Delta x \hat{s}_{LR}}{h^{\text{HLL}}} \right) + \mathcal{O}\left(\frac{1}{\lambda^2}\right). \end{aligned} \quad (3.15)$$

i) If  $\hat{s}_{LR}$  is consistent with  $\frac{1}{\Delta x} \int_{-\Delta x/2}^{\Delta x/2} (-gh \partial_x z) dx$  in the sens of Definition (2.2) then

$$\left( \hat{\eta}^{\text{HLL}} - \hat{\eta}(\hat{w}^{\text{HLL}}) + \frac{g}{8}(z_R - z_L)^2 \right) \Big|_{z_L = z_R} = \eta^{\text{HLL}} - \eta(w^{\text{HLL}}). \quad (3.16)$$

In addition, there exists  $\lambda > 0$  large enough such that

$$\eta^{\text{HLL}} - \eta(w^{\text{HLL}}) \geq 0. \quad (3.17)$$

*Proof.* For the statement i), using the notation  $[\cdot] = \cdot_R - \cdot_L$  then developing the quantities  $\hat{u}^{\text{HLL}}$ ,  $\hat{\eta}^{\text{HLL}}$ ,  $\hat{\eta}(\hat{w}^{\text{HLL}})$  defined in (3.4), we deduce

$$\begin{aligned} \hat{\eta}^{\text{HLL}} - \hat{\eta}(\hat{w}^{\text{HLL}}) + \frac{g}{8}[z]^2 = & \hat{\eta}^{\text{HLL}} - \frac{((hu)^{\text{HLL}} + \Delta x \hat{s}_{LR}/2\lambda)^2}{2h^{\text{HLL}}} - \frac{g}{2}(h^{\text{HLL}})^2 - gh^{\text{HLL}} \frac{z_L + z_R}{2} + \frac{g}{8}[z]^2, \\ = & \eta^{\text{HLL}} + g \frac{h_L z_L + h_R z_R}{2} - g \frac{[hu z]}{2\lambda} - \eta(w^{\text{HLL}}) - gh^{\text{HLL}} \frac{z_L + z_R}{2} \\ & - \frac{(hu)^{\text{HLL}}}{h^{\text{HLL}}} \frac{\Delta x \hat{s}_{LR}}{2\lambda} - \frac{1}{8h^{\text{HLL}}} \left( \frac{\Delta x \hat{s}_{LR}}{\lambda} \right)^2 + \frac{g}{8}[z]^2, \\ = & \eta^{\text{HLL}} - \eta(w^{\text{HLL}}) + \frac{g}{4}[z][h] + \frac{g}{8}[z]^2 \\ & - \frac{g}{4\lambda}[z](h_L u_L + h_R u_R) - \frac{(hu)^{\text{HLL}}}{h^{\text{HLL}}} \frac{\Delta x \hat{s}_{LR}}{2\lambda} - \frac{1}{8h^{\text{HLL}}} \left( \frac{\Delta x \hat{s}_{LR}}{\lambda} \right)^2. \end{aligned} \quad (3.18)$$

At this point, the right hand side of the above equality will be developed considering  $\lambda$  goes to infinity. Let denote  $q = hu$  and developing the quantities  $w^{\text{HLL}}$  and  $\eta^{\text{HLL}}$  respectively given by (3.4a)-(3.4d), we have at first

$$\eta^{\text{HLL}} - \eta(w^{\text{HLL}}) = \frac{q_L^2}{4h_L} + \frac{q_R^2}{4h_R} + \frac{g(h_L^2 + h_R^2)}{4} - \frac{[G]}{2\lambda} - \frac{g}{8} \left( h_R + h_L - \frac{[q]}{\lambda} \right)^2 - \frac{((hu)^{\text{HLL}})^2}{2h^{\text{HLL}}}. \quad (3.19)$$

Now, it is necessary to develop the quantity  $((hu)^{\text{HLL}})^2 / (2h^{\text{HLL}})$  in the above equality. Using the definition of  $h^{\text{HLL}}$  and  $(hu)^{\text{HLL}}$  given by (3.4a), we have

$$(hu)^{\text{HLL}} = \frac{q_L + q_R}{2} - \frac{1}{2\lambda} [hu^2 + gh^2/2], \quad (3.20a)$$

$$\frac{1}{2h^{\text{HLL}}} = \frac{1}{(h_L + h_R) \left( 1 - \frac{[q]}{\lambda(h_R + h_L)} \right)}. \quad (3.20b)$$

Considering the square of  $(hu)^{\text{HLL}}$  in the equality (3.20a) and interpreting the quantity  $1/(1 - \frac{[q]}{\lambda(h_R + h_L)})$  as a geometric series, we respectively deduce

$$((hu)^{\text{HLL}})^2 = \left( \frac{q_L + q_R}{2} \right)^2 - \frac{1}{2\lambda} (q_L + q_R) [hu^2 + \frac{g}{2}h^2] + \mathcal{O}\left(\frac{1}{\lambda^2}\right), \quad (3.21a)$$

$$\frac{1}{1 - \frac{[q]}{\lambda(h_R + h_L)}} = 1 + \frac{[q]}{\lambda(h_L + h_R)} + \mathcal{O}\left(\frac{1}{\lambda^2}\right). \quad (3.21b)$$

As a consequence, using the two above equalities to develop the quantity  $((hu)^{\text{HLL}})^2/(2h^{\text{HLL}})$  then considering the result in  $\eta^{\text{HLL}} - \eta(w^{\text{HLL}})$  given by (3.19), we have

$$\begin{aligned}\eta^{\text{HLL}} - \eta(w^{\text{HLL}}) &= \frac{q_L^2}{4h_L} + \frac{q_R^2}{4h_R} - \frac{(q_L + q_R)^2}{4(h_L + h_R)} + \frac{g}{8}[h]^2 - \frac{[G]}{2\lambda} + \frac{g}{4}(h_L + h_R)\frac{[q]}{\lambda} \\ &\quad - \frac{(q_L + q_R)^2[q]}{4\lambda(h_L + h_R)^2} + \frac{(q_L + q_R)[hu^2 + \frac{g}{2}h^2]}{2\lambda(h_L + h_R)} + \mathcal{O}\left(\frac{1}{\lambda^2}\right), \\ &= \frac{h_R h_L [u]^2}{4(h_L + h_R)} + \frac{g}{8}[h]^2 - \frac{[u]^2 h_R h_L (h_L u_R - h_R u_L)}{4\lambda(h_L + h_R)^2} + \mathcal{O}\left(\frac{1}{\lambda^2}\right).\end{aligned}$$

With the two equalities (3.20), it is also possible to develop  $(hu)^{\text{HLL}}/h^{\text{HLL}}$  and  $1/h^{\text{HLL}}$  in the equation (3.18) then using the above estimate in the result, we eventually deduce

$$\begin{aligned}\hat{\eta}^{\text{HLL}} - \hat{\eta}(\hat{w}^{\text{HLL}}) + \frac{g}{8}[z]^2 &= \\ \frac{h_R h_L [u]^2}{4(h_L + h_R)} + \frac{g}{8}[h + z]^2 - \frac{[u]^2 h_R h_L (h_L u_R - h_R u_L)}{4\lambda(h_L + h_R)^2} - \frac{h_L u_L + h_R u_R}{4\lambda} \left( g[z] + \frac{\Delta x \hat{s}_{LR}}{h^{\text{HLL}}} \right) + \mathcal{O}\left(\frac{1}{\lambda^2}\right),\end{aligned}$$

that achieves to show the statement *i*).

For the statement *ii*), if  $\hat{s}_{LR}$  is consistent then  $\hat{s}_{LR}|_{z_R=z_L} = 0$  and the equality (3.16) is a direct consequence of the computation (3.18). For the inequality (3.17), in the case of  $z_L = z_R$ , and since  $\hat{s}_{LR}$  is consistent, the estimate (3.15) reads

$$\begin{aligned}\left( \hat{\eta}^{\text{HLL}} - \hat{\eta}(\hat{w}^{\text{HLL}}) + \frac{g}{8}(z_R - z_L)^2 \right) \Big|_{z_L=z_R} &= \eta^{\text{HLL}} - \eta(w^{\text{HLL}}), \\ &= \frac{h_R h_L [u]^2}{4(h_L + h_R)} + \frac{g}{8}[h]^2 + \mathcal{O}\left(\frac{1}{\lambda}\right).\end{aligned}$$

As a consequence, as soon as  $[u] \neq 0$  or  $[h] \neq 0$ , there exists  $\lambda > 0$  large enough such that the inequality (3.17) holds. In the case of  $[u] = 0$  and  $[h] = 0$ , a direct computation gives  $\eta^{\text{HLL}} - \eta(w^{\text{HLL}}) = 0$ . Therefore, the inequality can be always enforced that concludes the proof.  $\square$

The statement *i*) of the above lemme highlights the role of the parameter  $\hat{s}_{LR}$  in the inequality (3.14). The choice of this parameter has to be motivated for the consistency but also to enforce the inequality (3.14) in the case  $u_L = u_R$  and  $h_L + z_L = h_R + z_R$ .

In the case  $z_L = z_R$ , the consistency imposes  $\hat{s}_{LR} = 0$  and the left hand side of the inequality (3.14) degenerates to  $\eta^{\text{HLL}} - \eta(w^{\text{HLL}})$ . The inequality  $\eta^{\text{HLL}} - \eta(w^{\text{HLL}}) \geq 0$  is proved in Lemma 3.2-*ii*) with algebraic computations but this inequality is a direct consequence of the entropy stability of the standard HLL solver [28]. Therefore, the inequality  $\eta^{\text{HLL}} - \eta(w^{\text{HLL}}) \geq 0$  holds under the standard CFL condition (3.2) and consequently, if  $z_L = z_R$  then the system (3.6) is well-posed. This system is so used at first to define an approximate Riemann solver (3.3) for the Shallow Water equations (1.1) in the flat regions characterized by  $z = \text{cste}$ .

## 4 A consistent, entropy stable numerical scheme in the flat regions ( $z = \text{cste}$ )

This section concerns the design of an approximate Riemann solver (3.3) defined by the system (3.6) and for the case  $z = \text{cste}$ . In the case  $z_L = z_R$ ,  $\hat{s}_{LR} = 0$  and the system (3.6) reads

$$\frac{h_L^* + h_R^*}{2} = h^{\text{HLL}}, \quad (4.1a)$$

$$\frac{h_L^* u_L^* + h_R^* u_R^*}{2} = h^{\text{HLL}} u^{\text{HLL}}, \quad (4.1b)$$

$$\frac{h_L^* h_R^*}{8h^{\text{HLL}}} (u_R^* - u_L^*)^2 + \frac{g}{8} (h_R^* - h_L^*)^2 = \eta^{\text{HLL}} - \eta(w^{\text{HLL}}), \quad (4.1c)$$

where  $w^{\text{HLL}}, \eta^{\text{HLL}}$  are respectively given by (3.4a), (3.4d) and with

$$u^{\text{HLL}} = \begin{cases} 0, & \text{if } h_L = 0 \text{ and } h_R = 0, \\ \frac{(hu)^{\text{HLL}}}{h^{\text{HLL}}}, & \text{otherwise.} \end{cases}$$

Since the system (4.1) is under-determined, we propose to complete it with the continuity of  $u$  in the intermediate states of the solver (3.3). Such continuity writes

$$u_L^* = u_R^*, \quad (4.2)$$

and it has already proposed in [45] for the HLLC scheme for instance. The next lemma deals with the solutions of the system (4.1) completed with the equation  $u_L^* = u_R^*$ .

**Lemma 4.1** (Entropic approximate Riemann solver for the homogeneous Shallow Water equations). *Consider  $w_L, w_R$  being two constant states in the interior of  $\Omega$  ( $h_L > 0, h_R > 0$ ). Consider a Godunov type scheme (2.1) that approximates the solution of the homogeneous Shallow Water equations and defined by an approximate Riemann solver  $\tilde{w}(\cdot, w_L, w_R) : \mathbb{R} \rightarrow \mathbb{R}^2$  of the form (3.3). Assume the CFL condition (3.2) holds. If the intermediate states  $w_L^*, w_R^*$  of the solver are defined by the system (4.1), (4.2) then, there exists two formulations for  $w_L^*$  and  $w_R^*$ . They are given by*

$$u_L^* = u_R^* = u^{\text{HLL}}, \quad (4.3a)$$

$$h_R^* = h^{\text{HLL}} \pm \sqrt{\frac{2}{g}(\eta^{\text{HLL}} - \eta(w^{\text{HLL}}))}, \quad (4.3b)$$

$$h_L^* = h^{\text{HLL}} \mp \sqrt{\frac{2}{g}(\eta^{\text{HLL}} - \eta(w^{\text{HLL}}))}. \quad (4.3c)$$

A Godunov type scheme (2.1) associated to one of the two above approximate Riemann solvers

- i) is consistent with the homogeneous Shallow Water equations given by (1.1) with  $z = \text{cste}$ ,
- ii) satisfies a discrete entropy inequality for the couple  $(\eta, G)$  defined in (2.8). The entropy numerical flux of this entropy inequality is given by (2.12) evaluated with  $z = \text{cste}$ .

In the definitions (4.3), both symbols  $\pm$  and  $\mp$  that mean  $+$  or  $-$  but they are self-dependent. If  $\pm$  is positive (*resp.* negative) then  $\mp$  is negative (*resp.* positive). This notation will be kept in the sequel. The proof of Lemma 4.1 is given below.

*Proof of Lemma 4.1.* At first, we show that the definitions (4.3) are the solutions of the system (4.1), (4.2). Since  $u_L^* = u_R^*$ , denoting  $u^* = u_L^* = u_R^*$ , the equation (4.1b) associated to (4.1a) gives

$$\begin{aligned} h^{\text{HLL}} u^{\text{HLL}} &= \frac{h_L^* u_L^* + h_R^* u_R^*}{2}, \\ &= \frac{h_L^* + h_R^*}{2} u^*, \\ &= h^{\text{HLL}} u^*. \end{aligned} \quad (4.4)$$

As  $w_L$  and  $w_R$  are in the interior of  $\Omega$ , we have  $h^{\text{HLL}} > 0$ . Multiplying the above equation by  $1/h^{\text{HLL}}$ , we deduce the result (4.3a) about  $u_L^*$  and  $u_R^*$ . Now, we prove the formulations of  $h_L^*, h_R^*$  given by (4.3c)-(4.3b). Since the equality  $u_L^* = u_R^*$  holds, the equation (4.1c) reads

$$\frac{g}{8}(h_R^* - h_L^*)^2 = \eta^{\text{HLL}} - \eta(w^{\text{HLL}}).$$

According to Lemma 3.2, the inequality  $\eta^{\text{HLL}} - \eta(w^{\text{HLL}}) \geq 0$  is ensured as soon as  $\lambda > 0$  is large enough. As a consequence the above equation is well-posed and applying the root-square on both sides, we deduce

$$h_R^* - h_L^* = \pm \sqrt{8(\eta^{\text{HLL}} - \eta(w^{\text{HLL}}))/g}.$$

Using the above result in the consistency equation (4.1a), we deduce the results (4.3c)-(4.3b) about  $h_L^*$ ,  $h_R^*$ .

Concerning the consistency property *i*), the definitions (4.3) lead to  $\tilde{w}(\cdot, w, w) = w$ . As the solver satisfies the integral consistency relation under the form (4.1a)-(4.1b) (see Lemma 3.1 for details), a Godunov type scheme (2.1) defined by (4.3) is consistent.

Eventually, the discrete entropy inequality *ii*) is a direct consequence of the equation (4.1c) as established in Lemma 2.2, 3.1, that concludes the proof.  $\square$

The intermediate heights  $(h_\alpha^*)_{\alpha \in \{L, R\}}$  of (4.3) are always well defined thanks to the HLL scheme property [28] that writes  $\eta^{\text{HLL}} - \eta(w^{\text{HLL}}) \geq 0$ . But, the schemes of Lemma 4.1 do not necessary preserve the convex set  $\Omega$ . As a consequence, we couple the solver 4.1 to a procedure described in [41] that ensures a conservative preservation of the convex set  $\Omega$  near the dry areas in which  $h_L$  or  $h_R$  are closed to zero.

