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# THE K41 THEORY AND TURBULENCE IN 1D BURGERS EQUATION.

SERGEI KUKSIN

ABSTRACT. This work is a review of a group of results on the stochastic Burgers equation with small viscosity, obtained during the last two decades. These results jointly show that the equation makes a good model of hydrodynamical turbulence. The model provides natural and rigorously justified analogies of a number of key predictions of the theory of turbulence, including the main assertions of the Kolmogorov approach to turbulence, known as the K41 theory. The presentation is based on the recent book Boritchev, Kuksin *One-Dimensional Turbulence and the Stochastic Burgers Equation* (AMS, 2021) and on two papers which develop the book's results. Part of the assertions which we review here together with their rigorous proofs, were earlier heuristically proved in works of physicists.

## 1. INTRODUCTION: KOLMOGOROV'S THEORY AND ITS 1D MODEL

A heuristic theory of turbulence, known now as the K41 theory, was created in three articles, published by A.N. Kolmogorov in 1941, and in two articles of his student Obukhov which appeared the same year. The theory for the first time put in formulas the thesis that *when the Reynolds number grows, the flow becomes chaotic, that is turbulent*. It describes statistical properties of flows with large Reynolds numbers, and probably now is the most popular theory of turbulence. Since taking into account turbulence is indispensable for numerous major applications and since still there is no rigorous theory of turbulence, then at the current stage of development of the field any mathematically correct theory which is consistently related to K41 and may be compared with it, is useful and important. In this paper we give a concise survey of such a theory which deals with turbulence in fictitious one-dimensional fluid, described by the space-periodic one-dimensional stochastic Burgers equation.

The Burgers equation as a 1d model of fluid motion was suggested by Burgers in late 1930's and since then was systematically used in this quality by him (e.g. see [6]) and by other experts in hydrodynamics. In 1980's-1990's Frisch studied on the physical level of rigour the equation with small viscosity and random initial data and/or random forcing, regarding this as a stochastic model of 1d turbulence, see [1] and references there. Motivated by this work, in 1990's Sinai, himself and with collaborators, started to examine the stochastic Burgers equation under the inviscid limit  $\nu \rightarrow 0$ . This research resulted in the influential paper [10] and then was continued by Sinai's students and followers (including E. Iturriaga, Khanin and others, e.g. see [15, 16]). Earlier this century, also in connection with 1d turbulence, the space-periodic Burgers equation with small positive viscosity was examined by two students of the author, Biriuk and Boritchev, using tools from nonlinear PDEs and some ideas from previous work of the author on nonlinear PDEs with small dissipation (see [5] for references). The study was continued by the two and the author and

led to the book [5], which shows that many basic statements of Kolmogorov’s theory allow rigorous interpretations in the Burgers framework in terms of solutions for the stochastic Burgers equation.<sup>1</sup>

The goal of this paper is to review main results of the book [5] and their recent developments in [17, 14] in a concise way, free from mathematical details, not needed to understand the assertions. Still we will explain main ideas, underlying the proofs of the results. Among other things we will show that the main laws of K41 have natural analogies for the 1d Burgers turbulence, with modified exponents and pre-exponential factors (if the latter is present in a law). Part of our results, given below, were proved earlier on the physical level of rigour, see a discussion after item **VIII** of Introduction. It is worth mentioning that the 2d turbulence, described by the stochastic space-periodic 2d Navier–Stokes system with small viscosity, behaves differently compare to the Burgers 1d turbulence which we talk about here, as well as compare to the 3d turbulence as is predicted by Kolmogorov. See a discussion on pages 94-95 of [5] and see [19, 20] for rigorous results on 2d turbulence.

Speaking about the K41 theory, as in the K41 papers, we assume that the turbulent 3d velocity fields  $u(t, x)$  under discussion are random fields, and will usually assume that they are stationary in time and homogeneous in space. In addition, unless otherwise stated, we suppose that these fields are space-periodic and normalise the period to be one. As in K41 we choose the units in such a way that the velocity fields are of order one, uniformly in small  $\nu$ :

$$(1.1) \quad \mathbb{E}|u(t, x)|^2 \sim 1.$$

Then their Reynolds numbers equal  $\nu^{-1}$ , where  $\nu$  is the viscosity. We also suppose (again as in K41) that the rates of energy dissipation  $\epsilon^K$  of the flows remains of order one as  $\nu \rightarrow 0$ ,

$$(1.2) \quad \epsilon^K := \nu \mathbb{E}|\nabla u(t, x)|^2 \sim 1.$$

Here and below  $A \sim B$  means that the ratio  $A/B$  is bounded from below and from above by positive constants, independent of  $\nu$  and of the indicated arguments of  $A$  and  $B$  (like  $\mathbf{k}$  in (1.12) below).

In the K41 papers Kolmogorov did not mention the equations which underly his theory. But starting from the work of Edwards in 1963 it became a dominant opinion that this should be the 3d Navier–Stokes system on a 3d torus with small viscosity, perturbed by a random force, smooth and homogeneous in  $x$ . Accordingly, talking about K41 and 3d turbulence we will sometimes assume that the corresponding velocity fields  $u(t, x)$  satisfy the stochastic 3d Navier–Stokes system since this assumption helps to explain better some assertions of the 3d theory.

In the rest of Introduction we will briefly present the main results of this work and will compare them with the corresponding assertions of K41. Section 2 of the paper is preliminary. There we develop notation and state main results about the Burgers equation with small positive viscosity, needed for the main part of the paper, formed by Sections 3-6. Here and below “Burgers equation” stands for the “space-periodic stochastic Burgers equation”. Unless otherwise stated, the random force in the equation is assumed to be not zero, delta-correlated in time and smooth in  $x$ .

**I. Rate of energy dissipation and intermittency.** In Section 2.2 we establish an asymptotic for second moments of Sobolev norms  $\|u^\nu(t, \cdot)\|_m$  of solutions  $u^\nu(t, x)$  for the Burgers equation with

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<sup>1</sup>We also mention the work [7], where without any relation to the Burgers equation is constructed a class of stationary processes  $u^\nu(x)$ ,  $x \in S^1$ , satisfying 1d versions of the main predictions of K41.

small positive viscosity  $\nu$ :

$$(1.3) \quad \langle\langle \|u^\nu(t, \cdot)\|_m^2 \rangle\rangle \sim \nu^{-(2m-1)}, \quad m \in \mathbb{N}.$$

Here  $\langle\langle \cdot \rangle\rangle$  stands for the averaging in ensemble and local averaging in time, see (2.10) in Section 2. As a consequence of (1.3), the energy of high frequency components of a solution for the Burgers equation with fixed initial data increases when the viscosity  $\nu$  decays to zero. In particular, taking  $m = 1$  we get from (1.3) that for the Burgers equation the rate of energy dissipation  $\epsilon^B$  is of order one uniformly in  $\nu$ :

$$(1.4) \quad \epsilon^B := \nu \langle\langle \|u_x\|^2 \rangle\rangle \sim 1.$$

We also show that for solutions of the equation the averaged energy is of order one, uniformly in  $\nu$ :

$$(1.5) \quad \langle\langle \|u\|^2 \rangle\rangle \sim 1$$

(here and below  $\|\cdot\|$  signifies the  $L_2$ -norm in  $x$ ).

By (1.4) any solution  $u(t, x)$  of the Burgers equation satisfies  $\langle\langle \int u_x^2 dx \rangle\rangle \sim \nu^{-1}$ . In the same time one of the basic estimates for the equation's solutions (see below (2.9)) implies that  $\langle\langle \int |u_x| dx \rangle\rangle \leq \text{Const}$ . In fact,  $\langle\langle \int |u_x| dx \rangle\rangle \sim 1$  and  $\langle\langle \int |u_x|^p dx \rangle\rangle \sim \nu^{1-p}$  for any  $p \geq 1$ , see [5, Section 2.4]. This demonstrates quantitatively that space-derivatives of 1d Burgers flows  $u$  with small viscosity are highly intermittent.

We stress that to prove (1.3) and (1.5), as well as other results, discussed below in items **II-VII**, we do not assume any non-degeneracy of the random force. In particular, the force may have only one non-zero Fourier coefficient; see below Section 2.1.

Note that analogies of (1.5) and (1.4) for 3d turbulent fluid flows – relations (1.1) and (1.2) – still are not rigorously proved (assuming that the velocity fields  $u$  are described by the stochastic 3d Navier–Stokes system).

**II. Dissipation scale.** We establish that for the Burgers equation the *dissipation scale*  $l_d^B$  (also known as the *inner scale*)<sup>2</sup> behaves as

$$(1.6) \quad l_d^B \sim \nu.$$

It means the following. Let us denote by  $\hat{u}^\nu(t, s)$  the Fourier coefficients of a solution  $u^\nu(t, x)$  of the space-periodic stochastic Burgers equation with a fixed initial data at  $t = 0$ , independent of  $\nu$ , and set  $E_k = \frac{1}{2} \langle\langle |\hat{u}(t, k)|^2 \rangle\rangle$  (this is “the averaged energy of a  $k$ -th mode”). Then for  $k \gg C\nu^{-1} \sim (l_d^B)^{-1}$  the energy  $E_k$  decreases with growing  $k$  faster than any negative power  $k^{-N}$ , and such decay is missing for smaller  $k$ 's. The scale (1.6) is justified in Theorem 3.2, also see the end of Section 3.1.

To compare, K41 predicts that for turbulence in real fluids the dissipation scale is

$$(1.7) \quad l_d^K \sim \nu^{3/4} \quad (\text{Kolmogorov's inner scale}).$$

**III. Moments of small-scale increments.** In Section 3.2 we discuss small-scale increments of velocity fields  $u(t, x)$ , which are the quantities  $u(t, x+l) - u(t, x)$  with  $|l| \ll 1$ . For solutions of the Burgers equation, we examine (up to equivalence  $\sim$ ) their averaged in  $x$  degrees  $s_{p,l}(u) = \int_{S^1} |u(t, x+l) - u(t, x)|^p dx$ ,  $p > 0$ , next averaged using the brackets  $\langle\langle \cdot \rangle\rangle$ . We prove that, in agreement with

<sup>2</sup>Note that in the book [5] by inner scale is called the inverse quantity  $(l_d^B)^{-1}$ .

K41, the behaviour of thus obtained functions  $l \mapsto \langle \langle s_{p,l} \rangle \rangle$  crucially depends on whether  $|l|$  is smaller or bigger than the dissipation scale  $l_d^B$ . The interval  $[l_d^B, c] = [c_1\nu, c] =: I_{inert}$ , where  $c$  and  $c_1$  are some constants, depending only on the random force, is called the *inertial range* (for the 1d Burgers turbulence). We show that for  $l$  in this range

$$(1.8) \quad \langle \langle s_{p,l}(u) \rangle \rangle \sim |l|^{\min(1,p)}, \quad \forall p > 0.$$

In particular, for such  $l$  for any solution  $u$  second moments of its increments  $\langle \langle s_{2,l}(u) \rangle \rangle$  behave as  $|l|$ .<sup>3</sup> The complementing interval  $[0, c_1\nu]$  is the *dissipation range*. For  $|l|$  from in range

$$(1.9) \quad \langle \langle s_{p,l}(u) \rangle \rangle \sim |l|^p \nu^{1-\max(1,p)}, \quad \forall p > 0.$$

Relation (1.8) with  $p = 2$  makes a rigorous analogy for turbulence in the Burgers equation of the celebrated 2/3-law of the K41 theory. The latter deals with longitudinal increments of velocity fields  $u$  of fluid  $(u(t, x+l) - u(t, x)) \cdot \frac{l}{|l|}$  and their moments  $S_{p,l}^{\parallel}(u)$ ,

$$(1.10) \quad S_{p,l}^{\parallel}(u) = \mathbb{E} s_{p,l}^{\parallel}(u(t, x)), \quad s_{p,l}^{\parallel}(u(t, x)) = \left( \frac{(u(t, x+l) - u(t, x)) \cdot l}{|l|} \right)^p, \quad p \in \mathbb{N}$$

(the moments do not depend on  $t$  and  $x$  since in K41  $u$  is assumed to be stationary in  $t$  and homogeneous in  $x$ ). The 2/3-law predicts the following behaviour of  $S_{2,l}^{\parallel}$  for  $|l|$  in Kolmogorov's inertial range:

$$(1.11) \quad S_{2,l}^{\parallel}(u) \sim (\epsilon^K |l|)^{2/3} \quad \text{if } |l| \in [C_1\nu^{3/4}, C_2],$$

see [21, 13]. We note that while recent experimental and numerical study of 3d turbulence indicates that the 2/3-law (1.11) holds only approximatively (i.e. the right exponent for  $\epsilon^K |l|$  is not exactly 2/3),<sup>4</sup> its analogy for the 1d Burgers fluid, stated above, is a completely rigorous result.

**IV. Distribution of energy along the spectrum.** By the definition of dissipation scale  $l_d^B$ , for  $|k| \gg C\nu^{-1}$  the averaged energy  $E_k = \frac{1}{2} \langle \langle \hat{u}(t, k)^2 \rangle \rangle$  decays faster than any negative power of  $|k|$ . For  $|k| < C\nu^{-1} = l_d^B$  the behaviour of  $E_k$  is quite different. Namely, Theorem 3.4 in Section 3.3 establishes that if for  $\mathbf{k} \geq 1$  we denote by  $E_{\mathbf{k}}^B$  the energy  $E_k$ , averaged over wave-vectors  $k'$  such that  $|k'|$  is close to  $\mathbf{k}$ , then

$$(1.12) \quad E_{\mathbf{k}}^B \sim \mathbf{k}^{-2} \quad \text{for } \mathbf{k}^{-1} \text{ in Burgers' inertial range, } \{c_1\nu \leq \mathbf{k}^{-1} \leq 1\}.$$

Such decay of the spectral energy of solutions for the Burgers equation was predicted by Burgers in [6], using heuristic argument.

