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# Asymptotic of the Bergman kernel on a projective complex manifold

Lu Yu

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| Lu Yu. Asymptotic of the Bergman kernel on a projective complex manifold. 2024. hal-04392390

**HAL Id: hal-04392390**

**<https://hal.science/hal-04392390v1>**

Preprint submitted on 13 Jan 2024

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Lu Yu

August 29, 2021

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# Chapter 1

## Introduction

We are interested in understanding the asymptotic behaviour of the Bergman kernel (restricted to the diagonal of a projective manifold) of twisted tensor powers of a positive line bundle. Although  $C^\infty$  convergence is known in this case, our approach, mainly following L.Charles [3], can only give  $C^0$  convergence. The central result of the paper is Theorem 7, from which we derive asymptotic Riemann-Roch theorem as an application.

The scheme of the proof (see Section 3.2) is divided into two parts: an upper bound and a lower bound. For the upper bound, basically we use mean value inequality, which is rather elementary. For the lower bound, we apply the "peak section" method originated by Tian in his thesis [26]. We illustrate this scheme again by strengthening a theorem of Liu (see Theorem 18). The last part is devoted to asymptotic behavior of tensor powers of a Nakano positive vector bundle of higher rank (at least when the base manifold is a Riemann surface, see Theorem 17).

### Notations

Throughout the paper, unless otherwise stated, Greek letters  $\alpha, \beta, \dots$ , run through  $1, \dots, r$  and letters  $i, j, k, l \dots$  run through  $1, \dots, n$ . We use  $\mu_0$  to denote the Lebesgue measure on Euclidean spaces. By  $[n]$  we mean the set  $\{1, 2, \dots, n\}$ . The permutation group on  $[n]$  is denoted by  $S_n$ .

We always use column vectors unless otherwise specified. The Landau notation  $O(\alpha)$  for matrices means each entry is a usual  $O(\alpha)$ . For a matrix  $A \in \mathbb{C}^{m \times n}$ , we put  $A^H := \bar{A}^T$ . If  $A$  is a hermitian positive definite (resp. semi-positive) matrix, we write  $A > 0$  (resp.  $A \geq 0$ ). For a square matrix  $A$ , let  $\rho(A)$  be the spectral radius of  $A$ .

Put  $\Delta^n(a) = \{z \in \mathbb{C}^n : |z_i| < a, i \in [n]\}$ , which is an open polydisc. All hermitian inner product is anti-linear about the second argument. For a complex manifold  $M$ , we write  $M^n$  to mean  $\dim_{\mathbb{C}} M = n$ . Its holomorphic tangent vector

bundle is denoted by  $T_h M$ . For a line bundle  $L$  and  $k \in \mathbb{Z}$ , to ease notation we write  $L^k$  for  $L^{\otimes k}$ . For a vector bundle  $E \rightarrow M$ , by abuse of notation, we denote the smooth sections of  $E$  by  $C^\infty(M, E)$ .

## Acknowledgements

This report is my "mémoire de recherche" for the master degree at University of Paris, under the guidance of Professor Julien Keller, to whom I wish to express my sincere gratitude, for all his support and advice. The Section 4 is finished with the help of Professor Jean-Pierre Demailly, Xieping Wang and Junsheng Zhang. I thank them for their patient and helpful answers to my questions.

# Chapter 2

## Preliminaries

We suggest the reader refer to this elementary section only as necessary when reading the main text.

### 2.1 Linear Algebra

**Proposition 1** (Rayleigh-Ritz). *If  $A \in M_n(\mathbb{C})$  is a hermitian matrix, and  $x \in \mathbb{C}^n, x \neq 0$ , we form the Rayleigh quotient  $R(A, x) = \frac{x^H A x}{x^H x}$ , then*

$$\max_{x \in \mathbb{C}^n, x \neq 0} R(A, x) = \lambda_{\max}(A)$$

*is the maximal eigenvalue of  $A$ . Idem for minimum. In particular, if further  $A \geq 0$ , then*

$$\max_{x \in \mathbb{C}^n, x \neq 0} R(A, x) = \rho(A).$$

Let  $V$  be a finite dimensional complex vector space whose dual is denoted by  $V^*$ .

**Definition 1** (Conjugate space). *The complex conjugate  $\bar{V}$  of  $V$  is a complex vector space satisfying: there is a morphism  $V \rightarrow \bar{V}$  by  $v \mapsto \bar{v}$  that is an  $\mathbb{R}$ -linear isomorphism, and  $(a + ib)\bar{v} = (a - ib)v$  for any  $v \in V; a, b \in \mathbb{R}$ .*

If further  $V$  is endowed with a hermitian inner product  $h$ , then  $\bar{V} \rightarrow V^*$  by  $\bar{v} \mapsto h(\cdot, v)$  is a  $\mathbb{C}$ -linear isomorphism. We put

$$\text{Herm}^+(V, h) = \{A \in \text{End}(V) : A \geq 0\}.$$

The partial order on  $\text{Herm}^+(V, h)$  is defined by:  $A \geq B$  means  $A - B \in \text{Herm}^+(V, h)$ .

**Exercise 1.** *Then  $V \times \bar{V} \rightarrow \mathbb{C}$  by  $(v, \bar{u}) \mapsto h(v, u)$  is bilinear, so induces a morphism  $V \otimes_{\mathbb{C}} \bar{V} \rightarrow \mathbb{C}$ .*

The morphism  $V \times \bar{V} \rightarrow \text{End}(V)$  by  $(v, \bar{u}) \mapsto A$  where  $A(w) = h(w, u)v$  is bilinear, so induces a morphism  $C : V \otimes \bar{V} \rightarrow \text{End}(V)$  which is an isomorphism. For any  $s \in V$ ,  $C(s \otimes \bar{s}) \in \text{Herm}^+(V, h)$ .

We have a commutative diagram

$$\begin{array}{ccc} V \times \bar{V} & \xrightarrow{(v, \bar{u}) \mapsto h(\cdot, u)v} & \text{End}(V) \\ & \searrow (v, \bar{u}) \mapsto h(v, u) & \swarrow \text{Tr} \\ & & \mathbb{C} \end{array}$$

In virtue of Exercise 1, we usually identify  $V \otimes \bar{V}$  with  $\text{End}(V)$  without mentioning the isomorphism  $C$  in between.

**Exercise 2.** If  $A \in M_n(\mathbb{C})$  and  $A > 0$ ,  $v \in \mathbb{C}^n$ , then

$$A^{-1}v^T A \bar{v} \geq vv^H.$$

*Proof.* Let  $(\cdot, \cdot)$  be the standard inner product on  $\mathbb{C}^n$ . Then  $(u, w)_A = (Au, w)$  defines another inner product. For any  $u \in \mathbb{C}^n$

$$(A^{-1}v^T A \bar{v} u, u) = v^T A \bar{v} (A^{-1}u, u) = (v, v)_A (A^{-1}u, A^{-1}u)_A,$$

$$(vv^H u, u) = v^H u (v, u) = (u, v)(v, u) = |(u, v)|^2 = |(A^{-1}u, v)_A|^2.$$

By Cauchy-Schwarz inequality,

$$|(A^{-1}u, v)_A|^2 \leq (v, v)_A (A^{-1}u, A^{-1}u)_A.$$

Thus we get the desired inequality

$$(A^{-1}v^T A \bar{v} u, u) \geq (vv^H u, u).$$

□

**Definition 2** (linear Bergman kernel). For a linear subspace  $U \subset V$ , we define  $B(U, h|_U) = \sum_{i=1}^n e_i \otimes \bar{e}_i$ , where  $\{e_1, \dots, e_n\}$  is an orthonormal basis of  $(U, h|_U)$ .

Linear Bergman kernel is well-defined by the following Exercise.

**Exercise 3.** 1. For another basis  $\{v_1, \dots, v_n\}$  of  $U$ ,

$$B = \sum_{kl} v_k \otimes \bar{v}_l A_{lk},$$

where  $A$  is the inverse of the size  $n$  Gram matrix  $G = (h(v_i, v_j))$ . In particular,  $B$  is independent of the choice of orthonormal basis.

2. Viewed as a linear operator on  $V$ ,  $B(U, h|_U) : V \rightarrow U$  is the orthogonal projection. In particular,

$$B(U, h|_U) \leq B(V, h).$$

*Proof.* 1. There is  $P \in \text{GL}_n(\mathbb{C})$  such that  $(e_1, \dots, e_n) = (v_1, \dots, v_n)P$ ,  
 $Id = h((e_1, \dots, e_n)^T, (e_1, \dots, e_n)) = h(P^T(v_1, \dots, v_n)^T, (v_1, \dots, v_n)P) = P^T G \bar{P}$ ,  
then  $G = P^{-T} \bar{P}^{-1}$ , so  $A^T = PP^H$ .

$$\begin{aligned} B(U, h|_U) &= (e_1, \dots, e_n) \otimes (e_1, \dots, e_n)^H \\ &= (v_1, \dots, v_n)P \otimes P^H(v_1, \dots, v_n)^H \\ &= \sum_{kl} (PP^H)_{kl} v_k \otimes \bar{v}_l \\ &= \sum_{kl} v_k \otimes \bar{v}_l A_{lk}. \end{aligned}$$

2. Take an orthonormal basis  $e_1, \dots, e_n$  of  $U$  and  $v \in V$ . By definition,

$$B(U, h|_U) = \sum_{i=1}^n e_i \otimes \bar{e}_i.$$

So,  $B(U, h|_U)v = \sum_{i=1}^n h(v, e_i)e_i$  is the orthogonal projection of  $v$  into  $U$ .  $\square$

## 2.2 Complex Geometry and Kähler Geometry

Recall that for a hermitian manifold  $(M^n, h)$ , in local coordinate  $\phi = (z_1, z_2, \dots, z_n) : U \rightarrow \mathbb{C}^n$ , its hermitian metric is determined by  $g_{i\bar{j}} = h(\frac{\partial}{\partial z_i}, \frac{\partial}{\partial \bar{z}_j})$ . The matrix  $(g^{i\bar{j}})$  is the transpose-inverse of the positive-definite matrix  $(g_{i\bar{j}})$ . Then  $g^{i\bar{j}} = h(dz^i, d\bar{z}^j)$ . Its fundamental form  $\omega = i \sum_{jk} g_{j\bar{k}} dz_j \wedge d\bar{z}_k$  and the volume form is  $\mu = \frac{\omega^n}{n!}$ .

**Exercise 4.** *Locally*

$$\mu = i^n \det(g_{i\bar{j}}) dz_1 \wedge d\bar{z}_1 \cdots \wedge dz_n \wedge d\bar{z}_n = 2^n \det(g_{i\bar{j}}) \phi^* \mu_0.$$

*Proof.*

$$\begin{aligned} \omega^n &= i^n \sum_{i_1, \bar{j}_1, \dots, i_n, \bar{j}_n} g_{i_1, \bar{j}_1} \cdots g_{i_n, \bar{j}_n} dz_{i_1} \wedge d\bar{z}_{j_1} \cdots \wedge dz_{i_n} \wedge d\bar{z}_{j_n} \\ &= i^n \sum_{i_*, j_* \in S_n} \epsilon(i_*, j_*) g_{i_1, \bar{j}_1} \cdots g_{i_n, \bar{j}_n} dz_1 \wedge d\bar{z}_1 \cdots \wedge dz_n \wedge d\bar{z}_n \\ &= n! i^n \sum_{j_* \in S_n} g_{1, \bar{j}_1} \cdots g_{n, \bar{j}_n} dz_1 \wedge d\bar{z}_1 \cdots \wedge dz_n \wedge d\bar{z}_n \\ &= i^n n! (\det g_{i\bar{j}}) dz_1 \wedge d\bar{z}_1 \wedge \cdots \wedge dz_n \wedge d\bar{z}_n. \end{aligned}$$

We get the first equation and the second follows since  $dz_j \wedge d\bar{z}_j = -2idz_j \wedge dy_j$ .  $\square$



**Definition 3** (Ricci form). Let  $(M, J)$  be an hermitian manifold whose Ricci tensor is noted by  $Ric$ , then  $\rho(X, Y) = Ric(JX, Y)$  defines a real 2-form  $\rho$ .

**Definition 4** (positive form). A smooth real  $(1, 1)$  form  $\alpha$  on a complex manifold  $(M, J)$  is called positive if the symmetric bilinear form  $(X, Y) \rightarrow \alpha(X, JY)$  for real tangent vectors  $X, Y$  is positive definite. A cohomology class in  $H_{dR}^2(M; \mathbb{R})$  is positive if it can be represented by a closed positive  $(1, 1)$  form.

Some central objects on which we will concentrate are the following.

**Definition 5** (positive line bundle). A holomorphic line bundle  $L \rightarrow M$  is called positive if its first Chern class  $c_1(L)$  is positive.

**Definition 6** (Kähler manifold). On a complex manifold  $(M, J)$ , a real  $(1, 1)$  form  $\omega$  is called Kähler if  $\omega$  is closed positive and non-degenerate. In that case,  $g(X, Y) = \omega(X, JY)$  defines a Riemann metric on  $M$  and  $\omega$  itself is a symplectic form on  $M$ .

To fix notation, we briefly discuss the following example.

**Example 1** (complex projective space). Set theoretically  $\mathbb{C}\mathbb{P}^n$  is the space of all complex line subspaces of  $\mathbb{C}^{n+1}$ . Using homogeneous coordinates  $[Z_0 : \dots : Z_n]$ , the standard atlas is  $\{U_0, \dots, U_n\}$ , where  $U_i = \{Z_i \neq 0\} \subset \mathbb{C}\mathbb{P}^n$ . On  $U_0$  the standard local holomorphic coordinate is  $z_i = \frac{Z_i}{Z_0}$  for  $i = 1, \dots, n$ , i.e. a biholomorphism  $\phi_0 : U_0 \rightarrow \mathbb{C}^n$  by  $[Z_0, \dots, Z_n] \mapsto (\frac{Z_1}{Z_0}, \dots, \frac{Z_n}{Z_0})$ .

**Example 2** (tautological line bundle). Let  $O(-1)$  be a line bundle on  $\mathbb{C}\mathbb{P}^n$  whose transition function from  $U_i$  to  $U_j$  is  $\frac{Z_j}{Z_i}$ . Concretely  $O(-1) = \{([l], w) \in \mathbb{C}\mathbb{P}^n \times \mathbb{C}^{n+1} : w \in l\}$  viewing  $\mathbb{C}\mathbb{P}^n$  as the set of lines in  $\mathbb{C}^{n+1}$ . This line bundle  $O(-1)$  is called the tautological line bundle of  $\mathbb{C}\mathbb{P}^n$ . With this point of view, the standard metric on  $\mathbb{C}^{n+1}$  induces a metric  $h^{-1}$  on  $O(-1)$ . We call  $\sigma_0^{-1} : U_0 \rightarrow O(-1)$  by  $[Z_0 : \dots : Z_n] \mapsto ([Z_0 : \dots : Z_n], (1, \frac{Z_1}{Z_0}, \dots, \frac{Z_n}{Z_0}))$  the standard frame of  $O(-1)$  over  $U_0$ . The local expression of  $h^{-1}$  on  $U_0$  is  $h^{-1}(\sigma_0^{-1}, \sigma_0^{-1}) = 1 + |z|^2$ .

Let  $O(1)$  be the dual of  $O(-1)$  and more generally set  $O(k) = O(1)^{\otimes k}$  for  $k \in \mathbb{Z}$ . Then  $h^k$  is a hermitian metric on  $O(k)$ . Then  $(O(k), h^k)$  is positive if and only if  $k > 0$  by Example 4. We similarly define the standard frame  $\sigma_i^k \in H^0(U_i, O(k))$  of  $O(k)$  on  $U_i$ .

Define a map  $\Sigma : \mathbb{C}[Z_0, \dots, Z_n]_k \rightarrow H^0(\mathbb{C}\mathbb{P}^n, O(k))$  by  $P \mapsto s_P$ , where

$$s_P|_{U_i} = \frac{P(Z_0, \dots, Z_n)}{Z_i^k} \sigma_i^k,$$

then  $\Sigma$  is a linear isomorphism. In particular,  $H^0(\mathbb{C}\mathbb{P}^n, O(k))$  has a basis

$$\{s_I : |I| = k, I \in \mathbb{N}^{n+1}\}, \quad s_I = \Sigma(Z^I).$$

As a result,  $\dim H^0(\mathbb{C}\mathbb{P}^n, O(k)) = \binom{k+n}{n}$ .

**Definition 7** (ample and very ample). *On a compact complex manifold  $M$ , a line bundle  $L$  is called:*

1. *very ample if there is a projective embedding  $j : M \rightarrow \mathbb{C}\mathbb{P}^n$  such that  $L$  is isomorphic to  $j^*O(1)$ ;*
2. *ample if there is  $m \geq 1$  such that  $L^m$  is very ample.*

Recall the following famous result.