**Lemma 4.2** (Robust and entropic approximate Riemann solver for the homogeneous Shallow Water equations). *Consider  $w_L, w_R$  being two constant states of  $\Omega$ . Consider a Godunov type scheme (2.1) that approximates the solution of the homogeneous Shallow Water equations and defined by an approximate Riemann solver  $\tilde{w}(\cdot, w_L, w_R) : \mathbb{R} \rightarrow \mathbb{R}^2$  of the form (3.3). Assume the CFL condition (3.2) holds. Let also consider the quantities  $w^{\text{HLL}}, \eta^{\text{HLL}}$  respectively given by (3.4a), (3.4d) and let denote*

$$\widetilde{h}_R^* = h^{\text{HLL}} \pm \sqrt{\frac{2}{g}(\eta^{\text{HLL}} - \eta(w^{\text{HLL}}))}, \quad \widetilde{h}_L^* = h^{\text{HLL}} \mp \sqrt{\frac{2}{g}(\eta^{\text{HLL}} - \eta(w^{\text{HLL}}))}. \quad (4.5)$$

If the intermediate states  $w_L^*, w_R^*$  of the approximate Riemann solver write

$$\begin{aligned} u_L^* &= u_R^* = u^{\text{HLL}}, \\ h_R^* &= \min\left(\max(\widetilde{h}_R^*, 0), 2h^{\text{HLL}}\right), \\ h_L^* &= \min\left(\max(\widetilde{h}_L^*, 0), 2h^{\text{HLL}}\right), \end{aligned} \quad (4.6)$$

then we have

- i) A Godunov type scheme (2.1) associated to a such solver is consistent with the homogeneous Shallow Water equations given by (1.1) with  $z = \text{cste}$  and it preserves the convex set  $\Omega$ .*
- ii) If  $\widetilde{h}_L^* > 0$  and  $\widetilde{h}_R^* > 0$  then a Godunov type scheme (2.1) associated to a such solver satisfies a discrete entropy inequality for the couple  $(\eta, G)$  defined in (2.8). The entropy numerical flux of this entropy inequality is given by (2.12) evaluated with  $z = \text{cste}$ .*

*Proof.* At first, if  $\widetilde{h}_L^*$  and  $\widetilde{h}_R^*$  given by (4.5) are both strictly positive then (4.6) coincides with (4.3). In this case, the consistency has already shown in Lemma 4.1. If the quantity  $\widetilde{h}_L^*$  given by (4.5) is such that  $\widetilde{h}_L^* \leq 0$  then the procedure  $\min(\max(\cdot, \cdot), \cdot)$  imposes  $h_L^* = 0$  and  $h_R^* = 2h^{\text{HLL}} \geq 0$ . As a consequence, the consistency equation  $h_L^* + h_R^* = 2h^{\text{HLL}}$  holds. Since a similar computation can be done in the case  $\widetilde{h}_R^* \leq 0$ , we deduce the consistency of the statement *i*).

For the preservation of the domain  $\Omega$ , it is sufficient to prove that  $h_L^* \geq 0$  and  $h_R^* \geq 0$  but according to the definitions (4.6), these inequalities always hold.

Finally, if the quantities  $\widetilde{h}_L^*$  and  $\widetilde{h}_R^*$  given by (4.5) are both strictly positive then (4.6) coincides with (4.3) and in this case, the discrete entropy inequality is shown in Lemma 4.1 that achieves the proof.  $\square$

The positiveness procedure  $\min(\max(\cdot, \cdot), \cdot)$  described in (4.6) ensures the robustness in the regions near the dry areas. This procedure still ensures the consistency relation (4.1a) when  $\widetilde{h}_L^* \leq 0$  or  $\widetilde{h}_R^* \leq 0$  but it not necessary ensures the entropy condition (4.1c). As a consequence, the discrete entropy stability might be locally lost in the regions close to dry areas. In the next section, we extend the design principles described in this section to the case where the bottom topography works.



## 5 A well-balanced entropy stable numerical scheme for the lake at rest

In this section, we consider the Shallow Water equations (1.1) for an arbitrary smooth given function  $z : \mathbb{R} \rightarrow \mathbb{R}$ . The aim is to design a consistent, entropic and well-balanced scheme for the lake at rest (1.6). In this regard, we propose to define an approximate Riemann solver (3.3) with the under-determined system (3.6) completed as follows:

$$u_L^* = u_R^*, \quad (5.1a)$$

$$\hat{s}_{LR} = \hat{s}_{LR}^{\text{WBAR}} = -gh^{\text{HLL}} \frac{z_R - z_L}{\Delta x} - \sqrt{\frac{g}{h^{\text{HLL}}}} \frac{h_L u_L + h_R u_R}{h^{\text{HLL}}} \frac{(z_R - z_L)^2}{\Delta x}, \quad (5.1b)$$

where the notation  $\hat{s}_{LR}^{\text{WBAR}}$  means *Well-Balanced At Rest*. This notation is defined in order to distinguish  $\hat{s}_{LR}^{\text{WBAR}}$  given by (5.1b) from a generic discretization  $\hat{s}_{LR}$ .

According to Lemma 2.2, the under-determined system (3.6) is a sufficient condition to the consistency and to the discrete entropy inequality (2.11) for the couple  $(\hat{\eta}, \hat{G})$  defined in (1.3). The closure (5.1) is sufficient to obtain the well-balanced property for the lake at rest (1.6). Before to prove this property, we show the existence of the solutions of the system (3.6), (5.1).

**Lemma 5.1.** *Consider  $\hat{w}_L, \hat{w}_R$  two constant states being in the interior of  $\hat{\Omega}$  and  $\tilde{w}(\cdot, \hat{w}_L, \hat{w}_R) : \mathbb{R} \rightarrow \mathbb{R}^2$  an approximate Riemann solver of the form (3.3). There exists  $\lambda > 0$  large enough such that if the CFL condition (3.2) holds and if the solvers intermediate states denoted  $w_L^*$  and  $w_R^*$  are defined by the system (3.6), (5.1) then the two possible formulations for  $w_L^*$  and  $w_R^*$  are*

$$u_L^* = u_R^* = \hat{u}^{\text{HLL}}, \quad (5.2a)$$

$$h_R^* = h^{\text{HLL}} + \frac{-(z_R - z_L) \pm \sqrt{8(\hat{\eta}^{\text{HLL}} - \hat{\eta}(\hat{w}^{\text{HLL}}))/g + (z_R - z_L)^2}}{2}, \quad (5.2b)$$

$$h_L^* = h^{\text{HLL}} - \frac{-(z_R - z_L) \pm \sqrt{8(\hat{\eta}^{\text{HLL}} - \hat{\eta}(\hat{w}^{\text{HLL}}))/g + (z_R - z_L)^2}}{2}. \quad (5.2c)$$

A Godunov type scheme (2.1) defined by such an approximate Riemann solver

i) is consistent with the Shallow Water equations (1.1),

ii) satisfies a discrete entropy inequality (2.11) for the couple  $(\hat{\eta}, \hat{G})$  defined in (1.3). The numerical entropy flux  $\hat{G}$  of this inequality is given by (2.12).

*Proof.* At first, we have to show that the definitions (5.2) are the solutions of the system (3.6), (5.1). We show below the equalities  $u_L^* = u_R^* = \hat{u}^{\text{HLL}}$ .

As  $u_L^* = u_R^*$ , let denote  $u^* = u_L^* = u_R^*$ . The equation (3.6b) associated to (3.6a) gives

$$\begin{aligned} h^{\text{HLL}} \hat{u}^{\text{HLL}} &= \frac{h_L^* u_L^* + h_R^* u_R^*}{2}, \\ &= \frac{h_L^* + h_R^*}{2} u^*, \\ &= h^{\text{HLL}} u^*. \end{aligned} \quad (5.3)$$

Since  $\hat{w}_L$  and  $\hat{w}_R$  are in the interior of  $\hat{\Omega}$ , the inequality  $h^{\text{HLL}} > 0$  holds. Therefore, multiplying the equation (5.3) by  $1/h^{\text{HLL}}$ , we deduce the result (5.2a).

Then, in order to show the formulation of  $h_R^*$  and  $h_L^*$  given by (5.2b)-(5.2c), we have to study the equation (3.6c). Using the equality  $u_L^* = u_R^*$  and the notation  $[\cdot] = \cdot_R - \cdot_L$ , this equation reads

$$\frac{g}{8} [h^* + z]^2 = \hat{\eta}^{\text{HLL}} - \hat{\eta}(\hat{w}^{\text{HLL}}) + \frac{g}{8} [z]^2. \quad (5.4)$$

Since the above equation is quadratic, it is now necessary to show that its right hand side is positive. To show this sign, we consider at first the case  $[z] = 0$ . In this case, since  $\hat{s}_{LR}^{\text{WBAR}}$  is consistent and since the CFL condition (3.2) holds, Lemma 3.2-ii) gives

$$\left(\hat{\eta}^{\text{HLL}} - \hat{\eta}(\hat{w}^{\text{HLL}}) + \frac{g}{8}[z]^2\right)\Big|_{[z]=0} = \eta^{\text{HLL}} - \eta(w^{\text{HLL}}) \geq 0.$$

In the case,  $[z] \neq 0$ , Lemma 3.2-i) gives the following estimate for all  $\hat{s}_{LR}$ :

$$\begin{aligned} \hat{\eta}^{\text{HLL}} - \hat{\eta}(\hat{w}^{\text{HLL}}) + \frac{g}{8}[z]^2 &= \\ \frac{h_R h_L [u]^2}{4(h_L + h_R)} + \frac{g}{8}[h+z]^2 - \frac{[u]^2 h_R h_L (h_L u_R - h_R u_L)}{4\lambda(h_L + h_R)^2} - \frac{h_L u_L + h_R u_R}{4\lambda} \left(g[z] + \frac{\Delta x \hat{s}_{LR}}{h^{\text{HLL}}}\right) &+ \mathcal{O}\left(\frac{1}{\lambda^2}\right). \end{aligned} \quad (5.5)$$

Therefore, if  $[u] \neq 0$  or  $[h+z] \neq 0$  then there exists  $\lambda > 0$  large enough such that the inequality  $\hat{\eta}^{\text{HLL}} - \hat{\eta}(\hat{w}^{\text{HLL}}) + g[z]^2/8 \geq 0$  is satisfied. However, a particular attention has to be paid for the cases  $[u] = 0$  and  $[h+z] = 0$ . For these particular cases, noting  $u = u_L = u_R$  and using the definition of  $\hat{s}_{LR}^{\text{WBAR}}$  given by (5.1b), we have

$$\begin{aligned} \left(g[z] + \frac{\Delta x \hat{s}_{LR}^{\text{WBAR}}}{h^{\text{HLL}}}\right)\Big|_{\substack{[u]=0, \\ [h+z]=0}} &= -\sqrt{g} \frac{(h_L + h_R)}{(h^{\text{HLL}})^{\frac{5}{2}}} u [z]^2, \\ &= -2^{\frac{5}{2}} \sqrt{g} (h_L + h_R)^{-\frac{3}{2}} u [z]^2 + \mathcal{O}\left(\frac{1}{\lambda}\right). \end{aligned}$$

Considering the above equality in the estimate (5.5), we deduce

$$\begin{aligned} \left(\hat{\eta}^{\text{HLL}} - \hat{\eta}(\hat{w}^{\text{HLL}}) + \frac{g}{8}[z]^2\right)\Big|_{\substack{[u]=0, \\ [h+z]=0}} &= -\frac{(h_L + h_R)u}{4\lambda} \left(g[z] + \frac{\Delta x \hat{s}_{LR}^{\text{WBAR}}}{h^{\text{HLL}}}\Big|_{\substack{[u]=0, \\ [h+z]=0}}\right) + \mathcal{O}\left(\frac{1}{\lambda^2}\right), \\ &= \sqrt{\frac{2g}{h_L + h_R}} \frac{u^2 [z]^2}{\lambda} + \mathcal{O}\left(\frac{1}{\lambda^2}\right). \end{aligned} \quad (5.6)$$

According to the above equation, if  $u \neq 0$ , there exists once again  $\lambda > 0$  large enough such that the inequality  $\hat{\eta}^{\text{HLL}} - \hat{\eta}(\hat{w}^{\text{HLL}}) + g[z]^2/8 \geq 0$  holds for the cases  $[h+z] = 0$  and  $u_L = u_R \neq 0$ . Before to conclude, we have to consider the case  $[h+z] = 0$  and  $u_L = u_R = 0$  which defines a local equilibrium for the lake at rest (2.6). In this last case, a direct computation using  $\hat{w}^{\text{HLL}}$  and  $\hat{s}_{LR}^{\text{WBAR}}$  respectively defined in (3.4c) and in (5.1b) gives

$$\begin{aligned} h^{\text{HLL}}\Big|_{\substack{u_L=u_R=0, \\ [h+z]=0}} &= \frac{h_L + h_R}{2}, \\ \hat{s}_{LR}^{\text{WBAR}}\Big|_{\substack{u_L=u_R=0, \\ [h+z]=0}} &= -g \frac{h_L + h_R}{2} \frac{z_R - z_L}{\Delta x}, \\ (h^{\text{HLL}} \hat{w}^{\text{HLL}})\Big|_{\substack{u_L=u_R=0, \\ [h+z]=0}} &= -\frac{g}{4\lambda}[h^2] - \frac{g}{4\lambda}(h_L + h_R)[z] = -\frac{g}{4\lambda}(h_L + h_R)[h+z] = 0. \end{aligned} \quad (5.7)$$

Considering the three above equations in the quantity  $\hat{\eta}^{\text{HLL}} - \hat{\eta}(\hat{w}^{\text{HLL}}) + g[z]^2/8$ , we obtain

$$\left(\hat{\eta}^{\text{HLL}} - \hat{\eta}(\hat{w}^{\text{HLL}}) + \frac{g}{8}[z]^2\right)\Big|_{\substack{u_L=u_R=0, \\ [h+z]=0}} = \frac{h_R h_L [u]^2}{4(h_L + h_R)} + \frac{g}{8}[h+z]^2 = 0. \quad (5.8)$$

Therefore, there always exists  $\lambda > 0$  large enough such that the following inequality is verified:

$$\hat{\eta}^{\text{HLL}} - \hat{\eta}(\hat{w}^{\text{HLL}}) + \frac{g[z]^2}{8} \geq 0. \quad (5.9)$$

As a consequence, the quadratic equation (5.4) is well-posed and we deduce

$$[h^* + z] = \pm \sqrt{8(\hat{\eta}^{\text{HLL}} - \hat{\eta}(w^{\text{HLL}}))/g + [z]^2}.$$

Associating the above equation to the equation (3.6a), we deduce the quantities  $h_R^*$ ,  $h_L^*$  respectively given by (5.2b)-(5.2c).

Concerning the statements *i*) and *ii*),  $\hat{s}_{LR}^{\text{WBAR}}$  is consistent and a direct computation shows  $\tilde{w}(\cdot, \hat{w}, \hat{w}) = w$ . In addition, as the intermediate states are defined by the system (3.6), Lemma 3.1 ensures that the approximate Riemann solver verifies the consistency integral relation (2.9) and the inequality (2.10). Therefore, the consistency of the Godunov type scheme *i*) and the discrete entropy stability *ii*) are direct consequences of Lemma 2.2 that concludes the proof.  $\square$

According to the above proof, the consistent discretization  $\hat{s}_{LR}^{\text{WBAR}}$  given by (5.1b) leads to the following equalities:

$$\hat{u}^{\text{HLL}} \Big|_{[z]=0} = u^{\text{HLL}}, \quad \left( \hat{\eta}^{\text{HLL}} - \hat{\eta}(\hat{w}^{\text{HLL}}) + \frac{g}{8}[z]^2 \right) \Big|_{[z]=0} = \eta^{\text{HLL}} - \eta(w^{\text{HLL}}).$$

As a consequence, the formulations (5.2) degenerate toward (4.3) when  $z_L = z_R$ . The approximate Riemann solvers of Lemma 5.1 are so the direct extensions of the solvers presented in Section 4 for the particular case  $z = \text{cste}$ .

The explicit formulations of  $w_L^*$  and  $w_R^*$  given by (5.2) entirely define an approximate Riemann solver (3.3). Nevertheless, it is necessary to complete these formulations by limitation techniques that ensure the robustness for the dry-wet transitions in which  $h_L$  or  $h_R$  are closed to zero. A limitation technique is given in the following theorem that also establishes the well-balanced property for the lake at rest.