The K41 theory claims the celebrated Kolmogorov–Obukhov law

$$(1.13) \quad E_{\mathbf{k}}^K \sim \mathbf{k}^{-5/3} \quad \text{for } \mathbf{k}^{-1} \text{ in Kolmogorov's inertial range } \{c\nu^{3/4} \leq \mathbf{k}^{-1} \leq c'\}.$$

Here the averaged energy  $E_{\mathbf{k}}^K$  is defined similarly to  $E_{\mathbf{k}}^B$ , but since the random fields  $u(t, x)$  in K41 are assumed to be stationary in  $t$ , then in the definition of  $E_{\mathbf{k}}^K$  the averaging  $\langle \langle \cdot \rangle \rangle$  may be replaced by the mathematical expectation.

<sup>3</sup>Earlier on the physical level of rigour these results on the moments of short-scale increments of solutions for the Burgers equation were obtained in [1].

<sup>4</sup>E.g. see Fig. 3 in [12].

If we model 3d turbulence in fluid by space-periodic solutions of the stochastic 3d Navier–Stokes system, then the validity of the laws (1.7), (1.11) and (1.13) of K41 becomes completely open mathematical problems.

**V. Statistical Equilibrium (the mixing).** We assume now that a turbulent flow is described by a random field  $u(t, x)$  which is a solution of the space-periodic stochastic 3d Navier–Stokes system. Then the distribution of  $u(t, \cdot)$  is a probability measure  $\mu_t$  on the space of divergence-free vector fields. The entire K41 theory concerns the stationarity turbulence which means that  $\mu_t \equiv \bar{\mu}^\nu$  is a statistical equilibrium of the system.

The existence of such an equilibrium measure  $\bar{\mu}^\nu$  easily follows by the well known Krylov–Bogolyubov argument (see in [25, 5]). On the other hand, its uniqueness is a highly nontrivial problem. The uniqueness is indispensable for the K41 theory since the latter operates exclusively in the equilibrium regimes. This central role of statistical equilibrium would be justified by the convergence

$$(1.14) \quad \mu_t \rightharpoonup \bar{\mu}^\nu, \quad t \rightarrow \infty,$$

for the distribution  $\mu_t$  of any solution  $u(t, \cdot)$  of the stochastic 3d Navier–Stokes system. If (1.14) holds, then the random process  $u(t, \cdot)$  is *mixing* (e.g. see in [9, 19]). Moreover, the convergence (1.14) must be uniform in small  $\nu > 0$  to justify the relevance of the equilibrium  $\bar{\mu}^\nu$  for turbulence (e.g. see in [2]).

The mixing (1.14) and its uniformity in small  $\nu > 0$  are challenging open problems in the framework of 3d turbulence. On the other hand, both these aspects are rigorously established for the Burgers equation, see Section 4.

**VI. Kolmogorov’s 4/5-law and the Landau objection.** This law of K41 deals with the third longitudinal moment of increments of a turbulent velocity field  $u$  (see (1.10)), assuming that  $u$  is a stationary in time, homogeneous and isotropic in space random field (so it is not space-periodic). The law predicts that

$$(1.15) \quad S_{3,l}^{\parallel}(u) = -\frac{4}{5}\epsilon^K |l| + o(\epsilon^K |l|),$$

if  $|l|$  belongs to the inertial range, see [21, 13, 11, 3]. In Section 5.1, for solutions  $u$  of the Burgers equation we examine the signed cubic moments of their increments in  $x$ ,  $s_{3,l}^s(u(t)) = \int (u(t, x + l) - u(t, x))^3 dx$ . We show that if  $u(t, x)$  is a stationary in time solution of the equation<sup>5</sup> and  $|l|$  belongs to a “strongly inertial range” which is just “a bit smaller” than the inertial range  $[c_1\nu, c_2]$  (see (5.5)), then

$$(1.16) \quad \mathbb{E}s_{3,l}^s(u(t)) = -12\epsilon^B l + o(l).$$

While for any solution  $u$  we get a weak form of this relation:  $\langle\langle s_{3,l}^s(u(t)) \rangle\rangle \sim -l$ .

The celebrated Landau objection to the universality of the 2/3-law (1.11) (which was originally claimed by Kolmogorov for homogeneous and isotropic velocity fields), states that this law cannot hold in the stronger form  $S_{2,l}^{\parallel}(u) = C^K(\epsilon^K |l|)^{2/3} + o((\epsilon^K |l|)^{2/3})$  with a universal constant  $C^K$ ,

<sup>5</sup>That is, for any  $t$  the law of  $u(t, \cdot)$  is a stationary measure of the stochastic Burgers equation, which is unique by item V.

independent of the random force. In Section 5.2 we show that the third moment of space-increments of stationary in time solutions for the Burgers equation indeed is the only one for which the power law, discussed above in item **III**, holds in a strong form like (1.16), with a universal pre-factor (which is -12 for the third moment).

**VII. Inviscid solutions** ( $\nu = 0$ ). In Section 6 we discuss the well known (e.g. see [10]) existence of an inviscid limit for solutions of the Burgers equation as  $\nu \rightarrow 0$ :  $u^\nu(t, x) \rightarrow u^0(t, x)$ , a.s. The limit  $u^0$  is a discontinuous function, bounded for bounded  $t$ , which satisfies the inviscid stochastic Burgers equation in the sense of generalised functions, and is traditionally called an *entropy*, or an *inviscid* solution.

We study moments of small-scale increments  $u^0(t, x + l) - u^0(t, x)$  of entropy solutions and establish for them an asymptotic behaviour which for any  $l > 0$  has the form (1.8). So – in a sense – for entropy solutions the dissipation scale is zero (and the dissipation range is empty). Moreover, we establish that for them the energy spectrum still has the form (1.12), now – for all  $\mathbf{k} \geq 1$ . We also show that the version of Kolmogorov’s 4/5 law as in item **VI** stays true for stationary entropy solutions, for all  $l$ .

**VIII.** In Appendix we discuss the Burgers equation with a discretised in time white noise. This model corresponds to the numerical schemes, used to calculate solutions for the stochastic Burgers equation. We prove the uniform in time convergence in law of solutions with discretised forces to a solution of the stochastic Burgers equation when the step of discretisation goes to zero.

*How new is this.* As we already mentioned, part of the results for the 1d Burgers turbulence which we discuss in this work jointly with their rigorous proofs, were earlier established on the physical level of rigour. Namely, relation (1.4) certainly was well familiar, but probably not its general analogy (1.3) with  $m \geq 2$ . The size (1.6) of the inner scale was known to Burgers. The “anomaly asymptotics” (1.8) for moments of increments of a Burgers velocity field  $u$  with  $|l|$  in the inertial range is a discovery of the work [1], but the asymptotics (1.9) with  $|l|$  in the dissipation range probably was not known before. The spectral power law (1.10) again is due to Burgers, but we do not know if the delicate question how the averaged in ensemble energy  $\mathbb{E}\|\hat{u}(k)\|^2$  should be next averaged in  $k$  for the asymptotics to hold, was discussed in the literature. The uniform in viscosity mixing (1.14) for 1d turbulence was not known before the work of our group. The version (1.16) of Kolmogorov’s 4/5-law for 1d turbulence with the universal pre-factor -12 was known to physicists, but the rigorous version of Landau’s argument for 1d turbulence seems to be new. The situation with the laws of turbulence for inviscid solutions of the Burgers equation is similar to that for solutions with small viscosity we have just discussed.

Saying that we mention that despite the K41 laws of 3d turbulence were proved by Kolmogorov on the physical level of rigour and were then re-proved by many physicists, the task of proper verification of the K41 theory remains an outstanding open problem. The author of this work is not in a position to discuss competently if the heuristic proofs in the existing physical literature of the laws of 1d turbulence are significantly more convincing than the proofs of the K41 laws, but he believes that the fact that the Burgers model of 1d turbulence in its stochastic setting is now rigorously justified, deserves attention of the physical and mathematical communities.

*About the proofs.* In Section 2 basic facts about the Burgers equation are given without demonstration, while proofs for the results in main Sections 3-6 are provided if they are short, and are briefly sketched otherwise. Complete demonstrations of all results may be found in book [5] and paper [14] (for Section 5). Paper [18], based on LN for a lecture course of the author, contains verifications of most results in this paper in a compressed form, with dropped technical details.

*Notation.* For a Banach space  $X$  and  $R > 0$  we denote by  $B_X^R$  the open ball  $\{u \in X : \|u\|_X < R\}$ . By  $\|\cdot\|$  we denote the  $L_2$ -norm for functions on  $S^1$ , by  $\|\cdot\|_m$  – their homogeneous  $m$ -th Sobolev norm (see (2.6)) and by  $|\cdot|_p$  – the norm in  $L_p(S^1)$ . For a function  $f$  and a measure  $\mu$  we write  $\langle f, \mu \rangle = \int f(u)\mu(du)$ . By  $\mathcal{P}(M)$  we denote the set of probability measures on  $M$ ,<sup>6</sup> and by the symbol  $\rightharpoonup$  denote the weak convergence of measures.<sup>7</sup> Sets of zero measure are called null-sets. For a r.v. (random variable)  $\eta$ , valued in  $M$ ,  $\mathcal{D}(\eta) \in \mathcal{P}(M)$  signifies its distribution.  $C(M)$  stands for the set of continuous functions on  $M$ . By  $\mathbb{N}$  (by  $\mathbb{Z}^*$ ) we denote the set of positive (of non-zero) integers.

## 2. STOCHASTIC BURGERS EQUATION

The initial-value problem for the space-periodic Burgers equation reads

$$(2.1) \quad \begin{cases} u_t(t, x) + uu_x - \nu u_{xx} = \eta(t, x), & t \geq 0, \\ u(0, x) = u_0(x). \end{cases} \quad \left| \quad x \in S^1 = \mathbb{R}/\mathbb{Z}. \right.$$

Here

$$0 < \nu \leq 1,$$

so everywhere below “for all  $\nu$ ” means “for all  $0 < \nu \leq 1$ ”. The force  $\eta$  is a random field  $\eta^\omega(t, x)$ , defined on a probability space  $(\Omega, \mathcal{F}, \mathbf{P})$ , and specified below.

We always assume that  $\int \eta^\omega(t, x) dx \equiv \int u_0^\omega(x) dx = 0$ . Since  $uu_x = \frac{1}{2} \frac{\partial}{\partial x} u^2$ , then integrating the Burgers equation (2.1) over  $S^1$ , we get that  $\frac{\partial}{\partial t} \int u(t, x) dx \equiv 0$ , so

$$\int u(t, x) dx \equiv 0, \quad t \geq 0.$$

Consider the space

$$H = \{u \in L_2(S^1) : \int u(x) dx = 0\},$$

equipped with the  $L_2$ -scalar product  $\langle \cdot, \cdot \rangle$  and the  $L_2$ -norm  $\|\cdot\|$  (so  $\|u\|^2 = \langle u, u \rangle = \int u^2 dx$ ). We will regard a solution  $u$  of the Burgers equation (2.1) either as a function  $u(t, x)$  of  $t, x$ , or as a curve  $t \mapsto u(t, \cdot) =: u(t) \in H$ , depending on the random parameter  $\omega$ . That is, either as a random field  $u^\omega(t, x)$ , or as a random process  $u^\omega(t) \in H$ .

Below in this section we show that eq. (2.1) is well posed and study properties of its solutions with small  $\nu$ . In particular, we obtain lower and upper bounds for the second moments of their Sobolev norms which are asymptotically sharp as  $\nu \rightarrow 0$  in the sense that they involve  $\nu$  in the same negative degree. Then in the latter sections we state one-dimensional versions of the main laws of the K41 theory and use the results of Section 2 to prove them rigorously for the fictitious 1d fluid whose motion is described by eq. (2.1).

<sup>6</sup>More precisely, except of the standard triad  $(\Omega, \mathcal{F}, P)$ , in such a context  $M$  is a complete metric space, and  $\mathcal{P}(M)$  signifies the space of probability Borel measures on  $M$ .

<sup>7</sup>That is, for measures  $\mu_n \in \mathcal{P}(M)$  we have  $\mu_n \rightharpoonup \mu$  if for each bounded continuous function  $f$  it holds  $\langle f, \mu_n \rangle \rightarrow \langle f, \mu \rangle$ .