**Theorem 1** (Kodaira). *On a compact complex manifold  $M$ , a line bundle is ample if and only if it is positive. If  $M$  admits such a line bundle, then  $M$  is projective.*

**Definition 8** (canonical line bundle). *For a complex manifold  $M^n$ , we denote  $K_M := \det(T_h^*M) = \Lambda^{(n,0)}T^*M$ . On a holomorphic chart  $(U, z_1, \dots, z_n)$ , it has a natural frame  $dz_1 \wedge \dots \wedge dz_n$ .*

**Example 3.** *The canonical bundle of  $\mathbb{C}\mathbb{P}^n$  is  $O(-n-1)$ .*

*Proof.* Let  $(U_0, z_1, \dots, z_n), (U_1, w_1, \dots, w_n)$  be the standard charts of  $\mathbb{C}\mathbb{P}^n$ , where  $w_1 = \frac{z_0}{z_1}$  and  $w_l = \frac{z_l}{z_1}$  for  $l > 1$ . Now  $\sigma_0 = dz_1 \wedge \dots \wedge dz_n$  (resp.  $\sigma_1 = dw_1 \wedge \dots \wedge dw_n$ ) is a local frame of the canonical bundle on  $U_0$  (resp. on  $U_1$ ). On  $U_0 \cap U_1$ ,  $z_1 = \frac{1}{w_1}$ ,  $dz_1 = -w_1^{-2}dw_1$  and for  $l > 1$ ,  $z_l = \frac{w_l}{w_1}$ .

$$dz_l = \frac{w_1 dw_l - w_l dw_1}{w_1^2}.$$

Then  $\sigma_0 = -w_1^{-(n+1)}\sigma_1 = -\left(\frac{z_1}{z_0}\right)^{(n+1)}\sigma_1$ . From this transition law, we see that the canonical bundle is  $O(-n-1)$ .  $\square$

**Definition 9** (Kähler potential). *Let  $(M, \omega)$  be a Kähler manifold, a local smooth real-valued function  $K$  is called a Kähler potential for  $(M, \omega)$  if  $\omega = \frac{i}{2}\partial\bar{\partial}K$ .*

By complex Poincaré lemma, for any Kähler manifold, locally there is a Kähler potential.

**Definition 10** (Kähler-Einstein manifold). *A Kähler manifold  $(M, \omega)$  is called Kähler-Einstein if there is a constant  $\lambda$  such that  $Ric = \lambda g$  or equivalently the Ricci form  $\rho = \lambda \omega$ .*

If  $M^n$  is a Kähler-Einstein manifold:  $Ric = \lambda g$ , then its scalar curvature

$$scal = g^{i\bar{j}} Ric_{i\bar{j}} = n\lambda \tag{2.1}$$

is constant.

**Example 4** (Fubini-Study). *On a complex projective space  $\mathbb{C}\mathbb{P}^n$ , the form  $\omega_{FS} := 2\pi c_1(O(1), h)$  is a Kähler form, which is called the Fubini-Study form. Its expression on the standard chart  $(U_0, \phi_0)$  is*

$$\omega_{FS} = i \left[ \sum_{j=1}^n \frac{dz_j \wedge d\bar{z}_j}{1 + |z|^2} - \frac{1}{(1 + |z|^2)^2} \left( \sum_{k=1}^n \bar{z}_k dz_k \right) \wedge \left( \sum_{l=1}^n z_l d\bar{z}_l \right) \right],$$

where  $K = 2 \log(1 + |z|^2)$  is a Kähler potential. Its local expression is

$$g_{i\bar{j}} = \frac{\delta_{ij}(1 + |z|^2) - \bar{z}_i z_j}{(1 + |z|^2)^2}$$

or rather

$$(g_{i\bar{j}}) = \frac{1}{1 + |z|^2} (Id_n - \frac{1}{1 + |z|^2} \bar{z}z^T),$$

from which we derive that  $\det(g_{i\bar{j}}) = (1 + |z|^2)^{-n-1}$  and its transpose-inverse  $(g^{i\bar{j}}) = (1 + |z|^2)(Id_n + zz^H)$ , so  $g^{i\bar{j}} = (1 + |z|^2)(\delta_{ij} + z_i \bar{z}_j)$ .

So,

$$Ric_{i\bar{j}} = -\frac{\partial^2}{\partial z_i \partial \bar{z}_j} \log \det(g_{k\bar{l}}) = (n + 1)g_{i\bar{j}},$$

i.e.,  $(\mathbb{C}\mathbb{P}^n, \omega_{FS})$  is Kähler-Einstein. The scalar curvature is  $n(n + 1)$ .

By Exercise 4, its volume form

$$\mu = \frac{2^n \mu_0}{(1 + |z|^2)^{n+1}}, \quad (2.2)$$

where  $\mu_0$  is the pullback of Lebesgue measure along  $\varphi_0$ . The volume of  $(\mathbb{C}\mathbb{P}^n, \omega_{FS})$  is

$$\int_{\mathbb{C}^n} \frac{2^n \mu_0}{(1 + |z|^2)^{n+1}} = 2^n \text{Area}(S^{2n-1}) \int_0^\infty \frac{r^{2n-1}}{(1 + r^2)^{n+1}} dr = \frac{(2\pi)^n}{n!}.$$

For a hermitian manifold  $M$  and  $f \in C^2(M)$ , we define the Laplacian by

$$\Delta f = 2 \sum_{i,j} g^{i\bar{j}} \frac{\partial^2 f}{\partial z_i \partial \bar{z}_j}. \quad (2.3)$$

For a hermitian manifold  $M$ , (cf.[27, Section 1.2]) its Chern curvature (corresponding to Riemann curvature if  $M$  is Kähler) components are

$$\begin{aligned} R_{i\bar{j}k\bar{l}}^M &:= g(R^M(\frac{\partial}{\partial z_i}, \frac{\partial}{\partial \bar{z}_j}) \frac{\partial}{\partial \bar{z}_l}, \frac{\partial}{\partial z_k}) \\ &= g_{k\bar{m}} \frac{\partial \Gamma_{\bar{j}l}^{\bar{m}}}{\partial z_i} \\ &= -\frac{\partial^2 g_{i\bar{j}}}{\partial z_k \partial \bar{z}_l} + g^{p\bar{q}} \frac{\partial g_{i\bar{q}}}{\partial z_k} \frac{\partial g_{p\bar{j}}}{\partial \bar{z}_l}. \end{aligned} \quad (2.4)$$

*Proof.*  $R(\partial_i, \partial_{\bar{j}})\partial_{\bar{l}} = \nabla_i \nabla_{\bar{j}} \partial_{\bar{l}} = \nabla_i (\overline{\Gamma_{j\bar{l}}^m} \partial_{\bar{m}}) = \frac{\partial \overline{\Gamma_{j\bar{l}}^m}}{\partial z_i} \partial_{\bar{m}}$ , so  $R_{i\bar{j}k\bar{l}} = g(\frac{\partial \overline{\Gamma_{j\bar{l}}^m}}{\partial z_i} \partial_{\bar{m}}, \partial_k) = g_{k\bar{m}} \frac{\partial \overline{\Gamma_{j\bar{l}}^m}}{\partial z_i}$ .  $\square$

They enjoy symmetries below (cf.[25, Exercise 1.21]):

$$R_{i\bar{j}k\bar{l}} = R_{k\bar{j}i\bar{l}} = R_{i\bar{l}k\bar{j}}. \quad (2.5)$$

Define  $R_{\bar{k}q, \bar{j}}^{\bar{m}}$  by  $R(\partial_{\bar{k}}, \partial_q) \frac{\partial}{\partial z_j} = \nabla_{\bar{k}} \nabla_q \partial_{\bar{j}} - \nabla_q \nabla_{\bar{k}} \partial_{\bar{j}} = R_{\bar{k}q, \bar{j}}^{\bar{m}} \partial_{\bar{m}}$ , we find that

$$R_{\bar{k}q, \bar{j}}^{\bar{l}} = -\frac{\partial \overline{\Gamma_{kj}^l}}{\partial z_q}.$$

The Ricci curvature components are

$$Ric_{k\bar{l}}^M = g^{i\bar{j}} R_{i\bar{j}k\bar{l}}^M = -\frac{\partial^2 \log \det(g_{i\bar{j}})}{\partial z_k \partial \bar{z}_l} = -\frac{\partial \Gamma_{ik}^k}{\partial \bar{z}_j}.$$

The scalar curvature is

$$\text{scal} = g^{i\bar{j}} Ric_{i\bar{j}}^M \quad (2.6)$$

. If  $M$  is Kähler, then We can define Christoffel symbols by

$$\nabla_{\frac{\partial}{\partial z_j}} \frac{\partial}{\partial z_k} = \sum_{i=1}^n \Gamma_{jk}^i \frac{\partial}{\partial z_i},$$

the symmetry  $\Gamma_{jk}^i = \Gamma_{kj}^i$ . By [25, Lemma 1.19]

$$\Gamma_{jk}^i = g^{i\bar{l}} \frac{\partial g_{k\bar{l}}}{\partial z_j}, \quad \Gamma_{\bar{j}k}^{\bar{l}} = \overline{\Gamma_{jk}^i} = g^{\bar{l}i} \frac{\partial g_{k\bar{i}}}{\partial \bar{z}_j}$$

and other Christoffel symbols of mixed type vanish. By [25, Example 1.17],

$$\nabla_{\frac{\partial}{\partial \bar{z}_i}} = -\sum_j \Gamma_{ij}^{\bar{k}} d\bar{z}^j. \quad (2.7)$$

**Definition 11.** For a complex manifold  $M$ , the first Chern class of  $M$  is defined to be  $c_1(M) := c_1(K_M^*)$ . Given moreover a hermitian form  $\omega$ , its first Chern form  $c_1(M, \omega) := c_1(K_M^*, h)$ , where  $h$  is the induced metric on  $K_M^*$ .

**Theorem 2** (Chern). Let  $(M, \omega)$  be a Kähler manifold, then its Ricci form  $\rho = 2\pi c_1(M, \omega)$ . In local expression

$$\rho = -i\partial\bar{\partial} \log \det(g_{i\bar{j}}) = i Ric_{k\bar{l}} dz_k \wedge d\bar{z}_l.$$

In particular, the class  $[\rho]$  is independent of the choice of Kähler form  $\omega$ .

*Proof.* Take a holomorphic chart  $(U, z_1, \dots, z_n)$  of  $M$ , then  $K_M^*$  has a holomorphic frame  $\sigma = \frac{\partial}{\partial z_1} \wedge \dots \wedge \frac{\partial}{\partial z_n}$ . Then  $|\sigma|_h^2 = \det(g_{i\bar{j}})$ . Therefore,

$$\rho = i\bar{\partial}\partial \log \det(g_{i\bar{j}}) = 2\pi c_1(M, \omega). \quad \square$$

**Theorem 3** (K-coordinate). [19, Proposition 2.2] Let  $(M, \omega)$  be a Kähler manifold with  $\omega$  real analytic, then given  $x \in M$ , there is a unique (up to affine transformation) local holomorphic coordinate  $(U; z_1, \dots, z_n)$  centered in  $x$  with a Kähler potential  $2\varphi$  such that in the Taylor expansion of  $\varphi$ , all the  $(0, l)$ ,  $(1, l)$ ,  $(l, 1)$ ,  $(l, 0)$  terms ( $l \in \mathbb{N}$ ) vanish except for  $(1, 1)$  term which equals to  $|z|^2$ . If  $\omega$  is only smooth, then for any  $A \in \mathbb{N}^+$ , the previous statement still holds with  $0 \leq l \leq A$ .

**Remark 1.** Tian used this theorem with  $A = 4$  in [26, (1.3)].

**Exercise 5.** [25, Section 7.2, p.132] In that case,

$$\varphi(z) = \varphi_4(z) + O(|z|^5)$$

where

$$\varphi_4 = |z|^2 - \frac{1}{4} \sum_{ijkl} R_{i\bar{j}k\bar{l}}^M(p) z_i \bar{z}_j z_k \bar{z}_l.$$

Moreover,

$$\det(g_{i\bar{j}}) = 1 - \sum_{k,l} Ric_{k\bar{l}} z_k \bar{z}_l + O(|z|^3).$$

*Proof.* By Theorem 3, we may find a chart such that  $(1, 1)$ -term of  $\varphi$  is  $|z|^2$  and all the other  $(0, l)$ ,  $(1, l)$ ,  $(l, 1)$ ,  $(l, 0)$  terms vanish for  $0 \leq l \leq 3$ . In particular,

$$\varphi = |z|^2 - \frac{1}{4} \sum_{ijkl} c_{ijkl} z_i \bar{z}_j z_k \bar{z}_l + O(|z|^5)$$

for certain constants  $c_{ijkl}$  satisfying symmetries  $c_{ijkl} = c_{kjil} = c_{ilkj}$ . Then

$$g_{i\bar{j}} = \frac{\partial^2 \varphi}{\partial z_i \partial \bar{z}_j} = \delta_{ij} - \sum_{k,l} c_{ijkl} z_k \bar{z}_l + O(|z|^3); \quad (2.8)$$

$$\frac{\partial^3 \varphi}{\partial z_i \partial \bar{z}_j \partial z_k} (p) = 0; \quad (2.9)$$

$$\frac{\partial^3 \varphi}{\partial z_i \partial \bar{z}_j \partial \bar{z}_l} (p) = 0; \quad (2.10)$$

$$\frac{\partial^4 \varphi}{\partial z_i \partial \bar{z}_j \partial z_k \partial \bar{z}_l} = -c_{ijkl} + O(|z|). \quad (2.11)$$

Thus by (2.4) we find  $c_{ijkl} = R_{i\bar{j}k\bar{l}}^M(p)$ . We calculate

$$\begin{aligned}
\det(g_{i\bar{j}}) &= \sum_{J=(j_1, \dots, j_n) \in \mathcal{S}_n} \epsilon(J) g_{1\bar{j}_1} \cdots g_{n\bar{j}_n} \\
&= \sum_J \epsilon(J) (\delta_{1j_1} - \sum_{k_1, l_1} R_{1\bar{j}_1 k_1 \bar{l}_1}(p) z_{k_1} \bar{z}_{l_1} + O(|z|^3)) \cdots (\delta_{nj_n} - \sum_{k_n, l_n} R_{n, \bar{j}_n, k_n, \bar{l}_n}(p) z_{k_n} \bar{z}_{l_n} + O(|z|^3)) \\
&= \sum_J \epsilon(J) (\delta_{1j_1} \cdots \delta_{nj_n} - \sum_{m=1}^n \sum_{k_m, l_m} R_{m, \bar{j}_m, k_m, \bar{l}_m} z_{k_m} \bar{z}_{l_m} \delta_{1j_1} \cdots \widehat{\delta_{m, j_m}} \cdots \delta_{nj_n} + O(|z|^3)) \\
&= 1 - \sum_{m=1}^n \sum_{k_m, l_m} R_{m, \bar{j}_m, k_m, \bar{l}_m} z_{k_m} \bar{z}_{l_m} + O(|z|^3) \\
&= 1 - \sum_{k, l} Ric_{k, \bar{l}}(p) z_k \bar{z}_l + O(|z|^3).
\end{aligned}$$

□

To avoid confusion, we recall some definitions.

**Proposition 2** (Chern connection). *For a holomorphic vector bundle  $E \rightarrow M$  with hermitian metric  $h^E$  on a complex manifold  $M$ , there is a unique connection  $\nabla$  on  $E$  which is*

1. hermitian: for any  $s, t \in C^\infty(M, E)$ ,

$$d(s, t)_{h^E} = (\nabla s, t)_{h^E} + (s, \nabla t)_{h^E}.$$

In local expression

$$\frac{\partial}{\partial z_k} (s, t)_{h^E} = (\nabla_{\frac{\partial}{\partial z_k}} s, t)_{h^E} + (s, \nabla_{\frac{\partial}{\partial z_k}} t)_{h^E}.$$

2. compatible with the holomorphic structure: for any smooth field  $U \in T^{0,1}M$ ,  $s \in C^\infty(M, E)$ , we have

$$\nabla_U s = \bar{\partial}^E s(U).$$

**Definition 12** (Chern curvature). *The curvature form  $R^E = \Theta(E, h^E)$  of the Chern curvature given by Proposition 2 is called the Chern curvature of  $(E, h^E)$ . Define the first Chern curvature form by  $c_1(E, h^E) = \frac{i \text{Tr}(R^E)}{2\pi} = c_1(\det(E), h^{\det(E)})$  and the first Chern class  $c_1(E) = [c_1(E, h^E)] \in H_{dR}^2(M; \mathbb{R})$ .*

In local expression,

$$R_{\alpha, \bar{\beta}, i, \bar{j}}^E = -\frac{\partial^2 h_{\alpha, \bar{\beta}}}{\partial z_i \partial \bar{z}_j} + h^{\lambda, \bar{\mu}} \frac{\partial h_{\alpha, \bar{\mu}}}{\partial z_i} \frac{\partial h_{\lambda, \bar{\beta}}}{\partial \bar{z}_j}.$$

Note that when  $E = T_h M$ , we recover last line of (2.4). Consider the particular case of a hermitian line bundle  $(L, h)$ . For a local frame  $\sigma \in H^0(U, L)$ ,

$\bar{\partial}\partial(\log|\sigma|_h^2)$  is a  $(1, 1)$ -form on  $U$  independent of the choice of  $s$  and over various  $U$  they glue to a global  $(1, 1)$ -form, which is exactly the Chern curvature of  $(L, h)$ .