**Theorem 5.1** (Robust, entropic, well-balanced Godunov type scheme for the lake at rest). *Consider  $\hat{w}_L, \hat{w}_R$  two constant states of  $\hat{\Omega}$  and  $\tilde{w}(\cdot, \hat{w}_L, \hat{w}_R) : \mathbb{R} \rightarrow \mathbb{R}^2$  an approximate Riemann solver of the form (3.3). Assume  $\lambda > 0$  is such that the CFL condition (3.2) holds and such that the system (3.6), (5.1) admit relaxed solutions. Let also consider the quantities  $\hat{w}^{\text{HLL}}, \hat{\eta}^{\text{HLL}}$  defined in (3.4c)-(3.4e) and the quantity  $(\tilde{h}_R^*, \tilde{h}_L^*)$  in  $\mathbb{R}^2$  such that*

$$\tilde{h}_R^* = h^{\text{HLL}} + \frac{-(z_R - z_L) \pm \sqrt{8(\hat{\eta}^{\text{HLL}} - \hat{\eta}(\hat{w}^{\text{HLL}}))/g + (z_R - z_L)^2}}{2}, \quad (5.10a)$$

$$\tilde{h}_L^* = h^{\text{HLL}} - \frac{-(z_R - z_L) \pm \sqrt{8(\hat{\eta}^{\text{HLL}} - \hat{\eta}(\hat{w}^{\text{HLL}}))/g + (z_R - z_L)^2}}{2}. \quad (5.10b)$$

If  $\hat{s}_{LR}$  verifies

$$\Delta x \hat{s}_{LR} = \begin{cases} 0, & \text{if } h_L = 0 \text{ and } h_R = 0, \\ gh_R^2/2, & \text{if } h_R u_R = 0 \text{ and } h_L = 0 \text{ and } h_R + z_R \leq z_L, \\ -gh_L^2/2, & \text{if } h_L u_L = 0 \text{ and } h_R = 0 \text{ and } h_L + z_L \leq z_R, \\ -g(h_L + h_R)(z_R - z_L)/2, & \text{if } h_L = 0 \text{ or } h_R = 0, \\ \Delta x \hat{s}_{LR}^{\text{WBAR}}, & \text{otherwise,} \end{cases} \quad (5.11)$$

with  $\hat{s}_{LR}^{\text{WBAR}}$  defined in (5.1b) and if the solver intermediate states denoted  $w_L^*$  and  $w_R^*$  write

$$\begin{pmatrix} h_L^* \\ u_L^* \\ h_R^* \\ u_R^* \end{pmatrix} = \begin{cases} (0, 0, 0, 0)^T, & \text{if } h_L = 0 \text{ and } h_R = 0, \\ (0, 0, 2h^{\text{HLL}}, \hat{u}^{\text{HLL}})^T, & \text{if } h_L = 0 \text{ and } h_R > 0, \\ (2h^{\text{HLL}}, \hat{u}^{\text{HLL}}, 0, 0)^T, & \text{if } h_L > 0 \text{ and } h_R = 0, \\ \begin{pmatrix} \min \left( \max(\widetilde{h}_L^*, 0), 2h^{\text{HLL}} \right) \\ \hat{u}^{\text{HLL}} \\ \min \left( \max(\widetilde{h}_R^*, 0), 2h^{\text{HLL}} \right) \\ \hat{u}^{\text{HLL}} \end{pmatrix}, & \text{otherwise,} \end{cases} \quad (5.12)$$

then a Godunov type scheme (2.1) defined by such an approximate Riemann solver

- i) is consistent with the Shallow Water equations (1.1),
- ii) preserves the convex set  $\hat{\Omega}$ , i.e.: if  $(\hat{w}_i^n)_{i \in \mathbb{Z}} \subset \hat{\Omega}$  then,  $(\hat{w}_i^{n+1})_{i \in \mathbb{Z}} \subset \hat{\Omega}$ ,
- iii) is robust for the dry-wet transitions,
- iv) is well-balanced for the lake at rest (1.6).

In addition, if  $\widetilde{h}_L^* > 0$  and  $\widetilde{h}_R^* > 0$  then a Godunov type scheme (2.1) associated to such an approximate Riemann solver satisfies a discrete entropy inequality (2.11) for the couple  $(\hat{\eta}, \hat{G})$  defined in (1.3). The entropy numerical flux  $\hat{G}$  of this inequality is given by (2.12).

The well-balanced property and the discrete entropy inequality detailed in the above theorem are also given in [12] with a relaxation scheme that needs to solve a cubic equation. The result of Theorem 5.1 overcomes this constrain and his main originality arises from the explicit solving of a quadratic equation. This explicit solving gives two numerical schemes distinguished with the symbol  $\pm$  in the definitions (5.12). Both schemes are entropic, well-balanced for the lake at rest and preserves the set  $\hat{\Omega}$  thank to the limitation techniques  $\min(\max(\cdot, \cdot), \cdot)$ . However, if these limitations work, the system (3.6) is no longer necessary verified and consequently, the entropy stability might be locally lost in the dry-wet transitions.

These transitions are also treated with the several cases in the equalities (5.11) and (5.12). According to [39, Section 3.1.2.4], theses equalities are robust but it is clear that they are not continuous with the numerical scheme defined in Lemma 5.1.

Next to this remarks, we now prove Theorem 5.1.

*Proof of Theorem 5.1.* Concerning *i*), the consistency has only to be proved in the wet regions. Therefore, if  $\widetilde{h}_L^*$  and  $\widetilde{h}_R^*$  given by (5.10) are both strictly positive then the formulations (5.12) coincide with the formulations (5.2). In this case, the consistency is shown in Lemma 5.1. If  $\widetilde{h}_L^* \leq 0$  or  $\widetilde{h}_R^* \leq 0$ , then the limitations techniques  $\min(\max(\cdot, \cdot), \cdot)$  work but, in this case, the proof of Lemma 4.2 shows that the consistency integral relation is preserved. As a consequence, we deduce the consistency of the numerical scheme.

For the preservation of the convex set  $\hat{\Omega}$  given by *ii*), it is sufficient to show  $h_L^* \geq 0$  and  $h_R^* \geq 0$  but, according to (5.12), these inequalities are ensured.

Concerning *iii*) and according to [39, Section 3.1.2.4], the formulations given in (5.11) for  $\hat{s}_{LR}$  and in (5.12) for the states  $w_L^*$ ,  $w_R^*$  ensure the robustness of the scheme in wet-dry transitions.

For the well-balanced property *iv*), and according to Lemma 2.2, it is sufficient to show that if  $\hat{w}_L$  and  $\hat{w}_R$  define a local lake at rest equilibrium (2.6) then  $w_L^* = w_L$  and  $w_R^* = w_R$ . Using the notation

$[\cdot] = \cdot_R - \cdot_L$ , if  $\hat{w}_L$  and  $\hat{w}_R$  define a local equilibrium (2.6) then  $u_L = u_R = 0$ ,  $[h + z] = 0$  and  $\hat{s}_{LR}$ ,  $w_L^*$ ,  $w_R^*$  are respectively given by (5.1b), (5.2).

As  $u_L^* = u_R^* = \hat{u}^{\text{HLL}}$ , the first aim is to show that  $\hat{u}^{\text{HLL}} = 0$ . If the local lake at rest equilibrium (2.6) holds then Lemma 5.1 gives  $h^{\text{HLL}} \hat{u}^{\text{HLL}} = 0$ . As a consequence, we have

$$\begin{aligned} 0 &= (h^{\text{HLL}} \hat{u}^{\text{HLL}}) \Big|_{\substack{u_L = u_R = 0, \\ [h+z] = 0}}, \\ &= h^{\text{HLL}} \Big|_{\substack{u_L = u_R = 0, \\ [h+z] = 0}}, \hat{u}^{\text{HLL}} \Big|_{\substack{u_L = u_R = 0, \\ [h+z] = 0}}, \\ &= \frac{h_L + h_R}{2} \hat{u}^{\text{HLL}} \Big|_{\substack{u_L = u_R = 0, \\ [h+z] = 0}}. \end{aligned} \tag{5.13}$$

According to the above equation, when the local lake at rest equilibrium (2.6) holds  $\hat{u}^{\text{HLL}} = 0$  that implies  $u_L^* = u_R^* = 0$ .

Now, we prove the equalities  $h_L^* = h_L$  et  $h_R^* = h_R$ . When the local lake at rest equilibrium (2.6) holds, Lemma 5.1 gives in (5.8) the equality  $\hat{\eta}^{\text{HLL}} - \hat{\eta}(\hat{w}^{\text{HLL}}) + g[z]^2/s = 0$ . Associating this equality to the formulations of  $h_R^*$ ,  $h_L^*$  respectively given by (5.2b) and (5.2c), we obtain

$$\begin{aligned} h_R^* \Big|_{\substack{u_L = u_R = 0, \\ [h+z] = 0}} &= \frac{h_L + h_R - [z]}{2} = \frac{h_L + h_R + [h]}{2} = h_R, \\ h_L^* \Big|_{\substack{u_L = u_R = 0, \\ [h+z] = 0}} &= \frac{h_L + h_R + [z]}{2} = \frac{h_L + h_R - [h]}{2} = h_L. \end{aligned}$$

Both above results conclude the proof of the well-balanced property *iv*).

Finally, for the discrete entropy inequality, if  $\hat{h}_L^*$  and  $\hat{h}_R^*$  given by (5.10) are both strictly positive then (5.12) coincide with (5.2). In this case, the discrete entropy inequality is a direct consequence of the approximate Riemann solver definition and this inequality is shown in Lemma 5.1-ii), that concludes the proof.  $\square$

The schemes of Theorem 5.1 are designed from the system (3.6) closed by the equations (5.1). An other closure is now presented in order to define an entropic well-balanced scheme for the moving equilibria.

## 6 A fully well-balanced entropy stable numerical scheme for the general equilibrium

In this section, a closure of the system (3.6) is proposed in order to reach a well-balanced property for the moving equilibria (1.5). Since the under-determined system (3.6) ensures the consistency and the discrete entropy inequality (2.11), this section concerns an entropic fully well-balanced scheme.

All the notations used in Section 4 and 5 are still used and in particular  $[\cdot] = \cdot_R - \cdot_L$ . These notations are completed by

$$\hat{B}(\hat{w}) = \frac{(hu)^2}{2h^2} + g(h + z), \quad [\hat{B}] = \hat{B}(\hat{w}_R) - \hat{B}(\hat{w}_L).$$

Considering the above definitions, the local moving equilibrium (2.5) now reads

$$h_L u_L = h_R u_R, \quad \text{and} \quad \hat{B}(\hat{w}_L) = \hat{B}(\hat{w}_R). \tag{6.1}$$

In addition, for all quantity  $X$ , if the above local equilibrium holds then  $X$  is denoted  $X^{\text{eq}}$  that writes

$$X \Big|_{[q]=0 [\hat{B}]=0} = X^{\text{eq}}.$$

Since the local moving equilibrium (6.1) imposes  $h_L u_L = h_R u_R$ , the notation  $q^{\text{eq}}$  in  $\mathbb{R}$  is now defined as follows:

$$(h_L u_L)^{\text{eq}} = (h_R u_R)^{\text{eq}} = q^{\text{eq}}.$$

Finally,  $\bar{h}$  denotes the arithmetic mean of  $h_L$  and  $h_R$  that writes  $\bar{h} = (h_L + h_R)/2$  in  $\mathbb{R}^+$ .

Associating these notations to a couple  $\lambda > 0$ ,  $\Delta t > 0$  and to  $w^{\text{HLL}}$  given by (3.4a), we now define the reals  $\zeta_{LR}$ ,  $q_{LR}$ ,  $\beta_{LR}$  et  $F_{LR}^2$  which take particular values when  $\hat{w}_L$  and  $\hat{w}_R$  define a local moving equilibrium (6.1). These reals are given by

$$\zeta_{LR} = [hu]^2 + \bar{h}[\hat{B}]^2/g, \quad (6.2a)$$

$$q_{LR} = \begin{cases} 0, & \text{if } \zeta_{LR} = 0 \text{ and } [z] = 0, \\ \frac{\min(h_L u_L, h_R u_R)(g[z]\Delta t)^2}{(g[z]\Delta t)^2 + \zeta_{LR}}, & \text{otherwise,} \end{cases}, \quad (6.2b)$$

$$\beta_{LR} = \begin{cases} 0, & \text{if } \zeta_{LR} = 0 \text{ and } [u] = 0 \text{ et } [h + z] = 0, \\ \frac{(\bar{h}[u])^2 + g\bar{h}[h + z]^2}{(\bar{h}[u])^2 + g\bar{h}[h + z]^2 + \zeta_{LR}}, & \text{otherwise,} \end{cases}, \quad (6.2c)$$

$$F_{LR}^2 = \frac{q_{LR}^2}{gh_L h_R h^{\text{HLL}}}. \quad (6.2d)$$

The lemma below gives the properties satisfied by  $\zeta_{LR}$ ,  $q_{LR}$  and  $F_{LR}^2$ .

**Lemma 6.1.** *Consider  $\hat{w}_L$ ,  $\hat{w}_R$  two constant states in the interior of  $\hat{\Omega}$  and  $\lambda > 0$ ,  $\Delta t > 0$  satisfying the CFL condition (3.2). Let also consider  $w^{\text{HLL}}$  given by (3.4a) and  $\zeta_{LR}$ ,  $q_{LR}$ ,  $F_{LR}^2$  respectively defined in (6.2a), (6.2b) and (6.2d). The following statements are satisfied.*

- i)  $\zeta_{LR} = 0$  if and only if  $\hat{w}_L$ ,  $\hat{w}_R$  verify a local equilibrium (6.1).*
- ii) If  $[z] \neq 0$  then  $q_{LR}^{\text{eq}} = q^{\text{eq}}$ .*
- iii) If  $[z] \neq 0$  then  $(F_{LR}^2)^{\text{eq}} = 2(q^{\text{eq}})^2/(gh_L h_R (h_L + h_R))$ .*

*Proof.* Concerning *i)*, as  $\hat{w}_L$ ,  $\hat{w}_R$  are in the interior of  $\hat{\Omega}$ , the strict inequality  $\bar{h} > 0$  is satisfied. As a consequence,  $\zeta_{LR}$  is an addition of positive quantities and the result is obvious.

For the statement *ii)* which concerns a property for a local equilibrium (6.1), as  $[z] \neq 0$ , the equality  $\zeta_{LR}^{\text{eq}} = 0$  gives

$$q_{LR}^{\text{eq}} = \left( \frac{\min(h_L u_L, h_R u_R)(g[z]\Delta t)^2}{(g[z]\Delta t)^2 + \zeta_{LR}} \right)^{\text{eq}} = (\min(h_L u_L, h_R u_R))^{\text{eq}} = q^{\text{eq}}.$$

Finally, for *iii)*, writing the definition of  $h^{\text{HLL}}$  given by (3.4a), we obtain

$$(h^{\text{HLL}})^{\text{eq}} = \frac{h_L + h_R}{2} - \frac{[hu]^{\text{eq}}}{2\lambda} = \frac{h_L + h_R}{2}.$$

Using the above equation and the statement *ii)* in the definition of  $F_{LR}^2$  given by (6.2d), we deduce the result that achieves the proof.  $\square$

Considering the definitions (6.2), the proposed closure of the system (3.6) is the following:

$$h_R^* h_L^* (u_R^* - u_L^*)^2 = \frac{q_{LR}^2}{h_R h_L} (h_R^* - h_L^*)^2, \quad (6.3a)$$

$$\hat{s}_{LR} = \hat{s}_{LR}^{\text{FWB}} = \hat{s}_{LR}^{\text{WBAR}} + \beta_{LR} \left( \frac{g\bar{h}F_{LR}^2}{4h_L h_R} \frac{(h_R - h_L)^3}{\Delta x} + \sqrt{g} \frac{h_L u_L + h_R u_R}{(h^{\text{HLL}})^{3/2}} \frac{(z_R - z_L)^2}{\Delta x} \right), \quad (6.3b)$$

The solving of the system (3.6), (6.3) is detailed in the above lemma which uses the usual symbols  $\pm$  that mean  $+$  or  $-$ . Two independent symbols  $\pm_1$  and  $\pm_2$  are necessary and the second one is completed by  $\mp_2$  such that if  $\pm_2$  is positive (*resp.* negative) then  $\mp_2$  is negative (*resp.* positive).

