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A Combinatorial Approach Towards Ramanujan's Partition Congruences Modulo 5, 7 and 11 In Terms Of Its Corresponding Cranks

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Abstract

The primary purpose of this article is to provide an explicit interpretation of *Ramanujan's* Partition Congruences, namely, $p(5n + 4) \equiv 0 \pmod{5}$, $p(7n + 5) \equiv 0 \pmod{7}$ and, $p(11n + 6) \equiv 0 \pmod{11}$, $p(n)$ being the *partition function* corresponding to any positive integer n , in terms of their corresponding *Cranks* using various combinatorial arguments implemented by *Dyson*, *Atkin*, *Swinnerton-Dyer* and later by *Andrews* and, *Garvan* to justify all the necessary concepts pertaining to this topic.

In addition to introducing readers to the Theory of *Partitions*, as well as defining formally the notion of *q-Series* and *Ramanujan's Theta Function*, we shall further deduce a rough estimate for $p(n)$, a priori applying the so called *Jacobi Triple Product Identity* and *Euler Pentagonal Theorem*, and subsequently, providing a thorough explanation for the special case, when $n \equiv 4 \pmod{5}$.

Important to mention that, a whole section in this article has been dedicated towards gaining an detailed understanding of *Rank* of a partition and *Crank* of a Vector Partition, both introduced by *Dyson*, due to its significance and relevance to our study. Furthermore, in the later half of the text, we shall indeed prove our main result with the help of an important property of the partition function $M(m, j, n)$ corresponding to any positive integer n with *crank* congruent to m modulo j . Exclusively, in the final section, rigorous derivations have been provided individually for each case to facilitate the motivated readers of this paper with

a better understanding of this subject. Moreover, adequate references have been included, considering the broader aspect of research in this area.

Keywords and Phrases: Partition, Rank, Crank, Vector Partition, q-Series, Ramanujan's Theta Function, Jacobi-Triple Product Identity, Euler's Pentagonal Theorem, Successive Rank.

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1 Theory of Partitions

1.1 Introduction

The theory of *partitions* first came to the foray when German mathematician *Gottfried Leibniz* (1646 – 1716) wrote a letter (1674) to *Jacob Bernoulli* seeking explanations about the sum of *divulsion* of integers. He observed that, there are 3 partitions of 3 (namely : 3, 2+1 and 1+1+1) and five partitions of 4 (namely : 4, 3 + 1 and 2 + 2, 2 + 1 + 1, and 1 + 1 + 1 + 1). Later on, the concept of partitions were generalized for arbitrary positive integers based on this very notion introduced by *Leibniz*. The following excerpt can be found in the book *Combinatorics: Ancient & Modern* [1, Chapt. 9],

“While Leibniz appears to have been the earliest to consider the partitioning of integers into sums, Euler was the first person to make truly deep discoveries. J. J. Sylvester was the next researcher to amke major contributions, followed by Fabian Franklin. The 20th Century saw an explosion of research with contributions from L. J. Rogers, G. H. Hardy, Percy MacMahon, Srinivasa Ramanujan and Hans Rademacher.”

- George E. Andrews.

1.2 Formal Definition

We can formally define a partition a follows:

Definition 1.2.1. (Partition) Given $n \in \mathbb{N}$, we define λ to be a partition of n , if λ is of the form,

$$\lambda := \{ (t_1, t_2, \dots, t_r) \mid t_i \geq t_{i+1}, \forall i = 1(1)r - 1, t_i \in \mathbb{N}, \forall i = 1(1)r, \sum_{i=1}^r t_i = n \}$$

and we denote it as $\lambda \vdash n$. Each t_i is called *summand/part*.

Remark 1.2.1. We denote the number of partitions of n by $p(n)$.

Example 1.2.2. For example, $2+1 \vdash 3$, $5+3+2 \vdash 10$ denotes partitions of 3 and 10 respectively.

A more pertinent question may pop up in a curious mind,

Are there infinitely many integers n such that, $p(n)$ is prime ?

Unsurprisingly, the problem is not as trivial as it seems and is still open.

There are certain aspects of partitions which mathematicians were intrigued about studying. For example, one can try to deduce any estimate or even an exact formula for the partition function $p(n)$. Moreover, we can also pursue further research to obtain any criteria for n such that, $p(n)$ is divisible by a specific prime number, or, the proportion of such n for which $p(n)$ is divisible by a particular integer, or even the behaviour of $p(n)$ if certain additional conditions are imposed.

Philippe Naudé asked *Euler* in a letter to him, to solve the problem of dividing a given integer n into certain number of partitions, denoted by m . To be more specific, *Naudé* inquired about the number of partitions of 50 into 7 **distinct** parts. Even after several efforts, the correct answer of 522 possibilities couldn't be derived by brute force calculations, and thereby listing all the possibilities. Hence, *Euler* came up with an alternate way of solving these problems by introducing the notion of *Generating Functions*.

Suppose, $D(m, n)$ denotes the number of partitions of n into m distinct parts. Furthermore, we choose q (called the *base*), such that, $|q| < 1$. Then, we can deduce the following identity,

$$\sum_{m, n \geq 0} D(m, n) z^m q^n = \prod_{j=1}^{\infty} (1 + z q^j) \quad (1.1)$$

For every such j , we index the distinct parts as (i_1, i_2, \dots, i_j) . Hence, R.H.S. of (1.1) will be equal to,

$$(1 + zq) \prod_{j=1}^{\infty} (1 + (zq)q^j)$$

Consequently,

$$\sum_{m, n \geq 0} D(m, n) z^m q^n = (1 + zq) \sum_{m, n \geq 0} D(m, n) z^m q^{n+m} \quad (1.2)$$

Comparing co-efficients of $z^m q^n$ on both sides,

$$D(m, n) = D(m, n - m) + D(m - 1, n - m) \quad (1.3)$$

2 Introduction to q-Series and Theta Functions

Definition 2.0.1. (q-Series) We define,

$$(a)_n := (a; q) := \prod_{k=0}^{n-1} (1 - a q^k) \quad , \quad n \geq 1.$$

$$(a)_\infty := (a; q) := \prod_{k=0}^{\infty} (1 - aq^k) \quad , \quad |q| < 1.$$

Using analogous definition of $(a; q)_n$ for negative integers n , we can extend the definition of q -Series to the form,

$$(a; q)_n = \frac{(a; q)_\infty}{(aq^n; q)_\infty} \quad (2.1)$$

Definition 2.0.2. (Ramanujan's Theta Function) *Ramanujan's General Theta Function* $f(a, b)$ is defined by,

$$f(a, b) := \sum_{n=-\infty}^{\infty} a^{n(n+1)/2} b^{n(n-1)/2} \quad , \quad |ab| < 1. \quad (2.2)$$