**Exercise 6.** *If in addition  $(M, \omega)$  is a hermitian manifold, we recall the Lefschetz operator  $L$  on  $\Lambda^{*,*}(T^*M) \otimes E$  by  $(\omega \wedge) \otimes Id$  and its adjoint  $\Lambda$  with respect to the hermitian metric therein induced by  $\omega$  and  $h^E$ . Then*

$$(i\Lambda R^E)_{\alpha,\beta} = g^{i\bar{j}} R_{\alpha,\beta,i,\bar{j}}^E.$$

*In particular, when  $E = T_h M$  and  $h^E$  is the hermitian metric of  $M$ , then  $\text{Tr}(i\Lambda R^E) = \text{scal}$ .*

*Proof.* Take  $\{w_j\}$  a smooth orthonormal frame of  $T_h M$ . Expand it in a local holomorphic chart  $(z_1, \dots, z_n)$ :

$$(w_1, \dots, w_n) = \left( \frac{\partial}{\partial z_1}, \dots, \frac{\partial}{\partial z_n} \right) P,$$

then  $Id = P^T G \bar{P}$ , where  $G = (g_{i\bar{j}})$ . Thus  $G^{-1} = \bar{P} P^T$ , so  $g^{k\bar{l}} = \sum_{j=1}^n \bar{P}_{lj} P_{kj}$ . By [15, (1.4.32)],

$$\begin{aligned} i\Lambda R^E &= \sum_{j=1}^n R^E(w_j, \bar{w}_j) = \sum_{j,k,l} R^E(P_{kj} \frac{\partial}{\partial z_k}, \bar{P}_{lj} \frac{\partial}{\partial \bar{z}_l}) \\ &= \sum_{k,l} R_{k\bar{l}}^E \left( \sum_{j=1}^n \bar{P}_{lj} P_{kj} \right) = \sum_{k,l} R_{k\bar{l}}^E g^{k\bar{l}}. \end{aligned}$$

We proved the first and the second follows from (2.6).  $\square$

**Proposition 3** (normal coordinate frame). [5, Ch.V, Proposition 12.10] *Let  $M$  be a complex manifold and  $(E, h)$  a hermitian holomorphic vector bundle of rank  $r$  on  $M$ . For any local holomorphic coordinates  $(z_1, z_2, \dots, z_n)$  centered in  $p \in M$ , there exists a holomorphic frame  $v_1, \dots, v_r$  of  $E$  in a neighborhood of  $p$  such that*

$$R^E(p) = \sum_{\alpha,\beta,j,l} R_{\alpha,\beta,j,\bar{l}}^E(p) v_\alpha \otimes \bar{v}_\beta \otimes (dz_j \wedge d\bar{z}_l)(p), \quad (2.12)$$

$$h(v_\alpha, v_\beta) = \delta_{\alpha,\beta} - \sum_{j,l=1}^n R_{\alpha,\beta,j,\bar{l}}^E(p) z_j \bar{z}_l + O(|z|^3). \quad (2.13)$$

*Let  $H^E$  be the Gram matrix  $(h(v_\alpha, v_\beta))$ . Let*

$$H_2^E(z) = \left( \delta_{\alpha,\beta} - \sum_{j,l=1}^n R_{\alpha,\beta,j,\bar{l}}^E(p) z_j \bar{z}_l \right), \quad (2.14)$$

*then we can rewrite (2.13) as  $H^E = H_2^E + O(|z|^3)$ .*

Let  $M$  be a hermitian manifold, then there is an induced hermitian metric on the space of global smooth  $(p, q)$  forms  $\Omega^{p,q}(M)$ . Let  $(E, h)$  be a holomorphic hermitian vector bundle on  $M$ .

**Definition 13** (anti-holomorphic Kodaira-Laplacian). [15, Definition 1.4.9] Let  $\bar{\partial}^{E,*}$  be the formal adjoint of the Dolbeault operator  $\bar{\partial}^E$ . Define an operator on the space of smooth forms  $\Omega^{*,*}(M, E)$  by

$$\square^E = [\bar{\partial}^E, \bar{\partial}^{E,*}].$$

**Theorem 4** (Hodge decomposition). [15, Thm 1.4.1] If  $M$  is in addition compact, then we have an orthogonal decomposition

$$C^\infty(M, E) = H^0(M, E) \oplus \text{Im}(\square^E|_{C^\infty(M, E)}).$$

Moreover,  $H^0(M, E)$  is a finite dimensional  $\mathbb{C}$ -vector space.

The following inequality relies on Bochner-Kodaira-Nakano identity.

**Theorem 5** (Nakano's inequality). [15, Theorem 1.4.14] If in addition  $M$  is Kähler, take  $s \in \Omega_c^{(p,q)}(M, E)$  an arbitrary compactly supported smooth  $E$ -valued  $(p, q)$ -form, then

$$\langle \square^E s, s \rangle \geq \langle [iR^E, \Lambda]s, s \rangle,$$

where the global scalar product is defined in (1.3.14) loc.cit.

## 2.3 Stationary phase method

**Theorem 6** (Hörmander). [18, Theorem 13.3.2][10, Theorem 7.7.5] Let  $U \subset \mathbb{R}^d$  be an open neighborhood of  $0 \in \mathbb{R}^d$  and  $\varphi, A$  be smooth functions on  $U$  such that  $\text{Re}(\varphi) \geq 0$  on  $U$ . Furthermore, suppose that on  $\text{supp}(A)$ ,  $\varphi$  has a unique stationary point at  $0$  and  $\varphi(0) = 0$ . Suppose the Hessian  $H$  of  $\varphi$  at  $0$  is nonsingular (i.e.,  $0$  is a non-degenerate critical point of  $\varphi$ .) Define a second order differential operator

$$D = \sum_{p,q=1}^d (H^{-1})_{p,q} \frac{\partial^2}{\partial x_p \partial x_q}$$

and  $\underline{\varphi} = \varphi(x) - (x, Hx)/2$ . Set

$$L_j(A\varphi) = (-1)^j \sum_{0 \leq l \leq 2j} \frac{D^{l+j}(A\underline{\varphi}^l)(0)}{2^{l+j} l! (l+j)!},$$

then for each  $M \in \mathbb{N}^+$ , let  $N = M + [d/2]$ , for  $k \in \mathbb{R}_+$ , we have

$$\left| \int_U A(x) e^{-k\varphi(x)} \mu_0 - \left[ \det\left(\frac{kH}{2\pi}\right) \right]^{-1/2} \sum_{j < M} k^{-j} L_j(A\varphi) \right| \leq C(\varphi) k^{-M} \sum_{|I| \leq 2N, I \in \mathbb{N}^d} \sup |\partial_x^I A|.$$

Here the constant  $C(\varphi)$  is bounded when  $\varphi$  stays in a bounded set in  $C^{3N+1}$ .



The following result play the same role in the proof of main theorem, but with integral domain shrinking with the growth of  $k$ .

**Lemma 1.** [14, Lemma 4.1] *Let  $A$  be a symmetric function on  $[n]^p \times [n]^p$ . Then*

$$\sum_{I, J \in [n]^p} \int_{|z| \leq \frac{\log k}{\sqrt{k}}} A_{I, J} z^I \bar{z}^J |z|^{2q} e^{-k|z|^2} \mu_0 = \pi^n \left( \sum_{I \in [n]^p} A_{I, I} \frac{p!(n+p+q-1)!}{(p+n-1)!k^{n+p+q}} \right) + O(e^{-\log^2 k}).$$

We collect some consequences of this lemma.

$$\int_{B(0, \frac{\log k}{\sqrt{k}})} e^{-k|z|^2} \mu_0 = \frac{\pi^n}{k^n} + O(e^{-(\log k)^2}) \quad (2.15)$$

$$\sum_{i, j} \int_{B(0, \frac{\log k}{\sqrt{k}})} Ric_{i, \bar{j}}^M(p) z_i \bar{z}_j e^{-k|z|^2} \mu_0 = \left(\frac{\pi}{k}\right)^n \text{scal}(p) + k^n + O(e^{-(\log k)^2}) \quad (2.16)$$

$$\int_{B(0, \frac{\log k}{\sqrt{k}})} e^{-k|z|^2} |z|^4 O(k) \left( \sum_{j, l} R_{\alpha, \beta, j, \bar{l}}^E z_j \bar{z}_l \det_2 \mu_0 \right) = O(k^{-n-2}) \quad (2.17)$$

$$\int_{B(p, \frac{\log k}{\sqrt{k}})} e^{-k|z|^2} \sum R_{\alpha, \beta, j, \bar{l}}^E z_j \bar{z}_l \mu_0 = (i\Lambda R^E)_{\alpha, \beta} \frac{\pi^n}{k^{n+1}} + O(e^{-(\log k)^2}) \quad (2.18)$$

$$\int_{B(0, \frac{\log k}{\sqrt{k}})} e^{-k\varphi_4} (|z|^5 O(k) + O(|z|^3)) \mu_0 = O(k^{-n-3/2}) \quad (2.19)$$

$$\sum_{i, \bar{j}} R_{\alpha, \beta, i, \bar{j}}^E \int_{B(p, \frac{\log k}{\sqrt{k}})} e^{-k|z|^2} z_i \bar{z}_j (O(|z|^3) + |z|^4 O(k)) \mu_0 = O(k^{-n-2}), \quad (2.20)$$

where for (2.18) we use Exercise 6.

## Chapter 3

# Bergman kernel

Let  $(M^n, g)$  be a compact hermitian manifold whose volume form is denoted by  $\mu$ , and  $P \rightarrow M$  be a holomorphic vector bundle with hermitian metric  $h = h^P$ . Endow  $C^\infty(M, P)$  a hermitian inner product by

$$\langle s, t \rangle = \int_M (s, t) \mu.$$

Define  $\|s\|_h^2 = \langle s, s \rangle$ . Let  $L^2(M, P)$  be the completion of  $(C^\infty(M, P), \langle \cdot, \cdot \rangle)$ . By Theorem 4,  $H^0(M, P)$  is finite dimensional. Choose an orthonormal basis  $(e_1, \dots, e_N)$  of  $H^0(M, P)$ .

**Definition 14** (Bergman kernel). *For  $p, q \in M$  the Bergman kernel of  $P$  is*

$$\Pi(p, q) = \sum_{i=1}^N e_i(p) \otimes \bar{e}_i(q) = \sum_{i=1}^N h_q(\cdot, e_i(q)) e_i(p) : P_q \rightarrow P_p.$$

Write

$$B(p) = \Pi(p, p) \in \text{End}(P_p),$$

where we use the identification given by Exercise 1.

In this m emoire, we are interested only in diagonal values. By Exercise 1,  $B(p) \in \text{Herm}^+(P_p, h_p)$ . Note that  $B \in C^\infty(M, \text{End}(P))$ . For  $\lambda$  a positive constant, if we use the metric  $\lambda g$  instead  $g$  on  $M^n$  (i.e., a constant scaling), the new volume form is  $\lambda^n \mu$ , the new scalar curvature is  $\frac{\text{scal}}{\lambda}$  and new Bergman kernel is  $\lambda^{-n} B$ .

**Exercise 7.** *Use another hermitian metric  $h'$  conformal to  $h$ , i.e., there is a real function  $f$  on  $M$  such that  $h'(v, v) = e^{2f} h(v, v)$  for any smooth section  $v$  of  $P$ . Let  $B'$  be the Bergman kernel of  $P$  with respect to  $h'$ . If  $h', h$  are of the same Chern curvature form, then  $f$  is a constant and the corresponding Bergman kernels  $B$  and  $B'$  coincide. (Note that if  $P$  is a line bundle, the conformal condition is automatic.)*

*Proof.* Recall that  $R(P, h') = \bar{\partial}(h'^{-1} \cdot \partial h') = 2(\bar{\partial}\partial f)Id + R(P, h)$ , so  $\bar{\partial}\partial f \equiv 0$ . By (2.3),  $\Delta f = 0$ . The manifold  $M$  being compact, by Stoke's formula,

$$\int_M (f\Delta f + |\nabla f|^2)\mu = 0.$$

We see that  $f$  is a constant. In particular,  $\{e^{-f}e_i | i = 1, \dots, N\}$  is an orthonormal basis of  $H^0(M, P)$  with respect to  $h'$ . For any  $p \in M$ ,

$$B'(p) = \sum_{i=1}^N h'_p(\cdot, e^{-f}e_i(p))e^{-f}e_i(p) = \sum_{i=1}^N h_p(\cdot, e_i(p))e_i(p) = B(p).$$

□

**Remark 2.** If  $s_1, \dots, s_N$  is a basis of  $H^0(M, P)$ , where  $s_{q+1}(p) = \dots = s_N(p) = 0$ , then by Exercise 3,

$$B(p) = \sum_{k,l=1}^q s_k(p) \otimes \bar{s}_l(p) A_{lk},$$

where  $A$  is the inverse of the size  $N$  Gram matrix  $((s_i, s_j))$ . This observation serves as a motivation of the peak section method firstly raised by Tian in [26]. In particular, if furthermore  $s_1(p), \dots, s_q(p)$  form an orthonormal basis of  $P_p$ , then  $B(p)_{kl} = A_{lk}$ . This approach is taken in [11, §3].

**Example 5.** If  $p \in M$  is a base point of  $P$ , i.e., all global sections of  $P$  vanish at  $p$ , then clearly  $B(p) = 0$ .

**Proposition 4.**  $\dim H^0(M, P) = \int_M \text{Tr}(B(p))\mu$ .

*Proof.* By Exercise 1,  $\text{Tr}(B(p)) = \sum_{i=1}^N |e_i|^2(p)$ . Taking integral

$$\int_M \text{Tr}(B(p))\mu = \sum_{i=1}^N \|e_i\|^2 = \sum_{i=1}^N 1 = \dim H^0(M, P).$$

□

As an analogue of Exercise 3 (2), we have an interpretation of Bergman kernel.

**Exercise 8.** The finite dimensional  $H^0(M, P)$  being a closed subspace of  $L^2(M, P)$ , there exists an orthogonal projection (so called Bergman projection)

$$\pi : L^2(M, P) \rightarrow H^0(M, P).$$

Then Bergman kernel  $\Pi$  is the reproducing kernel function of  $\pi$ , i.e., for any  $s \in L^2(M, P)$ ,

$$(\pi s)(p) = \int_M \Pi(p, q)s(q)\mu(q).$$

*Proof.* We calculate

$$\begin{aligned}\int_M \Pi(p, q) s(q) \mu(q) &= \int_M \sum_{i=1}^n (s, e_i)(q) e_i(p) \mu(q) \\ &= \sum_{i=1}^n e_i(p) \langle s, e_i \rangle = (\pi s)(p).\end{aligned}$$

□

We begin with a well-known generalization of [3, Lemma 3.1], which is an extremal characterization of Bergman kernel.

**Lemma 2.**

$$B(p) = \max_{s \in H^0(M, P) \setminus \{0\}} \frac{s(p) \otimes \bar{s}(p)}{\|s\|_h^2}$$

using the natural partial order of  $\text{Herm}^+(P_p, h_p)$ .

*Proof.* Take  $u \in P_p$ ,

$$(B(p)u, u) = \sum_{i=1}^N (u, e_i(p))(e_i(p), u) = \sum_{i=1}^N |(u, e_i(p))|^2.$$

For  $s \in H^0(M, P) \setminus 0$ , let  $s = \sum_{i=1}^N \lambda_i s_i$  with  $\lambda \in \mathbb{C}^N \setminus \{0\}$ , then  $\|s\|^2 = |\lambda|^2$ .

$$\begin{aligned}\left(\frac{s(p) \otimes \bar{s}(p)}{\|s\|_h^2}(u), u\right) &= \frac{1}{|\lambda|^2} \left( \left( \sum_i \lambda_i s_i(p) \right) \otimes \left( \sum_j \bar{\lambda}_j \overline{s_j(p)} \right) u, u \right) \\ &= \frac{1}{|\lambda|^2} \left( \sum_{i,j} \lambda_i \bar{\lambda}_j (u, s_j(p))(s_i(p), u) \right) \\ &= \frac{1}{|\lambda|^2} (\lambda^T A \bar{\lambda}),\end{aligned}$$

where

$$A = ww^H \quad w = ((s_1(p), u), \dots, (s_N(p), u))^t \in \mathbb{C}^N.$$

Therefore

$$\begin{aligned}\left( \max_{s \in H^0(M, P) \setminus 0} \frac{s(p) \otimes \bar{s}(p)}{\|s\|_h^2} u, u \right) &= \max_{s \neq 0} \left( \frac{s(p) \otimes \bar{s}(p) u, u}{\|s\|^2} \right) \\ &= \max_{\lambda \in \mathbb{C}^N \setminus 0} \frac{\lambda^T A \bar{\lambda}}{|\lambda|^2} \\ &= \rho(A) = \rho(w^H w) = w^H w \\ &= \sum_{i=1 \dots N} |(s_i(p), u)|^2 \\ &= (B(p)u, u),\end{aligned}$$

where we used Proposition 1. □

### 3.1 Main Theorem

The setting for Section 3.1 is the following: Let  $M^n$  be a projective manifold,  $(L, h^L)$  (resp.  $(E, h^E)$ ) be a hermitian holomorphic line (resp. vector) bundle with Chern curvature  $R^L$  (resp.  $R^E$ ). Assume that  $M$  is polarized by  $L$ , i.e.,  $iR^L$  is the Kähler form of  $M$  (so that  $(L, h^L)$  is positive). Let  $r$  be the rank of  $E$ ,  $\text{scal} : M \rightarrow \mathbb{R}$  be the scalar curvature. We state the main result of the paper.

**Theorem 7** (main theorem). *The Bergman kernel  $B_k$  of  $L^k \otimes E$  has an asymptotic expansion*

$$B_k(p) = \left(\frac{k}{2\pi}\right)^n (Id_r + k^{-1}[\frac{\text{scal}}{2}Id_r + i\Lambda R^E](p)) + O(k^{n-2}).$$

**Corollary 1.** *There is  $k_0 \geq 1$  such that for any  $k \geq k_0$ ,  $L^k \otimes E$  has no base points.*

We recall the following celebrated result.