**2.1. Function spaces and random force.** We denote by  $\{e_s(x) \in H : s \in \mathbb{Z}^* = \mathbb{Z} \setminus \{0\}\}$  the orthonormal trigonometric basis of  $H$ ,

$$(2.2) \quad e_s(x) = \begin{cases} \sqrt{2} \cos 2\pi s x, & s \geq 1, \\ \sqrt{2} \sin 2\pi |s|x, & s \leq -1. \end{cases}$$

Any  $u \in H$  decomposes as  $u(x) = \sum_{s \in \mathbb{Z}^*} u_s e_s(x)$ ,  $x \in S^1$ , and may be written as the Fourier series

$$u(x) = \sum_{s \in \mathbb{Z}^*} \hat{u}_s e^{2\pi i s x}, \quad \hat{u}_s = \bar{\hat{u}}_{-s} = \frac{1}{\sqrt{2}}(u_s - i u_{-s}), \quad s \in \mathbb{N}.$$

*The force and solutions.* We suppose that  $\eta(t, x)$  in (2.1) is a regular function of  $x$ , while as a function of  $t$  it is a distribution:

$$(2.3) \quad \eta = \eta^\omega(t, x) = \partial_t \xi^\omega(t, x), \quad \xi^\omega(t, x) = \sum_{s \in \mathbb{Z}^*} b_s \beta_s^\omega(t) e_s(x).$$

Here  $\{b_s\}$  are real numbers, and  $\{\beta_s\}$  are standard independent Wiener processes on some probability space  $(\Omega, \mathcal{F}, \mathbf{P})$ . Abusing language we also call random field  $\xi$  ‘‘a force’’. It is easy to see that

$$(2.4) \quad \text{if } b_s \equiv b_{-s}, \text{ then random field } \xi(t, x) \text{ is homogeneous in } x$$

(see [5, Section 1.5]). For a non-negative integer  $m$  we denote  $B_m = \sum |2\pi s|^{2m} b_s^2$ , and will always suppose that

$$(2.5) \quad B_0 \neq 0, \quad B_m < \infty \quad \text{for each } m$$

(e.g. the numbers  $b_s$  form a non-zero sequence which converges to zero exponentially fast). We do not assume any non-degeneracy of the random force  $\xi$ . So it may be that only one number  $b_s$  is non-zero.

Let  $m \in \mathbb{N}$ . The Hilbert space  $H^m$  is the Sobolev space  $\{v \in H : (\partial^m / \partial x^m)v \in H\}$ , equipped with the homogeneous Hilbert norm

$$(2.6) \quad \|v\|_m := \left\| \frac{\partial^m}{\partial x^m} v \right\|.$$

If  $v(x) = \sum v_s e_s(x)$ , then  $\|v\|_m^2 = \sum |2\pi s|^{2m} |v_s|^2$ . We also set  $H^\infty = \cap H^m = C^\infty(S^1) \cap H$ . Next, for  $0 < T < \infty$ , we introduce the Banach spaces

$$X_T^m = C(0, T; H^m).$$

For  $T = \infty$  we set  $X_\infty^m = C(0, \infty; H^m)$ . This is a metric space with the distance

$$\text{dist}(u, v) = \sum_{T=1}^{\infty} 2^{-T} \psi(|(u - v)|_{[0, T]}|_{X_T^m}), \quad \psi(r) := r/(1 + r), \quad r \geq 0.$$

The sum in (2.3) a.s. converges and defines random processes in the spaces  $H^m$  with the following properties:

**Theorem 2.1.** *If (2.5) holds, then there exists a null-set  $Q$  such that for each non-negative integer  $m$  we have:*

*i) for  $\omega \notin Q$  and  $t \geq 0$  the series in (2.3) converges in  $H^m$  to a limit  $\xi(t)$  which is a continuous process in  $H^m$ , vanishing at zero. For  $\omega \in Q$  we set  $\xi = 0$ .*

ii)  $\mathbb{E}\|\xi(t)\|_m^2 = tB_m \quad \forall t \geq 0$ .

iii) For any  $T < \infty$ ,  $\mathbb{E}e^{\alpha\|\xi\|_{X_T^m}^2} \leq 4e^{2T\alpha B_m} - 3$  if  $\alpha \leq \alpha_m(T) = 1/(4TB_m)$ .

Since  $\xi(t)$  is a Wiener process in the spaces  $H^m$ , then eq. (2.1) defines there Markov dynamics, see below Section 2.2. The Markov property is crucial for our approach. The latter applies to any equation (2.1) with a random force  $\eta$ , which defines Markov processes in function spaces (e.g. to equation (2.1) with  $\eta = (\partial/\partial t)\xi(t, x)$ , where  $\xi(t, \cdot)$  is a Lévy process, see [26]). It also applies to equations with stochastically time-periodic forces  $\eta$  which define Markov chains in functional spaces (see [5, Sections 10.1.2, 10.1.3] and [4]). We do not know how to study 1d turbulence in equations (2.1) without the Markov property. E.g. how to do this for the equations, where  $\eta(t, x)$  is a smooth random field whose correlations in  $t$  decays exponentially fast.

**2.2. Solutions of the equation.** Equation (2.1) with a non-random function  $\eta(t, x) = \partial_t \xi(t, x)$ , where  $\xi$  is continuous in  $t$  and smooth in  $x$ , is well posed in the spaces  $H^m$ : if  $u_0 \in H^m$ ,  $m \in \mathbb{N}$ , then (2.1) has a unique solution  $u \in X_\infty^m$ . These solutions posses an important property of non-expanding in the  $L_1$ -norm: if for  $j = 1, 2$ ,  $u^j(t, x) =: u^j(t)$  solves eq. (2.1) with  $u_0 = u_j$  and with the same  $\eta(t, x) = \partial_t \xi(t, x)$ , then for all  $T \geq 0$ ,

$$(2.7) \quad |u^1(T) - u^2(T)|_1 \leq |u_1 - u_2|_1.$$

Applying the existence of a solution result above to equation (2.1) with the random force  $\eta(t, x) = \partial_t \xi(t, x)$  as in (2.3) for each value of the random parameter, we obtain a solution of the stochastic equation which will be denoted

$$u^{\nu\omega}(t, x; u_0) =: u(t; u_0).$$

Here  $u_0(x)$  is either a non-random function from some space  $H^m$ ,  $m \geq 1$ , or a r.v. in  $H^m$ , independent of the random force. If  $u_0 \in H^m$  is non-random, then the second moments  $\mathbb{E}\|u(t; u_0)\|_m^2$  are bounded uniformly in  $t \geq 0$ . For any  $m \in \mathbb{N}$  the solutions  $u(t; u_0)$  define a Markov process in space  $H^m$

A remarkable property of solutions  $u(t; u_0)$  is given by the Oleinik maximum principle below. To get it one applies a variation of the argument, given in [22] for equation (2.1) with a smooth force  $\eta$ , to solutions of eq. (2.1) with the force  $\eta$  as in (2.3) and fixed  $\omega$ , and then take the expectation, using Theorem 2.1.iii). In this way one gets that for any r.v.  $u_0 \in H^1$ , independent of  $\xi(t, x)$ , any  $0 < \theta \leq 1$  and any  $t \geq \theta$ ,  $p \geq 1$  the solution  $u(t; u_0)$  satisfies the estimates

$$(2.8) \quad \mathbb{E}|u_x^+(t)|_\infty^p \leq C_p \theta^{-p},$$

$$(2.9) \quad \mathbb{E}[|u(t)|_\infty^p + |u_x(t)|_1^p] \leq C_p \theta^{-p}$$

where  $C_p$  depends only on the random force (and  $p$ ). In (2.8) for a real number  $a$ ,  $a^+$  stands for  $\max(a, 0)$ . The second estimate (2.9) is a consequence of the first one, but it is convenient to keep it as a separate assertion.

Further analysis of solutions for (2.1), (2.3), based on the Oleinik estimates, allows to establish equivalent lower and upper bounds for second moments of Sobolev norms (2.6) of solutions, locally averaged in time. To state the corresponding result we fix any  $\theta > 0$ , and for any random process

$f^\omega(t)$ ,  $t \geq 0$ , denote

$$(2.10) \quad \langle\langle f \rangle\rangle = \mathbb{E} \frac{1}{\sigma} \int_T^{T+\sigma} f(s) ds,$$

where the parameters  $T, \sigma$  of the brackets satisfy  $T \geq \theta$  and  $\sigma \geq \sigma_*$  with some  $\sigma_* > 0$  depending only on  $\theta$  and the random force  $\xi$ . (So if  $f^\omega(t)$  is a stationary process, then  $\langle\langle f \rangle\rangle = \mathbb{E}f(t)$  for any  $t$ ). Then for each r.v.  $u_0 \in H^1$ , independent of  $\xi$ , solution  $u^\nu(t; u_0)$  satisfies

$$(2.11) \quad \langle\langle \|u^\nu\|_m^2 \rangle\rangle \sim \nu^{-(2m-1)} \quad \text{for any } m \in \mathbb{N}.$$

Moreover, the upper bounds in these relations hold without the local averaging in time:

$$(2.12) \quad \mathbb{E}\|u^\nu(t)\|_m^2 \leq C_m \nu^{-(2m-1)} \quad \forall t \geq \theta, \text{ for any } m \in \mathbb{N}.$$

The constants  $C_m$  depend only on  $m, \theta$  and  $\xi$ , while the constants in the lower and upper bounds, corresponding to  $\sim$  in (2.11) (see the explanation of this symbol below (1.2)), depend only on  $m, \sigma_*, \theta$  and  $\xi$ . For  $m = 0$  the behaviour of the norm  $\|u^\nu\|_0$  is not given by (2.11) and instead

$$\langle\langle \|u^\nu\|_0^2 \rangle\rangle \sim 1.$$

Note that relation (2.11) with  $m = 1$  may be written as (1.2).

Relation (2.11) immediately implies that  $\frac{\ln \langle\langle \|u^\nu\|_m^2 \rangle\rangle}{\ln \nu^{-1}} = 2m - 1 + o(1)$  as  $\nu \rightarrow 0$ . But can (2.11) be improved to an asymptotic for  $\langle\langle \|u^\nu\|_m^2 \rangle\rangle$ ?

*Open problem.* Prove (or disprove) that for  $m \in \mathbb{N}$  the second moment  $\langle\langle \|u^\nu\|_m^2 \rangle\rangle$  admits an asymptotic expansion:

$$\langle\langle \|u^\nu\|_m^2 \rangle\rangle = C_m \nu^{-(2m-1)} + o(\nu^{-(2m-1)}) \quad \text{as } \nu \rightarrow 0,$$

for some  $C_m > 0$ , depending on the random force.

If  $u^\nu(t, x)$  is a stationary in time solution (see Section 4), then from the energy balance (see below (5.7)) we get that  $\langle\langle \|u^\nu\|_1^2 \rangle\rangle = B_0 \nu^{-1}$ . So in the stationary case the asymptotic above holds for  $m = 1$  with  $C_1 = B_0$ . This is the only case when we know its validity.

### 3. TURBULENCE IN 3D AND 1D

Now we will discuss the main heuristic laws of turbulence and the K41 theory in their relation with rigorous results for the 1d turbulence, described by the Burgers equation (2.1), (2.3). The latter will be derived from the results, presented in the previous section. In this section we assume that

$$(3.1) \quad u(t) = u^\nu(t, x; u_0) \text{ where } u_0 \in H^1 \text{ is a r.v., independent of } \xi.$$

We repeat that our restriction on the random force (2.3) is very mild; see relation (2.5) and below it.

While speaking about K41, we always assume that the corresponding flows  $u(t, x)$  are as in Section 1. See there relations (1.1), (1.2) and a discussion around them.

**3.1. Dissipation scale.** The essence of the K41 theory is analysis of properties of turbulent flows that are uniform in small values of viscosity  $\nu$  (in our choice of units, when the Reynolds number equals  $\nu^{-1}$ ). Crucial for Kolmogorov's analysis are the concepts of dissipation and inertial ranges.

The *dissipation range* is the region, corresponding to predominance of the dissipation term in the Navier–Stokes system, while the complementary *inertial range* is characterised by predominance of the inertial term. The two ranges may be defined in the Fourier- or in  $x$ -presentation. For our purposes we choose the first option.

*In 1d turbulence.* Now we present a rigorous theory of the ranges in context of the stochastic Burgers equation (2.1). Let us write its solution  $u(t, x)$  as Fourier series

$$u(t, x) = \sum_{s \in \mathbb{Z}^*} \hat{u}_s(t) e^{2\pi i s x}.$$

The ranges of  $u$ , regarded as a 1d turbulent flow with viscosity  $\nu$ , are defined via its *dissipation scale*  $l_d$ , a.k.a. the *inner scale*. We define it in the Fourier representation as the biggest number of the form

$$l_d(\nu) = \nu^\gamma, \quad \gamma > 0$$

(corresponding to the smallest possible exponent  $\gamma > 0$ ), such that for  $|s| \geq l_d^{-1}$ , the averaged squared Fourier coefficient  $\langle\langle |\hat{u}_s(t)|^2 \rangle\rangle$  as a function of  $|s|$  decays fast, *uniformly in*  $\nu$ .

To be precise, for a solution  $u = u^\nu(t, x)$  of (2.1) let  $\Gamma$  denote the set of all real numbers  $\gamma' > 0$  such that

$$\forall N \in \mathbb{N} \exists C > 0 \text{ such that } \langle\langle |\hat{u}_s|^2 \rangle\rangle \leq C |s|^{-N} \quad \forall |s| \geq \nu^{-\gamma'}, \quad \forall \nu \in (0, 1],$$

where  $C$  depends only on  $N$  and  $\gamma'$ .

**Definition 3.1.** *Mathematical dissipation scale*  $l_d = l_d(\nu)$  of  $u$  (if it exists) equals  $\nu^\gamma$ , where  $\gamma > 0$  is defined as  $\gamma = \inf \Gamma$ . If  $\Gamma = \emptyset$  or  $\inf \Gamma = 0$ , then  $l_d$  is not defined.