Ramanujan introduced notations for three special cases applying definition (2.0.2).

$$\phi(q) := f(q, q) = \sum_{n=-\infty}^{\infty} q^{n^2} = (-q; q^2)_\infty (q^2; q^2)_\infty \quad (2.3)$$

$$\psi(q) := f(q, q^3) = \sum_{n=0}^{\infty} q^{n(n+1)/2} = \frac{(q^2; q^2)_\infty}{(q; q^2)_\infty} \quad (2.4)$$

$$f(-q) := f(-q, -q^2) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n-1)/2} = (q; q)_\infty \quad (2.5)$$

Moreover, *Ramanujan* introduced another function beside other three Theta Functions defined above.

$$\chi(q) := (-q; q^2)_\infty \quad (2.6)$$

It can be verified that, χ is not a *Theta Function*. The expressions in R.H.S. of (2.3), (2.4), (2.5) follows from the *Jacobi-Triple Product Identity*.

Theorem 2.0.1. (*Jacobi-Triple Product Identity*) For $z \neq 0$ and $|q| < 1$,

$$\sum_{n=-\infty}^{\infty} z^n q^{n^2} = (-zq; q^2)_\infty \left(-\frac{q}{z}; q^2 \right)_\infty (q^2; q^2)_\infty$$

Combining *theorem*(2.0.1) and (2.5), we obtain the following result.

Theorem 2.0.2. (*Euler's Pentagonal Theorem*)

$$\sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n-1)/2} = \sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n+1)/2} = (q; q)_\infty \quad (2.7)$$

3 An Estimate for the Partition Function $p(n)$

A priori from *Theorem*(2.0.2), we can in fact conclude about the following recurrence relation which eventually leads us to approximate the value of $p(n)$.

Corollary 3.0.1. *For brevity, for every integer j , we set, $w_j = \frac{j(3j-1)}{2}$. Then,*

$$p(n) = \sum_{0 < w_j \leq n} (-1)^{j+1} p(n - w_j) \quad (3.1)$$

Ramanujan established in this paper [2] that,

$$p(5n + 4) \equiv 0 \pmod{5} \quad (3.2)$$

$$p(7n + 5) \equiv 0 \pmod{7} \quad (3.3)$$

$$p(11n + 6) \equiv 0 \pmod{11} \quad (3.4)$$

A priori given *Ramanujan's* generalizations of the results in (3.2), (3.3) and (3.4), the following conjectures have already been proved to be true.

$$p(5^j n + \delta_{5,j}) \equiv 0 \pmod{5^j} \quad (3.5)$$

$$p(7^j n + \delta_{7,j}) \equiv 0 \pmod{7^{\lfloor \frac{j}{2} \rfloor + 1}} \quad (3.6)$$

$$p(11^j n + \delta_{11,j}) \equiv 0 \pmod{11^j} \quad (3.7)$$

where, $j \geq 1$ and, $\delta_{l,j}$ denotes the l^j th reciprocal modulo 24.

Watson provided the proofs of (3.5) and (3.6) [3], whereas, *Hirschhorn* and *Hunt* derived (3.5) [interested readers can look up [4]]. Using similar techniques, *garvan* established the statement of (3.6) in his article [5]. More notably, the congruence (3.7) was established by *Atkin* [Ref. [6]].

4 Ramanujan's Congruence, $p(5n + 4) \equiv 0 \pmod{5}$

In this section, we intend to provide a detailed version of the proof of the *Ramanujan's Congruence* (3.2), a priori using the properties and fundamental results pertinent to the *q-Series* and *Theta Functions* described in **section 4** of this article.

It can also be observed as an adaptation of the proof proposed by *Ramanujan* in his papers [2] and [7, pp. 201 – 213], as mentioned in *Berndt's* book [8, pp. 31].

Theorem 4.0.1. *For each non-negative integer n ,*

$$p(5n+4) \equiv 0 \pmod{5}$$

We shall be needing the following result in order to understand the proof.

Theorem 4.0.2. (*Jacobi's Identity*)

$$\sum_{n=0}^{\infty} (-1)^n (2n+1) q^{n(n+1)/2} = (q; q)_{\infty}^3 \quad (4.1)$$

Proof. (**Proof of Theorem** (4.0.1))

A priori from definition of q -Series, we obtain,

$$q(q; q)_{\infty}^4 \cdot \frac{(q^5; q^5)_{\infty}}{(q; q)_{\infty}^5} = q \cdot \frac{(q^5; q^5)_{\infty}}{(q; q)_{\infty}} = (q^5; q^5)_{\infty} \sum_{m=0}^{\infty} p(m) q^{m+1} \quad (4.2)$$

Hence, a simple application of *Binomial Theorem* yields,

$$(q; q)_{\infty}^5 \equiv (q^5; q^5)_{\infty} \pmod{5}$$

In other words,

$$\frac{(q; q)_{\infty}^5}{(q^5; q^5)_{\infty}} \equiv 1 \pmod{5} \quad (4.3)$$

From (4.2) and (4.3), we can conclude,

$$(q; q)_{\infty}^4 \equiv (q^5; q^5)_{\infty} \sum_{m=0}^{\infty} p(m) q^{m+1} \pmod{5} \quad (4.4)$$

Thus it only suffices to show that, the co-efficients of q^{5n+5} on the L.H.S. of (4.4) are multiples of 5.

A priori from the statements of *Euler Pentagonal Theorem* (2.0.2) and *Jacobi's Identity* (4.0.2), (4.4) can also be interpreted as,

$$\begin{aligned} q(q; q)_{\infty}^4 &= q(q; q)_{\infty} (q; q)_{\infty}^3 = q \sum_{j=-\infty}^{\infty} (-1)^j q^{j(3j+1)/2} \sum_{k=0}^{\infty} (-1)^k (2k+1) q^{k(k+2)/2} \\ &= \sum_{j=-\infty}^{\infty} \sum_{k=0}^{\infty} (-1)^{j+k} (2k+1) q^{1+j(\frac{3j+1}{2})+k(\frac{k+1}{2})} \end{aligned} \quad (4.5)$$

Our objective is to determine when the exponents on the R.H.S. are multiples of 5. Observe that,

$$2(j+1)^2 + (2k+1)^2 = 8\left\{1 + \frac{1}{2}j(3j+1) + \frac{1}{2}k(k+1)\right\} - 10j^2 - 5$$