**Theorem 8** (Catlin-Zelditch). *[2, Theroem 2][31, Theorem][14, Theorem 1.1] The Bergman kernel admits a complete asymptotic expansion*

$$B_k(p) \sim k^n \left( \sum_{l \geq 0} a_l(p) k^{-l} \right),$$

where  $a_l$  are universal polynomials in the curvature of  $(M, g)$ ,  $R^E$  as well as their successive covariant derivatives at  $p$ . Each  $a_l(p)$  is a self-adjoint matrix with respect to  $h^E$ . More precisely, for each  $s, l \in \mathbb{N}$ , there is a constant  $C = C(s, l, M)$  such that for any  $p \in M$ ,

$$|B_k(p) - \sum_{j=0}^s a_j(p) k^{n-j}|_{C^l} \leq C k^{n-s-1}.$$

The convergence of Bergman metrics is one of the motivations for search of such expansion in the history. As a corollary of this theorem, we recover Tian's result ([26, Theorem A, Theorem B], for  $C^2$  convergence) and Ruan's result ([19] for  $C^\infty$  convergence). Say  $N_k = \dim H^0(M, L^k) - 1$  and choose an orthonormal basis  $\{s_{k,0}, \dots, s_{k,N_k}\}$  of  $H^0(M, L^k)$ , then when  $k$  is large enough, the Kodaira map  $\phi_k : M \rightarrow \mathbb{C}\mathbb{P}^{N_k}$  by

$$x \mapsto [s_{k,0}(x) : \dots : s_{k,N_k}(x)]$$

is a holomorphic embedding. We call  $\frac{1}{k}\phi_k^*\omega_{FS}$  a Bergman metric of  $M$  with respect to  $L$ .

**Theorem 9.** *[25, Corollary 7.5] The set of Bergman metrics is dense in the set of polarized Kähler metrics. In fact,*

$$\left\| \frac{1}{k}\phi_k^*\omega_{FS} - \omega \right\|_{C^\infty} = O(k^{-2}).$$

An informal proof! By assumption,

$$k\omega = ikR^L = iR^{L^k} = i\bar{\partial}\partial \log |s_{k,0}|^2.$$

On the open  $\phi_k^{-1}(Z_0 \neq 0) \subset M$ ,  $s_{k,0}$  is a frame of  $L^k$ . By Example 4,

$$\phi_k^* \omega_{FS} = i\bar{\partial}\partial \log \left(1 + \sum_{j=1}^{N_k} \left| \frac{s_{k,j}}{s_{k,0}} \right|^2\right) = i\bar{\partial}\partial \log \frac{B_k}{|s_{k,0}|^2} = i\bar{\partial}\partial \log B_k + k\omega.$$

(This is the content of [25, Lemma 7.3].) By Theorem 7 and Theorem 8,

$$\|i\bar{\partial}\partial \log B_k\|_{C^\infty} = O\left(\frac{1}{k}\right),$$

so

$$\left\| \frac{1}{k} \phi_k^* \omega_{FS} - \omega \right\|_{C^\infty} = \frac{1}{k} \|i\bar{\partial}\partial \log B_k\|_{C^\infty} = O\left(\frac{1}{k^2}\right).$$

□

We give a quick application of our main result.

**Definition 15** (Euler-Poincaré characteristic). *The Euler-Poincaré characteristic of  $E$  is*

$$\chi(M, E) = \sum_{j \geq 0} (-1)^j \dim_{\mathbb{C}} H^j(M, E).$$

**Theorem 10** (Serre vanishing theorem). [22, n.66, Théorème 2] *There is  $k_0 \geq 1$  such that any  $k \geq k_0$ , any  $q > 0$ ,  $H^q(M, L^k \otimes E) = 0$ .*

At present we have adequate tools for the famous result below.

**Corollary 2** (Asymptotic Riemann-Roch).

$$\chi(M, L^k \otimes E) = \frac{k^n \text{rk}(E)}{n!} \int_M c_1(L)^n + \frac{k^{n-1} \text{rk}(E)}{2(2\pi)^n} \int_M \text{scal} \mu + \frac{k^{n-1}}{(2\pi)^n} \int_M \text{Tr}(i\Lambda R^E) \mu + O(k^{n-2}).$$

*Proof.* By Theorem 10,  $\chi(M, L^k \otimes E) = \dim H^0(M, L^k \otimes E)$  when  $k$  is sufficiently large. Then Theorem 7 and Proposition 4 gives the result. □

### 3.1.1 Example

We illustrate Theorem 7 with a concrete example. On  $(\mathbb{C}\mathbb{P}^n, \omega_{FS})$ ,  $L = O(1)$ ,  $h^L = h$  the one given in Example 2.  $E = O(l)$  with  $h^E = h^l$ .

By Example 2, on  $U_0$ ,  $(s_I, s_J) = \frac{z^{I'} \bar{z}^{J'}}{(1+|z|^2)^k}$ , where  $I' = (i_1, \dots, i_n)$  is a truncation of the multi-index  $I = (i_0, \dots, i_n)$ . so by (2.2),

$$\langle s_I, s_J \rangle = \int_{U_0} (s_I, s_J) \mu = \int_{\mathbb{C}^n} \frac{z^{I'} \bar{z}^{J'}}{(1+|z|^2)^{n+1+k}} 2^n \mu_0.$$

Change variables:  $z_j = r_j e^{i\theta_j}$ , then  $\mu_0 = r_1 \dots r_n dr_1 \dots dr_n d\theta_1 \dots d\theta_n$ , By Exercise 9, when  $I' = J'$ ,

$$\begin{aligned} \int_{\mathbb{C}^n} \frac{z^{I'} \bar{z}^{J'}}{(1 + |z|^2)^{n+1+k}} 2^n \mu_0 &= 2^n (2\pi)^n \int_{\mathbb{R}_+^n} \frac{r^{2I'+\bar{1}}}{(1 + |r|^2)^{n+1+k}} dr_1 \dots dr_n \\ &= (2\pi)^n \frac{\Gamma(1 + i_1) \dots \Gamma(1 + i_n) \Gamma(1 + i_0)}{\Gamma(n + k + 1)}. \end{aligned}$$

When  $I' \neq J'$  we get  $\langle s_I, s_J \rangle = 0$ .

Fix  $p = [1 : 0 : \dots : 0] \in U_0$ , consider the special multi-index  $I_0 = (k, 0, \dots, 0)$ , then  $s_I(p) = 0$  when  $I \neq I_0$ .

By Remark 2, the Bergman kernel of  $L^k \otimes E = O(k+l)$  is  $B_k(p) = |s_{I_0}(p)|^2 A$ , where

$$A^{-1} = (2\pi)^n \frac{(k+l)!}{(n+k+l)!}.$$

So, we find the Bergman kernel of  $L^k \otimes E$

$$\begin{aligned} B_k(p) &= \frac{(n+k+l)!}{(2\pi)^n (k+l)!} = (2\pi)^{-n} (k+n+l)(k+n+l-1) \dots (k+l+1) \\ &= \left(\frac{k}{2\pi}\right)^n \left(1 + k^{-1} \left(\frac{n(n+1)}{2} + nl\right) + O(k^{-2})\right). \end{aligned}$$

Note that the scalar curvature is constant  $n(n+1)$  and  $i\Lambda R^E = nl$  in this case.

**Exercise 9.** For a real number  $A > 0$ , a multi-index  $I = (i_1, \dots, i_n) \in \mathbb{N}^n$ ,

1.

$$\int_{\mathbb{R}_+^n} \frac{r^I}{(A^2 + |r|^2)^b} dr = \frac{A^{n+|I|-2b}}{2^n} \frac{\Gamma(j_1) \dots \Gamma(j_n) \Gamma(b - \sum_k j_k)}{\Gamma(b)},$$

where  $j_i = \frac{1+i_i}{2}$ .

2.

$$\int_{\mathbb{C}^n} \frac{z^I \bar{z}^J}{(1 + |z|^2)^s} \mu_0 = 0.$$

when  $I \neq J$ .

*Proof.* 1. Induction on  $n$ . When  $n = 1$ , is

$$\int_{\mathbb{R}_+} \frac{r^i}{(A^2 + r^2)^b} dr = \int_0^{\pi/2} \frac{A \tan(t)^i}{(A^2 / \cos(t)^2)^b} A / \cos^2(t) dt = A^{1+i-2b} \frac{1}{2} B(j, b-j)$$

by  $r = A \tan(t)$ , done.

If true for  $n - 1$ . Integrating  $r_n$  first, we get

$$\frac{\Gamma(j_n) \Gamma(b - j_n)}{2\Gamma(b)} \int_{\mathbb{R}_+^{n-1}} \frac{r'^{I'}}{(A^2 + |r'|^2)^{b-j_n}} dr',$$

where  $r' = (r_1, \dots, r_{n-1})$  and  $I' = (i_1, \dots, i_{n-1})$ , then use induction hypothesis to conclude.

2. Mimic the proof of Exercise 10. □

## 3.2 Proof of Main Theorem

Once for all, we choose a local frame  $\sigma$  of  $L$  and a chart around  $p$ ,  $\varphi := -2 \log |\sigma|_h$  such that  $\varphi$  is as in Exercise 5 (cf.[26, (2.2)]); a frame  $\{v_1, \dots, v_r\}$  of  $E$  as in Proposition 3 and use notations therein. Let  $\det_2 = 1 - Ric_{k\bar{l}}(p)z_k\bar{z}_l$  given by Exercise 5. Note that at  $p$ ,

$$g_{i\bar{j}} = \delta_{ij}, \quad (3.1)$$

$$\sum_j R_{j\bar{j}k\bar{l}}^M = Ric_{k\bar{l}}, \quad (3.2)$$

$$\sum_i Ric_{i\bar{i}}^M = \text{scal}. \quad (3.3)$$

Use our proof below but with higher order approximation of  $H^E$  and  $\varphi$ , we can find the coefficient of  $k^{n-2}$  in the expansion of  $B_k(p)$  explicitly, cf.[28, Theorem 4.2].

### 3.2.1 Upper bound

The goal of this section is to derive an upper bound for  $B_k$ .

In each term of  $\varphi_4, H_2^E$ , the power of  $z$  and of  $\bar{z}$  are the same. This simple observation leads to the following vanishing identity, which should be compared to [26, Lemma 2.2].

**Exercise 10.** 1. For any multi-index  $I(\neq \vec{0}) \in \mathbb{N}^n$ ,  $\psi$  a bounded measurable function on  $\mathbb{C}^n$ . Assume  $\psi$  is rotationally invariant, i.e.,  $\psi(z)$  depending only in  $|z|$ . Then

$$\int_{\mathbb{C}^n} \psi e^{-k\varphi_4} z^I H_2^E \det_2 \mu_0 = 0. \quad (3.4)$$

2. Let  $B \subset \mathbb{C}^n$  be an open ball centered in 0, and  $f \in H(B)$  a holomorphic function with  $f(0) = 0$ , then

$$\int_B e^{-k\varphi_4} f(z) H_2^E \det_2 \mu_0 = 0.$$

*Proof.* 1. Take  $\lambda \in \mathbb{C}$  with  $|\lambda| = 1$  and  $\lambda^{|I|} \neq 1$ . Change variable  $z = \lambda w$ . Note that for the integral domain,  $\psi, \varphi_4, H_2^E, \det_2, \mu_0$  are formally invariant under this change. Denote the integral on the left of (3.4) by  $J$ . Under the change of variable,  $J = \lambda^{|I|} J$ , hence  $J = 0$ .

2. We apply 1. Take  $\psi$  to be the characteristic function of  $B$  and use the Taylor expansion of  $f$ . □



**Method 1: Tian**

The following generalization of [3, Lemma 3.3] will lead to the upper bound we seek. Note carefully we use this lemma to control the point difference of smooth and holomorphic peak sections in Section 3.2.2, to avoid applying [3, Theorem 2.1].

**Lemma 3.** *Notation as in Theorem 7. Put  $B = B(p, \frac{\ln k}{\sqrt{k}})$ . For any  $s \in H^0(B, L^k \otimes E)$ ,*

$$s(p) \otimes \bar{s}(p) \leq M(k) \int_B |s|^2 \mu,$$

where the matrix

$$M(k) = \left(\frac{k}{2\pi}\right)^n (Id_r + k^{-1} \left(\frac{\text{scal}}{2} Id_r + i\Lambda R^E\right) + O((\log k)^5 k^{-\frac{3}{2}})).$$

(See Exercise 6 for  $\Lambda$ .) Taking trace we get

$$|s|^2(p) \leq Ck^n \int_B |s|^2 \mu \tag{3.5}$$

All the implicit constants are uniform in  $p \in M$  and independent of  $k$ .

*Proof.* Put

$$C(k) = \frac{e^{-C(\log k)^5 k^{-\frac{3}{2}}}}{(1 + C(\log k)^3 k^{-\frac{3}{2}})^2} = 1 + O((\log k)^5 k^{-\frac{3}{2}}),$$

Write  $s = \sum_{\alpha=1}^r f_\alpha(\sigma^k \otimes v_\alpha)$ , where  $f_\alpha$  are holomorphic functions on  $B$ . Note  $\sigma \otimes \bar{\sigma} = |\sigma|^2 = e^{-\varphi}$ . Let  $\vec{f} = (f_1, \dots, f_r)^T$ .

The left hand side is

$$\sum_{\alpha} f_{\alpha}(\sigma^k \otimes v_{\alpha}) \otimes \overline{\left(\sum_{\beta} f_{\beta}(\sigma^k \otimes v_{\beta})\right)}(p) = \vec{f} \vec{f}^H(p).$$

The point-wise norm

$$|s|^2 = h^{L^k \otimes E} \left(\sum_{\alpha} f_{\alpha}(\sigma^k \otimes v_{\alpha}), \sum_{\beta} f_{\beta}(\sigma^k \otimes v_{\beta})\right) = e^{-k\varphi} \vec{f}^T H^E \vec{f}.$$

The integral on the right  $J(k) := \int_B |s|^2 \mu = \int_B e^{-k\varphi} \vec{f}^T H^E \vec{f} \mu$ .

There is a constant  $C > 0$  such that

$$-\varphi_4 \leq -\varphi + C|z|^5 \tag{3.6}$$

$$H_2^E \leq (1 + C|z|^3) H^E \tag{3.7}$$

$$\frac{2^n \det_2}{1 + C|z|^3} \leq \frac{\mu}{\mu_0}. \tag{3.8}$$

Moreover  $|z| \leq \frac{\ln k}{\sqrt{k}}$  in the domain of integral. Then (3.6),(3.7),(3.8) result in

$$J(k) \geq 2^n C(k) \int_B e^{-k\varphi_4} \vec{f}^T H_2^E \vec{f} \det_2 \mu_0. \quad (3.9)$$

Recall (2.14), when  $k$  is large enough,  $H_2^E(z)$  is positive definite at any  $z \in B(p, \frac{\ln k}{\sqrt{k}})$ , a fact used in the inequality (3.10). Hence

$$\begin{aligned} & \int_B e^{-k\varphi_4} \vec{f}^T H_2^E \vec{f} \det_2 \mu_0 \\ &= \int_B e^{-k\varphi_4} \det_2 [\vec{f}(p)^T H_2^E \vec{f}(p) + (\vec{f} - \vec{f}(p))^T H_2(\overline{\vec{f} - \vec{f}(p)})] \mu_0 \\ &\geq \vec{f}^T(p) \left( \int_B e^{-k\varphi_4} H_2^E \det_2 \mu_0 \right) \overline{\vec{f}(p)}, \end{aligned} \quad (3.10)$$

where the first equation is from Exercise 10.

We now calculate

$$\int_B e^{-k\varphi_4} H_{2;\alpha,\beta}^E \det_2 \mu_0 = S(k) \delta_{\alpha,\beta} - T_{\alpha,\beta}(k),$$

where  $S(k) := \int_B e^{-k\varphi_4} \det_2 \mu_0$  and  $T_{\alpha,\beta}(k) = \int_B e^{-k\varphi_4} \sum_{i,j} R_{\alpha,\beta,i,\bar{j}}^E(p) z_i \bar{z}_j \det_2 \mu_0$ .

By [3, (5)]

$$S(k) = \left(\frac{\pi}{k}\right)^n \left(1 - \frac{\text{scal}(p)}{2k} + O(k^{-2})\right).$$

(Note his  $\mu_{\text{Leb}} = 2^n \mu_0$ .)

Since

$$e^{-k\varphi_4} \det_2 = e^{-k|z|^2} (1 + |z|^4 O(k)) (1 + O(|z|^2)) = e^{-k|z|^2} (1 + |z|^4 O(k)),$$

$$T_{\alpha,\beta}(k) = \int_B e^{-k|z|^2} (1 + |z|^4 O(k)) \sum_{i,j} R_{\alpha,\beta,i,\bar{j}}^E z_i \bar{z}_j \mu_0 = (i\Lambda R^E)_{\alpha,\beta} \frac{\pi^n}{k^{n+1}} + O(k^{-n-2})$$

by (2.17) and (2.18).