**Lemma 6.2.** *Consider  $\hat{w}_L, \hat{w}_R$  two constant states in the interior of  $\hat{\Omega}$  and  $\tilde{w}(\cdot, \hat{w}_L, \hat{w}_R) : \mathbb{R} \rightarrow \mathbb{R}^2$  an approximate Riemann solver (3.3). There exists  $\lambda > 0$  large enough and  $\Delta t > 0$  small enough such that if the CFL condition (3.2) is satisfied and if the solver intermediate states denoted  $w_L^*$  and  $w_R^*$  are defined by the system (3.6), (6.3) then two formulations for  $h_L^*$  and  $h_R^*$  are possible. Considering the quantities  $\hat{w}^{\text{HLL}}$  and  $\hat{\eta}^{\text{HLL}}$  defined in (3.4c)-(3.4e), these formulations are*

$$h_L^* = h^{\text{HLL}} - \frac{[h^*]}{2}, \quad (6.4a)$$

$$h_R^* = h^{\text{HLL}} + \frac{[h^*]}{2}, \quad (6.4b)$$

$$\text{with, } [h^*] = -\frac{(z_R - z_L)}{1 + F_{LR}^2} \pm_1 \sqrt{\frac{8}{g} \frac{\hat{\eta}^{\text{HLL}} - \hat{\eta}(\hat{w}^{\text{HLL}})}{1 + F_{LR}^2} + \frac{(z_R - z_L)^2}{(1 + F_{LR}^2)^2}}. \quad (6.4c)$$

In addition, if the above quantities  $h_R^*$  and  $h_L^*$  are strictly positive then the quantities  $u_L^*$  and  $u_R^*$  respectively write

$$u_L^* = \hat{u}^{\text{HLL}} \mp_2 \frac{q_{LR}[h^*]}{2h^{\text{HLL}}\sqrt{h_L h_R}} \sqrt{\frac{h_R^*}{h_L^*}}, \quad (6.5a)$$

$$u_R^* = \hat{u}^{\text{HLL}} \pm_2 \frac{q_{LR}[h^*]}{2h^{\text{HLL}}\sqrt{h_L h_R}} \sqrt{\frac{h_L^*}{h_R^*}}. \quad (6.5b)$$

Finally, a Godunov type scheme (2.1) defined by such an approximate Riemann solver

i) is consistent with the Shallow Water equations (1.1),

ii) satisfies a discrete entropy inequality (2.11) for the couple  $(\hat{\eta}, \hat{G})$  defined in (1.3). The numerical entropy flux  $\hat{\mathcal{G}}$  of this inequality is given by (2.12).

The symbols  $\pm_1$  and  $\pm_2$  respectively used in (6.4) and in (6.5) may be interpreted as free parameters that will be fixed in the sequel to guarantee the well-balanced property.

The well-balanced property will be also reached with (6.3b) in which the notation  $\hat{s}_{LR}^{\text{FWB}}$  means *Fully Well-Balanced*. This notation is defined in order to distinguish  $\hat{s}_{LR}^{\text{WBAR}}$  given by (5.1b) from a generic discretization  $\hat{s}_{LR}$ . According to (5.1b),  $\hat{s}_{LR}^{\text{FWB}}$  is defined as a correction of the quantity  $\hat{s}_{LR}^{\text{WBAR}}$  given by (5.1b) and this property will be used in the sequel.

Before to prove Lemma 6.2, it is now necessary to give two intermediate results. The first one concerns the quantity  $\hat{s}_{LR}^{\text{FWB}}$  defined in (6.3b).

**Lemma 6.3** (Property of  $\hat{s}_{LR}^{\text{FWB}}$ ). *Consider  $\hat{w}_L, \hat{w}_R$  two constant states in the interior of  $\hat{\Omega}$ . If the CFL condition (3.2) holds and if  $\hat{s}_{LR} = \hat{s}_{LR}^{\text{FWB}}$  in the formulation of  $h^{\text{HLL}}\hat{u}^{\text{HLL}}$  given by (3.4b) then*

$$(h^{\text{HLL}}\hat{u}^{\text{HLL}})^{\text{eq}} = \left( (hu)^{\text{HLL}} + \frac{\Delta x \hat{s}_{LR}^{\text{FWB}}}{2\lambda} \right)^{\text{eq}} = q^{\text{eq}}.$$

*Proof.* The proof is led by three exhaustive cases given by

i)  $[z] = 0$ ,

ii)  $[z] \neq 0$  et  $q^{\text{eq}} = 0$ ,

iii)  $[z] \neq 0$  et  $q^{\text{eq}} \neq 0$ .

In the case *i*), and according to the definition (6.3b), we have  $\hat{s}_{LR}^{\text{FWB}} = 0$ . In addition, and according to the hypothesis (2.7), a local equilibrium defined by  $[z] = 0$  necessarily verifies  $\hat{w}_L = \hat{w}_R$ . As a consequence, using the definition of  $h^{\text{HLL}}\hat{u}^{\text{HLL}}$  given by (3.4b), we have

$$\begin{aligned} (h^{\text{HLL}}\hat{u}^{\text{HLL}})^{\text{eq}}|_{[z]=0} &= ((hu)^{\text{HLL}})^{\text{eq}}, \\ &= \frac{(h_L u_L)^{\text{eq}} + (h_R u_R)^{\text{eq}}}{2} - \frac{[hu^2 + gh^2/2]^{\text{eq}}}{2\lambda}, \\ &= q^{\text{eq}}. \end{aligned}$$

As the above equality gives the expected result, we now consider the second case of the proof.

The case *ii*) is defined by  $[z] \neq 0$  and  $q^{\text{eq}} = 0$ . In this case, a direct computation shows that the local moving equilibrium (6.1) degenerates toward the local lake at rest equilibrium that writes  $u_L = u_R = 0$  and  $[h + z] = 0$ . But, according to the definition of  $\beta_{LR}$  given by (6.2c), if the local lake at rest equilibrium occurs then  $\beta_{LR} = 0$ . Therefore, using the equation (6.3b), we have for the case *ii*)

$$(\hat{s}_{LR}^{\text{FWB}})^{\text{eq}}|_{q^{\text{eq}}=0} = (\hat{s}_{LR}^{\text{WBAR}})^{\text{eq}}|_{q^{\text{eq}}=0} = -g \frac{h_L + h_R z_R - z_L}{2 \Delta x}.$$

As a consequence, arguing the result (5.13) of Theorem 5.1-*iv*), we deduce  $h^{\text{HLL}}\hat{u}^{\text{HLL}} = 0$  and since  $h^{\text{HLL}} > 0$ , this last equality concludes the second case.

Let consider now the case *iii*) defined by  $[z] \neq 0$  and  $q^{\text{eq}} \neq 0$ . Since  $\hat{w}_L$  and  $\hat{w}_R$  define a local moving equilibrium (6.1) with  $q^{\text{eq}} \neq 0$ , usual computations that use (6.2c) and (6.1) show

$$\beta_{LR}^{\text{eq}}|_{q^{\text{eq}} \neq 0} = 1 \quad \text{and} \quad -\frac{(q^{\text{eq}})^2}{2h_L^2 h_R^2} [h^2] + g[h + z] = 0.$$

Using both above result and the equality  $(F_{LR}^2)^{\text{eq}} = 2(q^{\text{eq}})^2 / (gh_L h_R (h_L + h_R))$  given by Lemma 6.1-*iii*), we have

$$\begin{aligned} (\Delta x \hat{s}_{LR}^{\text{FWB}})^{\text{eq}}|_{q^{\text{eq}} \neq 0} &= -g \frac{h_L + h_R}{2} [z] + \frac{g}{8} [h]^3 \frac{h_L + h_R}{h_L h_R} F_{LR}^2|_{q^{\text{eq}} \neq 0}, \\ &= -g \frac{h_L + h_R}{2} [z] + \frac{[h]^2}{4} \frac{[h^2](q^{\text{eq}})^2}{h_L^2 h_R^2 (h_L + h_R)}, \\ &= -g \frac{h_L + h_R}{2} [z] + \frac{g[h]^2}{2} \frac{[h + z]}{h_L + h_R}. \end{aligned}$$

Writing the above equation in the definition of  $h^{\text{HLL}}\hat{u}^{\text{HLL}}$  given by (3.4b), we deduce

$$\begin{aligned} (h^{\text{HLL}}\hat{u}^{\text{HLL}})^{\text{eq}}|_{q^{\text{eq}} \neq 0} &= \left( h^{\text{HLL}} u^{\text{HLL}} + \frac{\Delta x \hat{s}_{LR}^{\text{FWB}}}{2\lambda} \right)^{\text{eq}}|_{q^{\text{eq}} \neq 0}, \\ &= q^{\text{eq}} - \frac{1}{2(h_L + h_R)\lambda} \left( -\frac{(q^{\text{eq}})^2 [h^2]}{h_R h_L} + \frac{g}{2} (h_L + h_R)^2 [h] \right) + \frac{g}{4\lambda} \frac{-(h_L + h_R)^2 [z] + [h]^2 [h + z]}{h_L + h_R}, \\ &= q^{\text{eq}} - \frac{g[h + z]}{4\lambda(h_L + h_R)} (-4h_L h_R + (h_L + h_R)^2 - [h]^2), \\ &= q^{\text{eq}}. \end{aligned}$$

The above equality achieves the proof.  $\square$

The second useful intermediate result for Lemma 6.2 is an inequality that defines a necessary and sufficient condition to the existence of reals  $h_L^*$  and  $h_R^*$  satisfying the system (3.6), (6.3).

**Lemma 6.4.** *Consider  $\hat{w}_L, \hat{w}_R$  two constant states in the interior of  $\hat{\Omega}$ . There exists  $\lambda > 0$  large enough and  $\Delta t > 0$  small enough such that if CFL condition (3.2) holds and if  $\hat{s}_{LR} = \hat{s}_{LR}^{\text{FWB}}$  in the formulation*



of  $h^{\text{HLL}}\hat{u}^{\text{HLL}}$  given by (3.4b) then, considering the quantities  $\hat{w}^{\text{HLL}}$ ,  $\hat{\eta}^{\text{HLL}}$  and  $F_{LR}^2$  respectively defined in (3.4c), (3.4e),(6.2d), the following inequality is satisfied:

$$\hat{\eta}^{\text{HLL}} - \hat{\eta}(\hat{w}^{\text{HLL}}) + \frac{g}{8}[z]^2 - \frac{g[z]^2}{8} \frac{F_{LR}^2}{1 + F_{LR}^2} \geq 0. \quad (6.6)$$

*Proof.* The proof is led by three exhaustive cases given by

- i)  $[z] = 0$ ,
- ii)  $[z] \neq 0$  et  $\zeta_{LR} = 0$ ,
- iii)  $[z] \neq 0$  et  $\zeta_{LR} \neq 0$ .

Let consider at first the case i) defined by  $[z] = 0$ . Using on the one hand the definition (6.2b), we have  $q_{LR} = 0$  and using on the other hand the equation (6.3b), we obtain  $\hat{s}_{LR}^{\text{FWB}} = 0$ . As a consequence, in the case  $[z] = 0$ , we deduce  $F_{LR}^2 = 0$  and  $\hat{u}^{\text{HLL}} = u^{\text{HLL}}$ . Using both last equalities, an analogous computation to (3.18) gives

$$\begin{aligned} & \left( \hat{\eta}^{\text{HLL}} - \hat{\eta}(\hat{w}^{\text{HLL}}) + \frac{g}{8}[z]^2 - \frac{g[z]^2}{8} \frac{F_{LR}^2}{1 + F_{LR}^2} \right) \Big|_{[z]=0} \\ &= \eta^{\text{HLL}} - \frac{(h^{\text{HLL}}\hat{u}^{\text{HLL}})^2}{2h^{\text{HLL}}} \Big|_{[z]=0} - \frac{g}{2}(h^{\text{HLL}})^2 + \left( \frac{g}{4}[z] \left( [h] - \frac{h_L u_L + h_R u_R}{\lambda} \right) \right) \Big|_{[z]=0} \\ &= \eta^{\text{HLL}} - \eta(w^{\text{HLL}}). \end{aligned}$$

Since the CFL condition (3.2) is satisfied, the inequality  $\eta^{\text{HLL}} - \eta(w^{\text{HLL}}) \geq 0$  is ensured (see Lemma 3.2-ii) for details).

The case ii) is defined by  $[z] \neq 0$  and  $\zeta_{LR} = 0$ . As  $\zeta_{LR} = 0$ ,  $\hat{w}_L$  and  $\hat{w}_R$  define a local moving equilibrium (6.1) and in this case, a direct computation shows  $\hat{G}(\hat{w}_R) - \hat{G}(\hat{w}_L) = [\hat{B}]q^{\text{eq}} = 0$ . According to Lemma 6.3, the equality  $(h^{\text{HLL}}\hat{u}^{\text{HLL}})^{\text{eq}} = q^{\text{eq}}$  is also verified. As a consequence, we have

$$\begin{aligned} (\hat{\eta}^{\text{HLL}})^{\text{eq}} - \hat{\eta}(\hat{w}^{\text{HLL}})^{\text{eq}} &= \frac{\eta(w_R)^{\text{eq}} + \eta(w_L)^{\text{eq}}}{2} + \frac{g}{2}(h_L z_L + h_R z_R) - \frac{\hat{G}(\hat{w}_R)^{\text{eq}} - \hat{G}(\hat{w}_L)^{\text{eq}}}{2\lambda} \\ &\quad - \left( \frac{(h^{\text{HLL}}\hat{u}^{\text{HLL}})^2}{2h^{\text{HLL}}} \right)^{\text{eq}} - \left( \frac{g(h^{\text{HLL}})^2}{2} \right)^{\text{eq}} - g(h^{\text{HLL}})^{\text{eq}} \frac{z_L + z_R}{2}, \\ &= \frac{(q^{\text{eq}})^2}{4} \left( \frac{1}{h_L} + \frac{1}{h_R} \right) + \frac{g}{4}(h_L^2 + h_R^2) + \frac{g}{2}(h_L z_L + h_R z_R) \\ &\quad - \frac{(q^{\text{eq}})^2}{h_L + h_R} - \frac{g(h_L + h_R)^2}{8} - \frac{g}{4}(h_L + h_R)(z_L + z_R). \\ &= \frac{(q^{\text{eq}})^2 h_L h_R}{4(h_L + h_R)} \left[ \frac{1}{h} \right]^2 + \frac{g}{8}[h]^2 + \frac{g}{4}[z][h], \\ &= \frac{g}{8}(1 + (F_{LR}^2)^{\text{eq}})[h]^2 + \frac{g}{4}[z][h]. \end{aligned}$$

Using the above equation, we obtain

$$\begin{aligned} \left( \hat{\eta}^{\text{HLL}} - \hat{\eta}(\hat{w}^{\text{HLL}}) + \frac{g}{8}[z]^2 - \frac{g[z]^2}{8} \frac{F_{LR}^2}{1 + F_{LR}^2} \right)^{\text{eq}} &= \frac{g}{8}(1 + (F_{LR}^2)^{\text{eq}})[h]^2 + \frac{g}{4}[h][z] + \frac{g}{8} \frac{[z]^2}{1 + (F_{LR}^2)^{\text{eq}}}, \\ &= \frac{g}{8} \left( \sqrt{1 + (F_{LR}^2)^{\text{eq}}}[h] + \frac{[z]}{\sqrt{1 + (F_{LR}^2)^{\text{eq}}}} \right)^2 \geq 0. \end{aligned}$$

Finally, the last case *iii*) is such that  $[z] \neq 0$  and for couples  $\hat{w}_L$  and  $\hat{w}_R$  that not define a local moving equilibrium (6.1). This last case writes  $[z] \neq 0$  and  $\zeta_{LR} \neq 0$  and Lemma 3.2-*i*) gives the following estimate:

$$\hat{\eta}^{\text{HLL}} - \hat{\eta}(\hat{w}^{\text{HLL}}) + \frac{g}{8}[z]^2 = \frac{h_R h_L [u]^2}{4(h_L + h_R)} + \frac{g}{8}[h+z]^2 - \frac{[u]^2 h_R h_L (h_L u_R - h_R u_L)}{4\lambda(h_L + h_R)^2} - \frac{h_L u_L + h_R u_R}{4\lambda} \left( g[z] + \frac{\Delta x \hat{s}_{LR}}{h^{\text{HLL}}} \right) + \mathcal{O}\left(\frac{1}{\lambda^2}\right). \quad (6.7)$$

As soon as  $[u] \neq 0$  or  $[h+z] \neq 0$ , there exists  $\lambda > 0$  large enough such that the following strict inequality is satisfied

$$\hat{\eta}^{\text{HLL}} - \hat{\eta}(\hat{w}^{\text{HLL}}) + \frac{g[z]^2}{8} > 0. \quad (6.8)$$

For the particular cases  $[u] = 0$  and  $[h+z] = 0$ , the following equations are verified:

$$\beta_{LR}|_{\substack{[u]=0, \\ [h+z]=0}} = 0, \quad \hat{s}_{LR}^{\text{FWB}}|_{\beta_{LR}=0} = \hat{s}_{LR}^{\text{WBAR}}.$$

Therefore, when  $[u] = 0$  and  $[h+z] = 0$ , the definition of  $\beta_{LR}$  given by (6.2c) commute  $\hat{s}_{LR}^{\text{FWB}}$  to  $\hat{s}_{LR}^{\text{WBAR}}$ . As a consequence, the existence of the strict inequality (6.8) for these particular cases is related to the proof of Lemma 5.1 that concerns the estimate (6.7) with  $\hat{s}_{LR} = \hat{s}_{LR}^{\text{WBAR}}$ . But, according to the computations (5.6) and (5.7) detailed in this lemma, the strict inequality (6.8) can be held except for the case where  $\hat{w}_L$  and  $\hat{w}_R$  define a local lake at rest equilibrium (2.6). Since, the lake at rest is a moving equilibrium with a null velocity, it has already been treated in the case *ii*). As a consequence, the strict inequality (6.8) can always be ensured for this case *iii*). In addition, according to the definitions (6.2b), (6.2d) and as  $\zeta_{LR} \neq 0$ ,  $F_{LR}^2$  is proportional to  $\Delta t^4$ . Therefore, if the strict inequality (6.8) is verified, the large inequality (6.6) can be held with  $\Delta t > 0$  such that

$$\hat{\eta}^{\text{HLL}} - \hat{\eta}(\hat{w}^{\text{HLL}}) + \frac{g}{8}[z]^2 \geq \frac{g[z]^2}{8} \frac{F_{LR}^2}{1 + F_{LR}^2}.$$

We deduce the expected result for this last case that concludes the proof.  $\square$

With both above lemma, Lemma 6.2 is now proved.