Relations (2.11) relatively easily imply (see in [5]):

**Theorem 3.2.** *The dissipation scale*  $l_d$  of any solution  $u^\nu(t; u_0)$  of (2.1) equals  $\nu$ .

In physics, the dissipation scale is defined modulo a constant factor, so for eq. (2.1) the physical dissipation scale is  $Cl_d = C\nu$ . For the Burgers equation, Burgers himself predicted its correct value.

*In 3d turbulence.* In K41 the hydrodynamical (Kolmogorov’s) dissipation scale is predicted to be  $l_d^K = C\nu^{3/4}$  (we recall (1.1) and (1.2)).

*The ranges.* The dissipation and inertial ranges are zones, specifying the sizes of the involved space-increments and Fourier modes. Namely, the *dissipation range* (in the  $x$ -presentation) is the interval  $I_{diss} = [0, cl_d]$ , and the *inertial range* is the interval  $I_{inert} = [cl_d, c_1]$ . The constants  $c$  and  $c_1$  certainly do not depend on  $\nu$ , and for the 1d turbulence, described by eq. (2.1), they depend only on the random force (2.3). These constants may change from one group of results to another.

Theorem 3.3 below implies that in the framework of 1d turbulence the dissipation range  $I_{diss}$  also may be defined in the  $x$ -representation as the largest closed interval to the right from zero, such that for  $|l| \in I_{diss}$  the increments  $u(t, x + l) - u(t, x)$  “statistically behave linearly in  $l$ ”.

**3.2. Moments of small-scale increments.** *In 1d turbulence.* Small-scale increments in  $x$  of a solution  $u(t, x)$  for eq. (2.1) are quantities  $u(t, x + l) - u(t, x)$ , where  $x \in S^1$  and  $|l| \ll 1$ . Their

absolute moments with respect to the averaging in  $x$  and brackets  $\langle\langle \cdot, \cdot \rangle\rangle$ , are

$$\langle\langle \int |u(t, x+l) - u(t, x)|^p dx \rangle\rangle =: S_{p,l} = S_{p,l}(u), \quad p > 0.$$

Function  $(p, l) \mapsto S_{p,l}$  is called *the (absolute) structure function* of a solution  $u$ . Naturally, if a solution  $u(t, x)$  is stationary in  $t$  (see below in Section 4), then  $S_{p,l}(u) = \mathbb{E}(\int |u(t, x+l) - u(t, x)|^p dx)$ . If in eq. (2.1) the force  $\xi(t, x)$  is homogeneous in  $x$  (see (2.4)) and  $u_0^\omega$  also is, then the random solution  $u^\omega(t, x; u_0)$  is homogeneous in  $x$ . In this case  $S_{p,l}(u) = \langle\langle |u(t, x+l) - u(t, x)|^p \rangle\rangle$ , for any  $x \in S^1$ . If in addition  $u^\omega(t, x; u_0)$  is stationary in time, then

$$S_{p,l}(u) = \mathbb{E}|u(t, x+l) - u(t, x)|^p,$$

for any  $t$  and  $x$ ; cf.  $S_{p,l}^{\parallel}$  in (1.10).

The function  $S_{p,l}(u)$ , calculated for any solution  $u$  of (2.1), (2.3) with  $u_0$  as in (3.1), obeys the following law:

**Theorem 3.3.** *If  $u = u(t; u_0)$  is a solution as above and  $0 < \nu \leq c_*$  for a sufficiently small  $c_* > 0$ , then for each  $p > 0$  there exists  $C'_p \geq 1$  such that for  $|l|$  in inertial range  $I_{inert} = [c_1\nu, c]$  with suitable  $c_1, c > 0$  the structure function  $S_{p,l} = S_{p,l}(u)$  satisfies:*

$$(3.2) \quad C_p'^{-1} |l|^{\min(1,p)} \leq S_{p,l} \leq C'_p |l|^{\min(1,p)}.$$

While for  $|l|$  in dissipation range  $I_{diss} = [0, c_1\nu]$ ,

$$(3.3) \quad C_p^{-1} |l|^p \nu^{1-\max(1,p)} \leq S_{p,l} \leq C_p |l|^p \nu^{1-\max(1,p)}, \quad \forall p > 0.$$

The constants  $c_*, C_p, C'_p, c, c_1$  depend on the force  $\xi$ .

In [1] U.Frisch with collaborators obtained assertion (3.2) by a convincing heuristic argument.

Theorem 3.3 follows from estimates (2.8), (2.9) and (2.11), (2.12), but the corresponding argument is rather technical. Roughly, to get the upper bounds in (3.2), (3.3) we use the above-mentioned estimates and combine them with Hölder's inequality and various interpolation inequalities, routinely used in nonlinear PDEs. To establish the lower bounds we consider the space  $\tilde{\Omega} = [T, T + \sigma] \times \Omega$  and provide it with the probability measure  $\tilde{\mathbf{P}} = (dt/\sigma) \times \mathbf{P}$ .<sup>8</sup> Firstly we show that estimates (2.8), (2.9), (2.11), (2.12) imply that there exists a positive constant  $\alpha$ , and for each  $\nu > 0$  there is an event  $Q_\nu \subset \tilde{\Omega}$ , such that  $((dt/\sigma) \times \mathbf{P})Q_\nu \geq \alpha$ , with the following property. For every  $(t, \omega) \in Q_\nu$  the function  $v(x) = u^{\nu\omega}(t, x)$  meets the inequalities

$$\begin{aligned} \alpha\nu^{-1/2} &\leq \|v\|_1 \leq \alpha^{-1}\nu^{-1/2}, \\ |v_x^+|_\infty + |v_x|_1 + \nu^{3/2}\|v\|_2 + \nu^{5/2}\|v\|_3 &\leq M, \end{aligned}$$

where  $M$  is a  $\nu$ -independent constant. Next we show that for each  $\nu$  every function  $v$  as above satisfies the lower estimates in (3.2), (3.3) with dropped averaging  $\langle\langle \cdot \rangle\rangle$ . This implies the lower bounds in Theorem 3.3.

<sup>8</sup>So for a process  $f^\omega(t)$  its average  $\langle\langle f \rangle\rangle$  equals the integral of  $f$  against  $\tilde{\mathbf{P}}$ .

The theory of turbulence also is interested in signed moments of increments of velocity fields, corresponding to the *skew structure function*

$$(3.4) \quad S_{p,l}^s = S_{p,l}^s(u) := \langle\langle \int (u(t, x+l) - u(t, x))^p dx \rangle\rangle, \quad p \in \mathbb{N}$$

(cf. (1.10)). Obviously  $S_{p,l}^s = S_{p,l}$  for an even integer  $p$ , but for odd  $p$ 's the moments (3.4) are different from  $S_{p,l}$ . The first signed moment  $S_{1,l}^s$  vanishes, and the third moment  $S_{3,l}^s$  is of special interest. Denote  $v_t(x) = (u(t, x+l) - u(t, x))^3$ . As  $v_t(x) = -|v_t(x)| + 2v_t(x)^+$  and since by (2.8)

$$\langle\langle \int_{S^1} (v_t(x))^+ dx \rangle\rangle \leq l^3 \langle\langle |u_x^+(t)|_\infty^3 \rangle\rangle \leq C_1 l^3,$$

then in view of (3.2) for  $|l|$  in the inertial range we have that

$$-C^{-1}l + 2C_1 l^3 \geq \langle\langle \int (u(t, x+l) - u(t, x))^3 dx \rangle\rangle \geq -Cl.$$

Thus, there exists  $c' > 0$  (independent of  $\nu$ ) such that

$$(3.5) \quad S_{3,l}^s(u) \sim -l \quad \text{for } l \in [c_1\nu, c'], \quad 0 < \nu \leq c_*.$$

This is a weak form for 1d turbulence of the 4/5-law from the K41 theory, which we discuss below in Section 5. Literally the same argument shows that relation (3.5) hold for all moments  $S_{p,l}^s(u)$  with odd integers  $p \geq 3$ .

*In 3d turbulence.* For a 3d velocity field  $u(x)$  the products  $(u_i(x+l) - u_i(x)) \frac{l_j}{|l|}$  make a 9-tensor, and the corresponding hydrodynamical structure function organises its moments or absolute moments. Instead, experts in turbulence often work with the scalar longitudinal structure function  $S_{p,l}^{\parallel}(u)$ , where  $p \in \mathbb{N}$ , defined in (1.10). For the second structure function (assuming, as usual, that a velocity field  $u$  is stationary in time and homogeneous in space), for  $|l|$  in the inertial range the K41 theory predicts the celebrated *2/3-law* (1.11). In the same time we have seen in (3.2) that in the 1d case in the inertial range  $S_{2,l} \sim |l|$ .

In the K41 papers the 2/3-law was stated in a stronger form: it was claimed there that in the inertial range

$$(3.6) \quad S_{2,l}^{\parallel}(u) = C^K (\epsilon^K |l|)^{2/3} + o((\epsilon^K |l|)^{2/3}),$$

where  $C^K$  is an absolute constant. But then, due to a criticism from Landau, it became clear that the asymptotic above cannot hold (at least, with an absolute constant  $C^K$ ). See in [21] a footnote at p. 126, see [13, Section 6.4]. We will return to this issue below in Section 5.2.

**3.3. Distribution of energy along the spectrum.** *In 1d turbulence.* For a solution  $u(t, x)$  of the Burgers equation, regarded as the velocity of a 1d flow, consider the halves of its averaged Fourier coefficients  $\frac{1}{2} \langle\langle |\hat{u}_s|^2 \rangle\rangle$  which describe the distribution of averaged energy of the 1d flow  $u$  along the spectrum. Another celebrated law of the K41 theory deals with similar quantities, calculated for 3d turbulent flows; we will return to this below.

For any  $\mathbf{k} \geq 1$  define  $E_{\mathbf{k}}^B(u)$  as the averaging of  $\frac{1}{2}\langle\langle|\hat{u}_s|^2\rangle\rangle$  along the layer  $J_{\mathbf{k}}^M$  around  $\pm\mathbf{k}$ , defined as  $J_{\mathbf{k}}^M = \{n \in \mathbb{Z}^* : M^{-1}\mathbf{k} \leq |n| \leq M\mathbf{k}\}$ ,  $M > 1$ . I.e.

$$(3.7) \quad E_{\mathbf{k}}^B(u) = \langle\langle e_{\mathbf{k}}^B(u) \rangle\rangle, \quad e_{\mathbf{k}}^B(u) = \frac{1}{|J_{\mathbf{k}}^M|} \sum_{n \in J_{\mathbf{k}}^M} \frac{1}{2} |\hat{u}_n|^2.$$

The function  $\mathbf{k} \mapsto E_{\mathbf{k}}^B$  is the *energy spectrum* of  $u$ .

It is immediate from the definition of the inner scale  $l_d$  that for  $\mathbf{k}^{-1}$  in the dissipation range,  $E_{\mathbf{k}}^B(u)$  decays faster than any negative degree of  $\mathbf{k}$  (uniformly in  $\nu$ ): for any  $N \in \mathbb{N}$ ,

$$E_{\mathbf{k}}^B \leq C_N \mathbf{k}^{-N} \quad \text{if } \mathbf{k} \gg l_d^{-1} = \nu^{-1}.$$

But for  $\mathbf{k}^{-1}$  in the inertial range the behaviour of  $E_{\mathbf{k}}^B$  is quite different:

**Theorem 3.4.** *There exists  $M' > 1$  (depending on the random force) such that if in the definition of energy spectrum we use layers  $J_{\mathbf{k}}^M$  with  $M \geq M'$ , then for  $\mathbf{k} \in \mathbb{N}$  with  $\mathbf{k}^{-1}$  in the inertial range  $I_{inert}$ , i.e. for  $1 \leq \mathbf{k} \leq c^{-1}\nu^{-1}$ , we have:*

$$(3.8) \quad E_{\mathbf{k}}^B(u^\nu) \sim \mathbf{k}^{-2}.$$

For solutions of the Burgers equation, Burgers already in 1948 predicted (3.8) for  $\mathbf{k} < \text{Const } \nu^{-1}$ , i.e. exactly the spectral power law above, see [6].

*Open problem.* Is the assertion of the theorem true for any  $M > 1$  if  $\mathbf{k}_0 \leq \mathbf{k} \leq c^{-1}\nu^{-1}$  with a suitable  $\mathbf{k}_0(M) \geq 1$ , independent of  $\nu$ ?