Therefore

$$\begin{aligned} A &:= 2^n C(k) \int_B e^{-k\varphi_4} H_2^E \det_2 \mu_0 \\ &= C(k) \left(\frac{2\pi}{k}\right)^n \left( Id_r - k^{-1} (i\Lambda R^E + \frac{\text{scal}(p)}{2} Id_r) + O(k^{-2}) \right) = M(k)^{-1}. \end{aligned} \quad (3.11)$$

We conclude as follows.

$$\begin{aligned} & M(k) \int_B |s|^2 \mu = M(k) J(k) \\ &\geq M(k) 2^n C(k) \int_B e^{-k\varphi_4} \vec{f}^T H_2 \vec{f} \det_2 \mu_0 \\ &\geq M(k) 2^n C(k) \vec{f}^T(p) \left( \int_B e^{-k\varphi_4} H_2^E \det_2 \mu_0 \right) \overline{\vec{f}(p)} \\ &= A^{-1} v^T A \bar{v} \\ &\geq v v^H = s(p) \otimes \bar{s}(p), \end{aligned} \quad (3.12)$$

where the inequalities are from (3.9),(3.10) and Exercise 2.  $\square$

### Method 2: L.Charles

We can prove a weaker version of Lemma 3 using Theorem 6.

**Lemma 4.** *Notation as in Theorem 7. Fix an open neighborhood  $U$  of  $p \in M$ . There for any  $s \in H^0(U, L^k)$ ,*

$$s(p) \otimes \bar{s}(p) \leq M(k) \int_U |s|^2 \mu,$$

*Proof.* We may shrink  $U$  to find a holomorphic chart  $(z_1, \dots, z_n)$  over  $U$ . There is  $a > 0$  such that the ball  $B(p, a) \subset U$ . Take a smooth cutoff function  $\eta : \mathbb{R} \rightarrow [0, 1]$  supported in  $(-a, a)$  with  $\eta \equiv 1$  near 0. Define  $\psi : M \rightarrow [0, 1]$  by  $\psi(z) = \eta(|z|)$  for  $z \in U$  and  $\psi(q) = 0$  for  $q \notin U$ . Then  $\psi$  is smooth. Put

$$J(k) := \int_M \psi |s|^2 \mu = \int_U \psi e^{-k\varphi} \vec{f} H^E \vec{f} \mu.$$

Note that  $J(k) \leq \int_U |s|^2 \mu$ . As in (3.9),(3.10), we get

$$J(k) \geq 2^n C(k) \vec{f}^T(p) \left( \int_U \psi e^{-k\varphi_4} H_2^E \det_2 \mu_0 \right) \overline{\vec{f}(p)}.$$

We calculate with the help of Theorem 6

$$\int_U \psi e^{-k\varphi_4} H_2^E \det_2 \mu_0 = \left(\frac{\pi}{k}\right)^n (Id_r - k^{-1} \left(\frac{\text{scal}}{2} Id_r + i\Lambda R^E\right) + O(k^{-2})).$$

The rest part is similar to that of Lemma 3, which we omit.  $\square$

**Corollary 3** (upper bound).

$$B_k(p) \leq \left(\frac{k}{2\pi}\right)^n (Id_r + k^{-1} \left[\frac{\text{scal}}{2} Id_r + i\Lambda R^E\right](p) + O(k^{-3/2} \log^5 k))$$

*Proof.* Combine Lemma 2 and 4.  $\square$

### 3.2.2 Lower bound and peak sections

We derive a Kodaira-Hörmander type estimate, which is essential to the construction of peak sections.

**Theorem 11.** *Let  $\square_k$  be the Kodaira-Laplacian (see Definition 13) on  $\Omega^{*,*}(M, L^k \otimes E)$ , then there is  $C > 0$  such that any  $k \geq 0$ ,  $0 \leq q \leq n$ ,*

1.  $\square_k \geq (kq - C) Id$  on  $\Omega^{n,q}(M, L^k \otimes E)$ ;
2.  $\square_k \geq (kq - C) Id$  on  $\Omega^{0,q}(M, L^k \otimes E)$ ,

where we use  $L^2$  inner products.

*Proof.* 1. Since  $M$  is compact, there is  $C = C(M, \omega, E, h^E) > 0$  such that  $[iR^E, \Lambda] \geq -CId$ . As  $iR^L = \omega$ ,  $[iR^L, \Lambda] = q$  on  $\Omega^{n,q}(M, L^k \otimes E)$ . Because  $R^{L^k \otimes E} = kR^L + R^E$ , we have

$$\begin{aligned} \square_k &\geq [iR^{L^k \otimes E}, \Lambda] \\ &= k[iR^L, \Lambda] + [iR^E, \Lambda] \\ &\geq (kq - C)Id_r, \end{aligned} \tag{3.13}$$

where the first line is from Theorem 5.

2. Note that  $\Omega^{n,q}(M, L^k \otimes E \otimes K_M^{-1}) = \Omega^{0,q}(M, L^k \otimes E)$  and the action of  $\square^{L^k \otimes E \otimes K_M^{-1}}$  on the left is compatible with the action of  $\square_k$  on the right. Then 2 follow from the 1 using this identification.  $\square$

The following theorem is a generalization of [3, Theorem 2.2]. It will play the role of classical  $\bar{\partial}$ -estimates.

**Theorem 12.** *There is  $C = C(M, \omega, E, h^E) > 0$  such that for any  $s \in C^\infty(M, L^k \otimes E)$  orthogonal to  $H^0(M, L^k \otimes E)$ , we have*

$$(k - C)\|s\|^2 \leq \|\bar{\partial}s\|^2.$$

*Proof.* Let  $C$  be given by Theorem 11. We may assume  $k > C$ . When  $1 \leq q \leq n$ , always on  $\Omega^{0,q}(M, L^k \otimes E)$ ,  $\square_k \geq (kq - C)Id$  is positive-definite. By a variant of [25, Theorem 2.13],  $\square_k$  is invertible.

When  $q = 1$ , this fact gives  $t \in \Omega^{0,1}(M, L^k \otimes E)$  such that  $\square_k t = \bar{\partial}s$ . As

$$\square_k \bar{\partial}t = \bar{\partial}\square_k t = \bar{\partial}\bar{\partial}s = 0,$$

the fact with  $q = 2$  implies  $\bar{\partial}t = 0$ . Therefore,

$$\bar{\partial}\bar{\partial}^*t = \square_k t = \bar{\partial}s,$$

or equivalently  $s - \bar{\partial}^*t \in H^0(M, L^k \otimes E)$  is holomorphic. In particular,  $\langle s, s - \bar{\partial}^*t \rangle = 0$  which gives (\*) below.

$$\|s\|^2 = \langle s, s \rangle \stackrel{(*)}{=} \langle s, \bar{\partial}^*t \rangle = \langle \bar{\partial}s, t \rangle = \langle \square_k t, t \rangle.$$

By Cauchy-Schwarz inequality  $\|\square_k t\| \|t\| \geq \langle \square_k t, t \rangle \geq (k - C)\|t\|^2$ , so  $\|\square_k t\| \geq (k - C)\|t\|$ . Now

$$\|\bar{\partial}s\|^2 = \|\square_k t\|^2 \geq (k - C)\|\square_k t\| \|t\| \geq (k - C)\langle \square_k t, t \rangle = (k - C)\|s\|^2.$$

$\square$

### Method 1: Tian

The following construction is a twisted version of [26, Lemma 1.2].

Fix  $p \in M$  and a holomorphic chart  $(U, z_1, \dots, z_n)$  centered in  $p$ , for  $v$  a local section of  $E$  on  $U$ , we will construct a peak section.

Take a smooth cutoff function  $\eta : \mathbb{R}_+ \rightarrow [0, 1]$  with  $\eta(x) = 1$  on  $x < 1$  and  $\eta(x) = 0$  when  $x \geq 1$  and  $-3 < \eta' \leq 0$ . Let

$$\psi_k(z) = \eta\left(\frac{|z|^2 k}{\log^2 k}\right)$$

for  $z \in U$  and  $\psi_k(q) = 0$  for  $q \in M \setminus U$ . Then  $\psi_k : M \rightarrow [0, 1]$  is a smooth function.  $\psi_k$  is supported in  $B(p, \sqrt{2} \frac{\log k}{\sqrt{k}})$  and  $\psi \equiv 1$  on  $B(p, \frac{\log k}{\sqrt{k}})$ .

So,  $\bar{\partial}\psi_k$  is supported in the annulus  $\frac{\log k}{\sqrt{k}} \leq |z| \leq \sqrt{2} \frac{\log k}{\sqrt{k}}$ . On  $U$ ,

$$\bar{\partial}\psi_k = \frac{k}{\log^2 k} \eta'\left(\frac{k|z|^2}{\log^2 k}\right) \sum_{i=1}^n z_i d\bar{z}_i.$$

For each  $i$ ,  $|d\bar{z}_i| \leq C(M)$  in the annulus. Thus, on  $M$

$$|\bar{\partial}\psi_k| \leq C \frac{\sqrt{k}}{\log k} \quad (3.14)$$

Note that  $\psi_k(\sigma^k \otimes v) \in C^\infty(M, L^k \otimes E)$ . We call it a smooth section peaked at  $p$ . Now we construct a holomorphic peak section from it.

Let

$$\psi_k(\sigma^k \otimes v) = s_k + t_k \quad (3.15)$$

be the Hodge decomposition given by Theorem 4, where  $s_k \in H^0(M, L^k \otimes E)$ .

Because near  $p$ ,

$$\varphi(z) = |z|^2 + O(|z|^4), \quad \frac{\mu}{\mu_0} = 2^n + O(|z|^2),$$

when  $k$  is large,

$$\varphi(z) \geq |z|^2/2, \quad \mu \leq C\mu_0 \quad (3.16)$$

in  $B(p, \sqrt{2} \frac{\log k}{\sqrt{k}})$ . An important estimate in the annulus is

$$e^{-k\varphi} \leq k^{-\frac{\log k}{2}},$$

By (3.15),  $\bar{\partial}t_k = (\bar{\partial}\psi_k)(\sigma^k \otimes v)$ ,

$$|\bar{\partial}t_k|^2 = |\bar{\partial}\psi_k|^2 e^{-k\varphi} |v|_{h^E}^2 \leq C(M, v, h^E) \frac{k}{\log^2 k} k^{-(\log k)/2}.$$

Therefore,

$$\begin{aligned}
\|\bar{\partial}t_k\|^2 &= \int_M |\bar{\partial}t_k|^2 \mu = \int_{\frac{\log k}{\sqrt{k}} \leq |z| \leq \sqrt{2} \frac{\log k}{\sqrt{k}}} |\bar{\partial}t_k|^2 \mu \\
&\leq C(M, v, h^E) \frac{k}{\log^2 k} k^{-(\log k)/2} \int_{\frac{\log k}{\sqrt{k}} \leq |z| \leq \sqrt{2} \frac{\log k}{\sqrt{k}}} \mu_0. \quad (3.17) \\
&\leq C(M, v, h^E) \frac{k}{\log^2 k} k^{-(\log k)/2} \left(\frac{\log k}{\sqrt{k}}\right)^{2n}
\end{aligned}$$

By Theorem 12,  $\|t_k\| \leq \frac{\|\bar{\partial}t_k\|}{\sqrt{k-C}}$ .

$$\|t_k\| \leq \frac{C}{\sqrt{k-C}} k^{\frac{2-2n-\log k}{4}} \log^{n-1} k \leq C k^{-n/2-(\log k)/4} \log^{n-1} k. \quad (3.18)$$

Estimate the point difference between smooth and holomorphic peak section. Note that

$$t_k \in H^0(B(p, \frac{\log k}{\sqrt{k}}), L^k \otimes E),$$

by (3.5),

$$\begin{aligned}
&|s_k(p) - (\sigma^k \otimes v)(p)|^2 \\
&= |t_k(p)|^2 \\
&\leq C k^{-(\log k)/2} \log^{2n-2} k. \quad (3.19)
\end{aligned}$$

Estimate the outside  $L^2$  norm of smooth peak section.

$$\begin{aligned}
&\int_{M-B(p, \frac{\log k}{\sqrt{k}})} |\psi_k \sigma^k \otimes v|^2 \mu \\
&= \int_{\frac{\log k}{\sqrt{k}} \leq |z| \leq \sqrt{2} \frac{\log k}{\sqrt{k}}} |\psi_k|^2 e^{-k\varphi} |v|^2 \mu \\
&\leq \int_{\frac{\log k}{\sqrt{k}} \leq |z| \leq \sqrt{2} \frac{\log k}{\sqrt{k}}} C k^{-(\log k)/2} \mu_0 \\
&\leq C k^{-(\log k)/2} \left(\frac{\log k}{\sqrt{k}}\right)^{2n}. \quad (3.20)
\end{aligned}$$

We use (3.18),(3.20) to get

$$\begin{aligned}
&\int_{M-B(p, \frac{\log k}{\sqrt{k}})} |s_k|^2 \mu \\
&\leq \int_{M-B(p, \frac{\log k}{\sqrt{k}})} |\psi_k(\sigma^k \otimes v)|^2 \mu + 2\|t_k\| \cdot \|\psi_k(\sigma^k \otimes v)\|_{L^2(M-B(p, \frac{\log k}{\sqrt{k}}))} + \|t_k\|^2 \\
&\leq C k^{-n-(\log k)/2} \log^{2n} k \quad (3.21)
\end{aligned}$$

Estimate the global  $L^2$  norm of smooth peak section.

$$\begin{aligned}
\|\psi_k(\sigma^k \otimes v)\|^2 &= \int_M |\psi_k(\sigma^k \otimes v)|^2 \mu \\
&= \int_{B(p, \frac{\log k}{\sqrt{k}})} |\psi_k|^2 e^{-k\varphi} |v|^2 \mu + \int_{M \setminus B(p, \frac{\log k}{\sqrt{k}})} |\psi_k|^2 e^{-k\varphi} |v|^2 \mu \\
&\leq Ck^{-n} + Ck^{-(\log k)/2} \left(\frac{\log k}{\sqrt{k}}\right)^{2n} \\
&\leq Ck^{-n},
\end{aligned} \tag{3.22}$$

where we used (3.20), (3.29).

*Proof of Theorem 7, Tian's method.* Let

$$\psi_k(\sigma^k \otimes v_\alpha) = s_{k,\alpha} + t_{k,\alpha}$$

be the Hodge decomposition given by Theorem 4, where  $s_{k,\alpha} \in H^0(M, L^k \otimes E)$ . then by Exercise 3.(2),

$$B_k(p) \geq \sum_{\alpha} s_{k,\alpha} \otimes \bar{s}_{k,\beta}(p) A_{\beta,\alpha}, \tag{3.23}$$

where  $A$  is the inverse of the Gram matrix  $G = (\langle s_{k,\alpha}, s_{k,\beta} \rangle)$ .

$$\begin{aligned}
\langle s_{k,\alpha}, s_{k,\beta} \rangle &= \int_{B(p, \frac{\log k}{\sqrt{k}})} (\psi_k(\sigma^k \otimes v_\alpha), \psi_k(\sigma^k \otimes v_\beta)) \mu \\
&\quad + \int_{M \setminus B(p, \frac{\log k}{\sqrt{k}})} (\psi_k(\sigma^k \otimes v_\alpha), \psi_k(\sigma^k \otimes v_\beta)) \mu \\
&\quad + \int_M (t_{k,\alpha}, t_{k,\beta}) \mu - \int_M (\psi_k(\sigma^k \otimes v_\alpha), t_{k,\beta}) \mu - \int_M (t_{k,\alpha}, \psi_k(\sigma^k \otimes v_\beta)) \mu.
\end{aligned} \tag{3.24}$$

Apply Cauchy-Schwarz inequality several times. By (3.20),

$$\left| \int_{M \setminus B(p, \frac{\log k}{\sqrt{k}})} (\psi_k(\sigma^k \otimes v_\alpha), \psi_k(\sigma^k \otimes v_\beta)) \mu \right| \leq Ck^{-(\log k)/2} \left(\frac{\log k}{\sqrt{k}}\right)^{2n}. \tag{3.25}$$

By (3.18),

$$\left| \int_M (t_{k,\alpha}, t_{k,\beta}) \mu \right| \leq \|t_{k,\alpha}\| \cdot \|t_{k,\beta}\| \leq Ck^{-n-(\log k)/2} \log^{2n-2} k. \tag{3.26}$$

By (3.18), (3.22),

$$\left| \int_M (\psi_k(\sigma^k \otimes v_\alpha), t_{k,\beta}) \mu \right| \leq \|t_{k,\beta}\| \|\psi_k(\sigma^k \otimes v_\alpha)\| \leq Ck^{-n-(\log k)/4} \log^{n-1} k. \tag{3.27}$$

Because

$$\begin{aligned} e^{-k\varphi} H^E \mu &= e^{-k\varphi_4} (1 + |z|^5 O(k)) H_2^E \cdot (Id_r + O(|z|^3)) \det_2(1 + O(|z|^3)) 2^n \mu_0 \\ &= 2^n e^{-k\varphi_4} H_2 \det_2(1 + |z|^5 O(k) + O(|z|^3)) \mu_0, \end{aligned} \quad (3.28)$$

we find its integral

$$\begin{aligned} &\int_B e^{-k\varphi} H^E \mu \\ &= \int_B 2^n e^{-k\varphi_4} H_2 \det_2(1 + |z|^5 O(k) + O(|z|^3)) \mu_0 \\ &= \left(\frac{2\pi}{k}\right)^n (Id_r - k^{-1}(i\Lambda R^E + \frac{scal(p)}{2} Id_r) + O(k^{-\frac{3}{2}})), \end{aligned} \quad (3.29)$$

where the last equality is from (3.11),(2.19). Therefore,

$$\begin{aligned} G_{\alpha,\beta} &= \langle s_{k,\alpha}, s_{k,\beta} \rangle \\ &= \int_{B(p, \frac{\log k}{\sqrt{k}})} (\psi_k(\sigma^k \otimes v_\alpha), \psi_k(\sigma^k \otimes v_\beta)) \mu + O(k^{-n-(\log k)/4} \log^{n-1} k) \\ &= \int_{B(p, \frac{\log k}{\sqrt{k}})} e^{-k\varphi} H_{\alpha,\beta}^E \mu + O(k^{-n-(\log k)/4} \log^{n-1} k) \end{aligned} \quad (3.30)$$

By (3.29),(3.30), we get an important expansion of  $G$ :

$$\begin{aligned} G &= \int_B e^{-k\varphi} H \mu + O(k^{-n-\frac{\log k}{4}} \log^{n-1} k) \\ &= \left(\frac{2\pi}{k}\right)^n (Id_r - k^{-1}(i\Lambda R^E + \frac{scal(p)}{2} Id_r) + O(k^{-\frac{3}{2}})). \end{aligned} \quad (3.31)$$

Therefore,

$$A = G^{-1} = \left(\frac{k}{2\pi}\right)^n (Id_r + k^{-1}(\frac{scal(p)}{2} Id_r + i\Lambda R^E) + O(k^{-\frac{3}{2}})). \quad (3.32)$$

By (3.19), (3.23) and (3.32), we get the lower bound

$$B_k(p) \geq \left(\frac{k}{2\pi}\right)^n (Id_r + k^{-1}(\frac{scal(p)}{2} Id_r + i\Lambda R^E) + O(k^{-\frac{3}{2}})). \quad (3.33)$$

By Theorem 8,  $B_k(p)$  has only integral power of  $k$ . Corollary 3 together with (3.33) implies

$$B_k(p) = \left(\frac{k}{2\pi}\right)^n (Id_r + k^{-1}(\frac{scal(p)}{2} Id_r + i\Lambda R^E) + O(k^{-2})).$$

□

**Remark 3.** From the proof we see that the smoothness conditions can be relaxed to:  $h^L$  is  $C^4$  (then  $\omega$  and  $\mu$  are  $C^2$ ) and  $h^E$  is  $C^2$ . Instead of this construction of peak sections, we can also apply [29, Proposition 3.6.1] which is closer to Tian's original way of construction.