*Proof of Lemma 6.2.* At first, we show that the formulations (6.4)-(6.5) are solutions of the system (3.6), (6.3). Using the equation (6.3a) in the equation (3.6c) and according to the definition of  $F_{LR}^2$  given by (6.2d), we deduce the two following equations satisfied by  $h_L^*$  and  $h_R^*$ :

$$\frac{h_L^* + h_R^*}{2} = h^{\text{HLL}}, \quad (6.9a)$$

$$\frac{g}{8}(1 + F_{LR}^2)[h^*]^2 + \frac{g}{4}[z][h^*] = \hat{\eta}^{\text{HLL}} - \hat{\eta}(\hat{w}^{\text{HLL}}). \quad (6.9b)$$

Now, we have to show that the above quadratic equation is well-posed. Dividing this quadratic equation (6.9b) on both sides by  $g/8(1+F_{LR}^2)$  then adding  $[z]^2/(1+F_{LR}^2)^2$ , we obtain

$$\begin{aligned} \left( [h^*] + \frac{[z]}{1 + F_{LR}^2} \right)^2 &= \frac{8 \hat{\eta}^{\text{HLL}} - \hat{\eta}(\hat{w}^{\text{HLL}})}{g} \frac{1}{1 + F_{LR}^2} + \frac{[z]^2}{(1 + F_{LR}^2)^2}, \\ &= \frac{8}{g(1 + F_{LR}^2)} \left( \hat{\eta}^{\text{HLL}} - \hat{\eta}(\hat{w}^{\text{HLL}}) + \frac{g}{8}[z]^2 - \frac{g[z]^2}{8} \frac{F_{LR}^2}{1 + F_{LR}^2} \right). \end{aligned}$$

According to Lemma 6.4, if  $\lambda$  (*resp.*  $\Delta t$ ) is large (*resp.* small) enough then the right hand side of the above equation is positive. As a consequence, if  $\lambda$  and  $\Delta t$  are well-chosen, the previous equation is well-posed and a direct computation leads to the formulation of  $[h^*]$  given by (6.4c). Coupling the

formulation of  $[h^*] = h_R^* - h_L^*$  to the equation (6.9a), we deduce the quantities  $h_L^*$  and  $h_R^*$  respectively given by (6.4a),(6.4b).

Since the quantities  $h_R^*$  et  $h_L^*$  are known and since they are assumed to be strictly positive, the equation (6.3a) re-writes

$$[u^*] = \pm \frac{q_{LR}}{\sqrt{h_L h_R}} \frac{[h^*]}{\sqrt{h_L^* h_R^*}}. \quad (6.10)$$

Using the above equation and the equation (6.9a) in the relation  $(h_L^* u_L^* + h_R^* u_R^*)/2 = h^{\text{HLL}} \hat{u}^{\text{HLL}}$ , we obtain

$$\begin{aligned} h^{\text{HLL}} \hat{u}^{\text{HLL}} &= \frac{h_L^* u_L^* + h_R^* u_R^*}{2}, \\ &= \frac{h_L^* + h_R^*}{2} \frac{u_L^* + u_R^*}{2} + \frac{[h^*][u^*]}{4}, \\ &= h^{\text{HLL}} \frac{u_L^* + u_R^*}{2} \pm \frac{q_{LR}}{4\sqrt{h_L h_R}} \frac{[h^*]^2}{\sqrt{h_L^* h_R^*}}. \end{aligned} \quad (6.11)$$

The above equation associated to the equalities (6.10) and (6.9a) yields

$$\begin{aligned} u_R^* &= \hat{u}^{\text{HLL}} \pm \frac{q_{LR}[h^*]}{2\sqrt{h_L h_R h_L^* h_R^*}} \mp \frac{q_{LR}[h^*]^2}{4h^{\text{HLL}}\sqrt{h_L h_R}\sqrt{h_R^* h_L^*}}, \\ &= \hat{u}^{\text{HLL}} \pm \frac{q_{LR}[h^*]}{2\sqrt{h_L h_R h_L^* h_R^*}} \left(1 - \frac{[h^*]}{2h^{\text{HLL}}}\right), \\ &= \hat{u}^{\text{HLL}} \pm \frac{q_{LR}[h^*]}{2\sqrt{h_L h_R h_L^* h_R^*}} \frac{h_L^*}{h^{\text{HLL}}}. \end{aligned}$$

This last equality gives the formulation of  $u_R^*$  presented in (6.5b). Since the formulation of  $u_L^*$  (6.5a) can be derived from an analogous computation, this achieves to show (6.4)-(6.5).

Concerning the statements *i*) and *ii*),  $\hat{s}_{LR}^{\text{FWB}}$  is consistent and a direct computation shows  $\tilde{w}(\cdot, \hat{w}, \hat{w}) = w$ . In addition, as the intermediate states are defined by the system (3.6), Lemma 3.1 ensures that the approximate Riemann solver verifies the consistency integral relation (2.9) and the inequality (2.10). Therefore, the consistency of the Godunov type scheme *i*) and the discrete entropy stability *ii*) are direct consequences of Lemma 2.2 that concludes the proof.  $\square$

The equation (6.3a) decomposes the equations that define  $w_L^*$  and  $w_R^*$  into two system respectively given by (6.9) and by (6.3a), (6.11). Each of this system is made of a linear equation and a quadratic equation.

If the implicit condition (6.6) that restricts the couple  $(\lambda, \Delta t)$  is satisfied then the first quadratic equation (6.9b) is well-posed. The well-posed property of the second quadratic equation (6.3a) is obtained from the assumption that the quantities  $h_L^*$  and  $h_R^*$  given by (6.4) are strictly positive. If this assumption is not satisfied then the inequalities  $h_L^* > 0$  and  $h_R^* > 0$  can be imposed with a limitation technique. As a consequence, the system (3.6), (6.3) always admit solutions.

According to the formulations (6.4)-(6.5), these solutions are not unique and they depend on the choices of the symbols  $\pm_1$  and  $\pm_2$ . The selection of one solution has to be done according to the well-balanced property.

In order to satisfy this property,  $\pm_2$  has to be negative (that imposes  $\mp_2$  positive) and a direct computing shows that the choice of  $\pm_1$  has to be done according a condition obtained when  $\hat{w}_L$  and  $\hat{w}_R$  define a local equilibrium (6.1). This condition formally writes

$$\pm_1^{\text{eq}} = \text{sign}((1 + (F_{LR}^2)^{\text{eq}})[h] + [z]).$$

As a consequence, a simple formulation for the symbol  $\pm_1$  is given by

$$\pm_1 = \text{sign}((1 + (F_{LR}^2))[h] + [z]).$$

The above formulation is considered in the equality (6.13) but others choices are possible. This formulation leads to an expression of  $(h_\alpha^*, u_\alpha^*)_{\alpha \in \{L, R\}}$  that is detailed in the below theorem in which a limitation procedure is also given.

**Theorem 6.1** (Robust, entropic, well-balanced Godunov type scheme for all regular equilibrium). *Consider  $\hat{w}_L, \hat{w}_R$  two constant states of  $\hat{\Omega}$  and  $\tilde{w}(\cdot, \hat{w}_L, \hat{w}_R) : \mathbb{R} \rightarrow \mathbb{R}^2$  an approximate Riemann solver of the form (3.3). Assume  $\lambda > 0$  and  $\Delta t > 0$  are such that the CFL condition (3.2) holds and such that the system (3.6), (6.3) admits real solutions. Let also consider the quantities  $\hat{w}^{\text{HLL}}, \hat{\eta}^{\text{HLL}}, q_{LR}$  and  $F_{LR}^2$  respectively defined in (3.4c), (3.4e) (6.2b), (6.2d) and the quantities  $(\tilde{h}_L^*, \tilde{u}_L^*, \tilde{h}_R^*, \tilde{u}_R^*)$  in  $\mathbb{R}^4$  such that*

$$\tilde{h}_L^* = h^{\text{HLL}} - \frac{[h^*]}{2}, \quad (6.12a)$$

$$\tilde{u}_L^* = \hat{u}^{\text{HLL}} + \frac{q_{LR}[h^*]}{2h^{\text{HLL}}\sqrt{h_L h_R}} \sqrt{\frac{h_R^*}{h_L^*}}, \quad (6.12b)$$

$$\tilde{h}_R^* = h^{\text{HLL}} + \frac{[h^*]}{2}, \quad (6.12c)$$

$$\tilde{u}_R^* = \hat{u}^{\text{HLL}} - \frac{q_{LR}[h^*]}{2h^{\text{HLL}}\sqrt{h_L h_R}} \sqrt{\frac{h_L^*}{h_R^*}}, \quad (6.12d)$$

with

$$[h^*] = -\frac{(z_R - z_L)}{1 + F_{LR}^2} + \text{sign}((1 + F_{LR}^2)[h] + [z]) \sqrt{\frac{8\hat{\eta}^{\text{HLL}} - \hat{\eta}(\hat{w}^{\text{HLL}})}{g} + \frac{(z_R - z_L)^2}{(1 + F_{LR}^2)^2}}. \quad (6.13)$$

Let consider in addition  $\hat{s}_{LR}$  under the form

$$\Delta x \hat{s}_{LR} = \begin{cases} 0, & \text{if } h_L = 0 \text{ and } h_R = 0, \\ gh_R^2/2, & \text{if } h_R u_R = 0 \text{ and } h_L = 0 \text{ and } h_R + z_R \leq z_L, \\ -gh_L^2/2, & \text{if } h_L u_L = 0 \text{ and } h_R = 0 \text{ and } h_L + z_L \leq z_R, \\ -g(h_L + h_R)(z_R - z_L)/2, & \text{if } h_L = 0 \text{ or } h_R = 0, \\ \Delta x \hat{s}_{LR}^{\text{FWB}}, & \text{otherwise,} \end{cases} \quad (6.14)$$

with  $\hat{s}_{LR}^{\text{FWB}}$  defined by (6.3b). For  $\varepsilon > 0$ , assume the solver intermediate states denoted  $w_L^*$  and  $w_R^*$  write

$$\begin{pmatrix} h_L^* \\ u_L^* \\ h_R^* \\ u_R^* \end{pmatrix} = \begin{cases} (0, 0, 0, 0)^T, & \text{if } h_L = 0 \text{ and } h_R = 0, \\ (0, 0, 2h^{\text{HLL}}, \hat{u}^{\text{HLL}})^T, & \text{if } h_L = 0 \text{ and } h_R > 0, \\ (2h^{\text{HLL}}, \hat{u}^{\text{HLL}}, 0, 0)^T, & \text{if } h_L > 0 \text{ and } h_R = 0, \\ \begin{pmatrix} \min\left(\max(\tilde{h}_L^*, \varepsilon), 2h^{\text{HLL}} - \varepsilon\right) \\ \tilde{u}_L^* \\ \min\left(\max(\tilde{h}_R^*, \varepsilon), 2h^{\text{HLL}} - \varepsilon\right) \\ \tilde{u}_R^* \end{pmatrix}, & \text{otherwise.} \end{cases} \quad (6.15)$$

A Godunov type scheme (2.1) defined by such an approximate Riemann solver

- i) is consistent with the Shallow Water equations (1.1),
- ii) preserves the convex set  $\hat{\Omega}$ , i.e: if  $(\hat{w}_i^n)_{i \in \mathbb{Z}} \subset \hat{\Omega}$  then,  $(\hat{w}_i^{n+1})_{i \in \mathbb{Z}} \subset \hat{\Omega}$ ,
- iii) is robust for the dry-wet transitions,

*iv)* is well-balanced for all moving regular equilibrium (1.5).

In addition, if  $\widetilde{h}_L^* > 0$  and  $\widetilde{h}_R^* > 0$  then a Godunov type scheme (2.1) associated to such an approximate Riemann solver satisfies a discrete entropy inequality (2.11) for the couple  $(\hat{\eta}, \hat{G})$  defined in (1.3). The entropy numerical flux  $\hat{G}$  of this inequality is given by (2.12).

*Proof.* Concerning *i)*, the consistency has only to be proved in the wet regions. Therefore, if  $\widetilde{h}_L^*$  and  $\widetilde{h}_R^*$  given by (6.12a), (6.12c) are both strictly positive then (6.15) coincide with (6.12). In this case, the consistency is shown in Lemma 6.2-*i)*. If  $\widetilde{h}_L^* \leq 0$  or  $\widetilde{h}_R^* \leq 0$ , then the limitation techniques  $\min(\max(\cdot, \cdot), \cdot)$  work and an analogous computation to one done in Lemma 4.2-*i)* shows that these procedures preserve the consistency integral relation for all  $\varepsilon > 0$  small enough. As a consequence, we deduce the consistency of the statement *i)*.

For the statement *ii)* related to the preservation of the convex set  $\hat{\Omega}$ , it is sufficient to prove the inequalities  $h_L^* \geq 0$  and  $h_R^* \geq 0$  but according to (6.15), these inequalities are obviously ensured for  $\varepsilon > 0$  small enough.

Concerning *iii)* and according to [39, Section 3.1.2.4], the formulations given in (6.14) for  $\hat{s}_{LR}$  and in (6.15) for the states  $w_L^*$ ,  $w_R^*$  ensure the robustness of the scheme in wet-dry transitions.