Therefore, $\{1 + j\left(\frac{3j+1}{2}\right) + k\left(\frac{k+1}{2}\right)\}$ is a multiple of 5 iff,

$$2(j+1)^2 + (2k+1)^2 \equiv 0 \pmod{5} \quad (4.6)$$

Equivalently,

$$2(j+1)^2 \equiv 0, 2 \text{ or } 3 \pmod{5}$$

and,

$$(2k+1)^2 \equiv 0, 1 \text{ or } 4 \pmod{5}$$

thus, (4.6) is true iff,

$$2(j+1)^2 \equiv 0 \pmod{5} \quad \text{and,} \quad (2k+1)^2 \equiv 0 \pmod{5}$$

In particular, we shall have, $(2k+1) \equiv 0 \pmod{5}$, which, by (4.5), further implies that, the co-efficient of q^{5n+5} (where, $n > 0$) in the term $q(q; q)_\infty^4$ is a multiple of 5. The co-efficient of q^{5n+5} on the R.H.S. of (4.4) is therefore, also a multiple of 5, i.e., $p(5n+4)$ is a multiple of 5, and our result is proved. \square

Remark 4.0.3. Andrews provided an alternative proof of *Theorem* (4.0.1). [Interested readers can read further from Berndt's book [8, pp. 33]]

Remark 4.0.4. As for a third version of the proof of *Theorem* (4.0.1), we can further study Ramanujan's work [2], as cited in [8, pp. 34]. It can be observed that, the proof follows from a beautiful identity. As mentioned by Ramanujan, throughout the whole course of the proof, we denote $J_k(q)$, $k = 1, 2$ as the power series with integral powers & integral co-efficients, not necessarily identical at each appearance. The precise identities of $J_1(q)$ and $J_2(q)$ are not pertinent to the proof.

Remark 4.0.5. Worth mentioning that, we can, in fact, prove *Theorem* (4.0.1) using the concept of *Eisenstein Series* and *Generating Function* for $p(n)$. Curious readers can refer to [8, pp. 102].

5 Combinatorial Interpretation for Ramanujan Congruences

5.1 Rank of a Partition

Now that, we have already discussed about the *partition congruences*,

$$p(an + b) \equiv 0 \pmod{a}$$

Where, $(a, b) \in \{(5, 4), (7, 5), (11, 6)\}$, a more instinctive question which should arise is,

Is there any method to classify the partitions of integers of the form $(an + b)$ in a number of groups ?

F. J. Dyson first gave an answer to this question in his paper[9], when he established some significant empirical combinatorial results regarding (3.2) and (3.3).

He gave rise to the notion of *rank of a partition*, and defined it as :

Definition 5.1.1. (Rank of a Partition) Given any $n \in \mathbb{N}$, and $\lambda \vdash n$, suppose, we introduce the following notations :

1. $l(\lambda) :=$ The largest part of λ .
2. $\#(\lambda) :=$ Number of parts of λ .

Then, the Rank of λ , denoted by $r(\lambda)$, is defined as,

$$r(\lambda) := l(\lambda) - \#(\lambda)$$

Remark 5.1.1. The number of partitions of n ($\in \mathbb{N}$) with rank m is denoted by $N(m, n)$.

Remark 5.1.2. The number of partitions of n with rank congruent to m modulo j is denoted by $N(m, j, n)$.

Example 5.1.3. Consider the following example of a partition $\lambda := 5 + 3 + 2 \vdash 10$. Then, from definition, we can in fact deduce that, $r(\lambda) = 5 - 3 = 2$.

Example 5.1.4. As for classifying partitions of 9 according to their respective ranks modulo 5, one can indeed derive the following table.

Table 1: **Ranks of the Partitions of 9**

Partition	Rank	Rank(mod 5)	Partition	Rank	Rank(mod 5)
9	$9 - 1 = 8$	3	$4 + 2 + 2 + 1$	$4 - 4 = 0$	0
$8 + 1$	$8 - 2 = 6$	1	$4 + 2 + 1_3$	-1	4
$7 + 2$	$7 - 2 = 5$	0	$4 + 1_5$	-2	3
$7 + 1 + 1$	$7 - 3 = 4$	4	3_3	0	0
$6 + 3$	$6 - 2 = 4$	4	$3 + 3 + 2 + 1$	-1	4
$6 + 2 + 1$	$6 - 3 = 3$	3	$3 + 3 + 1_3$	-2	3
$6 + 1_3$	$6 - 4 = 2$	2	$3 + 2_3$	-1	4
$5 + 4$	$5 - 2 = 3$	3	$3 + 2 + 2 + 1 + 1$	-2	3
$5 + 3 + 1$	$5 - 3 = 2$	2	$3 + 2 + 1_4$	-3	2
$5 + 2 + 2$	$5 - 3 = 2$	2	$3 + 1_6$	-4	1
$5 + 2 + 1 + 1$	$5 - 4 = 1$	1	$2_4 + 1$	-3	2
$4 + 4 + 1$	$4 - 3 = 1$	1	$2 + 2 + 1_5$	-5	0
$4 + 3 + 2$	$4 - 3 = 1$	1	$2 + 1_7$	-6	4
$4 + 3 + 1 + 1$	$4 - 4 = 0$	0	1_9	-8	2

Consequently, we obtain the following classification of partitions of 9 corresponding to their respective ranks provided in the table below.

 Table 2: **Partitions of 9 with Rank $\equiv a \pmod{5}$**

$a = 0$	$a = 1$	$a = 2$	$a = 3$	$a = 4$
$4 + 2 + 2 + 1$	$8 + 1$	$6 + 1_3$	9	$4 + 2 + 1_3$
$7 + 2$	$3 + 1_6$	$5 + 3 + 1$	$4 + 1_5$	$7 + 1 + 1$
3_3	$5 + 2 + 2 + 1$	$3 + 2 + 1_4$	$6 + 2 + 1$	$6 + 3$
$5 + 1_4$	$2_3 + 1_3$	$5 + 2 + 2$	$3 + 3 + 1_3$	$3 + 2 + 2 + 1$
$2 + 2 + 1_5$	$4 + 4 + 1$	$2_4 + 1$	$5 + 4$	$3 + 2 + 2 + 2$
$4 + 3 + 1 + 1$	$4 + 3 + 2$	1_9	$3 + 2 + 2 + 1 + 1$	$2 + 1_7$

Where, we followed the notation, $a_b := \underbrace{a + a + \dots + a}_{b\text{-times}}$.

Furthermore, *Dyson* (1944) conjectured the following.