## Method 2: L.Charles

We present another construction of peak sections. Fix a holomorphic chart  $(U, z_1, \dots, z_n)$  centered at  $p$  with a frame  $\sigma \in H^0(U, L)$  and section  $v \in H^0(U, E)$ .

Like (3.16), we fix  $r_2 > 0$  such that  $B(p, r_2) \subset U$  and  $\varphi(z) \geq |z|^2/2$  on  $B(p, r_2)$ . Fix a bump function  $\psi \in C^\infty(M, [0, 1])$  with  $\psi \equiv 1$  on  $B(p, r_1)$  for some  $r_1 \in (0, r_2)$  and  $\psi \equiv 0$  on  $M \setminus B(p, r_2)$ . So,  $\bar{\partial}\psi$  is supported in the annulus  $B(p, r_2) \setminus B(p, r_1)$  and there is a constant  $C = C(M)$  such that on the whole  $M$ ,  $|\bar{\partial}\psi|^2 \leq C(M)$ . In the annulus,  $e^{-k\varphi} \leq e^{-kr_1^2/2}$ .

Then  $\psi(\sigma^k \otimes v) \in C^\infty(M, L^k \otimes E)$ . Let

$$\psi(\sigma^k \otimes v) = s_k + t_k$$

be the Hodge decomposition given by Theorem 4, where  $s_k \in H^0(M, L^k \otimes E)$ . Note that

$$\bar{\partial}t_k = (\bar{\partial}\psi)(\sigma^k \otimes v). \quad (3.34)$$

For  $z \in B(p, r_2)$ ,

$$|\bar{\partial}t_k|^2(z) = |\bar{\partial}\psi|^2 e^{-k\varphi} |v|^2(z) \leq C e^{-kr_1^2/2} \sup |v|^2 \leq C(v, M) e^{-kr_1^2/2}. \quad (3.35)$$

Therefore,

$$\begin{aligned} \|\bar{\partial}t_k\|^2 &= \int_M |\bar{\partial}t_k|^2 \mu = \int_{B(p, r_2)} |\bar{\partial}t_k|^2 \mu \\ &\leq \int_{B(p, r_2)} C(v, M) e^{-kr_1^2/2} \mu_0 \\ &\leq C(v, M, \sigma, r_1, r_2) e^{-kr_1^2/2}. \end{aligned} \quad (3.36)$$

Apply Theorem 12 to  $t_k$  we get the estimate

$$\|t_k\| \leq \frac{\|\bar{\partial}t_k\|}{\sqrt{k-C}} \leq \frac{C' e^{-kr_1^2/4}}{\sqrt{k-C}}. \quad (3.37)$$

We derive a replacement of (3.19). Now the important observation is that

$$t_k \in H^0(B(p, r_1), L^k \otimes E),$$

so by (3.5) for large  $k$  we have

$$\begin{aligned} |s_k(p) - (\sigma^k \otimes v)(p)| &= |t_k(p)| \\ &\leq C k^{n/2} \|t_k\| \leq k^{n/2} \frac{C'}{\sqrt{k-C}} e^{-kr_1^2/4}, \end{aligned} \quad (3.38)$$

where we used (3.37).

By Theorem 6,

$$\|\psi(\sigma^k \otimes v)\|^2 = \int_M |\psi|^2 e^{-k\varphi} |v|^2 \mu = O(k^{-n}). \quad (3.39)$$

*Proof of Theorem 7, L.Charles' method.* The proof is parallel to that via Tian given in Section 3.2.2. Recall the frame  $\{v_1, \dots, v_r\}$  of  $E$ . Let

$$\psi(\sigma^k \otimes v_\alpha) = s_{k,\alpha} + t_{k,\alpha}$$

be the Hodge decomposition given by Theorem 4, where  $s_{k,\alpha} \in H^0(M, L^k \otimes E)$ . Then by Exercise 3.(2),

$$B_k(p) \geq \sum_{\alpha, \beta \in [r]} s_{k,\alpha} \otimes \bar{s}_{k,\beta}(p) A_{\beta,\alpha}, \quad (3.40)$$

where  $A$  is the inverse of the Gram matrix  $G = (\langle s_{k,\alpha}, s_{k,\beta} \rangle)$ .

**Claim:**

$$G_{\alpha,\beta} = \left( \int_M (\psi(\sigma^k \otimes v_\alpha), \psi(\sigma^k \otimes v_\beta)) \mu \right) (1 + O(k^{-\frac{n-1}{2}} e^{-kr_1^2/4})). \quad (3.41)$$

In fact,

$$\begin{aligned} \langle s_{k,\alpha}, s_{k,\beta} \rangle &= \int_M (\psi(\sigma^k \otimes v_\alpha), \psi(\sigma^k \otimes v_\beta)) \mu + \int_M (t_{k,\alpha}, t_{k,\beta}) \mu \\ &\quad - \int_M (\psi(\sigma^k \otimes v_\alpha), t_{k,\beta}) \mu - \int_M (t_{k,\alpha}, \psi(\sigma^k \otimes v_\beta)) \mu. \end{aligned} \quad (3.42)$$

Apply Cauchy-Schwarz inequality several times. By (3.37),

$$\left| \int_M (t_{k,\alpha}, t_{k,\beta}) \mu \right| \leq \frac{C e^{-kr_1^2/2}}{k - C}. \quad (3.43)$$

By (3.39),

$$\left| \int_M (\psi(\sigma^k \otimes v_\alpha), t_{k,\beta}) \mu \right| \leq \frac{C k^{-n/2} e^{-kr_1^2/4}}{\sqrt{k - C}}. \quad (3.44)$$

The claim is thus proved.

$$\int_M (\psi(\sigma^k \otimes v_\alpha), \psi(\sigma^k \otimes v_\beta)) \mu = \int_U |\psi|^2 e^{-k\varphi} H_{\alpha,\beta}^E \mu. \quad (3.45)$$

We just need to apply Theorem 6 to estimate the right hand side of (3.45), where we take

$$A = |\psi|^2 \frac{\mu}{\mu_0} \sum_{j,l} R_{\alpha,\beta,j,l}^E(p) z_j \bar{z}_l = 2^n \sum_{j,l} R_{\alpha,\beta,j,l}^E(p) z_j \bar{z}_l + O(|z|^4).$$

Note  $H = 2\text{Id}_{2n}$ , then  $D = 2 \sum_{j=1}^n \frac{\partial^2}{\partial z_j \partial \bar{z}_j}$  is half of the Euclidean Laplacian.  $L_0(A\varphi) = A(0) = 0$ .

$$\underline{\varphi} = -\frac{1}{4} R_{ij\bar{k}\bar{l}}^M(p) z_i \bar{z}_j z_k \bar{z}_l + O(|z|^5).$$

vanishes of order 3 at  $0 \in U$ . so  $D^2(A\varphi)(0) = A(0)D^2(\varphi)(0) = 0$  and  $D^2(A\varphi)(0) = 0$ .

$$D(A)(0) = 2 \sum_{i=1}^n \frac{\partial^2}{\partial z_i \partial \bar{z}_i} [2^n \sum_{j,l} R_{\alpha,\beta,j,l}^E(p) z_j \bar{z}_l + O(|z|^4)] = 2^{n+1} (i\Lambda R^E)_{\alpha,\beta}.$$

$$L_1(A\varphi) = -\frac{D(A)(0)}{2} = -2^n (i\Lambda R^E)_{\alpha,\beta}$$

We get

$$\int_U |\psi|^2 e^{-k\varphi} \sum_{j,l} R_{\alpha,\beta,j,l}^E(p) z_j \bar{z}_l \mu = -\left(\frac{2\pi}{k}\right)^n [k^{-1} (i\Lambda R^E)_{\alpha,\beta} + O(k^{-2})]. \quad (3.46)$$

From [3, Proof of Lemma 3.2],

$$\int_U |\psi|^2 e^{-k\varphi} \mu = \left(\frac{2\pi}{k}\right)^n \left(1 - \frac{scal}{2k} + O(k^{-2})\right). \quad (3.47)$$

Combine (3.46),(3.47) we get

$$\int_U |\psi|^2 e^{-k\varphi} H^E \mu = \left(\frac{2\pi}{k}\right)^n (Id_r - k^{-1} \left(\frac{scal}{2} Id_r + i\Lambda R^E\right) + O(k^{-2})). \quad (3.48)$$

Then (3.42),(3.43),(3.44),(3.45),(3.48) result in

$$G = \left(\frac{2\pi}{k}\right)^n (Id_r - k^{-1} \left(\frac{scal}{2} Id_r + i\Lambda R^E\right) + O(k^{-2})).$$

The remaining part is identical to the proof via Tian's method.  $\square$

## Chapter 4

# $L^2$ existence theorem

For a hermitian holomorphic vector bundle  $(E, h^E)$ , we define a section  $\dot{R}^E$  of  $\text{End}(E \otimes T_h M)$  by

$$(R^E(u, \bar{v})\xi, \eta)_{h^E} = (\dots R^E(\xi \otimes u), \eta \otimes v)$$

for any  $p \in M; u, v \in T_p^{0,1}M; \xi, \eta \in E_p$ . Then for each  $p \in M$ ,  $\dot{R}_p^E$  is a hermitian operator.

**Definition 16** (Nakano positivity). [15, Definition 1.1.6][12, Definition 6.1.24 (i)] *The bundle  $(E, h^E)$  is called Nakano positive if  $\dot{R}^E$  is positive definite everywhere.*

For example, for holomorphic line bundles, the notions of positivity and Nakano positivity are equivalent.

Following Hörmander, we use some functional analysis theory to derive  $L^2$  existence theorem. For a densely defined closed linear (unbounded) operator  $T : \text{Dom}(T) \subset H_1 \rightarrow H_2$  between Hilbert spaces  $H_1$  and  $H_2$ , its adjoint  $T^* : \text{Dom}(T^*) \subset H_2 \rightarrow H_1$  is also densely defined closed.  $\text{Im}(T)^\perp = \ker(T^*)$  and  $T^{**} = T$ . For proof, see [20, Theorem 13.9, Theorem 13.12].

**Theorem 13.** *Let  $F \subset H_2$  be a closed linear subspace with  $\text{Im}T \subset F$ , and assume there is  $C > 0$  st  $|T^*y|_1 \geq C|y|_2$  for any  $y \in F \cap \text{Dom}(T^*)$ , then for any  $v \in F$ , there is  $u \in \text{Dom}(T)$  st  $Tu = v$  and  $C|u|_1 \leq |v|_2$ . In particular,  $F = \text{Im}T$ .*

*Proof.* Since  $F \subset H_2$  is closed, we have an orthogonal decomposition  $H_2 = F \oplus F^\perp$ . For any  $y \in \text{Dom}(T^*)$ , write  $y = y_1 + y_2$  where  $y_1 \in F, y_2 \in F^\perp$ . Because  $\text{Im}T \subset F$ ,  $F^\perp \subset (\text{Im}T)^\perp = \ker(T^*) \subset \text{Dom}(T^*)$ , we find  $y_1 = y - y_2 \in F \cap \text{Dom}(T^*)$ . Now  $|(y, v)_2| = |(y_1, v)_2| \leq |y_1|_2 |v|_2 \leq C^{-1} |T^*y_1|_1 |v|_2 = C^{-1} |T^*y|_1 |v|_2$ . Therefore, the assignment  $T^*y \mapsto (y, v)_2$  defines a bounded linear functional  $\text{Im}(T^*) \rightarrow \mathbb{C}$  of norm at most  $C^{-1}|v|_2$ . By Hahn-Banach theorem, this functional extends to the whole  $H_1$  with norm at most  $C^{-1}|v|_2$ . By Riesz Representation theorem, there is  $u \in H_1$  such that for any  $y \in \text{Dom}(T^*)$ ,

$$(T^*y, u)_1 = (y, v)_2.$$

Therefore,  $u \in \text{Dom}(T^{**})$  and  $T^{**}u = v$ . Note that  $T^{**} = T$ .  $\square$

For a measurable, bounded from above function  $\psi$  on  $M$ , consider the point-wise wighted inner product  $(s, t)_\psi = (s, t)e^{-\psi}$  on  $\Omega^{*,*}(M, E)$  and global one  $\langle s, t \rangle_\psi = \int_M (s, t)_\psi \mu$ . Let  $L_{p,q}^2(M, E; \psi)$  be the completion of  $\Omega^{p,q}(M, E)$  under  $\langle \cdot, \cdot \rangle_\psi$ . Put some Hilbert spaces  $H_i = L_{0,i-1}^2(M, E; \psi)$  for  $i = 1, 2, 3$ . Define  $T : \text{Dom}(T) (\subset H_1) \rightarrow H_2$  and  $S : \text{Dom}(S) (\subset H_2) \rightarrow H_3$  to be the maximal weak differential operator extension of  $\bar{\partial}^E$ . Note that  $\text{Im}(T) \subset \text{Dom}(S)$  and  $ST = 0$  on  $\text{Dom}(T)$ . If  $\psi \in C(M)$ , then  $T, S$  are densely defined closed operators, cf [9, p99].

**Theorem 14** (twisted Bochner–Kodaira–Morrey–Kohn identity). *For  $\psi \in C^2(M)$  and  $\eta \in \Omega^{0,1}(M, E)$ , we have.*

$$\begin{aligned} |S\eta|_\psi^2 + |T^*\eta|_\psi^2 &= |\bar{\nabla}\eta|_\psi^2 \\ &+ \int_M (i(R^E + \text{Ricci}(\omega) + \partial\bar{\partial}\psi)\eta, \eta)_\omega e^{-\psi} \mu. \end{aligned} \quad (4.1)$$

Here  $\text{Ricci}(\omega)$  is the Ricci form of  $(M, \omega)$ . This result is essentially the Weitzenböck formula. Please compare this result to [17, Theorem 2.10, Theroem 3.1], [30, (2.36)], [21, Proposition 5.1] and [23, (1.3.3)].