For the well-balanced property *iv)*, according to Lemma 2.2, it is sufficient to show that if  $\hat{w}_L$  and  $\hat{w}_R$  define a local moving equilibrium (6.1) then  $w_L^* = w_L$  and  $w_R^* = w_R$ . In the case of a such local equilibrium,  $\hat{s}_{LR}$ ,  $w_L^*$  and  $w_R^*$  are given by (6.3b), (6.12) and the proof of Lemma 6.3 yields

$$(h^{\text{HLL}})^{\text{eq}}(\hat{u}^{\text{HLL}})^{\text{eq}} = \frac{h_L + h_R}{2}(\hat{u}^{\text{HLL}})^{\text{eq}} = q^{\text{eq}}.$$

Denoting  $[\cdot] = \cdot_R - \cdot_L$  and according to the proof of Lemma 6.4-*ii)*, the above equation infers

$$\begin{aligned} \frac{g}{8} \frac{1}{1 + (F_{LR}^2)^{\text{eq}}} \left(1 + (F_{LR}^2)^{\text{eq}}[h] + [z]\right)^2 &= \left(\hat{\eta}^{\text{HLL}} - \hat{\eta}(\hat{w}^{\text{HLL}}) + \frac{g}{8}[z]^2 - \frac{g[z]^2}{8} \frac{F_{LR}^2}{1 + F_{LR}^2}\right)^{\text{eq}}, \\ &= \frac{(\hat{\eta}^{\text{HLL}})^{\text{eq}} - \hat{\eta}(\hat{w}^{\text{HLL}})^{\text{eq}}}{1 + (F_{LR}^2)^{\text{eq}}} + \frac{g}{8} \frac{[z]}{1 + (F_{LR}^2)^{\text{eq}}}. \end{aligned}$$

Using this last formulation in (6.13), we deduce that  $[h^*]^{\text{eq}}$  verifies

$$\begin{aligned} [h^*]^{\text{eq}} &= \\ &= -\frac{[z]}{1 + (F_{LR}^2)^{\text{eq}}} + \left(\text{sign}((1 + (F_{LR}^2)^{\text{eq}})[h] + [z])\right)^{\text{eq}} \sqrt{\frac{8(\hat{\eta}^{\text{HLL}})^{\text{eq}} - \hat{\eta}(\hat{w}^{\text{HLL}})^{\text{eq}}}{g} + \frac{[z]^2}{(1 + (F_{LR}^2)^{\text{eq}})^2}}, \\ &= -\frac{[z]}{1 + (F_{LR}^2)^{\text{eq}}} + \text{sign}((1 + (F_{LR}^2)^{\text{eq}})[h] + [z]) \frac{|((1 + (F_{LR}^2)^{\text{eq}})[h] + [z])|}{|1 + (F_{LR}^2)^{\text{eq}}|}, \\ &= \frac{-[z] + (1 + (F_{LR}^2)^{\text{eq}})[h] + [z]}{1 + (F_{LR}^2)^{\text{eq}}}, \\ &= [h]. \end{aligned} \tag{6.16}$$

Associating the above equality to the definitions (6.12a), (6.12c), we eventually have  $(h_\alpha^*)^{\text{eq}} = h_\alpha$  for all  $\alpha \in \{L, R\}$ .

Now, we have to show the equalities

$$q^{\text{eq}} = (h_R^*)^{\text{eq}}(u_R^*)^{\text{eq}} = (h_L^*)^{\text{eq}}(u_L^*)^{\text{eq}}.$$

Since the result  $(h_\alpha^*)^{\text{eq}} = h_\alpha$  has been shown, the above equalities are equivalent to  $(u_\alpha^*)^{\text{eq}} = q^{\text{eq}}/h_\alpha$  for all  $\alpha \in \{L, R\}$ . According to Lemma 6.1,  $q_{LR}^{\text{eq}} = q^{\text{eq}}$  but using this result in the equality  $(h_\alpha^*)^{\text{eq}} = h_\alpha$ ,

we have

$$\begin{aligned}
(u_L^*)^{\text{eq}} &= (\hat{u}^{\text{HLL}})^{\text{eq}} + \frac{q_{LR}^{\text{eq}}[h^*]^{\text{eq}}}{2(h^{\text{HLL}})^{\text{eq}}\sqrt{h_L h_R}} \sqrt{\frac{(h_R^*)^{\text{eq}}}{(h_L^*)^{\text{eq}}}}, \\
&= \frac{2q^{\text{eq}}}{h_L + h_R} + \frac{q^{\text{eq}}[h]}{(h_L + h_R)h_L}, \\
&= q^{\text{eq}} \frac{2h_L + [h]}{(h_L + h_R)h_L}, \\
&= \frac{q^{\text{eq}}}{h_L} = u_L.
\end{aligned}$$

As a similar computation gives  $(u_R^*)^{\text{eq}} = u_R$ , we deduce the well-balanced property.

Concerning the discrete entropy inequality, if  $\widetilde{h}_L^*$  and  $\widetilde{h}_R^*$  respectively given by (6.12a), (6.12c) are both strictly positive then the formulations (6.15) coincide with (6.12). In this case, the discrete entropy inequality is a direct consequence of the Riemann solver definition and this inequality is shown in Lemma 6.2 that concludes the proof.  $\square$

The dry-wet transitions in the above theorem are done with several cases in the formulations (6.14) and (6.15). These case distinctions are robust (see [39, Section 3.1.2.4]) but their formulations are not continue with the numerical scheme defined in Lemma 6.2.

The limitation technique  $\min(\max(\cdot, \cdot), \cdot)$  used in (6.15) is an  $\varepsilon$ -parametrized version of the procedure defined in Sections 4 and 5. The parameter  $\varepsilon > 0$  guarantees the strict inequalities  $h_L^* > 0$  and  $h_R^* > 0$  that are essential for the system (3.6), (6.3) (see Lemma 6.2).

However, the limit case  $\varepsilon = 0$  is usually considered in the literature [40, 41]. This limit case can be simply treated in Theorem's 6.1 scheme because the approximate Riemann solver (3.3) only needs the programming of  $h_L^*$ ,  $h_R^*$  and  $h_L^* u_L^*$ ,  $h_R^* u_R^*$ . But, if  $h_L^* > 0$  and  $h_R^* > 0$  in the definitions (6.15), then

$$h_L^* u_L^* = h_L^* \hat{u}^{\text{HLL}} + \frac{q_{LR}[h^*]}{2h^{\text{HLL}}} \sqrt{\frac{h_L^* h_R^*}{h_L h_R}}, \quad h_R^* u_R^* = h_R^* \hat{u}^{\text{HLL}} - \frac{q_{LR}[h^*]}{2h^{\text{HLL}}} \sqrt{\frac{h_L^* h_R^*}{h_L h_R}}. \quad (6.17)$$

Since both above formulations are continue when  $h_L^* \rightarrow 0$  or  $h_R^* \rightarrow 0$ , the programming of Theorem's 6.1 scheme can be done with the quantities  $h_L^*$  and  $h_R^*$  considering the formal limit  $\varphi = 0$  in (6.15) then with  $h_L^* u_L^*$  and  $h_R^* u_R^*$  given by (6.17). Finally, if the limitation techniques work, the system (3.6) is not necessary verified and consequently, the discrete entropy stability might be locally lost in this case.

Theorem 6.1 shows the existence of an entropic well-balanced numerical scheme for the the moving equilibrium. This scheme generalizes the entropic well-balanced scheme for the lake at rest established in Theorem 5.1 because a direct computation infers if  $q_{LR} = 0$  then the formulations (6.15) degenerate toward (5.12). As a consequence, Theorem 6.1 unifies the numerical schemes introduced in Theorem 5.1 (well-balanced for the lake at rest) and in Lemma 4.2 ( $z = \text{cste}$ ).

The next section is devoted to the numerical experiments.

## 7 Numerical results

For all test cases, we fix  $g = 9.81$ . The subsection 7.1 concerns the flat regions and we focus on the schemes given by Lemma 4.2. In the subsection 7.2, we illustrate the entropy stable schemes 5.1 that are only well balanced for the lake at rest (1.8). Finally, the numerical tests of the subsection 7.3 deal with the fully well-balanced entropy stable scheme described in Theorem 6.1.

### 7.1 Two states entropy stable approximate Riemann solvers in the flat regions

In this section, we consider the two schemes described in Lemma 4.1 and their limited versions given by Lemma 4.2. For all interfaces having the states  $w_L$ ,  $w_R$  on its either sides, the numerical artificial

viscosity  $\lambda > 0$  and the time step  $\Delta t > 0$  are taken equals to

$$\lambda = \max_{\alpha \in \{L, R\}} |u_\alpha \pm \sqrt{gh_\alpha}|, \quad \frac{\lambda \Delta t}{\Delta x} \leq \frac{1}{2}, \quad \forall (L, R). \quad (7.1)$$

The results are compared to the Suliciu relaxation scheme and to the solver proposed in [28, Section b.ii]. The two constants  $C_1$  and  $C_2$  required for the scheme [28, Section b.ii] are fixed to  $C_1 = C_2 = 10^{-7}$ .

At first, we illustrate the influence of the choice of the intermediate states  $h^*$  given by Lemma 4.1. In this regard, we set

$$\begin{aligned} \text{EC}_1 : \begin{cases} h_R^* &= h^{\text{HLL}} + \sqrt{\frac{2}{g}(\eta^{\text{HLL}} - \eta(w^{\text{HLL}}))}, \\ h_L^* &= h^{\text{HLL}} - \sqrt{\frac{2}{g}(\eta^{\text{HLL}} - \eta(w^{\text{HLL}}))}. \end{cases} \\ \text{EC}_2 : \begin{cases} h_R^* &= h^{\text{HLL}} - \sqrt{\frac{2}{g}(\eta^{\text{HLL}} - \eta(w^{\text{HLL}}))}, \\ h_L^* &= h^{\text{HLL}} + \sqrt{\frac{2}{g}(\eta^{\text{HLL}} - \eta(w^{\text{HLL}}))}. \end{cases} \\ \text{EC}_3 : \text{random}(\text{EC}_1, \text{EC}_2), \end{aligned} \quad (7.2)$$

with  $w^{\text{HLL}}, \eta^{\text{HLL}}$  given by (3.4a)-(3.4d) and where  $\text{random}(\text{EC}_1, \text{EC}_2)$  denotes a random choice between the two configurations  $\text{EC}_1, \text{EC}_2$ . The domain  $[-1, 1]$  is discretized with 400 cells and we consider the following initial condition:

$$h_0(x) = \begin{cases} 3 & \text{if } x < 0.5, \\ 1 & \text{otherwise,} \end{cases} \quad u_0(x) = 0. \quad (7.3)$$

We impose homogeneous Neumann boundary conditions on both sides. The exact solution consists in a rarefaction wave and a shock wave. Figure 1 shows the compared results.

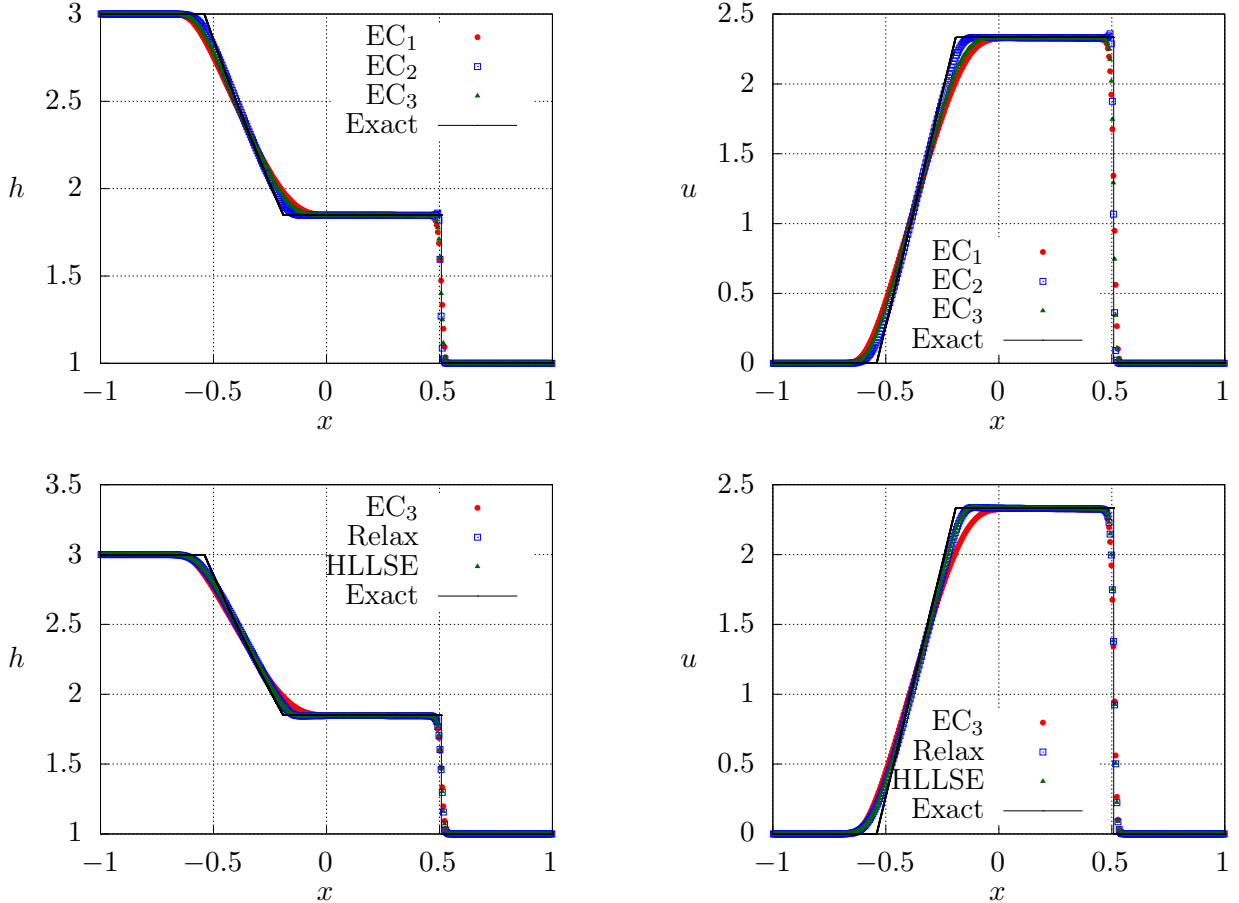


Figure 1: Numerical results at time 0.1 for the initial condition (7.3), with the legend  $(EC_i)_{i \in \{1,2,3\}}$ : Lemma’s 4.1 solvers with the configurations (7.2), Relax: Suliciu relaxation scheme and HLLSE: solver [28, Section b.ii].

The three configurations  $(EC_i)_{i \in \{1,2,3\}}$  provide very close results. As a consequence, we only keep the  $EC_3$  configuration. The second numerical test concerns the role of the limitation techniques for the dry areas described in Lemma 4.2. The domain  $[-1, 1]$  is discretized with 400 cells and we consider the following initial condition:

$$h_0(x) = 0.1, \quad u_0(x) = \begin{cases} 10 & \text{if } x < 0.5, \\ 0 & \text{otherwise.} \end{cases} \quad (7.4)$$

We lay down homogeneous Neumann boundary conditions on both sides. The exact solution is composed of two strong shock waves near dry areas. The CFL condition is given by (7.1). The final time is 0.1. Figure 2 displays the results.

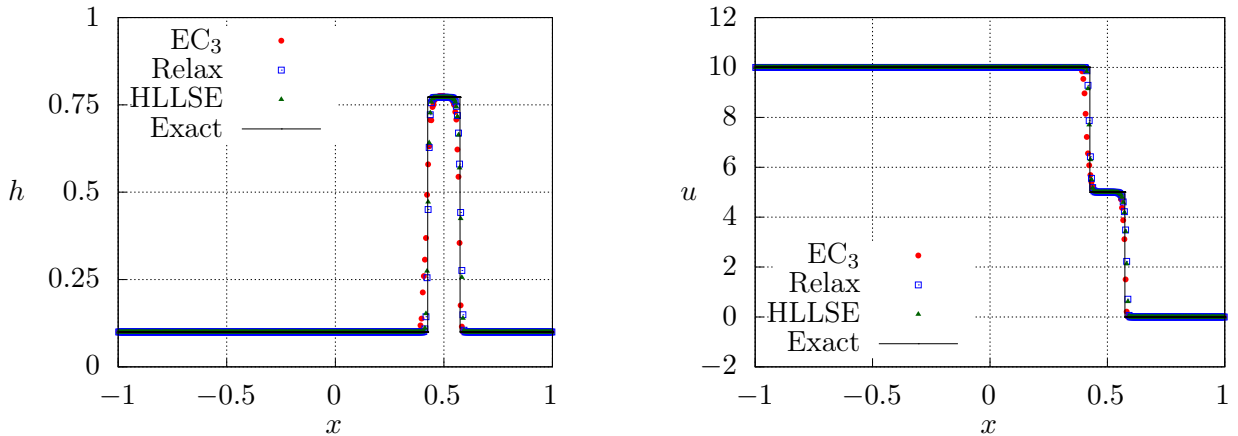


Figure 2: Numerical results at time 0.1 for the initial condition (7.4), with the legend  $EC_3$ : Lemma’s 4.2 solvers, Relax: Suliciu relaxation and HLLSE: solver [28, Section b.ii].