Theorem 5.1.5. *Suppose, $N(m, j, n)$ be as defined in (5.1.2). Then, the following holds true.*

$$N(m, 5, 5n + 4) = \frac{1}{5}p(5n + 4), \quad 0 \leq m \leq 4 \quad (5.1)$$

$$N(m, 7, 7n + 5) = \frac{1}{7}p(7n + 5), \quad 0 \leq m \leq 6 \quad (5.2)$$

Example 5.1.6. One can in fact verify the result (5.1) from the table (5.1.4) in the example (5.1.4).

Important to note that, these conjectures were later derived by *Atkin* and *Swinnerton-Dyer* [12]. Although their proof is analytic, and relies significantly on the properties of *modular functions*, we must admit that, no such combinatorial proof is known as such. All we can comment in combinatorial context about the rank are the following.

Theorem 5.1.7. *The following holds for every chosen positive integer n .*

1. $N(m, n) = N(-m, n)$
2. $N(m, j, n) = N(j - m, j, n)$
3. $N(m, j, n) = \sum_{r=-\infty}^{\infty} N(m + rq, n)$

Proof. (1) follows from the fact that, the operation of conjugation reverses the sign of the rank. A trivial consequence of the above argument helps us conclude (2). A priori, combining (1) and (2), we obtain (3). \square

Atkin and *Swinnerton-Dyer* [12] further provided justifications in support of validity of other conjectures, such as,

$$N(1, 5, 5n + 1) = N(2, 5, 5n + 1)$$

Their intention was to derive *generating functions* for $N(m, a, an + b) - N(l, a, an + b)$ in special cases like, $a = 5, 7$ and for all possible values of m, b and l .

Similar concepts were incorporated by *Atkin* and *Hussain* [13] to study the case when $a = 11$, and, *O'Brien* (1965) [14] too opted for the same technique to evaluate the case, when $a = 13$.

5.2 Generalization of Dyson's Rank

We must acknowledge the fact that, *Atkin* [15] generalized the definition (5.1.1) of rank provided by *Dyson*. Geometrically, given any $n \in \mathbb{N}$, we can represent any partition of n as a set of *nested right angles* of nodes.

For example, the partition, $5 + 4 + 3 + 3 + 1 \vdash 16$ has the following representation by 3 number of right angles. (These specific diagrams are also known as *Ferrers Graph*)

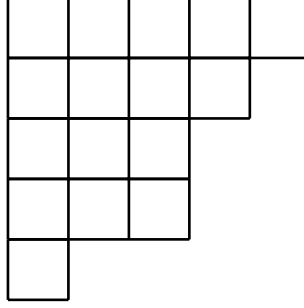


Figure 1: Ferrers Graph

5.3 Successive Ranks of a Partition

Given a partition λ , suppose $d_i(\lambda)$ denotes the number of *nodes* in the horizontal section of the i^{th} right angle in the graph of λ subtracted by the number of nodes in the vertical section of the same.

Thus, *Dyson's rank* is equal to $d_1(\lambda)$ and, $d_i(\lambda) = 0$, if λ doesn't have an i^{th} right angle. These $d_i(\lambda)$'s are termed as the **successive ranks** of λ .

In his article [15], *Atkin* provided alternate combinatorial interpretations of (3.2) and (3.3), which are analogous to (5.1) and (5.2) respectively.

We define $N^*(m, n)$ to be the number of partitions λ of any chosen n ($\in \mathbb{N}$) such that, $d_1(\lambda) - 2d_2(\lambda) = m$.

Similarly, denote the number of partitions $\lambda \vdash n$ such that, $d_1(\lambda) - 2d_2(\lambda) \equiv m \pmod{j}$ by $N^*(m, j, n)$.

Therefore, *Atkin* asserted that,

Theorem 5.3.1.

$$N^*(m, 5, 5n + 4) = \frac{1}{5}p(5n + 4) , 0 \leq m \leq 4 \quad (5.3)$$

$$N^*(m, 7, 7n + 5) = \frac{1}{7}p(7n + 5) , 0 \leq m \leq 6 \quad (5.4)$$

Moreover, he claimed that, given an i -conjugacy operation (denoted as C_i) acting on partitions satisfying,

$$d_1(C_i \lambda) = d_1(\lambda) - 2d_i(\lambda)$$

We can therefore conclude that,

$$N^*(m, n) = N(m, n) \quad (5.5)$$

Remark 5.3.2. A priori from the above identity, one can observe that, the statements (5.3) and (5.4) described in the *Theorem*(5.3.1) can be rendered as a trivial consequence of (5.1) and (5.2) respectively.

Remark 5.3.3. Later on, *Andrews* extensively studied *Atkin's Successive Ranks*. [16, §9.3]

5.4 Cranks of a Partition

The notion of *Crank* of a partition was first implemented by *Dyson* [9] when he claimed that, it isn't possible to classify the partition of any integer of the form $11n + 6$ in terms of their ranks mod 11, even though *Ramanujan's Congruence* (3.4) holds true.

His conjecture late on led to the existence of some other significant partition statistic (called as "*crank*"), which indeed provides a combinatorial interpretation of $\frac{1}{11}p(11n + 6)$ in similar manner as defined in (5.1) and (5.2).

Definition 5.4.1. (Crank of a Partition) Let, $\lambda \vdash n$ be any partition of some positive integer n . Suppose, we denote,

1. $l(\lambda) :=$ The largest part of λ .
2. $\omega(\lambda) :=$ Number of ones in λ .
3. $\mu(\lambda) :=$ Number of parts of λ (strictly) larger than $\omega(\lambda)$.

We define the *crank* of λ (denoted by $c(\lambda)$) as follows :

$$c(\lambda) := \begin{cases} l(\lambda) & \text{if, } \omega(\lambda) = 0 \\ \mu(\lambda) - \omega(\lambda) & \text{if, } \omega(\lambda) > 0. \end{cases} \quad (5.6)$$

If we denote the number of partitions of any positive integer n with *crank* congruent to m mod j as $M(m, j, n)$.

Proposition 5.4.1. *We shall have,*

$$M(m, j, n) = M(j - m, j, n)$$

Dyson [9] further conjectured the following in his article.

Theorem 5.4.2. (*Dyson*, 1944)

$$M(m, 11, 11n + 6) = \frac{1}{11}p(11n + 6), \quad \forall \quad 0 \leq m \leq 10 \quad (5.7)$$

Garvan proposed a combinatorial interpretation of (3.4) in his paper [Ref. [11]]. His deductions doesn't actually serves the purpose of classifying partitions of numbers of the form $11n + 6$ according to their *cranks* mod 11, but rather it illustrates the expression $\frac{1}{11}p(11n + 6)$ combinatorially in terms of the crank of the so called *vector partitions*.