*Proof.* This proof is a modification of [25, Proof of Lemma 7.7]. We work at a point  $x \in M$  in K-coordinates and the preferred frame of  $E$ , as long as the final answer is coordinate invariant. Then

$$|\bar{\partial}\eta|_\psi^2 = g^{p\bar{j}} g^{q\bar{k}} ((\nabla_{\bar{j}}\eta)_{\bar{k}}, (\nabla_{\bar{p}}\eta)_{\bar{q}} - (\nabla_{\bar{q}}\eta)_{\bar{p}})_\psi.$$

Let  $H = H^E$  be the Gram matrix of  $\{v_\alpha\}$  Define Christoffel symbols  $C_{q,\alpha}^\gamma$  by  $\nabla_q v_\alpha = C_{q,\alpha}^\gamma v_\gamma$ ,

$$(\partial_q H)_{\alpha,\beta} = (\nabla_q v_\alpha, v_\beta) = C_{q,\alpha}^\gamma H_{\gamma,\beta}$$

so  $C_{q,\alpha}^\beta = (\frac{\partial H}{\partial z_q} H^{-1})_{\alpha,\beta}$ . Note that

$$R_{j,q}^E v_\alpha = R^E(\partial_{\bar{j}}, \partial_q)v_\alpha = \nabla_{\bar{j}}^E \nabla_q^E v_\alpha = \nabla_{\bar{j}}^E (C_{q,\alpha}^\beta v_\beta) = \frac{\partial C_{q,\alpha}^\beta}{\partial \bar{z}_j} v_\beta.$$

At  $x$  all the Christoffel symbols vanish

$$\Gamma_{ij}^k = 0, \quad C_{q,\alpha}^\beta = 0. \quad (4.2)$$

For the adjoint operator,

$$T^*\eta = -e^\psi g^{j\bar{k}} \nabla_{\bar{j}}(e^{-\psi} \eta_{\bar{k}}).$$

Note that

$$\int_M g^{p\bar{j}} g^{q\bar{k}} ((\nabla_{\bar{j}}\eta)_{\bar{k}}, (\nabla_{\bar{p}}\eta)_{\bar{q}})_\psi \mu = \|\bar{\nabla}\eta\|_\psi^2 \quad (4.3)$$

Integration by part

$$\begin{aligned}
& \int_M g^{p\bar{j}} g^{q\bar{k}} ((\nabla_{\bar{j}}\eta)_{\bar{k}}, -(\nabla_{\bar{q}}\eta)_{\bar{p}})_{\psi} \mu \\
&= \int_M g^{p\bar{j}} g^{q\bar{k}} (e^{\psi} \nabla_q [e^{-\psi} (\nabla_{\bar{j}}\eta)_{\bar{k}}], \eta_{\bar{p}})_{\psi} \mu \\
&= \int_M g^{p\bar{j}} g^{q\bar{k}} (\nabla_q [(\nabla_{\bar{j}}\eta)_{\bar{k}}] - \frac{\partial \psi}{\partial z_q} (\nabla_{\bar{j}}\eta)_{\bar{k}}, \eta_{\bar{p}})_{\psi} \mu
\end{aligned} \tag{4.4}$$

and

$$\begin{aligned}
& \int_M g^{p\bar{j}} g^{q\bar{k}} (e^{\psi} (\nabla_q (e^{-\psi} \eta))_{\bar{k}}, e^{\psi} (\nabla_j (e^{-\psi} \eta))_{\bar{p}})_{\psi} \mu \\
&= - \int_M g^{p\bar{j}} g^{q\bar{k}} (\nabla_{\bar{j}} [e^{\psi} (\nabla_q (e^{-\psi} \eta))_{\bar{k}}], \eta_{\bar{p}})_{\psi} \mu \\
&= - \int_M g^{p\bar{j}} g^{q\bar{k}} (\nabla_{\bar{j}} [(\nabla_q \eta)_{\bar{k}}] - \frac{\partial^2 \psi}{\partial \bar{z}_j \partial z_q} \eta_{\bar{k}} - \frac{\partial \psi}{\partial z_q} \nabla_{\bar{j}} (\eta_{\bar{k}}), \eta_{\bar{p}})_{\psi} \mu
\end{aligned} \tag{4.5}$$

By (2.7)

$$\nabla_{\bar{j}} \eta = \nabla_{\bar{j}} (\eta_{\bar{l}} d\bar{z}^l) = (\nabla_{\bar{j}} (\eta_{\bar{l}})) d\bar{z}^l - \eta_{\bar{l}} \overline{\Gamma_{jm}^l} d\bar{z}^m, \tag{4.6}$$

so

$$(\nabla_{\bar{j}} \eta)_{\bar{k}} = \nabla_{\bar{j}} (\eta_{\bar{k}}) - \eta_{\bar{l}} \overline{\Gamma_{jk}^l} = \left( \frac{\partial \eta_{\bar{k}}^{\alpha}}{\partial \bar{z}_j} - \eta_{\bar{l}}^{\alpha} \overline{\Gamma_{jk}^l} \right) v_{\alpha}$$

and

$$\nabla_q [(\nabla_{\bar{j}} \eta)_{\bar{k}}] = \nabla_q \left( \left( \frac{\partial \eta_{\bar{k}}^{\alpha}}{\partial \bar{z}_j} - \eta_{\bar{l}}^{\alpha} \overline{\Gamma_{jk}^l} \right) v_{\alpha} \right) = \left( \frac{\partial^2 \eta_{\bar{k}}^{\alpha}}{\partial z_q \partial \bar{z}_j} - \eta_{\bar{l}}^{\alpha} \frac{\partial \overline{\Gamma_{jk}^l}}{\partial z_q} \right) v_{\alpha}, \tag{4.7}$$

where we use (4.2) for last equation.

Similarly,

$$\nabla_q \eta = \nabla_q (\eta_{\bar{l}} d\bar{z}^l) = (\nabla_q (\eta_{\bar{l}})) d\bar{z}^l,$$

so

$$(\nabla_q \eta)_{\bar{k}} = \nabla_q (\eta_{\bar{k}}) = \frac{\partial \eta_{\bar{k}}^{\alpha}}{\partial z_q} v_{\alpha} + \eta_{\bar{k}}^{\alpha} C_{q\alpha}^{\beta} v_{\beta},$$

and

$$\nabla_{\bar{j}} [(\nabla_q \eta)_{\bar{k}}] = \nabla_{\bar{j}} \left( \frac{\partial \eta_{\bar{k}}^{\alpha}}{\partial z_q} v_{\alpha} + \eta_{\bar{k}}^{\alpha} C_{q\alpha}^{\beta} v_{\beta} \right) = \frac{\partial^2 \eta_{\bar{k}}^{\alpha}}{\partial \bar{z}_j \partial z_q} v_{\alpha} + \eta_{\bar{k}}^{\alpha} \frac{\partial C_{q\alpha}^{\beta}}{\partial \bar{z}_j} v_{\beta} = \frac{\partial^2 \eta_{\bar{k}}^{\alpha}}{\partial \bar{z}_j \partial z_q} v_{\alpha} + R_{j,q}^E \eta_{\bar{k}}. \tag{4.8}$$

Note

$$R_{q\bar{j},\bar{k}}^{\bar{l}} \partial_{\bar{l}} = \nabla_q \nabla_{\bar{j}} \partial_{\bar{k}} = \nabla_q (\overline{\Gamma_{jk}^l} \partial_{\bar{l}}) = \frac{\partial \overline{\Gamma_{jk}^l}}{\partial z_q} \partial_{\bar{l}},$$

so  $R_{q\bar{j},\bar{k}}^{\bar{l}} = \frac{\partial \overline{\Gamma_{jk}^l}}{\partial z_q}$ . All in all,

$$\nabla_q [(\nabla_{\bar{j}} \eta)_{\bar{k}}] - \nabla_{\bar{j}} [(\nabla_q \eta)_{\bar{k}}] = -\eta_{\bar{l}} R_{q\bar{j},\bar{k}}^{\bar{l}} + R_{q,\bar{j}}^E \eta_{\bar{k}}, \tag{4.9}$$

where  $R^E = R_{q\bar{j}}^E dz^q \wedge d\bar{z}^j$ .

Now,

$$g^{p\bar{j}} g^{q\bar{k}} (-\eta_{\bar{l}} R_{q\bar{j}, \bar{k}}^l, \eta_{\bar{p}})_{\psi} = -g^{p\bar{j}} g^{q\bar{k}} g^{t\bar{l}} R_{q\bar{j}t\bar{k}}(\eta_{\bar{j}}, \eta_{\bar{q}})_{\psi} = -g^{p\bar{j}} g^{q\bar{k}} Ric_{p\bar{k}}(\eta_{\bar{j}}, \eta_{\bar{q}})_{\psi}.$$

To sum up,

$$g^{p\bar{j}} g^{q\bar{k}} (\nabla_q [(\nabla_{\bar{j}} \eta)_{\bar{k}}] - \nabla_{\bar{j}} [(\nabla_q \eta)_{\bar{k}}], \eta_{\bar{p}})_{\psi} = g^{p\bar{j}} g^{q\bar{k}} ((-Ric_{p\bar{k}} Id_r + R_{p\bar{k}}^E)(\eta_{\bar{j}}), \eta_{\bar{q}})_{\psi} \quad (4.10)$$

Therefore,

$$\begin{aligned} & \int_M g^{p\bar{j}} g^{q\bar{k}} [(\nabla_{\bar{j}} \eta)_{\bar{k}}, -(\nabla_{\bar{q}} \eta)_{\bar{p}}]_{\psi} + (e^{\psi} (\nabla_q (e^{-\psi} \eta))_{\bar{k}}, e^{\psi} (\nabla_{\bar{j}} (e^{-\psi} \eta))_{\bar{p}})_{\psi} \mu \\ &= \int_M g^{p\bar{j}} g^{q\bar{k}} \left( (-Ric_{p\bar{k}} + \frac{\partial^2 \psi}{\partial z_p \partial \bar{z}_k}) Id_r + R_{p\bar{k}}^E \right) (\eta_{\bar{j}}), \eta_{\bar{q}})_{\psi} \mu \end{aligned} \quad (4.11)$$

Finally we get

$$\begin{aligned} & |S\eta|_{\psi}^2 + |T^* \eta|_{\psi}^2 \\ &= \int_M g^{p\bar{j}} g^{q\bar{k}} [(\nabla_{\bar{j}} \eta)_{\bar{k}}, (\nabla_{\bar{p}} \eta)_{\bar{q}} - (\nabla_{\bar{q}} \eta)_{\bar{p}}]_{\psi} + (e^{\psi} (\nabla_q (e^{-\psi} \eta))_{\bar{k}}, e^{\psi} (\nabla_{\bar{j}} (e^{-\psi} \eta))_{\bar{p}})_{\psi} \mu \\ &= \|\bar{\nabla} \eta\|_{\psi}^2 + \int_M g^{p\bar{j}} g^{q\bar{k}} \left( (-Ric_{p\bar{k}}^M + \frac{\partial^2 \psi}{\partial z_p \partial \bar{z}_k}) Id_r + R_{p\bar{k}}^E \right) (\eta_{\bar{j}}), \eta_{\bar{q}})_{\psi} \mu \end{aligned} \quad (4.12)$$

□

**Corollary 4** (a priori estimate). *If there is  $C > 0$  st  $i(R^E + Ricci(\omega) + \partial\bar{\partial}\psi) \geq C\omega$  on  $C^\infty(M, E)$ , then for any  $\eta$  a smooth  $E$  valued  $(0, 1)$ -form, we have*

$$|S\eta|_{\psi}^2 + |T^* \eta|_{\psi}^2 \geq C|\eta|_{\psi}^2.$$

We remark that the most important term in (4.12), for our purpose, is

$$\int_M g^{p\bar{j}} g^{q\bar{k}} (R_{p\bar{k}}^E \eta_{\bar{j}}, \eta_{\bar{q}})_{\psi} \mu.$$

Under K-coordinate it becomes

$$\sum_{j,q} (R_{j\bar{q}}^E \eta_{\bar{j}}, \eta_{\bar{q}}) = (\dots R^E (\sum_{j=1}^n \eta_{\bar{j}} \partial_j), \sum_{q=1}^n \eta_{\bar{q}} \partial_q) \geq \lambda |\sum_{j=1}^n \eta_{\bar{j}} \partial_j|^2 = \lambda |\eta|^2, \quad (4.13)$$

where the function  $\lambda \in C(M)$  is the minimal eigenvalue of  $\dots R^E$ . (Here we used Proposition 1.)

**Theorem 15** (Hörmander density). *[9, Proposition 2.1.1 and p.121][4, Lemma 4.3.2] If  $\psi \in C^2(M)$ , then  $\Omega^{0,1}(M, E) \subset Dom(S) \cap Dom(T^*)$  is dense with respect to the graph norm*

$$\eta \mapsto |\eta|_{\psi} + |S\eta|_{\psi} + |T^* \eta|_{\psi}^2$$

**Theorem 16** ( $L^2$  existence). *If  $\psi \in C^2(M)$  and there is  $C > 0$  such that*

$$i(R^E + \text{Ricci}(\omega) + \partial\bar{\partial}\psi) \geq C\omega,$$

*then*

*for every  $\eta \in L^2_{0,1}(M, E; \psi)$  with  $\bar{\partial}\eta = 0$ , there is a  $\xi \in \text{Dom}(\bar{\partial}) \subset L^2_{0,0}(M, E; \psi)$  st  $\bar{\partial}\xi = \eta$ , which means for any  $\chi \in \text{Dom}(T^*)$ ,*

$$\int_M (\xi, T^*\chi)_{\psi\mu} = \int_M (\eta, \chi)_{\psi\mu}. \quad (4.14)$$

*and*

$$C|\xi|_{\psi}^2 \leq |\eta|_{\psi}^2.$$

Please compare this to [24, Proposition 2.1(ii)] and [26, Proposition 1.1].

*Proof.* By Corollary 4, we have the a priori estimate for  $\eta \in \Omega^{0,1}(M, E)$ . By Theorem 15, the same estimate holds for any  $\eta \in \text{Dom}(S) \cap \text{Dom}(T^*)$ . Take  $F = \ker(S)$ , then  $\eta \in F$ . By Theorem 13, there is  $\xi \in H_1$  st  $\bar{\partial}\xi = \eta$  and

$$C|\xi|_{\psi}^2 \leq |\eta|_{\psi}^2.$$

□

To show regularity of  $\bar{\partial}$ -solution, we introduce some Sobolev spaces. For  $l \in \mathbb{N}$  and  $U \subset \mathbb{C}^n$  an open, let  $H^l(U)$  be as in [6, Section 5.2.2, Remarks (i)] and similarly, let  $H^l(U; \text{loc})$  be the space of all  $f \in L^2(U; \text{loc})$  such that for each multi-index  $I, J \in \mathbb{N}^n$  with  $|I| + |J| \leq l$ , the weak derivative  $(\frac{\partial}{\partial z})^I (\frac{\partial}{\partial \bar{z}})^J f \in L^2(U; \text{loc})$ .

**Lemma 5.** *If  $f \in L^2(U)$  and its weak derivatives  $\frac{\partial f}{\partial \bar{z}_j} \in L^2(U)$  for all  $1 \leq j \leq n$ , then  $f \in H^1(U)$  and for any  $j$ ,  $|\frac{\partial f}{\partial z_j}|_{L^2(U)} = |\frac{\partial f}{\partial \bar{z}_j}|_{L^2(U)}$ .*

Let  $H^l_{p,q}(M, E; \text{loc})$  be the space of measurable  $E$ -valued  $(p, q)$ -forms, for which there is an atlas of  $M$  and a set of frame of  $E$  over each chart  $U$ , such that the representation functions are in  $H^l(U; \text{loc})$ . (In fact, due to compactness of  $M$ ,  $H^l_{p,q}(M, E; \text{loc}) = H^l_{p,q}(M, E)$ .) By Sobolev embedding theorem, [6, Section 5.6.3 (ii), Theorem 6], if  $k > n$  and  $0 < \lambda < 1$ , then  $H^k_{p,q}(M, E; \text{loc}) \subset C^{k-n-1, \lambda}_{p,q}(M, E; \text{loc})$  the latter being space of locally  $\lambda$ -Hölder continuous  $E$ -valued  $(p, q)$ -forms.

**Lemma 6.** *Let  $\psi$  be a measurable function on  $M$  and  $\{\psi^{(l)}\}$  be a decreasing sequence in  $C^2(M)$  with  $\psi^{(l)} \rightarrow \psi$ . Assume there is  $C > 0$  such that for any  $l \geq 1$*

$$i(R^E + \text{Ricci}(\omega) + \partial\bar{\partial}\psi) \geq C\omega. \quad (4.15)$$

*Then for  $\eta \in W^k_{0,1}(M, E; \psi)$ , there is  $\xi \in W^{k+1}_{0,0}(M, E)$  st  $\bar{\partial}\xi = \eta$  in the distribution sense with*

$$C|\xi|_{\psi}^2 \leq |\eta|_{\psi}^2.$$

*In particular, if furthermore  $\eta \in \Omega^{0,1}(M, E)$ , then this  $\xi \in C^\infty(M, E)$  and  $\bar{\partial}\xi = \eta$  in the classical sense.*



*Proof.* Because  $e^{-\psi^{(l)}} \rightarrow e^{-\psi}$  is an increasing sequence, by Levi convergence theorem  $\|\eta\|_{\psi^{(l)}}^2 \rightarrow \|\eta\|_{\psi}^2$ . By Theorem 16, there is  $\xi^{(l)} \in \text{Dom}(T_l) \subset L_{0,0}^2(M, E; \psi^{(l)})$  such that  $T_l \xi^{(l)} = \eta$  and

$$C \|\xi^{(l)}\|_{\psi^{(l)}}^2 \leq \|\eta\|_{\psi^{(l)}}^2,$$

where  $T_l = \bar{\partial}^E : \text{Dom}(T_l) \subset L_{0,0}^2(M, E; \psi^{(l)}) \rightarrow L_{0,1}^2(M, E; \psi^{(l)})$  as usual. There is  $C' > 0$  such that  $\psi^{(l)} \leq C'$  for any  $l \geq 1$ , so

$$\|\xi^{(l)}\|^2 \leq e^S \|\xi^{(l)}\|_{\psi^{(l)}}^2 \leq C^{-1} e^{C'} \|\eta\|_{\psi^{(l)}}^2.$$

Thus the sequence  $\{\xi^{(l)}\}$  is bounded in  $L^2(M, E)$ . By weak compactness [7, Theorem 1.42], we can find a subsequence of  $\xi^{(l)}$ , still denoted by  $\xi^{(l)}$ , converging weakly to some  $\xi \in L^2(M, E)$ .