*Sketch of the proof.* We have to establish for  $\mathbf{k}^{-1} \in I_{inert}$  the bounds

$$(3.9) \quad C\mathbf{k}^{-2} \geq E_{\mathbf{k}}^B \geq C^{-1}\mathbf{k}^{-2}$$

for some  $C > 1$ . The upper bound is easy since

$$\hat{u}_k = \int u(x) e^{-2\pi i k x} dx = \frac{1}{2\pi i k} \int u'(x) e^{-2\pi i k x} dx,$$

and so by (2.9)

$$(3.10) \quad \langle\langle |\hat{u}_k|^2 \rangle\rangle \leq (2\pi k)^{-2} \langle\langle |u_x|_1^2 \rangle\rangle \leq Ck^{-2}.$$

To get the lower bound in (3.9) it suffices to properly estimate from below the quantity

$$\Psi_{\mathbf{k}} = \sum_{|n| \leq M\mathbf{k}} |n|^2 \langle\langle |\hat{u}_n|^2 \rangle\rangle$$

and estimate from above the sum  $\sum_{|n| < M^{-1}\mathbf{k}} |n|^2 \langle\langle |\hat{u}_n|^2 \rangle\rangle$ . An estimate for the latter immediately follows from (3.10). To bound from below  $\Psi_{\mathbf{k}}$  we note that as  $|\alpha| \geq |\sin \alpha|$ , then

$$(3.11) \quad \Psi_{\mathbf{k}} \geq \frac{\mathbf{k}^2}{\pi^2} \left( \sum_n \sin^2(n\pi\mathbf{k}^{-1}) \langle\langle |\hat{u}_n|^2 \rangle\rangle - \sum_{|n| > M\mathbf{k}} \langle\langle |\hat{u}_n|^2 \rangle\rangle \right).$$

Since by Parseval's identity  $\|u(t, \cdot + \mathbf{k}^{-1}) - u(t, \cdot)\|^2 = 4 \sum_n \sin^2(n\pi\mathbf{k}^{-1}) |\hat{u}_n(t)|^2$ , then the second structure function with  $l = \mathbf{k}^{-1}$  may be written as

$$S_{2, \mathbf{k}^{-1}}(u) = 4 \sum_n \sin^2(n\pi\mathbf{k}^{-1}) \langle\langle |\hat{u}_n(t)|^2 \rangle\rangle.$$

Thus a lower bound for the first terms in the r.h.s. of (3.11) follows from (3.2) with  $p = 2$ . An upper bound for the second term follows from (3.10). For  $M$  sufficiently big this implies the lower bound in (3.9) after a small work.  $\square$

*In 3d turbulence.* Let us consider a turbulent flow  $u(t, x)$  of water (as in (1.1), (1.2) and around them) with Fourier coefficients  $\hat{u}(t, s)$ . Next for  $\mathbf{k} \geq 1$  denote by  $E_{\mathbf{k}}^K$  the average of energies  $\frac{1}{2}|\hat{u}(t, s)|^2$  in  $s$  from a suitable layer around the sphere  $\{|s| = \mathbf{k}\}$  and in ensemble. The celebrated Kolmogorov–Obukhov law predicts that

$$E_{\mathbf{k}}^K \sim \mathbf{k}^{-5/3} \quad \text{for } \mathbf{k}^{-1} \text{ in Kolmogorov's inertial range,}$$

see [21, 13].

*Open problem.* We saw that the Oleinik estimate and theorem on moments of small-scale increments of solutions for (2.1) jointly imply the spectral power law of 1d turbulence. Under what assumption on  $u(x)$  the latter is equivalent to the theorem on moments of small-scale increments? More interesting is this question, asked for the laws of K41: under what restriction on a field  $u(x)$  the Kolmogorov 2/3-law is equivalent to the Kolmogorov–Obukhov law? Or at least one of them implies another? On the physical level of rigour the two laws are claimed to be equivalent, e.g. see in [21, 13]. But mathematically the situation is not clear.<sup>9</sup> See Section 3.4 of [17] for a discussion of this question.

**3.4. Energy dissipation and vorticity stretching.** *In 3d turbulence.* As a mechanism, responsible for appearance of turbulence, Kolmogorov suggested the energy cascade to high frequencies. Namely, a smooth external force, applied to fluid with a space-periodic velocity field  $u(t, x)$ , pumps energy to low frequencies (i.e. to low Fourier modes in  $x$  of  $u(t, x)$ ). Then the Euler nonlinearity spread the energy to high frequencies, where the kinetic friction dissipates it to heat. This leads to growth of  $\nabla_x u$ , while the total energy of the fluid stays of order one when  $\nu$  decays to zero, cf. (1.1) and (1.2).

The vorticity of a 3d flow  $u(t, x)$  is  $\text{curl} u(t, x) =: \zeta$ . As a manifestation of the energy cascade experts in turbulence often consider the “stretching of vorticity”, which they understand as follows. Postulating that  $u(t, x)$  satisfies the 3d Navier–Stokes system with a random force, we get for the “energy of vorticity”  $\frac{1}{2}\mathbb{E}|\zeta(t)|_{L_2}^2$  the equation (e.g. see in [23, (C.16)])

$$(3.12) \quad \frac{1}{2} \frac{d}{dt} \mathbb{E}|\zeta(t)|_{L_2}^2 + \nu \mathbb{E}|\nabla \zeta(t)|_{L_2}^2 = \mathbb{E} \sum_{i,j=1}^3 \int (\zeta_i \zeta_j s_{ij}) dx + \dots$$

Here  $s_{ij} = \frac{1}{2}(\partial u_i / \partial x_j + \partial u_j / \partial x_i)$  is the rate of strain, and the dots stand for terms, coming from the force, which stay bounded as  $\nu \rightarrow 0$ . The term  $\sum_{i,j} \int (\zeta_i \zeta_j s_{ij}) dx =: \text{Str}$  is called the *vorticity stretching*. Numerical studies of the 3d Navier–Stokes system with small  $\nu$  clearly show that the expectation of Str is a large positive quantity. In view of (3.12), to keep the “energy of vorticity”  $\mathbb{E} \frac{1}{2}|\zeta(t)|_{L_2}^2$  from indefinite growth, the “rate of dissipation of vorticity”  $\nu \mathbb{E}|\nabla \zeta(t)|_{L_2}^2$  should be large.

<sup>9</sup>Recent numerical and experimental studies of 3d turbulence suggest that the Kolmogorov–Obukhov law holds for it with high precision (e.g. see in [13]), while the 2/3-law holds only approximatively (e.g. see in [12]). This casts more doubts on the claim that the two properties of 3d velocity fields are equivalent.

So

$$\mathbb{E}|\nabla\zeta(t)|_{L_2}^2 \gg \nu^{-1} \sim \mathbb{E}|\nabla u(t)|_{L_2}^2 \gg \mathbb{E}|u(t)|_{L_2}^2 \sim 1.$$

These relations show that a large part of energy of a flow  $u(t, x)$  with small viscosity is contained in high frequencies. Unfortunately no rigorous theory in support of the numerical fact that  $\mathbb{E}(\text{Str}) \gg 1$  currently exists.

*In 1d turbulence.* The vorticity of a “1d velocity field”  $u(t, x)$  should be defined as its derivative  $\zeta := u_x(t, x)$ . Applying Ito’s formula to  $\|u(t, \cdot)\|_1^2$ , where  $u(t, x)$  is a solutions of (2.1), (2.3), then taking expectation and using equality  $-\langle \Delta u, \partial_x u^2 \rangle = \int u_x^3 dx$  we get that in the 1d case an analogy of relation (3.12) is

$$\frac{d}{dt} \mathbb{E} \left( \frac{1}{2} \int \zeta^2(t, x) dx \right) + \nu \mathbb{E} \int \zeta_x^2(t, x) dx = -\frac{1}{2} \mathbb{E} \int \zeta^3(t, x) dx + \frac{1}{2} B_1$$

(for  $B_1$  see (2.5)). So the 1d vorticity stretching is  $-\frac{1}{2} \mathbb{E} \int \zeta^3(t, x) dx$ . To estimate it let us average the equality above in time from  $T$  to  $T + \sigma$ :

$$\frac{1}{\sigma} \mathbb{E} \frac{1}{2} \int \zeta^2(T + \sigma, x) dx - \frac{1}{\sigma} \mathbb{E} \frac{1}{2} \int \zeta^2(T, x) dx = -\nu \langle \langle \int \zeta_x^2(t, x) dx \rangle \rangle - \frac{1}{2} \langle \langle \int \zeta^3(t, x) dx \rangle \rangle + \frac{1}{2} B_1.$$

By (2.12) with  $m = 1$  the l.h.s. above is bounded by  $C\nu^{-1}$ , while the first term in the r.h.s. is  $\sim \nu^{-2}$ . We conclude that

$$(3.13) \quad -\frac{1}{2} \langle \langle \int \zeta^3(t, x) dx \rangle \rangle = -\frac{1}{2} \langle \langle \int u_x^3(t, x) dx \rangle \rangle \sim \nu^{-2}.$$

I.e. in 1d turbulence the vorticity stretching indeed is a large positive quantity.

#### 4. STATISTICAL EQUILIBRIUM (THE MIXING)

It is a general believe in the theory of turbulence that as time grows, statistical characteristics of a turbulent flow  $u(t, \cdot)$  converge to a universal statistical equilibrium. E.g. see in [2] pages 6-7 and 109. Mathematically it means that if we regard a space-periodic turbulent flow  $u(t, x)$  as a random process  $u(t, \cdot)$  in a function space  $\mathcal{H}$  of 1-periodic non-compressible vector fields, then for any bounded continuous functional  $f$  on  $\mathcal{H}$  we have

$$(4.1) \quad \mathbb{E}f(u(t, \cdot)) \rightarrow \langle f, \mu_\nu \rangle \quad \text{as } t \rightarrow \infty,$$

where  $\mu_\nu$  is a measure on  $\mathcal{H}$ , describing the equilibrium ( $\nu$  is the viscosity). If the initial data  $u(0, x)$  of a solution  $u(t, x)$  is a r.v. in  $\mathcal{H}$ , distributes as  $\mu_\nu$  (and independent of the applied random force), then

$$\mathcal{D}u(t, \cdot) = \mu_\nu \quad \forall t.$$

The property, manifested by relation (4.1), is called *the mixing*, and the measure  $\mu_\nu$  is called a *stationary measure*, see [9, 19]. Since the K41 theory deals with stationary in time turbulent velocity fields, then there convergence (4.1) trivialises to an equality which holds for all  $t$ .

In the 1d case, if  $u(t, x)$  is a solution of (2.1) and  $\mathcal{H}$  is a space  $H^m$  with some  $m \in \mathbb{N}$ , then the validity of convergence (4.1) with a suitable measure  $\mu_\nu$  on  $H^m$  may be derived from general results for SPDEs (e.g. see in [19]). But then the rate of convergence would depend on  $\nu$ . At the same

time, in the theory of turbulence it should not depend on  $\nu$ , and, as we show in this section, for eq. (2.1) it does not:

**Theorem 4.1.** *For any  $\nu$  equation (2.1), (2.3) has a unique stationary measure  $\mu_\nu$ . It is supported by smooth functions,  $\mu_\nu(H^\infty) = 1$ , and if  $f$  is a bounded continuous function on a space  $L_p(S^1)$  for some  $1 \leq p < \infty$  such that*

$$(4.2) \quad \text{Lip } f \leq 1, \quad |f| \leq 1,$$

then for any random initial data  $u_0 \in H^1$ , independent of  $\xi$ ,

$$(4.3) \quad |\mathbb{E}f(u(t; u_0)) - \langle f, \mu_\nu \rangle| \leq C_p (\ln t)^{-\kappa_p}, \quad t \geq 3,$$

where the positive constants  $C_p$  and  $\kappa_p$  depend only on  $\xi$  (and  $p$ ).

*Sketch of the proof.* The proof follows from a combination of three results: a lower bound for the probability that during a time  $T$  the Wiener process  $\xi(t)$  stays inside the ball  $B_{H^m}^\epsilon$  of a small radius  $\epsilon > 0$ , the  $L_1$ -nonexpanding property (2.7) of eq. (2.1), and the Oleinik maximum principle.

The probability in question is  $\mathbf{P}\{\|\xi\|_{X_T^m} < \epsilon\} =: \gamma_{\epsilon, T}^m$ , where  $\xi$  is the Wiener process (2.3). For a standard Brownian motion  $\beta(t)$  the function  $\hat{\gamma}_{\epsilon', T} = \mathbf{P}\{\sup_{0 \leq t \leq T} |\beta(t)| < \epsilon'\}$  is well known in probability. It is given by a converging series and admits a lower bound  $\hat{\gamma}_{\epsilon', T} \geq e^{-\pi^2 T / 8\epsilon'^2}$ . Based on this inequality and using (2.5) we get that

$$(4.4) \quad \gamma_{\epsilon, T}^m \geq f_m(\epsilon / \sqrt{T}), \quad f_m(a) = 2e^{-\kappa_m(a^{-3} + a^{-2})}.$$

With this estimate in hands, to prove (4.3) we argue as follows. Let us first assume that in the theorem  $p = 1$ , and for any non-random  $u_1, u_2 \in H^1$  consider solutions  $u^1(t) = u(t; u_1)$  and  $u^2(t) = u(t; u_2)$ . We cut the time-axe  $[0, \infty)$  to segments of length  $N \in \mathbb{N}$ . Since values of the force  $\eta = \partial_t \xi$  on different segments are independent, then estimate (4.4) and a suitable version of Oleinik's inequality (already used to establish (2.8) and (2.9)), allow to find an explicit function  $G(N)$ , constructed in terms of  $f_m$ , and independent of  $u_1, u_2$  and  $\nu$ , such that

$$\mathbf{P}\left(\min_{0 \leq k \leq G(N)} (\max(|u^1(k)|_1, |u^2(k)|_1)) > 24N^{-1}\right) \leq N^{-1}.$$

Denote by  $Q(N)$  the event above, and for an  $\omega \notin Q(N)$  let  $k_0 = k_0^\omega \leq G(N)$  be such that  $|u^{1\omega}(k_0)|_1, |u^{2\omega}(k_0)|_1 \leq 24N^{-1}$ . Then by (2.7) for this  $\omega$ ,

$$|u^{1\omega}(t) - u^{2\omega}(t)|_1 \leq 48N^{-1} \quad \forall t \geq k_0.$$

From here for any functional  $f$  as in (4.2) we have that

$$|\mathbb{E}(f(u(t; u_1) - f(u(t; u_2)))| \leq 48N^{-1} + 2N^{-1} = 50N^{-1} \quad \text{for } t \geq G(N).$$

Here the first term in the r.h.s. comes from the integrating over  $Q(N)$  (since  $\text{Lip } f \leq 1$ ) and the second – over its complement (since  $|f| \leq 1$ ). As this estimate holds for any integer  $N \geq 1$ , then in view of the explicit form of function  $G$  we derive from it that

$$(4.5) \quad |\mathbb{E}(f(u(t; u_1) - f(u(t; u_2)))| \leq C(\ln t)^{-1/8}, \quad t \geq 3.$$

This estimate implies the theorem assertion with  $p = 1$ . Indeed, some stationary measure  $\mu_\nu \in \mathcal{P}(H^1)$  exists by the Krylov–Bogolyubov argument in view of estimates (2.12). Then (4.5) implies (4.3) with  $p = 1$  and  $\kappa_1 = 1/8$  by a well known argument; e.g. see [5, Section 3.3]. Estimate (4.3) with any  $p \geq 1$  follows from the estimate with  $p = 1$  and (2.9) by suitable interpolation.  $\square$

It is easy to see that if the random field  $\xi(t, x)$  is homogeneous in  $x$  (see (2.4)), then the stationary measure  $\mu_\nu$  also is.