5.5 Construction of Cranks relative to Vector Partitions

Andrews and *Garvan* [10] [11] derived an explicit method in order to compute cranks relative to *vector partitions*.

For any partition $\lambda \vdash n$, define, $\#(\lambda)$ to be the number of parts of λ and $\sigma(\lambda)$ to be the sum of parts of λ (or, the number ' λ ' is partitioning).

As per convention, us assume that, $\#(\phi) = \sigma(\phi) = 0$ for empty partition ϕ of 0. Furthermore, we denote,

$$V := \{(\lambda_1, \lambda_2, \lambda_3) \mid \lambda_1 \text{ is a partition into distinct parts, } \lambda_2, \lambda_3 \text{ are unrestricted partitions} \}$$

We define elements of V to be the *vector partitions*.

Definition 5.5.1. For every $\lambda \in V$, suppose,

1. $\mathfrak{S}_V :=$ The sum of parts.
2. $\mathfrak{W}_V :=$ A weight.
3. $\mathfrak{R} :=$ crank.

A priori, using the above notations, we define,

$$\mathfrak{S}(\vec{\lambda}) := \sigma(\lambda_1) + \sigma(\lambda_2) + \sigma(\lambda_3). \quad (5.8)$$

$$\mathfrak{W}(\vec{\lambda}) := (-1)^{\#(\lambda_1)}. \quad (5.9)$$

$$\mathfrak{R}(\vec{\lambda}) := \#(\lambda_2) - \#(\lambda_3). \quad (5.10)$$

Thus, we say that, $\vec{\lambda}$ is a *vector partition* of a positive integer n provided,

$$\mathfrak{S}(\vec{\lambda}) = n$$

Example 5.5.1. Consider, $\vec{\lambda} = (5 + 3 + 2, 2 + 2 + 1, 2 + 1 + 1)$. Then, $\mathfrak{S}(\vec{\lambda}) = 19$, $\mathfrak{W}(\vec{\lambda}) = -1$, $\mathfrak{R}(\vec{\lambda}) = 0$.

Consequently, $\vec{\lambda}$ is a vector partition of 19.

Suppose, $N_V(m, n)$ denotes the number of vector partitions of n (counted in accordance to the weight \mathfrak{W}) having $crank = m$.

Hence,

$$N_V(m, n) = \sum_{\substack{\vec{\lambda} \\ \mathfrak{S}(\vec{\lambda})=n \\ \mathfrak{R}(\vec{\lambda})=m}} \mathfrak{W}(\vec{\lambda}) \quad (5.11)$$

Moreover, assuming that, $N_V(m, j, n)$ yields the number of vector partitions of n (counting in terms of the weight \mathfrak{W}) with $crank$ congruent to $m \pmod j$.

Therefore, the following result follows.

Proposition 5.5.2.

$$N_V(m, j, n) = \sum_{k=-\infty}^{\infty} N_V(kj + m, n) = \sum_{\substack{\vec{\lambda} \\ \mathfrak{S}(\vec{\lambda})=n \\ \mathfrak{R}(\vec{\lambda})=m \pmod j}} \mathfrak{W}(\vec{\lambda})$$

An application of transformation formula (interchanging the roles of λ_2 and λ_3) helps us assert that,

Corollary 5.5.3.

$$N_V(m, n) = N_V(-m, n) \quad (5.12)$$

We shall prove the above Corollary in the next section.

As a consequence, we conclude,

Corollary 5.5.4.

$$N_V(j - m, j, n) = N_V(m, j, n) \quad (5.13)$$

Lemma 5.5.5. *The following identity does indeed illustrate a generating function for $N_V(m, n)$.*

$$\sum_{m=-\infty}^{\infty} \sum_{n=0}^{\infty} N_V(m, n) z^m q^n = \prod_{n=1}^{\infty} \frac{(1 - q^n)}{(1 - zq^n)(1 - z^{-1}q^n)} = \frac{(q; q)_{\infty}}{(zq; q)_{\infty} \left(\frac{q}{z}; q\right)_{\infty}} \quad (5.14)$$

Whenever $z \neq 0$, $|q| < 1$ and, $|z| < \frac{1}{|q|}$.

Remark 5.5.6. Putting $z = 1$ in (5.14) gives,

$$\sum_{m=-\infty}^{\infty} N_V(m, n) = p(n). \quad (5.15)$$

Apparently, we can thus derive an important relation regarding cranks of vector partitions and the partition function corresponding to integers of the form $5n + 4$, $7n + 5$ and $11n + 6$.

Theorem 5.5.7.

$$N_V(m, 5, 5n + 4) = \frac{1}{5}p(5n + 4) \quad (5.16)$$

$$N_V(m, 7, 7n + 5) = \frac{1}{7}p(7n + 5) \quad (5.17)$$

And,

$$N_V(m, 11, 11n + 6) = \frac{1}{11}p(11n + 6) \quad (5.18)$$

One may subsequently pose this question :

Does there exists a crank for every ordinary partition ?

A priori from the definition (5.4.1) of $crank(s)$ of a partition, we can infer that, the answer is "yes" indeed.

Its obvious to know comprehend the subsequent result.

Theorem 5.5.8. For every partition, say, $\lambda \vdash n$, given some $n \in \mathbb{N}$ and with,

$$c(\lambda) = m$$

Then, $N_V(m, n)$ is equal to the number of partitions $\lambda \vdash n$ (as defined in (5.11)) with $crank = m$, denoted by $M(m, n)$. In other words,

$$N_V(m, n) = M(m, n). \quad (5.19)$$

Proof. A priori from the definition of q -series (2.0.1) and (5.14),

$$\sum_{m=-\infty}^{\infty} \sum_{n=0}^{\infty} N_V(m, n) z^m q^n = \frac{(1-q)}{(zq)_{\infty}} \cdot \frac{(q^2; q)_{\infty}}{\left(\frac{q}{z}\right)_{\infty}} \quad (5.20)$$

$$= \frac{(1-q)}{(zq)_{\infty}} \sum_{j=0}^{\infty} \frac{(zq)_j \left(\frac{q}{z}\right)^j}{(q)_j} \quad (\text{ref. [16, pp. 17]}) \quad (5.21)$$

$$= \frac{(1-q)}{(zq)_{\infty}} + \sum_{j=1}^{\infty} \frac{q^j z^{-j}}{(q^2; q)_{j-1} (zq^{j+1})_{\infty}} \quad (5.22)$$