We observe a very good agreement with the exact solution. Without the conservative limitations techniques used in Lemma 4.2, this problem can not be carried out with the solvers of Lemma 4.1.

## 7.2 Well-balanced two states entropy stable approximate Riemann solvers

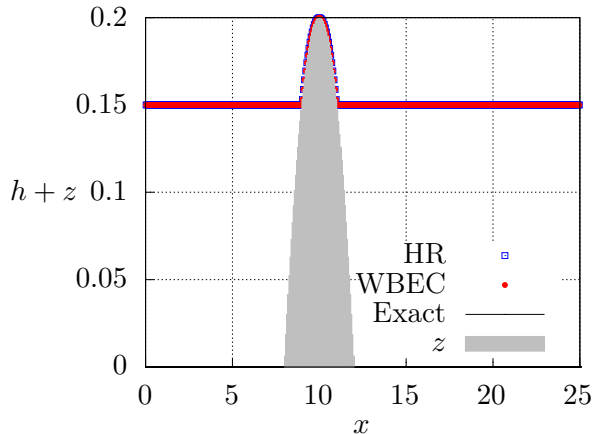
In this section, we consider Theorem’s schemes 5.1. For these solvers, the numerical artificial viscosity  $\lambda > 0$  has to be selected to ensure the existence of solutions of the system (3.6)-(5.1). We adopt the following selection procedure. Starting from the equation (7.1), we increase  $\lambda$  until the system (3.6)-(5.1) admits solutions. The time step  $\Delta t > 0$  is then selected according to the standard CFL condition  $\frac{\lambda \Delta t}{\Delta x} \leq \frac{1}{2}$ . To run the following simulations, we select randomly one of two solvers defined in Theorem 5.1 and we compare the results to the standard hydro-static reconstruction [3] coupled to the standard Rusanov numerical flux [44].

The first experiment is devoted to a flow at rest with emerging bottom as introduced in [22]. The space domain  $[0, 25]$  is discretized with 400 cells. The initial condition and the topography are given by

$$h_0(x) = \max(0.15 - z(x), 0), \quad u_0(x) = 0, \quad \text{with } z(x) = \max(0, 0.2 - 0.05(x - 10)^2). \quad (7.5)$$

We prescribe periodic boundary conditions. The exact solution is a lake at rest equilibrium (1.6). The final time is 100. Figure 3 shows the results.





Errors on $(h, hu)$ for the lake at rest.			
	$h$		
	$L^1$	$L^2$	$L^\infty$
WBEC	1.69E-21	4.79E-21	2.29E-41
HR	6.55E-18	8.44E-18	1.25E-17
	$hu$		
	$L^1$	$L^2$	$L^\infty$
WBEC	9.88E-20	3.43E-20	1.14E-22
HR	3.17E-17	2.52E-17	3.07E-17

Figure 3: On the left: numerical results at time 100.0 for the lake at rest problem (7.5) with the legend WBEC: entropic well-balanced solvers for the lake at rest given by Theorem 5.1, HR: hydro-static reconstruction [3] with Rusanov numerical flux [44]. On the right: errors between the exact and the numerical solutions at time 100.0 for the variables  $h, hu$ .

Thanks to the well-balanced property and to the transition toward dry areas, the steady state at rest (7.5) is preserved up to the machine precision.

The second numerical experiment concerns the three Goutal and Maurel's test cases [26]. For these three experiments, the space domain  $[0, 25]$  is discretized with 400 cells. Using the superscript  $GM_1$ ,  $GM_2$  and  $GM_3$  to denote each problems, the initial conditions are

$$h_0^{GM_k}(x) = h^{GM_k}, \quad (hu)_0^{GM_k}(x) = q^{GM_k}, \quad \forall k \in \{1, 2, 3\}, \quad (7.6)$$

where  $(h^{GM_k})_{k \in \{1, 2, 3\}}$  and  $(q^{GM_k})_{k \in \{1, 2, 3\}}$  are given in Table 1. The bottom topography  $z$  is given by (7.5). On the left boundary, the water height satisfies a homogeneous Neumann condition and the discharge  $q = hu$  is set to  $(q^{GM_k})_{k \in \{1, 2, 3\}}$ . On the right boundary, the water height is set to  $h^{GM_k}$  when the flow is sub-critical and a homogeneous Neumann boundary condition is prescribed otherwise. The discharge follows a homogeneous Neumann boundary condition.

Parameters used for the Goutal and Maurel test cases.			
	$GM_1$	$GM_2$	$GM_3$
Final time	500	125	1000
Initial height $h^{GM_k}$	2	0.66	0.33
Boundary discharge $q^{GM_k}$	4.42	1.53	0.18

Table 1: Final times, initial values and boundary conditions for the Goutal and Maurel's test cases 7.6.

Such initial and boundary conditions provide a transient state followed by a steady state made of an uniform discharge. For  $GM_1$  and  $GM_2$ , this steady state is continuous whereas  $GM_3$  involves a stationary shock. The final times are given in the Table 1. The exact solutions are computed with the software SWASHES [19] and Figure 4 shows the compared results.

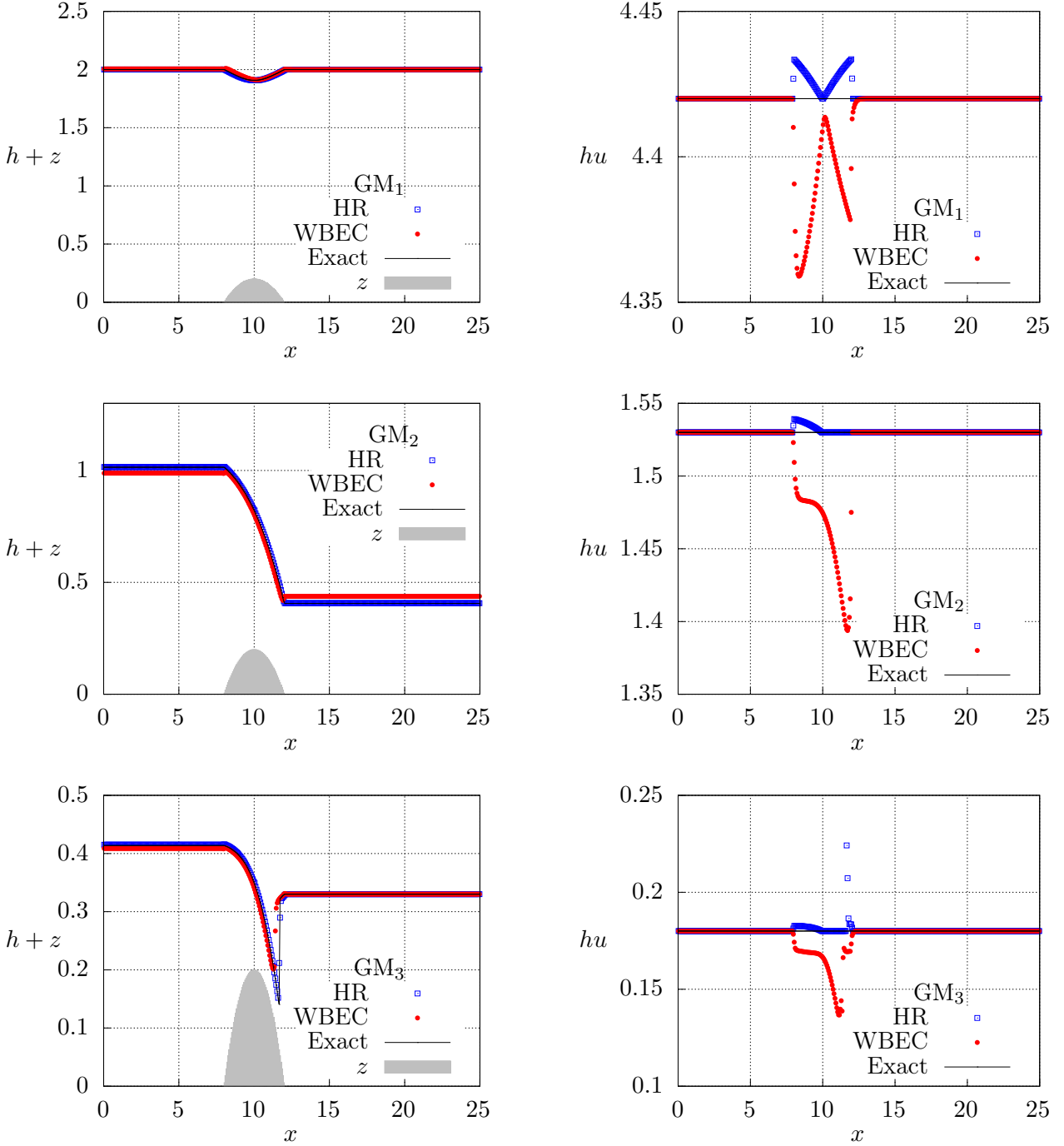


Figure 4: Numerical results for the Goutal and Maurel's problems (7.6) on a mesh composed of 400 cells. The legend is WBEC: entropic well-balanced solvers for the lake at rest given by Theorem 5.1, HR: hydro-static reconstruction [3] with the standard Rusanov numerical flux [44].

The results are corrects but the free surface  $h + z$  may be misplaced: it is particularly obvious for the  $GM_2$  problem. For the  $GM_3$  problem, the free surface is undervalued before the stationary shock wave and it is sharp after. Despite the fully discrete entropy stability verified by the Theorem's 5.1 schemes, the numerical solutions may converge to a non admissible weak solution. Such wrong arbitrary convergences have already been observed [13, 15, 17]. In addition, the schemes of Theorem 5.1 produce some spurious oscillations for the variable  $hu$ . For these three test cases, the standard hydro-static reconstruction [3] coupled to the Rusanov numerical flux [44] is more relevant.

The last test case of this section is devoted to the break dam problem as described in [41]. The

domain  $[-1, 1]$  is discretized with 400 cells. We consider the following initial condition and topography:

$$h_0(x) + z(x) = \begin{cases} 3 & \text{if } x < 0, \\ 1 & \text{otherwise,} \end{cases} \quad u_0(x) = 0, \quad \text{with,} \quad z(x) = \frac{1}{2} \cos^2(\pi x). \quad (7.7)$$

We prescribe homogeneous Neumann boundary conditions on both boundaries. The final time is 0.1. A reference solution is computed with the standard HLL scheme [28] on a fine grid made of 50 000 cells. Figure 5 displays the results.

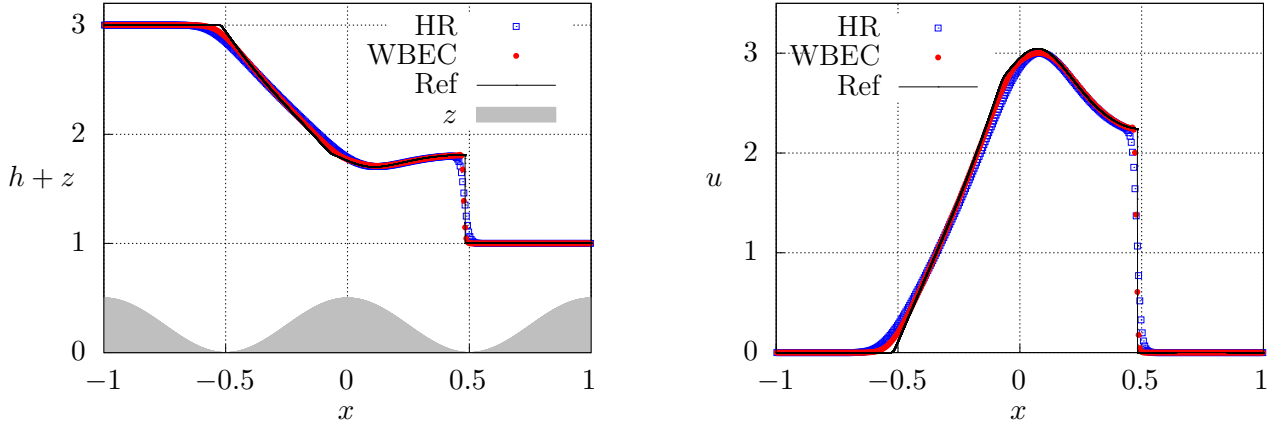


Figure 5: Numerical results at time 0.1 for the break dam problem (7.7) on a mesh made of 400 cells with the legend WBEC: Theorem’s 5.1 solvers, HR: hydro-static reconstruction [3] with the Rusanov numerical flux [44].

We observe a very good agreement to the reference solution. The shock waves is sharper with the schemes detailed in Theorem 5.1.

### 7.3 Fully well-balanced two states entropy approximate stable Riemann solvers

In this section, we consider the fully well-balanced entropy stable scheme defined in Theorem 6.1. For this scheme, the numerical artificial viscosity  $\lambda > 0$  and the time step  $\Delta t > 0$  have to be selected to guarantee the existence solutions of the system (3.6)-(6.3). To select a such couple, we initialize  $\lambda > 0$  with the value given by the equation (7.1) and we set  $\Delta t > 0$  according to  $\frac{\lambda \Delta t}{\Delta x} = \frac{1}{2}$ . Then, we increase  $\lambda$  and we decrease  $\Delta t$  until the system (3.6)-(6.3) admits solutions. For the sign function required in the equation (6.13) we use the following regularized version:

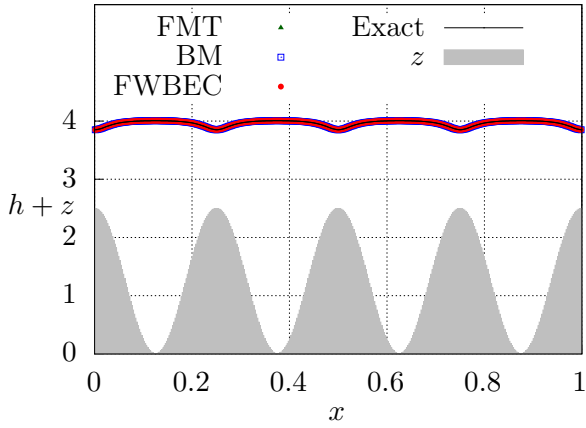
$$\text{sign}(r) \approx \frac{r}{|r| + \zeta_{LR}/(g\bar{h})}, \quad (7.8)$$

where  $\zeta_{LR}$  and  $\bar{h}$  are defined in (6.2). We compare the Theorem’s 6.1 scheme to the schemes detailed in [21, Section 3.2] and in [14] that will be denoted FMT and BM respectively. The scheme BM is associated to the standard Rusanov numerical flux [44].

The first test case concerns the fully well-balanced property. We consider the domain  $[0, 1]$  discretized with 400 cells. The initial conditions and the bottom topography verify

$$(hu)_0(x) = q_0, \quad \frac{u_0^2(x)}{2} + g(h_0 + z)(x) = B_0, \quad \text{with} \quad z(x) = \frac{5}{2} \cos^2(4\pi x), \quad (7.9)$$

where  $q_0 = \frac{5}{2}$ ,  $B_0 = \frac{25}{98} + 4g$ . The initial values  $(h_0, u_0)(x)$  are computed from the equations (7.9) with a Newton method. We lay down periodic boundary conditions on both sides. The exact solution is a moving steady state (1.5). The final time is 1.0. The results are reported in Figure 6. We also report the errors between the numerical and the exact solution for several norms.