Let  $M$  be some complete separable metric space,<sup>10</sup> and let  $\mu_1, \mu_2 \in \mathcal{P}(M)$ . Then the dual-Lipschitz distance between these two measures (also known as the Kantorovich-Rubinstein distance) is

$$\|\mu_1 - \mu_2\|_L^* = \|\mu_1 - \mu_2\|_{L,M}^* := \sup_{f \in C(M), |f|_L \leq 1} |\langle f, \mu_1 \rangle - \langle f, \mu_2 \rangle| \leq 2,$$

where  $|f|_L = |f|_{L,M} = \text{Lip } f + \sup_m |f(m)|$ . This distance converts  $\mathcal{P}(M)$  to a complete metric space, and the convergence with respect to it is equivalent to the weak convergence of measures (see in [24, 19, 5]).

Then Theorem 4.1 means that for any r.v.  $u_0 \in H^1$  (independent of  $\xi$ ),

$$(4.6) \quad \|\mathcal{D}u(t; u_0) - \mu_\nu\|_{L,L^p}^* \leq C(\ln t)^{-\kappa p}, \quad t \geq 3.$$

So  $\mathcal{D}u(t; u_0) \rightarrow \mu_\nu$  as  $t \rightarrow \infty$ , uniformly in  $\nu$ .

Arguing similarly to the above and using in addition estimates (2.12) we find that equation (2.1) also is mixing in each space  $H^M$ :

$$\|\mathcal{D}u(t; u_0) - \mu_\nu\|_{L,H^M}^* \leq C_M(\nu)(\ln t)^{-\kappa_M(\nu)}, \quad t \geq 3.$$

But now the rate of mixing depends on the viscosity  $\nu$ .

**4.1. Energy spectrum and structure function of the stationary measure.** Stationary solution  $u^{stat}(t)$  of eq. (2.1) is a solution such that  $\mathcal{D}u^{stat}(t) = \mu_\nu$  for all  $t$ , where  $\mu_\nu$  is the stationary measure. Energy spectrum of  $\mu_\nu$  is the function  $\mathbf{k} \mapsto E_{\mathbf{k}}^B(\mu_\nu) = \int e_{\mathbf{k}}^B(u) \mu_\nu(du)$  ( $e_{\mathbf{k}}^B$  is defined in (3.7)). Obviously,

$$E_{\mathbf{k}}^B(\mu_\nu) = \langle \langle e_{\mathbf{k}}^B(u^{stat}(\cdot)) \rangle \rangle = \mathbb{E} e_{\mathbf{k}}^B(u^{stat}(t)) \quad \forall t \geq 0.$$

Since  $\langle \langle e_{\mathbf{k}}^B(u^{stat}(t)) \rangle \rangle$  satisfies the spectral power law (3.8), then  $E_{\mathbf{k}}^B(\mu_\nu)$  also does:

$$E_{\mathbf{k}}^B(\mu_\nu) \sim \mathbf{k}^{-2}, \quad 1 \leq \mathbf{k} \leq C_1^{-1} \nu^{-1}.$$

The map  $u \mapsto \hat{u}_n$  is a continuous linear functional on  $L_1$  of unit norm. Moreover, by (2.9) all moments of the  $L_1$ -norm of a solution  $u(t; u_0)$ ,  $u_0 \in H^1$ , are bounded uniformly in  $t \geq 1$ . Hence, from (4.6) we get that

$$\mathbb{E} e_{\mathbf{k}}^B(u(t; u_0)) \rightarrow E_{\mathbf{k}}^B(\mu_\nu) \quad \text{as } t \rightarrow \infty,$$

where the rate of convergence does not depend on  $\nu$  and  $\mathbf{k}$ . So asymptotically as  $t \rightarrow \infty$  the instant energy spectrum  $\mathbb{E} e_{\mathbf{k}}^B(u(t; u_0))$  of any solution  $u(t; u_0)$  also satisfies the spectral power law.

<sup>10</sup>E.g.  $M$  is some  $\mathbb{R}^N$ , or a Hilbert space with a countable base, or a space  $L_p(S^1)$  with  $1 \leq p < \infty$ , or a closed subset of one of these spaces.

Writing the structure function of a solution  $u = u(t; u_0)$  as

$$S_{p,l}(u) = \langle\langle s_{p,l}(u) \rangle\rangle, \quad s_{p,l}(v) = \int |v(x+l) - v(x)|^p dx,$$

we define  $S_{p,l}(\mu_\nu) := \langle s_{p,l}, \mu_\nu \rangle$ . Similar to the above,  $S_{p,l}(\mu_\nu)$  satisfies the relations in Theorem 3.3. Noting that  $s_{p,l}$  is continuous on the space  $L_{\max(p,1)}$  we derive from (4.3) that  $\mathbb{E}s_{p,l}(u(t; u_0)) \rightarrow S_{p,l}(\mu_\nu)$  as  $t \rightarrow \infty$ , uniformly in  $l$  and  $\nu$ . So asymptotically as  $t \rightarrow \infty$  the instant structure function  $\mathbb{E}s_{p,l}(u(t; u_0))$  also satisfies (3.2) and (3.3).

As we pointed out, if the random force is homogeneous, then stationary measure  $\mu_\nu$  also is. In this case

$$S_{p,l}(\mu_\nu) = \langle |u(x+l) - u(x)|^p, \mu_\nu \rangle = \mathbb{E}|u^{stat}(t, x+l) - u^{stat}(t, x)|^p,$$

for any  $x$  and  $t$ . This is in close agreement with the objects, treated by K41, where velocity fields  $u$  are assumed to be stationary and homogeneous (see in Section 3.2).

## 5. THE 4/5-LAW AND LANDAU OBJECTION

In this section we follow paper [14]. Talking about K41, as in the K41 papers, we now suppose that the involved velocity fields  $u(t, x)$  are homogeneous and isotropic in  $x$  (and so are not space-periodic). We do not assume that they are stationary in time, unless otherwise stated.

**5.1. The 4/5-law. In 3d turbulence.** Apart from the second moments of longitudinal increments  $S_{2,l}^{\parallel}(u)$  of velocity fields of turbulent flows (see (1.10)), K41 studies the cubic moments  $S_{3,l}^{\parallel}(u)$ . Concerning this quantities K41 makes a very precise prediction, called *Kolmogorov's 4/5-law*:

$$(5.1) \quad S_{3,l}^{\parallel} = -(4/5)\epsilon^K |l| + o(\epsilon^K |l|) \quad \text{when } |l| \text{ is in the inertial range,}$$

where  $\epsilon^K$  is the rate of energy dissipation (1.2) (which is  $\sim 1$  by assumption). The law was intensively discussed by physicists and was re-proved by them a number of times, using physical arguments, always related to that in the K41 papers. Recently a progress in rigorous verification of the law was achieved in [3]. There the relation in (5.1) is established for stationary solutions  $u(t, x)$  of the stochastic 3d Navier–Stokes system on a torus, assuming that they meet the assumption  $\nu \mathbb{E}|u|_{L^2}^2 = o(1)$  as  $\nu \rightarrow 0$ , and that  $|l|$  belongs to some interval in  $\mathbb{R}_+$  whose left edge converges to 0 with  $\nu$ , but whose relation with the inertial range is not clear.

*In 1d turbulence.* Following K41, proofs of the 4/5-law in physical works, as well as in the rigorous paper [3], crucially use the Karman–Howard–Monin formula (rather a class of formulae with this name). The formula relates time-derivative of  $S_{2,l}^{\parallel}$  with derivatives of  $S_{3,l}^{\parallel}$  in  $l$ . Variants of this formula, e.g. the one in [11], instead of second moments  $S_{2,l}^{\parallel}$  use correlations  $\mathbb{E}(u(t, x) \cdot u(t, x+l))$ , closely related to  $S_{2,l}^{\parallel}$ . Thus motivated let us examine time-derivatives of “correlations in  $x$ ”  $f^l$  for a solution  $u(t) = u(t, x; u_0)$  of (2.1) with a random  $u_0$  (independent from the force),

$$f^l(u(t)) := \int u(t, x)u(t, x+l)dx.$$

To simplify notation a bit, in this section and in the next one we assume that in the 1d case the increments  $l$  are positive.

Applying Ito's formula to  $f^l(u(t))$  and noting that  $d^2 f^l(u)(e, e) = 2f^l(e)$  we arrive at the equality

$$\frac{d}{dt} \mathbb{E} f^l(u(t)) = \mathbb{E} \left( -df^l(u)(uu_x) + \nu df^l(u)(u_{xx}) + \sum b_s^2 f^l(e_s) \right) =: \mathbb{E}(-I_1(t) + I_2(t) + I_3(t)).$$

Noting that  $df^l(u)(v) = \int (u(x)v(x+l) + u(x+l)v(x)) dx$  and that, trivially,  $(\partial/\partial l)u(x+l) = u_x(x+l)$ , we calculate that

$$\begin{aligned} I_1(t) &= -\frac{1}{6} \frac{\partial}{\partial l} s_{3,l}^s(u(t)), & s_{3,l}^s(v(x)) &= \int (v(x+l) - v(x))^3 dx; \\ I_2(t) &= 2\nu \frac{\partial^2}{\partial l^2} f^l(u(t)); & I_3(t) &= \sum b_s^2 \cos(2\pi sl) =: \tilde{B}_0(l). \end{aligned}$$

So

$$(5.2) \quad \frac{d}{dt} \mathbb{E} f^l(u(t)) = \frac{1}{6} \mathbb{E} \frac{\partial}{\partial l} s_{3,l}^s(u(t)) + 2\nu \mathbb{E} \frac{\partial^2}{\partial l^2} f^l(u(t)) + \tilde{B}_0(l).$$

This is a version of the Karman–Howard–Monin formula for the Burgers equation.

Now let  $u(t) = u^{st}(t)$  be a stationary solution of (2.1), (2.3) (see Section 4). Then the l.h.s. of (5.2) vanishes. Since  $s_{3,0}^s = 0$  and  $(\partial/\partial l)f^l(u)|_{l=0} = 0$ , then integrating (5.2) in  $dl$  and multiplying the result by six we find that

$$(5.3) \quad \mathbb{E}(s_{3,l}^s(u^{st}(t))) = -12\nu(\partial/\partial l)\mathbb{E}(f^l(u^{st}(t))) - 6 \int_0^l \tilde{B}_0(r) dr.$$

Consider the first term in the r.h.s. of (5.3). Abbreviating  $u^{st}$  to  $u$  we get:

$$\begin{aligned} |(\partial/\partial l)\mathbb{E}(f^l(u(t)))| &= \left| \mathbb{E} \int u(t, x) u_x(t, x+l) dx \right| = \left| \mathbb{E} \int (u(t, x) - u(t, x+l)) u_x(t, x+l) dx \right| \\ &\leq \left[ \mathbb{E} \int (u(t, x) - u(t, x+l))^2 dx \right]^{1/2} \left[ \mathbb{E} \int u_x(t, x)^2 dx \right]^{1/2}. \end{aligned}$$

Since  $u$  is a stationary solution, then the first factor in the r.h.s. equals  $S_{2,l}^{1/2}$ . Estimating it from Theorem 3.3 and bounding the second factor as in (2.12) we find that the first term in the r.h.s. of (5.3) is  $O(\sqrt{l}\sqrt{\nu})$ . As  $\tilde{B}_0(l)$  is a  $C^2$ -smooth even function and  $\tilde{B}_0(0) = B_0$ , then  $\int_0^l \tilde{B}_0(r) dr = B_0 l + O(l^3)$ . We have seen that

$$(5.4) \quad \mathbb{E}(s_{3,l}^s(u^{st}(t))) = -6B_0 l + O(l^3) + O(\sqrt{l}\sqrt{\nu}).$$

Now assume that  $l$  belongs to a “strongly inertial range”:

$$(5.5) \quad l \in [L(\nu)\nu, C_1],$$

where  $L(\nu)$  is some fixed positive function of  $\nu > 0$  such that

$$L(\nu)\nu \rightarrow 0 \text{ and } L(\nu) \rightarrow \infty \text{ as } \nu \rightarrow 0.$$

Then  $\sqrt{l}\sqrt{\nu} = o(l)$  as  $\nu \rightarrow 0$  and we get from (5.4)

**Theorem 5.1.** *Let  $u^{st}(t)$  be a stationary solution of (2.1), (2.3) and  $l$  satisfies (5.5). Then*

$$(5.6) \quad \mathbb{E}(s_{3,l}^s(u^{st}(t))) = -6B_0 l + o(l) \text{ as } \nu \rightarrow 0,$$

where  $o(l)$  depends on the function  $L(\nu)$  and the random force  $\xi$ .