Using similar arguments as in (5.5.6), $z = 1$ implies that, the L.H.S. of (5.20) reduces to the generating function for $p(n)$. Furthermore, assuming $j > 0$, we can in fact conclude that, the j^{th} term in the sum on the R.H.S. is equal to,

$$\frac{z^{-j} q^{\overbrace{1+1+\dots+1}^{j\text{-times}}}}{\prod_{i=2}^j (1-q^i) \cdot \prod_{l=j+1}^{\infty} (1-zq^l)}$$

The standard techniques of partition theory [16, Chapt. 1] yields that the above expression surely generates partitions λ with, $\omega(\lambda) = j$, and the exponent on z can be computed as $\mu(\lambda) - \omega(\lambda) = c(\lambda)$ [$\because j > 0$]

In turn, we can in fact interpret the expression,

$$\frac{(1-q)}{\prod_{i=1}^{\infty} (1-zq^i)}$$

with respect to the generating function for the partitions without 1's. Considering conjugate partition, we obtain,

$$\frac{1}{\prod_{i=1}^{\infty} (1-zq^i)}$$

generates all the partitions with the exponent on z counting the largest part, and for integers larger than 1,

$$\frac{q}{\prod_{i=1}^{\infty} (1-zq^i)}$$

generates partition with at least one 1 appearing again with the exponent on z counting the largest part.

Important to observe that,, this interpretation doesn't hold true for 1, since this is the unique instance in which introducing a 1 into the partitions of $(n-1)$ alters the largest part. Hence, $\frac{1-q}{(zq)_{\infty}}$ counts (for $n > 1$) the number of partitions with no 1's and with the exponent of z being the largest part of the partition, suggesting, $l(\lambda) = c(\lambda)$.

Subsequently, in the double series expansion of,

$$\frac{1-q}{(zq)_{\infty}} + \sum_{j=1}^{\infty} \frac{q^j z^{-j}}{(q^2; q)_{j-1} (zq^{j+1})_{\infty}}$$

Using the above information, we end up with the following identity,

$$\sum_{m \in \mathbb{Z}} \sum_{n=0}^{\infty} N_V(m, n) z^m q^n = \sum_{m \in \mathbb{Z}} \sum_{n=0}^{\infty} M(m, n) z^m q^n$$

Observe that, the co-efficient of $(z^m q^n)$ ($n > 1$) is in fact equal to the number of ordinary partitions λ of n provided, $c(\lambda) = m$. Hence the proof is done.

Ultimately, (5.20) implies that, the statement for the *theorem* (5.5.8) holds true.

Remark 5.5.9. Ardent readers can read [11, pp. 52-53] as reference for further explicit illustrations regarding Crank of Vector partitions.

□

6 An Important Property of $M(m, j, n)$

Let, n be any positive integer. Fix a positive integer j . Then, we intend to establish the following relation regarding the number of partitions of any positive integer n with *crank* congruent to $m \pmod j$, usually denoted as $M(m, j, n)$.

Theorem 6.0.1.

$$M(m, j, n) = M(-m, j, n) \tag{6.1}$$

for every $m = 0, 1, 2, \dots, j-1$.

Before we proceed to the proof, it is absolutely imperative to establish following identities for the case when, $n > 1$ and $j > 1$. In other words, we intend to divide the whole proof into simpler steps.

Proposition 6.0.2.

$$N_V(m, n) = N_V(-m, n) \tag{6.2}$$

Proof. A priori for every integer m ,

$$\mathcal{A}_m(n) = \{V \ni v = (v_1, v_2, v_3) \vdash n \mid c_V(v) = m\}$$

and,

$$\mathcal{B}_m(n) = \{V \ni v = (v_1, v_2, v_3) \vdash n \mid c_V(v) = -m\}$$

Our primary intention is to construct a *linear transformation*, $T : \mathcal{A}_m(n) \longrightarrow \mathcal{B}_m(n)$, defined as :

$$T((v_1, v_2, v_3)) = (v_1, v_3, v_2), \quad \forall (v_1, v_2, v_3) \in \mathcal{A}_m(n),$$

We claim that, T is indeed a bijective map. Consequently, $|\mathcal{A}_m(n)| = |\mathcal{B}_m(n)|$. Therefore, $N_V(m, n) = N_V(-m, n)$, and we are done. \square

Our next aim is to study the behavior of the function $N_V(m, j, n)$ when we consider the vector partitions with crank congruent to $-m$ as compared to $m \pmod j$.

Proposition 6.0.3.

$$N_V(m, j, n) = N_V(-m, j, n)$$

For every $n \in \mathbb{N}$.

Proof. A priori from *proposition* (5.5.2), we obtain,

$$\begin{aligned} N_V(m, j, n) &= \sum_{k \in \mathbb{Z}} N_V(jk + m, n) = \sum_{k \in \mathbb{Z}} N_V(-(jk + m), n) \quad (\text{ref. (6.0.2)}) \\ &= \sum_{k \in \mathbb{Z}} N_V(jk - m, n) = N_V(-m, j, n). \end{aligned}$$

Therefore, the proof is done. \square

Finally, we does in fact have all the necessary tools to justify the result stated in *theorem* (6.0.1).

Proof. (**Proof of Theorem** (6.0.1))

The case : $n = 1$ and $j = 1$ holds trivially, as one can observe,

$$M(m, j, n) = M(0, 1, 1) = 1 = M(-0, 1, 1) = M(-m, j, n).$$

Consider the case when, $n > 1$ and $j > 1$.

We can deduce,

$$\begin{aligned} M(m, j, n) &= \sum_{k \in \mathbb{Z}} M(jk + m, n) = \sum_{k \in \mathbb{Z}} N_V(jk + m, n) \quad (\text{ref. (5.5.8)}) \\ &= N_{m, j, n} = N_V(-m, j, n) \quad (\text{ref. (6.0.3)}) \\ &= \sum_{k \in \mathbb{Z}} N_V(jk - m, n) = \sum_{k \in \mathbb{Z}} M(jk - m, n) = M(-m, j, n). \end{aligned}$$

This completes the proof. \square

7 Interpretation of the Partition Congruences $\pmod{5, 7}$ and 11 in terms of their corresponding Cranks

7.1 The Main Result

In this section, our main objective is to establish the following relations regarding partition congruences $\pmod{5, 7}$ and 11 in terms of partitions of any positive integer n with crank congruent to $m \pmod{j}$, denoted by $M(m, j, n)$.

Theorem 7.1.1.

$$M(j, 5, 5n + 4) = \frac{1}{5}p(5n + 4) , \quad 0 \leq j \leq 4 \quad (7.1)$$

$$M(j, 7, 7n + 5) = \frac{1}{7}p(7n + 5) , \quad 0 \leq j \leq 6 \quad (7.2)$$

$$M(j, 11, 11n + 6) = \frac{1}{11}p(11n + 6) , \quad 0 \leq j \leq 10 \quad (7.3)$$

Garvan made a significant observation in his article [11] by representing the identities deduced in (5.5.7) in a more compact form.