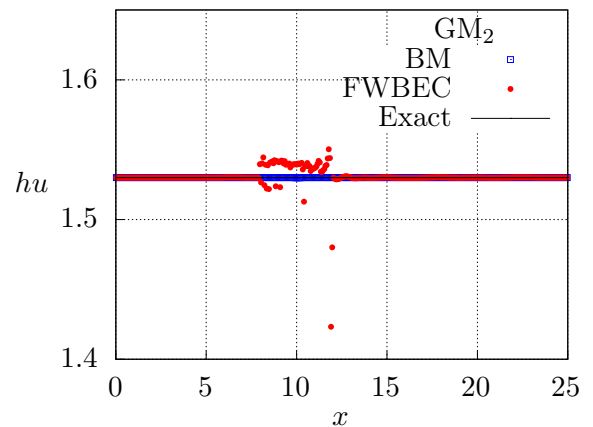
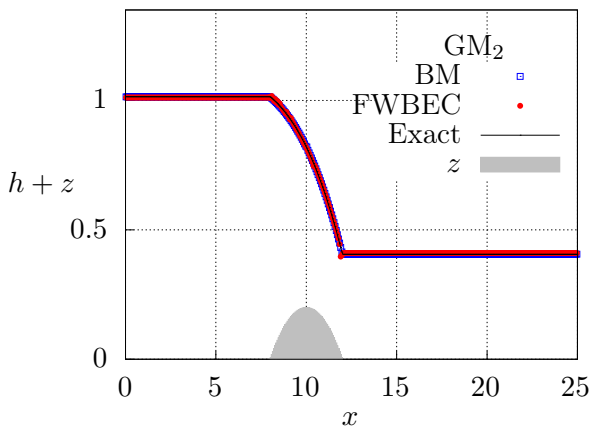
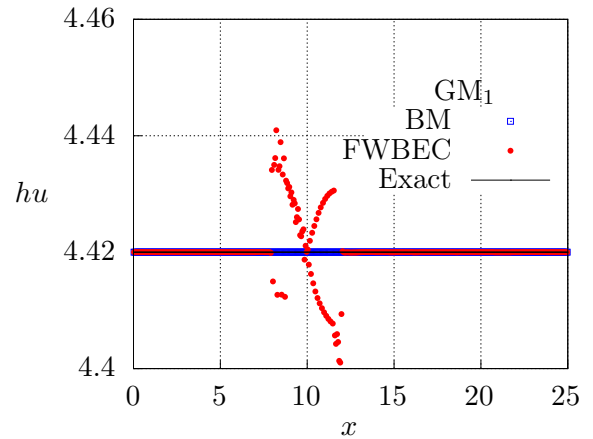
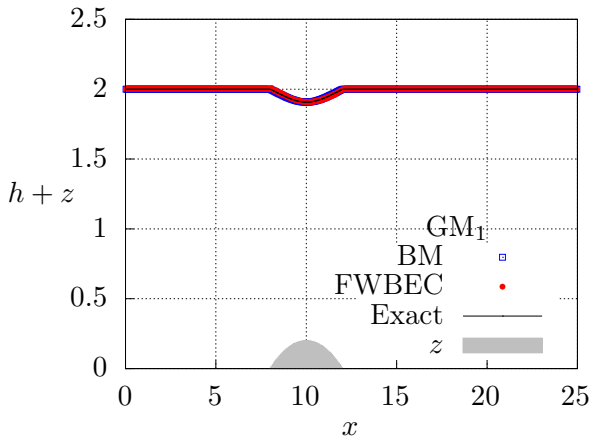


Errors on $(h, hu)$ for the steady state (7.9).			
	$h$		
	$L^1$	$L^2$	$L^\infty$
FWBEC	0.00E-00	0.00E-00	0.00E-00
BM	2.49E-17	4.41E-17	1.27E-17
FMT	1.84E-13	2.03E-13	3.28E-13
	$hu$		
	$L^1$	$L^2$	$L^\infty$
FWBEC	1.13E-20	8.74E-20	7.64E-39
BM	8.58E-17	1.29E-16	9.19E-17
FMT	2.87E-13	3.75E-13	4.10E-13

Figure 6: On the left: numerical results at time 1.0 for the steady state problem (7.9) with the legend FWBEC: fully well-balanced entropic solver given by Theorem 6.1, BM: scheme derived in [14] associated to the Rusanov numerical flux [44], FMT: scheme derived in [21, Section 3.2]. On the right: errors between the exact and the numerical solutions at time 1.0 for the variables  $h, hu$ .

Thanks to the well-balanced property, the moving equilibrium is preserved up to the machine precision.

Now we repeat the three Goutal and Maurel's test cases [26] and we refer to Section 7.2 for the details of the test cases. Figure 7 shows the results and Tables 8 show the errors between the numerical and the exact solutions for each problems and for several norms.



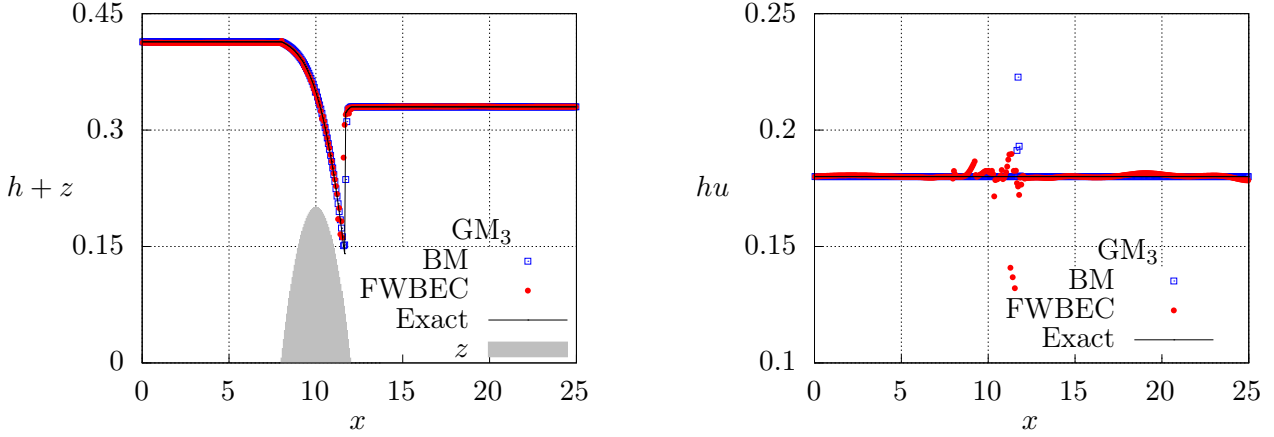


Figure 7: Numerical results for the Goutal and Maurel's problems (7.6) on a mesh composed of 400 cells. The legend is FWBEC: fully well-balanced entropic solver given by Theorem 6.1, BM: scheme derived in [14] associated to the Rusanov numerical flux [44].

Despite our efforts, we was not able to run these three problems with the scheme FMT. Theorem's 6.1 scheme generates spurious oscillations, in particular for the variable  $hu$ . Nevertheless, the wrong convergences observed for the  $GM_2$ ,  $GM_3$  problems with the schemes derived in Theorem 5.1 and tested in Section 7.2, do not occur with the fully well-balanced scheme in Figure 7. However, the free surface for the  $GM_1$  problem is once again slightly misplaced.

Errors on $(h, hu)$ for the $GM_1$ problem.				Errors on $(h, hu)$ for the $GM_2$ problem.			
	$h$				$h$		
	$L^1$	$L^2$	$L^\infty$		$L^1$	$L^2$	$L^\infty$
FWBEC	3.86E-02	1.41E-02	2.00E-04	FWBEC	9.35E-02	2.13E-02	4.46E-03
BM	1.91E-15	4.16E-16	6.11E-17	BM	9.06E-03	2.37E-03	5.12E-04
	$hu$				$hu$		
	$L^1$	$L^2$	$L^\infty$		$L^1$	$L^2$	$L^\infty$
FWBEC	3.77E-02	2.10E-02	4.439E-04	FWBEC	4.89E-02	3.53E-02	1.25E-03
BM	7.29E-15	1.49E-15	2.91E-16	BM	7.78E-05	3.38E-05	4.33E-08

Errors on $(h, hu)$ for the $GM_3$ problem.			
	$h$		
	$L^1$	$L^2$	$L^\infty$
FWBEC	4.28E-02	5.11E-02	2.62E-03
BM	8.23E-03	2.20E-02	4.86E-04
	$hu$		
	$L^1$	$L^2$	$L^\infty$
FWBEC	2.57E-02	2.00E-02	1.59E-03
BM	4.21E-03	1.15E-02	1.32E-04

Figure 8: Errors between the numerical and the exact solutions for each Goutal and Maurel's problems (7.6) and for several norms. The legend is FWBEC: fully well-balanced entropic solver given by Theorem 6.1, BM: scheme derived in [14] coupled to the Rusanov numerical flux [44]

Regardless of its fully well-balanced property, the scheme derived in Theorem 6.1 does not reach the exact solution up to the machine precision. This default could be due to the implementation of the sign function required in the equation (6.13). The choice proposed in the equation (7.8) is a regular version of the sign function but other versions are possible and each of them could influence the result. For the three Goutal and Maurel's problems the BM scheme is more accurate.

We conclude this section with the break dam problem as done in Section 7.2. The initial condition and the bottom topography are given by the equations (7.7). The final time is 0.1 and Figure 9 displays the results.

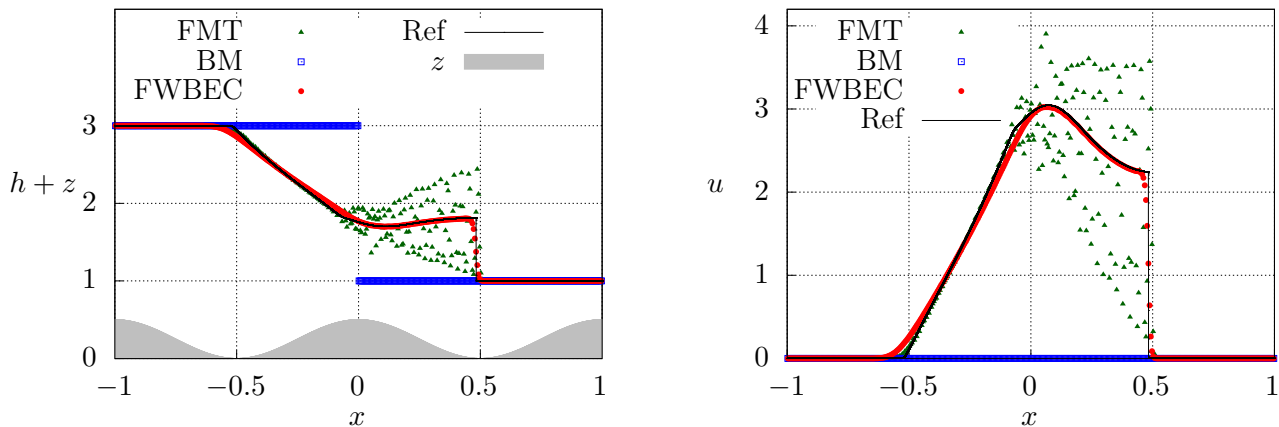


Figure 9: Numerical results at time 0.1 on a mesh made of 400 cells for the break dam problem (7.7) with the legend FWBEC: Theorem’s 6.1 solver, BM: scheme derived in [14] associated to the Rusanov numerical flux [44], FMT: scheme derived in [21, Section 3.2].

The scheme [21, Section 3.2] generates spurious oscillations near the shock wave and we failed to achieve better results. The scheme [14] associated to the Rusanov numerical flux [44] preserves the initial condition and captures two steady states at rest. For the scheme of Theorem 6.1, we observe a good agreement to the reference solution.

## 8 Conclusion

We have presented three explicit entropy stable Godunov type scheme for the Shallow Water equations. The first one concerns the flat regions, the second scheme is well-balanced for lake at rest (1.8) and the third one is well-balanced for all regular stationary solutions defined by (1.9). The discrete entropy inequality is reached from sufficient conditions used in the scheme design. These conditions lead to quadratic equations that are always well-posed under restrictions for the artificial viscosity and for the time step. These restrictions are implicit for the well-balanced schemes.

From a numerical point of view, the scheme devoted to the flat regions provides good results. The well-balanced schemes yield satisfying results in particular in the presence of shock waves. But, they may converge to weak solutions made of non admissible stationary contact waves. The study of the reasons of this wrong convergence should be investigated.

## Acknowledgements

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## A Smooth stationary solutions of the Shallow Water system

This appendix concerns the smooth stationary solutions of the Shallow Water equations (1.1). The algebraic equations and a property verified by these solutions are given below.

**Lemma A.1** (Smooth stationary solutions of the Shallow Water equations). *Consider  $I$  a bounded, closed interval included in  $\mathbb{R}$ . For a given smooth function  $z : \mathbb{R} \rightarrow \mathbb{R}$ , let also consider the Shallow Water equations given by*

$$\partial_t \begin{pmatrix} h \\ q \end{pmatrix} + \partial_x \begin{pmatrix} q \\ \frac{q^2}{h} + \frac{gh^2}{2} \end{pmatrix} = \begin{pmatrix} 0 \\ -gh \end{pmatrix} \partial_x z, \quad \forall (x, t) \in I \times ]0, +\infty[. \quad (\text{A.1})$$

The vector  $(h(x, t), q(x, t))^T$  belongs to  $\Omega$  a convex set defined by

$$\Omega = \{(h, q) \in \mathbb{R}^2 \mid h > 0, q \in \mathbb{R}\}.$$

For  $x \mapsto (h(x), q(x))$ , a smooth stationary solution of the Shallow Water system (A.1), the following statements hold.

i) There exists  $(q_0, B_0)$  in  $\mathbb{R} \times \mathbb{R}_+^*$  such that for all  $x$  in  $I$

$$q(x) = q_0, \quad (\text{A.2a})$$

$$\frac{q_0}{2h^2(x)} + g(h(x) + z(x)) = B_0. \quad (\text{A.2b})$$

ii) Considering the above definition of  $q_0$  in  $\mathbb{R}$  and the set  $J_{q_0} \subset I$  such that

$$J_{q_0} = \left\{ x \in I \mid 1 - \frac{q_0^2}{gh^3(x)} \neq 0 \right\}, \quad (\text{A.3})$$

the derivative of  $h$  with respect to  $x$  satisfies

$$\partial_x h = -\frac{\partial_x z}{1 - \frac{q_0^2}{gh^3(x)}}, \quad \forall x \in J_{q_0}. \quad (\text{A.4})$$

*Proof.* The stationary functions  $x \mapsto h(x)$  and  $x \mapsto q(x)$  satisfy the two following equations for all  $x$  in  $I$ :

$$\partial_x q = 0, \quad (\text{A.5a})$$

$$\partial_x \left( \frac{q^2}{h} + \frac{gh^2}{2} \right) = -gh\partial_x z. \quad (\text{A.5b})$$

For the statement *i*), and according to the equation (A.5a), the result (A.2a) is obvious. Using this result in the equation (A.5b), the usual derivation rules give

$$\begin{aligned} 0 &= \partial_x \left( \frac{q_0^2}{h} + \frac{gh^2}{2} \right) + gh\partial_x z, \\ &= -(\partial_x h) \frac{q_0^2}{h^2} + gh\partial_x (h + z), \\ &= h \left( -(\partial_x h) \frac{q_0^2}{h^3} \right) + gh\partial_x (h + z), \\ &= h\partial_x \left( \frac{q_0^2}{2h^2} \right) + gh\partial_x (h + z). \end{aligned}$$

Multiplying the above equation by  $1/h > 0$ , we deduce

$$\partial_x \left( \frac{q_0^2}{2h^2} + g(h + z) \right) = 0,$$

that concludes the proof of the result (A.2b) and the proof of the statement *i*).

For the statement *ii*), using the usual derivations rules, the equation (A.5b) also writes for all  $x$  in  $I$

$$\begin{aligned} -gh\partial_x z &= \partial_x \left( \frac{q_0^2}{h} + \frac{gh^2}{2} \right), \\ &= -\partial_x h \frac{q_0^2}{h^2} + gh\partial_x h, \\ &= gh\partial_x h \left( 1 - \frac{q_0^2}{gh^3} \right). \end{aligned}$$

Dividing the above equation by  $gh > 0$ , we have

$$-\partial_x z = \partial_x h \left( 1 - \frac{q_0^2}{gh^3} \right). \quad (\text{A.6})$$

Since the above equation is defined on the set  $J_{q_0}$  given by (A.3),  $1 - q_0^2/gh^3 \neq 0$  that yields the result (A.4) and that concludes the proof of the statement *ii*.  $\square$

According to the statement *ii*) of the above lemma, in stationary states and for  $x \in J_{q_0}$ , the derivatives of  $h$  is proportional to the derivative of  $z$ . As a consequence, if  $z$  is locally constant then  $h$  is also constant. In addition, the result (A.2a) establishes that  $q$  is also constant in stationary states. Therefore, for  $x \in J_{q_0}$  the statement *ii*) shows that if  $z$  is locally constant then the only smooth stationary solutions of the system (A.1) are the constants of  $\Omega$ .

If  $x \notin J_{q_0}$ , the equation (A.6) is singular. In this case, and as  $z$  is a given function, the stationary equations written under the form of (A.5) could be ill-posed.