Applying Ito's formula to  $\|u^{st}(t)\|^2$  and taking expectation we get that  $0 = -2\nu\mathbb{E}\|u^{st}(t)\|_1^2 + B_0$ . Thus the rate of energy dissipation for  $u^{st}$ ,  $\epsilon^B = \nu\mathbb{E}\|u^{st}(t)\|_1^2$ , is given by

$$(5.7) \quad \epsilon^B = \frac{1}{2}B_0.$$

So relation (5.6) may be written as

$$(5.8) \quad \mathbb{E}(s_{3,l}^s(u^{st}(t))) = -12\epsilon^B l + o(l) \quad \text{as } \nu \rightarrow 0.$$

In this form the 4/5-law for turbulence in the 1d Burgers equation appears in works of physicists, justified by a heuristic argument.

Combining the last theorem with Theorem 4.1 we get that for any r.v.  $u_0 \in H^1$ , independent of  $\xi$ , and for  $l$  as in (5.5), solution  $u(t; u_0)$  satisfies

$$\lim_{t \rightarrow \infty} \mathbb{E}(s_{3,l}^s(u(t; u_0))) = -6B_0 l + o(l) = -12\epsilon^B l + o(l) \quad \text{as } \nu \rightarrow 0.$$

An easy calculation shows that if  $u(t, x)$  is an  $L$ -periodic in  $x$  solution of the equation in (2.1) with any  $L > 0$ , where  $\xi$  has the form (2.3) with  $e_s(x)$  replaced by  $e_s(x/L)$ , then relation (5.8) still holds.

**5.2. The Landau objection.** As we already mentioned in Section 3.2, Landau suggested a physical argument, implying that a relation for a moment  $S_{p,l}^{\parallel}$  of velocity increments like relation (3.6) for the second moment, may hold with a universal constant  $C^K$ , independent of the random force  $\xi$ , only if the value of the moment, suggested by this relation, is linear in the rate of energy dissipation  $\epsilon^K$ ; like relation (5.1) for the third moment. So the 2/3-law (1.11) cannot hold in the stronger form (3.6) with a universal constant  $C^K$ . The goal of this section is to show that in 1d turbulence, indeed, the only universal relation for the moments  $S_{p,l}^s$  (see (3.4)) is relation (5.8) for the cubic one (which is linear in  $\epsilon^B$ ).

Namely, for a stationary solution  $u^{\nu st}(t, x)$  of the Burgers equation (2.1) and for an integer  $p \geq 2$  consider the following hypothetical relation for the  $p$ -th moment of  $u^{\nu st}$ :

$$(5.9) \quad S_{p,l}^s(u^{\nu st}(t)) = C_*(\epsilon^B l)^q + o(\epsilon^B l)^q \quad \text{as } \nu \rightarrow 0,$$

where  $l$  is any number from the inertial range  $[c_1\nu, c]$  and  $q > 0$ . We address the following question: for which  $p$  and which  $q$  relation (5.9) holds with a *universal* constant  $C_*$ , independent of the random force  $\xi$ ?

**Theorem 5.2.** *If relation (5.9) holds for any random force  $\xi$ , satisfying (2.5), with a  $C_*$  independent of  $\xi$ , then*

$$p = 3, \quad q = 1, \quad C_* = -12.$$

*Proof.* Let us abbreviate  $u^{\nu st}(t)$  to  $u(t)$ . We take some real number  $\mu > 1$  and define  $\tilde{\xi}(\tau) := \mu^{-\frac{1}{2}}\xi(\mu\tau)$ . This also is a process as in (2.3) (with another set of independent Wiener processes  $\beta_s$ ). Denote  $w(\tau, x) := \mu u(\mu\tau, x)$ . Then  $w$  is a stationary solution of the equation

$$(5.10) \quad w_\tau(\tau, x) + w(\tau, x)w_x(\tau, x) - \nu^\mu w_{xx}(\tau, x) = \mu^{\frac{3}{2}}\partial_\tau \tilde{\xi}(\tau, x), \quad \nu^\mu = \nu\mu,$$

which is eq. (2.1), (2.3) with scaled viscosity  $\nu$  and force  $\xi$ .

Consider inertial range  $J^1 = [c_1\nu, c]$  for eq. (2.1) and inertial range  $J^\mu = [c_1^\mu\nu, c^\mu]$  for eq. (5.10). For small  $\nu$  their intersection  $J = J^1 \cap J^\mu$  is not empty. For any  $l \in J$  relation (5.9) holds for  $u$  which solves eq. (2.1) and for  $w$ , solving eq. (5.10). Since  $S_{p,l}^s(w) = \mu^p S_{p,l}^s(u)$  and as by (5.7)  $\epsilon_w^B = \mu^3 \epsilon_u^B$ , then from here we get that

$$\mu^p C_* (\epsilon_u^B l)^q + o(\epsilon_u^B l)^q = C_* (\mu^3 \epsilon_u^B l)^q + o(\epsilon_u^B l)^q$$

for any  $l \in J$  and all small  $\nu$ . As  $\mu > 1$ , then by this equality  $q = p/3$ .<sup>11</sup> On the other hand, it follows from Theorem 3.3 if  $p$  is even and from relation (3.5) and a discussion after it if  $p$  is odd that  $|S_{p,l}^s(u)| \sim |l|$  for any integer  $p \geq 2$ . Thus in (5.9)  $q = 1$ , and so  $p = 3q = 3$ . Then by Theorem 5.1  $C_* = -12$  and the theorem is proved.  $\square$

**Remark 5.3.** 1) The result of Theorem 5.2 remains true with the same proof if relation (5.9) is claimed to hold not for all  $l$  from the inertial range, but only for  $l$  from a strongly inertial range as in (5.5). In this form asymptotic (5.9) with  $p = 3$  and  $q = 1$  indeed is valid by Theorem 5.1.

2) We do not know if for some integer  $p \geq 2$ , different from 3, asymptotical expansion for  $S_{p,l}^s(u^{\nu st}(t))$  of the form (5.9) may be valid for  $l$  from the inertial range (or from a strongly inertial range) with a constant  $C_*$  which depends on the random force  $\xi$ .

## 6. INVISCID 1D TURBULENCE

In this section we study the asymptotics of solutions for the Burgers equation (2.1), (2.3) as  $\nu \rightarrow 0$ , define the limiting *entropy solutions*, corresponding to  $\nu = 0$ , and establish their properties.

**6.1. Asymptotics of solutions as  $\nu \rightarrow 0$ .** For  $m \in \mathbb{N} \cup 0$  we denote by  $\overline{H}^m$  the Sobolev space of order  $m$  of functions on  $S^1$  with **any** mean value and set  $\overline{X}_T^m = C(0, T; \overline{H}^m)$ . In (2.1) we considered the problem

$$(6.1) \quad \left\{ \begin{array}{l} u_t(t, x) + uu_x - \nu u_{xx} = \eta(t, x) = \partial_t \xi(t, x), \quad t \geq 0, \\ u(0, x) = u_0(x) \end{array} \right\}, \quad x \in S^1.$$

Now let us also consider another one

$$(6.2) \quad \left\{ \begin{array}{l} \varphi_t(t, x) + \varphi_x^2/2 - \nu \varphi_{xx} = \eta(t, x) = \partial_t \zeta(t, x), \quad t \geq 0, \\ \varphi(0, x) = \varphi_0(x) \end{array} \right\}, \quad x \in S^1.$$

For solutions  $\varphi$  of (6.2) the mean value  $\int \varphi(x) dx$  is **not** an integral of motion. But obviously, if  $\varphi(t, x)$  solves (6.2), then  $u(t, x) = \varphi_x(t, x)$  has zero mean-value and solves (6.1) with  $\xi = \partial_x \zeta$ ,  $u_0 = \partial_x \varphi_0$ . Conversely, if  $u(t, x)$  is a solution of (6.1), then  $\varphi(t, x) = \int_0^x u(t, y) dy - \theta(t)$  with a suitable  $\theta(t)$  (which is explicit in terms of  $u$ ) solves (6.2) with  $\varphi_0 = \int_0^x u_0(y) dy$  and  $\zeta(t, x) = \int_0^x \xi(t, y) dy$ . Obviously,

$$\varphi \in \overline{X}_T^m \Leftrightarrow u \in X_T^{m-1}, \quad \varphi_0 \in \overline{H}^m \Leftrightarrow u_0 \in H^{m-1}, \quad \zeta \in \overline{X}_T^m \Leftrightarrow \xi \in X_T^{m-1}.$$

<sup>11</sup>This is in line with relation  $|u(t, x+r) - u(t, x)| \asymp (\epsilon|r|)^{1/3}$  which appears in the theory of turbulence due to a basic dimension argument, without any relation to the equations, describing the fluid. See [21, (32,1)].

So, essentially, (6.1) and (6.2) is the same problem. As we will now show, this isomorphism between the two problems is a tool to study the asymptotics of solutions for (6.1) as  $\nu \rightarrow 0$ .

We need a version of the Oleinik estimates for solutions of (6.1) with non-random  $\xi$  (see [5] for a proof). Let  $u$  be a solution of (6.1) with

$$(6.3) \quad u_0 \in H^2, \quad \xi \in X_T^4, \quad 0 < \nu \leq 1.$$

Then

$$\sup_{0 \leq t \leq T} |u(t)|_\infty \leq B_T, \quad \sup_{0 \leq t \leq T} |u_x(t)|_1 \leq 2B_T, \quad \sup_{0 \leq t \leq T} |u_x^+(t)|_\infty \leq B_T,$$

where

$$(6.4) \quad B_T = B_T(u_0, \xi) = \max [ |u_{0x}^+|_\infty + |\xi|_{X_T^4}, 4|\xi|_{X_T^4} + |\xi|_{X_T^4}^{1/2} ].$$

So if  $\varphi^\nu(t, x)$  is a corresponding solution of problem (6.2), then

$$(6.5) \quad |\varphi_x^\nu(t)|_\infty = |u^\nu(t)|_\infty \leq B_T, \quad |\varphi_{xx}^\nu(t)|_1 = |u_x^\nu(t)|_1 \leq 2B_T, \quad |\varphi_{xx}^{\nu+}(t)|_\infty = |u_x^{\nu+}(t)|_\infty \leq B_T,$$

for any  $0 \leq t \leq T$ .

**Theorem 6.1.** (*S. Kruzkov, see [5] for references*). *Let  $0 < \nu_1 < \nu_2 \leq 1$ ,  $1 \leq p < \infty$ ,  $T < \infty$  and (6.3) holds. Let  $u^\nu$  solves (6.1). Then*

$$(6.6) \quad |u^{\nu_2}(t) - u^{\nu_1}(t)|_p \leq C_p B_T^{1-\alpha_p} \bar{\nu}^{\alpha_p} e^{Bt\alpha_p}, \quad 0 \leq t \leq T,$$

where  $\bar{\nu} = \nu_2 - \nu_1 > 0$ ,  $\alpha_p = \min(\frac{1}{4}, \frac{1}{3p})$ , and  $B_T = B_T(u_0, \xi)$  is defined in (6.4).