Theorem 7.1.2. *Given a prime number s and r_s be the reciprocal of 24 modulo s . Then,*

$$N_V(j, s, sn + r_s) = \frac{1}{s}p(sn + r_s) , \quad 0 \leq j \leq s - 1 \quad (7.4)$$

iff, $a_{sn+r_s} = 0$, where,

$$\sum_{n \geq 0} a_n q^n = \frac{(q; q)_\infty}{(\zeta_s q; q)_\infty (\zeta_s^{-1} q; q)_\infty} \quad (7.5)$$

ζ_s being the primitive s^{th} root of unity.

Equivalently, one can also study (7.4) in the following manner.

Proposition 7.1.3. *Suppose, s be a prime, proving the statement (7.4) is equivalent to estimating the co-efficient of q^{sn+r_s} in the product,*

$$\prod_{n=1}^{\infty} \frac{(1 - q^n)}{(1 - \zeta_s q^n)(1 - \zeta_s^{-1} q^n)} \quad (7.6)$$

to be equal to 0, where, $\zeta_s = \exp(2\pi i/s)$ is the primitive s^{th} root of unity.

Proof. A priori, considering any prime s , our aim is to rewrite (7.6) in terms of $N_V(k, s, n)$. Following the substitution, $z = \zeta_s$ into L.H.S. of (5.14),

$$\begin{aligned} \sum_{m=-\infty}^{\infty} \sum_{n=0}^{\infty} N_V(m, n) \zeta_s^m q^n &= \sum_{k=0}^{s-1} \sum_{\substack{m \equiv k \\ \pmod{s}}} \sum_{n=0}^{\infty} N_V(m, n) \zeta_s^m q^n \\ &= \sum_{k=0}^{s-1} \zeta_s^k \sum_{n=0}^{\infty} \left(\sum_{\substack{m \equiv k \\ \pmod{s}}} N_V(m, n) \right) q^n = \sum_{k=0}^{s-1} \zeta_s^k \sum_{n=0}^{\infty} N_V(k, s, n) q^n \quad (\text{ref. (5.5.2)}) \end{aligned}$$

Therefore,

$$\frac{(q)_{\infty}}{(\zeta_s q)_{\infty} (\zeta_s^{-1} q)_{\infty}} = \sum_{k=0}^{s-1} \zeta_s^k \sum_{n=0}^{\infty} N_V(k, s, n) q^n \quad (7.7)$$

Important to note that,

$$\sum_{k=0}^{s-1} N_V(k, s, sn + r_s) \zeta_s^k$$

equals to the co-efficient of q^{sn+r_s} in the expression on the L.H.S. of (7.7), i.e.,

$$\sum_{k=0}^{s-1} N_V(k, s, sn + r_s) \zeta_s^k = N_V(0, s, sn + r_s) \sum_{k=0}^{s-1} \zeta_s^k = 0$$

as expected.

Conversely, assuming that, the co-efficient of q^{sn+r_s} in L.H.S. of (7.7) is 0, it implies,

$$\sum_{k=0}^{s-1} N_V(k, s, sn + r_s) \zeta_s^k = 0 \quad (7.8)$$

Note that, L.H.S. of (7.8) is indeed a polynomial in ζ_s over \mathbb{Z} . Subsequently,

$$N_V(j, s, ns + r_s) = N_V(j+1, s, ns + r_s) \quad \forall \quad j = 0, 1, 2, \dots, s-2. \quad (7.9)$$

A priori using the fact that, s is a prime and the minimal polynomial for ζ_s over \mathbb{Q} being,

$$p(x) = 1 + x + \dots + x^{s-1} = \left(\frac{x^s - 1}{x - 1} \right)$$

In conclusion, combining (5.15) and (7.9), we obtain,

$$p(sn + r_s) = \sum_{k=0}^{s-1} N_V(k, s, sn + r_s) = s N_V(0, s, sn + r_s)$$

Hence, (7.4) holds true. □

In the subsequent subsections, we shall establish each of the identities stated in (7.1), (7.2) and (7.3) in order to justify our claim.

7.2 Special Case when $s = 5$

A priori using (7.1.2), we'll prove that, (7.1) is true. It only suffices to show that, $a_{5n+4} = 0$.

Suppose, ζ_5 be the primitive 5th root of unity. By the *Euler Pentagonal Theorem* (2.0.2), we obtain,

$$\prod_{n \geq 1} (1 - q^n) = \sum_{n \in \mathbb{Z}} (-1)^n q^{n(3n-1)/2} = 1 + \sum_{n \geq 1} (-1)^n q^{n(3n-1)/2} (1 + q^n),$$

and,

$$\prod_{n \geq 1} (1 - q^n)(1 - zq^n)(1 - z^{-1}q^n) = \sum_{n \geq 0} (-1)^n q^{n(n+1)/2} z^{-n} \left(\frac{1 - z^{2n+1}}{1 - z} \right)$$

It can be deduced that,

$$\begin{aligned} \sum_{n \geq 0} a_n q^n &= \frac{(q; q)_\infty}{(\zeta_5 q; q)_\infty (\zeta_5^{-1} q; q)_\infty} \\ &= \frac{\sum_{n \in \mathbb{Z}} \sum_{m \geq 0} (-1)^{n+m} q^{\{\frac{n(3n-1)}{2} + \frac{m(m+1)}{2}\}} \zeta_5^{-2m} \left(1 - \zeta_5^{2(2m+1)} \right)}{(q^5; q^5)_\infty (1 - \zeta_5^2)} \end{aligned}$$

Since, $\frac{n(3n-1)}{2} \equiv 0, 1, 2 \pmod{5}$ and, $\frac{m(m+1)}{2} \equiv 0, 1, 3 \pmod{5}$, the power of q is congruent to 4 $\pmod{5}$ only when,

$$\frac{n(3n-1)}{2} \equiv 1 \pmod{5}, \text{ and, } \frac{m(m+1)}{2} \equiv 3 \pmod{5}$$

Therefore, we conclude that, $m \equiv 2 \pmod{5}$, and so, $2m + 1 \equiv 0 \pmod{5}$. Consequently,

$$((\zeta_5^2)^{2m+1} - 1) = 0$$

Hence, $a_{5n+4} = 0$. By *theorem* (7.1.2),

$$N_V(i, 5, 5n + 4) = \frac{1}{5} p(5n + 4), \quad 0 \leq i \leq 4$$

Since, $N_V(m, j, n) = M(m, j, n)$ (**ref.**(5.5.8)), we thus conclude,

$$M(i, 5, 5n + 4) = \frac{1}{5} p(5n + 4), \quad 0 \leq i \leq 4$$

as desired.

7.3 Special Case when $s = 7$

A priori using (7.1.2), we'll show that, (7.2) is satisfied. It only suffices to show that, $a_{7n+5} = 0$.

Suppose, ζ_7 be the primitive 7^{th} root of unity. A priori implementing similar arguments as in section 7.2, we obtain,

$$\begin{aligned} \sum_{n \geq 0} a_n q^n &= \frac{(q; q)_\infty}{(\zeta_7 q; q)_\infty (\zeta_7^{-1} q; q)_\infty} \\ &= \frac{\sum_{n, m \geq 0} (-1)^{n+m} q^{\{\frac{n(n+1)}{2} + \frac{m(m+1)}{2}\}} \zeta_7^{-2n-3m} \left(1 - \zeta_7^{2(2n+1)}\right) \left(1 - \zeta_7^{3(2m+1)}\right)}{(q^7; q^7)_\infty (1 - \zeta_7^5)(1 - \zeta_7^3)} \end{aligned} \quad (7.10)$$

Since, $\frac{n(n+1)}{2} \equiv 0, 1, 3, 6 \pmod{7}$, implies that, the power of q in R.H.S. of (7.10) is in fact congruent to 5 modulo 7 provided,

$$\frac{n(n+1)}{2} \equiv \frac{m(m+1)}{2} \equiv 6 \pmod{7}$$

therefore, $n \equiv m \equiv 3 \pmod{7} \implies 2m+1 \equiv 0 \pmod{7}$, and, $2n+1 \equiv 0 \pmod{7}$.

As a consequence,

$$((\zeta_7^2)^{2n+1} - 1) = 0, \text{ and, } ((\zeta_7^3)^{2m+1} - 1) = 0$$

Thus, $a_{7n+5} = 0$. In conclusion,

$$N_V(i, 7, 7n+5) = \frac{1}{7} p(7n+5), \quad 0 \leq i \leq 6$$

A priori applying similar arguments as in Case : $s = 5$, we thus conclude,

$$M(i, 7, 7n+5) = \frac{1}{7} p(7n+5), \quad 0 \leq i \leq 6$$

as required.

7.4 Special Case when $s = 11$

A priori using (7.1.2), we'll show that, (7.3) is satisfied. It only suffices to show that, $a_{11n+6} = 0$.

Suppose, ζ_{11} be the primitive 11^{th} root of unity. We shall require *Winquist's Identity* [17].

Theorem 7.4.1. (*Winquist's Identity*)

$$\begin{aligned} &(q; q)_\infty^2 (y; q)_\infty \left(\frac{q}{y}; q\right)_\infty (z; q)_\infty \left(\frac{q}{z}; q\right)_\infty \left(\frac{y}{z}; q\right)_\infty \left(\frac{zq}{y}; q\right)_\infty (yz; q)_\infty \left(\frac{q}{yz}; q\right)_\infty \\ &= \sum_{i \geq 0} \sum_{j \in \mathbb{Z}} (-1)^{i+j} ((y^{-3i} - y^{3i+3})(z^{-3j} - z^{3j+1}) + (y^{-3j+1} - y^{3j+2})(z^{3i+2} - z^{-3i-1})) q^{\{\frac{3i(i+1)}{2} + \frac{j(3j+1)}{2}\}} \end{aligned} \quad (7.11)$$

Replacing y by ζ_{11}^9 and z by ζ_{11}^5 in (7.4.1) yields,

$$\sum_{n \geq 0} a_n q^n = \frac{(q; q)_\infty}{(\zeta_{11} q; q)_\infty (\zeta_{11}^{-1} q; q)_\infty}$$

$$= \frac{\sum_{i \geq 0} \sum_{j \in \mathbb{Z}} (-1)^{i+j} \left((\zeta_{11}^{6i} - \zeta_{11}^{5i+5})(\zeta_{11}^{7j} - \zeta_{11}^{4j+5}) + (\zeta_{11}^{6j+9} - \zeta_{11}^{5j+7})(\zeta_{11}^{4i+10} - \zeta_{11}^{7i+6}) \right) q^{\left\{ \frac{3i(i+1)}{2} + \frac{j(3j+1)}{2} \right\}}}{(1 - \zeta_{11}^3)(1 - \zeta_{11}^4)(1 - \zeta_{11}^5)(1 - \zeta_{11}^9)(q^{11}; q^{11})_\infty} \quad (7.12)$$

Since,

$$\frac{3i(i+1)}{2} \equiv 0, 1, 3, 7, 8, 9 \pmod{11}$$

and,

$$\frac{j(3j+1)}{2} \equiv 0, 1, 2, 4, 5, 7 \pmod{11}$$

Hence, the power of q is indeed congruent to 6 modulo 11 provided, $i \equiv 5 \pmod{11}$ and, $j \equiv 9 \pmod{11}$.

Therefore,

$$(\zeta_{11}^{6i} - \zeta_{11}^{5i+5}) = 0, \text{ and, } (\zeta_{11}^{6j+9} - \zeta_{11}^{5j+7}) = 0$$

Consequently, $a_{11n+6} = 0$. Accordingly,

$$N_V(i, 11, 11n+6) = \frac{1}{11} p(11n+6), \quad 0 \leq i \leq 10$$

A priori applying similar arguments as in **Case** : $s = 5$ and, in **Case** : $s = 7$, we thus conclude,

$$M(i, 11, 11n+6) = \frac{1}{11} p(11n+6), \quad 0 \leq i \leq 10$$

Remark 7.4.2. We can in fact, infer from the results proven in the above three special cases in section 7.2, 7.3 and, 7.4 that, the *Crank* does indeed provides an explicit and thorough explanation to the Partition Congruences modulo 5, 7 and 11.

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Statements and Declarations

Conflicts of Interest Statement

I as the sole author of this article certify that I have no affiliations with or involvement in any organization or entity with any financial interest (such as honoraria; educational grants; participation in speakers' bureaus; membership, employment, consultancies, stock ownership, or other equity interest; and expert testimony or patent-licensing arrangements), or non-financial interest (such as personal or professional relationships, affiliations, knowledge or beliefs) in the subject matter or materials discussed in this manuscript.

Data Availability Statement

I as the sole author of this aarticle confirm that the data supporting the findings of this study are available within the article [and/or] its supplementary materials.

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