*Sketch of the proof.* Let  $u^\nu(t; u_0, \xi) \in X_T^4$  be a solution of (6.1), and  $\varphi^\nu(t, x) \in X_T^5$  be the corresponding solution of (6.2). Denote  $b(t, x) = \varphi^{\nu_1}(t, x) - \varphi^{\nu_2}(t, x)$ . Then subtracting the equation for  $\varphi^{\nu_1}$  from that for  $\varphi^{\nu_2}$  we get that

$$b_t + (\varphi_x^{\nu_1} + \varphi_x^{\nu_2})b_x/2 = \nu_1 \varphi_{xx}^{\nu_1} - \nu_2 \varphi_{xx}^{\nu_2}.$$

Denote  $E(t) = |b(t)|_2^2$ . Then in view of (6.5) and the equation above,

$$(d/dt)E \leq B_T E(t) + 8B_T^2 \bar{\nu}, \quad E(0) = 0,$$

(see [5]) for the calculation). So by Gronwall's inequality,

$$|b(t)|_2^2 = E(t) \leq 8B_T \bar{\nu} e^{B_T t}.$$

Since  $\frac{\partial}{\partial x} b(t, x) = u^{\nu_1}(t, x) - u^{\nu_2}(t, x)$ , then by (6.5) we have  $|\partial_x b(t)|_2 \leq 4B_T$ . From here

$$|u^{\nu_1}(t) - u^{\nu_2}(t)|_{4/3} = |\partial_x b(t, \cdot)|_{4/3} \leq |\partial_{xx}^2 b(t, \cdot)|_1^{1/2} |b(t, \cdot)|_2^{1/2} \leq C_1 B_T^{1/2} (B_T^{1/4} e^{B_T t/4} \bar{\nu}^{1/4})$$

(we have used the Gagliardo–Nirenberg inequality and (6.5)). This proves (6.6) with  $p \leq 4/3$ . To get (6.6) for  $4/3 \leq p < \infty$ , we apply the Riesz–Thorin interpolation inequality to  $v = u^{\nu_1}(t) - u^{\nu_2}(t)$  to get that  $|v|_p \leq |v|_\infty^{1-\frac{4}{3}p} |v|_{4/3}^{\frac{4}{3}p}$  for  $p \geq \frac{4}{3}$ , where  $|v|_\infty \leq 2B_T$  by (6.5). This completes the proof.  $\square$

By (6.6) there exists a function  $u^0(t, x)$  such that

$$(6.7) \quad u^\nu \xrightarrow{\nu \rightarrow 0} u^0 \text{ in } C(0, T; L_p), \quad \forall p < \infty.$$

Passing to the limits in the last estimate in (6.5) and in (6.6) we get:

**Corollary 6.2.** *There exists  $u^0 = u^0(t, x; u_0, \xi) \in \cap_{p < \infty} C(0, T; L_p)$  such that (6.7) holds, and (6.6) holds for  $\nu_1 = 0$  and  $0 < \nu_2 \leq 1$ . Moreover,  $|u^0|_{C(0, T; L_p)} \leq B_T$  for all  $p < \infty$ .*

Consider equation (2.1) with  $\nu = 0$ :

$$(6.8) \quad u_t(t, x) + \frac{1}{2} \partial_x u^2 = \partial_t \xi(t, x), \quad u(0, x) = u_0(x).$$

It follows immediately from (6.7) that  $u^0(t; u_0, \xi)$  with  $u_0, \xi$  as in (6.3) solves (6.8) in the sense of generalized functions. A generalized solution of (6.8) is **not unique**, and the construction above single out among various solutions a **unique** one. It is called an *entropy*, or an *inviscid* solution of (6.8), e.g. see in [8].

Now let  $\xi$  be the random force (2.3). Let  $u_0 \in H^2$  be a r.v., independent from  $\xi$ .

**Definition 6.3.**  $u^{0\omega}(t, x; u_0, \xi)$ , where for each  $\omega$   $u^{0\omega}$  is an entropy solution of (6.8) with  $\xi = \xi^\omega$ ,  $u_0 = u_0^\omega$ , is called an *entropy solution* for problem (6.8), (2.3).

We will usually write entropy solutions from this definition as  $u^0(t, x; u_0)$  or  $u^0(t; u_0)$ .

Let  $u_0 \in H^2$ ,  $\theta > 0$ ,  $1 \leq p < \infty$  and  $a > 0$ . Then for each  $t \geq \theta$  estimate (2.9), convergence (6.7) and Fatou's lemma imply that

$$\mathbb{E}|u^0(t; u_0)|_p^a \leq C(a, B_4)\theta^{-a} \quad \forall t \geq \theta.$$

Due to convergence (6.7) with  $p = 1$ , for any fixed force  $\xi$  and any  $t \geq 0$  the function on  $H^2$  which sends any  $u_0 \in H^2$  to  $u^0(t; u_0)$ , inherits estimate (2.7) and extends by continuity to a 1-Lipschitz mapping  $L_1 \rightarrow L_1$ . Accordingly entropy solutions  $u^0(t; u_0)$  extend to a Markov process in  $L_1$ . The latter is mixing: there is a measure  $\mu_0 \in \mathcal{P}(L_1)$  ("the inviscid stationary measure"), satisfying  $\mu_0(\cap_{q < \infty} L_q) = 1$ , such that for any r.v.  $u_0 \in L_1$ , independent of  $\xi$ ,

$$\mathcal{D}u^0(t; u_0) \rightharpoonup \mu_0 \quad \text{in } \mathcal{P}(L_p) \quad \text{as } t \rightarrow \infty,$$

for any  $p < \infty$ . If  $\mathcal{D}u_0 = \mu_0$ , then  $u^{0st}(t) := u^0(t; u_0)$  is a stationary entropy solution,  $\mathcal{D}u^{0st}(t) \equiv \mu_0$ . Moreover, the viscous stationary measures  $\mu_\nu$  weakly converge to  $\mu_0$  as  $\nu \rightarrow 0$ :

$$(6.9) \quad \mu_\nu \rightharpoonup \mu_0 \quad \text{as } \nu \rightarrow 0, \quad \text{on each space } \mathcal{P}(L_p), \quad p < \infty.$$

See [5, Chapter 8.5]. The limiting entropy Markov process in  $L_1$  admits an elegant presentation in terms of stochastic Lagrangians, see [10] and [15].

**6.2. Moments of small-scale increments and energy spectra of entropy solutions.** As before we define the (absolute) structure function of an entropy solution  $u^0(t, x)$  as  $S_{p,l}(u^0) = \langle\langle s_{p,l}(u^0) \rangle\rangle$ , where  $s_{p,l}(v) = \int |v(x+l) - v(x)|^p dx$ .

By Theorem 3.3, for suitable  $C_1, c, c_* > 0$ , for any  $0 < \nu \leq c_*$  and for every  $p > 0$  we have

$$(6.10) \quad S_{p,l}(u^\nu(\cdot; u_0)) = \langle\langle s_{p,l}(u^\nu(\cdot, \cdot; u_0)) \rangle\rangle \sim |l|^{\min(1,p)} \quad \text{if } |l| \in [C_1\nu, c].$$

Since functional  $s_{p,l}$  is continuous on the space  $L_{\max(1,p)}$  and  $|s_{p,l}(v)| \leq C_p |v|_{\max(1,p)}^p$ , then convergence (6.7) and estimate

$$\mathbb{E}|u^\nu(t)|_p^a \leq C(a, \theta) \quad \forall \nu > 0, \quad \forall t \geq \theta > 0, \quad \forall a > 0,$$

which follows from (2.9), allow to pass to a limit in (6.10) as  $\nu \rightarrow 0$ ,  $\nu \leq |l|/C_1$ , and prove the following result.

**Theorem 6.4.** *Let  $c$  be as in (6.10). Then for any  $u_0 \in H^2$  entropy solution  $u^0(t; u_0)$  of (6.8) satisfies*

$$S_{p,l}(u^0(\cdot; u_0)) \sim |l|^{\min(1,p)}, \quad \forall p > 0,$$

for  $|l| \leq c$ .

Since the assertion holds for all  $|l| \leq c$ , then for entropy solutions there is no dissipation range!

Now let us turn to the 4/5-law (5.1). Consider relation (5.3). Its l.h.s. equals  $\langle s_{3,l}^s, \mu_\nu \rangle$ . The functional  $s_{3,l}^s$  is continuous on  $L_3$ , and by (6.9),  $\mu_\nu \rightharpoonup \mu_0$  in  $\mathcal{P}(L_3)$ , where  $\mu_0$  is the stationary measure for the inviscid Burgers equation. So passing to a limit as  $\nu \rightarrow 0$  in relation (5.4) we get that

$$\mathbb{E}(s_{3,l}^s(u^{0st}(t))) = \langle s_{3,l}^s, \mu_0 \rangle = -6B_0l + O(l^3),$$

where  $u^{0st}(t)$  is the stationary entropy solution. This relation is the 4/5-law for inviscid 1d turbulence.

Similarly, one can pass to a limit in the energy-spectrum Theorem 3.4 and get

**Theorem 6.5.** *For  $M \geq M'$  as in Theorem 3.4 and any  $u_0 \in H^2$ , the energy spectrum of the entropy solution  $u^0(t; u_0)$  satisfies  $E_{\mathbf{k}}^B(u^0) \sim \mathbf{k}^{-2}$  for  $\mathbf{k} \geq 1$ .*

In 3d turbulence no analogies of Theorems 6.4 and 6.5 are known. That is, for the moment of writing the inviscid 3d turbulence is missing.

#### APPENDIX A. HIGH-FREQUENCY KICK-FORCES

Results, presented in our paper, show that the Burgers equation makes a good and relatively non-complicated model for the extremely complicated phenomenon of water turbulence, how the latter is treated by the K41 theory. But up to now no physical phenomenon, described by the Burgers equation is known. So all our knowledge about the equation's solutions comes not from physical experiments, but from computer numerics, where the white in time random force (2.3) (necessarily) is replaced by a discrete-time white force

$$(A.1) \quad \eta^\theta(t, x) = \sum_{k=1}^{\infty} \delta(t - k\theta) \eta_k^\omega(x), \quad \theta \in (0, 1].$$

Here  $\delta(t - l\theta)$  stands for the Dirac delta-function at a time-moment  $l\theta$ , and  $\eta_k^\omega(x)$ ,  $k = 1, 2, \dots$  are independent identically distributed random functions of the form

$$\eta_k^\omega(x) = \sum_{s \in \mathbb{Z}^*} b_s \xi_{sk}^\omega e_s(x).$$

In this relation  $\{b_s\}$  are real numbers as in (2.5),  $\{e_s(x)\}$  is the trigonometric basis (2.2) and  $\xi_{sk}$  are independent identically distributed random variables. Concerning them we assume that

- i)  $\mathbb{E}\xi_{sk} = 0$  and  $\mathbb{E}|\xi_{sk}|^2 = 1$ ;
- ii)  $|\xi_{sk}| \leq C_*$  for all  $\omega$ , with some  $C_* > 0$ ;

iii)  $\mathcal{D}(\xi_{sk}) = p(x) dx$ , where  $p$  is a Lipschitz function on  $[-C_*, C_*]$  such that  $p(0) \neq 0$ .

Here and below in this appendix  $k$  always stands for non-negative integers.

The force  $\eta^\theta(t, x)$  as in (A.1) is called a *kick-force of frequency*  $\theta^{-1}$ , and in the context of (A.1) the r.v.'s  $\eta_k$  are called *kicks*. We will discuss the Burgers equation

$$(A.2) \quad u_t + uu_x - \nu u_{xx} = \sqrt{\theta} \eta^\theta(t, x), \quad x \in S^1,$$

where  $\eta^\theta$  is a kick-force (A.1), and

$$(A.3) \quad u(0, x) = u_0(x),$$

$u_0$  being a r.v. in  $H^1$ , independent of  $\eta^\theta$ . A solution  $u(t) = u(t, x)$  of the problem (A.2), (A.3) will be denoted  $u^\theta(t; u_0)$ . The solutions  $u^\theta$  are normalised to be right-continuous in time  $t$ . Then between the kick-moments  $t = k\theta$ ,  $u^\theta$  satisfies the free Burgers equation (A.2) $_{|\eta^\theta=0}$ , and at the moments  $k\theta$  it instantly increases by the kicks  $\eta_k^\omega(x)$ .

The discrete time random process

$$u^\theta(k\theta; u_0), \quad k = 0, 1, 2, \dots,$$

defines a Markov chain in  $H^1$ . This chain is mixing. That is,

$$\mathcal{D}u^\theta(k\theta; u_0) \rightarrow \mu^\theta \quad \text{in } \mathcal{P}(H^1) \quad \text{as } k \rightarrow \infty.$$

Here  $\mu^\theta$  is the chain's unique stationary measure, so  $\mathcal{D}u^\theta(k\theta; u_0) \equiv \mu^\theta$  when  $\mathcal{D}u_0 = \mu^\theta$ . If the numbers  $b_s$  all are non-zero, the mixing follows from an abstract theorem [19, Theorem 3.2.5]. For a general kick-force (A.1) the result may be obtained by repeating the proof of Theorem 4.1.

For  $0 < \theta \ll 1$  solutions  $u^\theta(t; u_0)$  of (A.2), (A.3) and stationary measures  $\mu^\theta$  are closely related to solutions  $u(t; u_0)$  of (2.1), (2.3) and the stationary measure  $\mu_\nu$  of eq. (2.1). Namely, in [5, Section 10.2] the results below are obtained:

**Theorem.** *For any  $m \in \mathbb{N}$  and every  $u_0 \in H^m$  the following holds:*

- 1) for each  $T < \infty$ ,  $\sup_{0 \leq t \leq T} \|\mathcal{D}u^\theta(t; u_0) - \mathcal{D}u(t; u_0)\|_{L(H^m)}^* \rightarrow 0$  as  $\theta \rightarrow 0$ ;
- 2)  $\|\mathcal{D}u^\theta(t; u_0) - \mathcal{D}u(t; u_0)\|_{L(H^{m-1})}^* \rightarrow 0$  as  $\theta \rightarrow 0$ , uniformly in  $t \geq 0$ .
- 3)  $\mu^\theta \rightarrow \mu_\nu$  in  $\mathcal{P}(H^m)$  as  $\theta \rightarrow 0$ ;
- 4)  $\|\mathcal{D}u^\theta(t; u_0) - \mu_\nu\|_{L(H^m)}^* \rightarrow 0$  as  $t \rightarrow \infty$  and  $\theta \rightarrow 0$ ;

For a fixed  $\theta$  solutions  $u^\theta(t; u_0)$  of (A.2), (A.3) qualitatively are similar to solutions of (2.1), (2.3) and also may be used as a model of 1d turbulence with properties, similar to those in Section 3. See [4] and [5, Section 10.1.2].

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