We check  $\bar{\partial}\xi = \eta$  in the sense of distribution. In fact, for any  $t \in \Omega^{0,1}(M, E)$ , let  $\chi = e^{\psi^{(l)}} t \in L_{0,1}^2(M, E; \psi^{(l)})$  in (4.14), then

$$\int_M (\xi, \bar{\partial}^* t) \mu = \int_M (\xi, T_l^* \chi)_{\psi} \mu = \int_M (\eta, t) \mu.$$

Now we treat the regularity. Suppose that  $\xi \in W_{0,0}^l(M, E)$  for some  $0 \leq l \leq k$  (this is true for  $l = 0$ ), we will show  $\xi \in W_{0,0}^{l+1}(M, E)$ . Thus by induction on  $l$  we find  $\xi \in W_{0,0}^{k+1}(M, E)$ . In fact, for any chart  $(U, z_1, \dots, z_n) \subset M$  and any frame  $\{v_\alpha\}$  of  $E$ , write  $\xi = \sum_{i=1}^r \xi^\alpha v_\alpha$ . For any  $\chi \in C_c^\infty(U)$ , any multi-indices  $I, J \in \mathbb{N}^n$  with  $|I| + |J| \leq l$ ,

$$\partial_{\bar{j}} \left( \frac{\partial}{\partial z} \right)^I \left( \frac{\partial}{\partial \bar{z}} \right)^J (\chi \xi^\alpha) = \left( \frac{\partial}{\partial z} \right)^I \left( \frac{\partial}{\partial \bar{z}} \right)^J (\chi \eta_{\bar{j}}^\alpha + \xi^\alpha \partial_{\bar{j}} \chi) \in L^2(U)$$

By Lemma 5,  $\left( \frac{\partial}{\partial z} \right)^I \left( \frac{\partial}{\partial \bar{z}} \right)^J (\chi \xi^\alpha) \in H^1(U)$ . So  $\xi^\alpha \in H^{l+1}(U; \text{loc})$  or equivalently  $\xi \in H^{l+1}(M, E; \text{loc})$ .

By weak lower semi-continuity of norm, for any  $j \geq 1, l \geq j, e^{-\psi^{(j)}} \leq e^{-\psi^{(l)}}$  we thus have

$$\|\xi\|_{\psi^{(j)}}^2 \leq \liminf_l \|\xi^{(l)}\|_{\psi^{(j)}}^2 \leq \liminf_l \|\xi^{(l)}\|_{\psi^{(l)}}^2 \leq \liminf_l C^{-1} \|\eta\|_{\psi^{(l)}}^2 = C^{-1} \|\eta\|_{\psi}^2.$$

Let  $j \rightarrow \infty$ , by Levi convergence theorem again,  $\|\xi\|_{\psi^{(j)}} \rightarrow \|\xi\|_{\psi}$ , thus  $\|\xi\|_{\psi}^2 \leq C^{-1} \|\eta\|_{\psi}^2$ . □

**Remark 4.** Note that by Weyl lemma, any solution  $\xi_0$  of  $\bar{\partial}\xi_0 = 0$  is holomorphic. Then any  $\bar{\partial}$ -solution is in  $W_{0,0}^{k+1}(M, E)$

## Chapter 5

# High rank Bergman kernel

Let  $P \rightarrow M$  be another hermitian holomorphic vector bundle with  $rk(P) = s$ , we are going to find the first term in the asymptotic of  $B_k$  the Bergman kernel of  $P^k \otimes E$ . However, we restrict ourselves to the case of Riemann surface in Section 5.3 for simplicity of calculation.

We fix a frame  $\{u_1, \dots, u_s\}$  of  $P$  as in Proposition 3 and write  $H_P$  for the corresponding Gram matrix. For  $a \in [s]$ , let  $H_{2,P;a,a} = e^{-\sum_{j,l=1}^n R_{j\bar{l},aa}^P z_j \bar{z}_l}$  and for  $a \neq b \in [s]$ , let  $H_{2,P;a,b} = -\sum_{j,l=1}^n R_{j\bar{l},ab}^P z_j \bar{z}_l$ , then  $H_P = H_{2,P}(Id_s + O(|z|^3))$ . Recall that as usual  $rk(E) = r$ . Put  $v_I := u_{i_1} \otimes \dots \otimes u_{i_k} \otimes v_\alpha$  for each multi-index  $I = (i_1, \dots, i_k; \alpha) \in [s]^k \times [r]$ . Then  $\{v_I : I \in [s]^k \times [r]\}$  form a frame of  $P^k \otimes E$  of Gram matrix  $H_P^{\otimes k} \otimes H_E$ . In particular, this preferred frame is orthonormal at  $p$ . We assume that  $(P, h^P)$  is Nakano positive.

### 5.1 Upper bound

**Lemma 7.**  $s \in H^0(B(p, \frac{\log k}{\sqrt{k}}), P^k \otimes E)$ , write  $B = B(p, \frac{\log k}{\sqrt{k}})$ . Then

$$s \otimes \bar{s}(p) \leq M(k) \int_B |s|^2 \mu,$$

where the matrix

$$M(k) = 2^{-n} (1 + Ck^{-1/2} \log^3 k) \left[ \int_B H_{2,P}^{\otimes k} \mu_0 \right]^{-1} \otimes Id_r. \quad (5.1)$$

Taking trace,  $|s|^2(p) \leq Tr(M(k)) \int_B |s|^2 \mu$ .

*Proof.* Write  $s = \sum_{I \in [s]^k \times [r]} f_I v_I$  and let  $y = (f_I : I \in [s]^k \times [r])$  be a column vector (of holomorphic functions on  $B$ ).

Then

$$|s|^2 = y^T (H_P^{\otimes k} \otimes H_E) \bar{y}$$

and  $s \otimes \bar{s}(p) = yy^H(p)$ . There is  $C > 0$  and a neighborhood of  $p$  (independent of  $k$ ) over which

$$H_P^{\otimes k} \otimes H_E \geq \frac{H_{2,P}^{\otimes k} \otimes Id_r}{(1 + C|z|^3)^k(1 + C|z|^2)}; \quad (5.2)$$

$$\frac{\mu}{\mu_0} \geq \frac{2^n}{1 + C|z|^2}. \quad (5.3)$$

We estimate the integrand on the right hand side. By (5.2) and (5.3), we have

$$\begin{aligned} |s|^2 \mu &\geq y^T \frac{H_{2,P}^{\otimes k} \otimes Id_r}{(1 + C|z|^3)^k(1 + C|z|^2)} y^H \frac{2^n \mu_0}{1 + C|z|^2} \\ &\geq 2^n (1 + Ck^{-1/2} \log^3 k)^{-1} y^T (H_{2,P}^{\otimes k} \otimes Id_r) \bar{y} \mu_0, \end{aligned} \quad (5.4)$$

where for the last step we used  $|z| \leq \frac{\log k}{\sqrt{k}}$ .

Therefore

$$\int_B |s|^2 \mu \geq 2^n (1 + Ck^{-1/2} \log^3 k)^{-1} \int_B y^T (H_{2,P}^{\otimes k} \otimes Id_r) \bar{y} \mu_0 \quad (5.5)$$

$$\geq 2^n (1 + Ck^{-1/2} \log^3 k)^{-1} y^T(p) \left[ \int_B (H_{2,P}^{\otimes k} \otimes Id_r) \mu_0 \right] \bar{y}(p), \quad (5.6)$$

where for (5.6) we used some analogue of vanishing identity (Exercise 10). Finally we apply Exercise 2 to conclude.  $\square$

**Corollary 5.** *The Bergman kernel  $B_k$  of  $P^k \otimes E$  satisfies  $B_k(p) \leq M(k)$ .*

## 5.2 Lower bound: A third construction of peak section

We present yet another construction of peak sections, which is a modification of [26, Proof of Lemma 1.2].

Recall  $U, \psi_k, v$  from Section 3.2.2. For each  $I \in [s]^k$ , put  $u_I = u_{i_1} \otimes \cdots \otimes u_{i_k} \otimes v$  to be a holomorphic section of  $P^k \otimes E$ . We define a weight function

$$\psi(z) = n\chi\left(\frac{k|z|^2}{\log^2 k}\right) \log\left(\frac{k|z|^2}{\log^2 k}\right), (\chi(x) = \eta(2x)) \quad (5.7)$$

The factor  $n$  in definition of  $\psi$  is to ensure vanishing of  $\xi$  (to be constructed) at  $p$ . Note carefully that  $\psi$  is supported in the ball  $B(p, \frac{\log k}{\sqrt{k}})$  and admits a logarithmic pole at  $p \in M$ . In addition,  $\psi \leq 0$  on  $M$ . To apply Theorem 16, we approximate  $\psi$  by smooth functions. Let

$$\psi^{(l)} = n\chi\left(\frac{k|z|^2}{\log^2 k}\right) \log\left(\frac{k}{\log^2 k} \left(\frac{1}{l} + |z|^2\right)\right),$$

then this decreasing sequence converges to  $\psi$ .

We check condition (4.15). We calculate

$$n^{-1}\partial\bar{\partial}\psi = \tag{5.8}$$

$$\left[\left(\frac{k}{\log^2 k}\right)^2 \chi''\left(\frac{k}{\log^2 k}|z|^2\right) \log\left(\frac{k}{\log^2 k}|z|^2\right) + 2\frac{k}{\log^2 k}|z|^{-2} \chi'\left(\frac{k}{\log^2 k}|z|^2\right)\right] \partial|z|^2 \wedge \bar{\partial}|z|^2 \tag{5.9}$$

$$+ \frac{k}{\log^2 k} \chi'\left(\frac{k}{\log^2 k}|z|^2\right) \log\left(\frac{k}{\log^2 k}|z|^2\right) \partial\bar{\partial}(|z|^2) \tag{5.10}$$

$$+ \chi\left(\frac{k}{\log^2 k}|z|^2\right) \partial\bar{\partial} \log(|z|^2). \tag{5.11}$$

Firstly, the (1, 1)-form

$$i\partial|z|^2 \wedge \bar{\partial}|z|^2 = i(\bar{z}_j z_k dz^j \wedge d\bar{z}^k)$$

is semi-positive, and there is  $C > 0$  st

$$C|z|^2\omega \geq i\partial|z|^2 \wedge \bar{\partial}|z|^2$$

for  $|z| \leq 1$ . We get

$$i \times (5.9) \geq -C \frac{k}{\log^2 k} \omega$$

over  $M$ .

Secondly, The form  $i\partial\bar{\partial}|z|^2 = idz_j \wedge d\bar{z}_j$  is positive, so  $i \times (5.10) \geq 0$ .

Thirdly, by Exercise 2,  $|z|^2 Id_n \geq \bar{z}z^T$ , so the form

$$i\partial\bar{\partial} \log |z|^2 = i|z|^{-4}(|z|^2 \delta_{jk} - z_j \bar{z}_k) dz_k \wedge d\bar{z}_j$$

is semi-positive, so  $i \times (5.11) \geq 0$ . The estimation with  $\psi$  replace by  $\psi^{(l)}$  is similar.

To sum up, there is  $C > 0$  such than for any  $l \geq 1$ ,

$$i\partial\bar{\partial}\psi^{(l)} \geq -C \frac{k}{\log^2 k} \omega$$

on  $M$ .

Because  $P$  is Nakano positive, by (4.13) there is  $\delta > 0$  and  $k_0 \geq 1$  such that when  $k \geq k_0, l \geq 1$  we have

$$Ricci(\omega) + i\partial\bar{\partial}\psi^{(l)} + iR^{P^k \otimes E} \geq \delta k \omega.$$

Note that  $(\bar{\partial}\psi_k)u_I$  vanishes at  $p$ , so  $\|\bar{\partial}\psi_k\|_\psi$  is finite. By Lemma 6, there is  $w_I \in C^\infty(M, P^k \otimes E)$  such that  $\bar{\partial}w_I = (\bar{\partial}\psi_k)u_I$  with

$$\|w_I\|_\psi^2 \leq (k\delta)^{-1} \|(\bar{\partial}\psi_k)u_I\|_\psi^2 < \infty,$$

in particular  $w_I(p) = 0$ .

Let  $s_I = \psi_k u_I - w_I$  then  $s_I \in H^0(M, P^k \otimes E)$  and  $s_I(p) = u_I(p)$ . The  $s_I$  is the peak section we are looking for.

We need some estimates as in Section 3.2.2. For any  $a \in [s]$ ,

$$\sum_{j,l=1}^n R_{j\bar{l},aa}^P z_j \bar{z}_l = (\dots R^E(u_a \otimes (\sum_j z_j \partial_j)), u_a \otimes (\sum_l z_l \partial_l)).$$

Because  $P$  is Nakano positive,  $R_{*,*,aa}^P$  is a positive definite matrix, so there is  $\lambda > 0$  such that on  $B(p, \sqrt{2} \frac{\log k}{\sqrt{k}})$ , for each  $a \in [s]$ ,

$$\sum_{j,l=1}^n R_{j\bar{l},aa}^P z_j \bar{z}_l \geq \lambda |z|^2. \quad (5.12)$$

By (5.12), there is  $\lambda > 0$  such that on  $B(p, \sqrt{2} \frac{\log k}{\sqrt{k}})$  we have

$$H_{\alpha,\alpha}^P = e^{-R_{\alpha,\alpha,i\bar{j}}^P z_i \bar{z}_j} (1 + O(|z|^3)) \leq e^{-\lambda |z|^2}.$$

Then

$$|u_I|^2 = |v|^2 \prod_{j=1}^k H_{i_j, i_j}^P \leq C e^{-k\lambda |z|^2}.$$

Let  $A$  be the annulus  $\frac{\log k}{\sqrt{k}} \leq |z| \leq \sqrt{2} \frac{\log k}{\sqrt{k}}$ . Then

$$\begin{aligned} k\delta \|w_I\|^2 &\leq k\delta \|w_I\|_{\psi}^2 \leq \|(\bar{\partial}\psi_k)u_I\|_{\psi}^2 \\ &= \int_A |(\bar{\partial}\psi_k)u_I|^2 e^{-\psi} \mu \\ &= \int_A |(\bar{\partial}\psi_k)u_I|^2 \mu \\ &\leq C \int_A \frac{k}{\log^2 k} e^{-k\delta |z|^2} \mu_0 \leq C \left(\frac{\log^2 k}{k}\right)^n \frac{k}{\log^2 k} e^{-\delta \log^2 k} \end{aligned} \quad (5.13)$$

Thus

$$\|w_I\|^2 = O(k^{-n} e^{-\delta \log^2 k} \log^{2n-2} k). \quad (5.14)$$

For the outside  $L^2$  norm of smooth peak section, as in (3.20),

$$\int_{M-B} |\psi_k u_I|^2 \mu \leq C k^{-n} e^{-\lambda \log^2 k} \log^{2n} k. \quad (5.15)$$

Similarly we have  $\|\psi_k u_I\|^2 = O(k^{-n})$ .

Finally we arrive at the lower bound of Bergman kernel. For  $I = (i_1, \dots, i_k; \alpha) \in [s]^k \otimes [r]$ , let  $s_I$  to be the peak section constructed from  $v_\alpha$ . Then  $\{s_I : I \in [s]^k \times [r]\}$  is orthonormal at  $p$ . By Exercise 3,

$$B_k(p) \geq \sum_{I,J} s_I \otimes \bar{s}_J(p) A_{I,I} = A^T, \quad (5.16)$$

where  $A$  is the inverse of the Gram matrix  $G$ .

By the same argument for (3.31), we find

$$G = G'(Id_{s^k r} + O(e^{-\lambda/2 \log k}) \log^{n-1} k),$$

where

$$G'_{I,J} = \int_B (\psi_k u_I, \psi_k u_J) \mu$$

so

$$A = (G')^{-1}(Id_{s^k r} + O(\log^{n-1} k e^{-\delta/2 \log k})) \quad (5.17)$$

### 5.3 Calculation

**Exercise 11.** Let  $B = B(0, \frac{\log k}{\sqrt{k}}) \subset \mathbb{C}$  be an open ball,  $a > 0$ , then

$$\int_B e^{-a|z|^2} \mu_0 = \frac{\pi}{a} (1 - e^{-a \frac{\log^2 k}{k}})$$

In this section, **let  $M$  be a Riemann surface.** In this case, for  $a \neq b \in [s]$ ,

$$H_{2,P,a,a} = e^{-F_{aa}|z|^2}; \quad H_{2,P,a,b} = -F_{ab}|z|^2,$$

where  $F = \dots R^P$  is a square matrix of size  $s$  such that the Chern curvature  $R^P(p) = F dz \wedge d\bar{z}$ . Due to our assumption,  $F$  is positive definite.

**Theorem 17.** The Bergman kernel of  $P^k \otimes E$  satisfies

$$B_k(p) = \frac{D \otimes Id_r}{2\pi} (Id_{s^k \otimes r} + O(k^{-1/2} \log^3 k)),$$

where  $D$  is a size  $s^k$  diagonal matrix whose  $(i_1, \dots, i_k)$  entry is  $\sum_{j=1}^k F_{i_j, i_j}$ .

**Example 6.** When  $s = 1$ , i.e.,  $P$  is a line bundle, Theorem 17 agrees with [15, Theorem 4.1.1].

*Proof.* We compute the upper bound first. We calculate  $\int_B H_{2,P}^{\otimes k} \mu_0$  inside (5.1). By Exercise 11, the diagonal entry at  $(i_1, \dots, i_k; i_1, \dots, i_k)$  position is

$$\int_B e^{-\sum_{j=1}^k F_{i_j, i_j} |z|^2} \mu_0 = \frac{\pi}{\sum_{j=1}^k F_{i_j, i_j}} (1 - e^{-\frac{\log^2 k}{k} \sum_{j=1}^k F_{i_j, i_j}}) = \frac{\pi}{\sum_{j=1}^k F_{i_j, i_j}} (1 + O(e^{-\lambda \log^2 k})).$$

Consider an off-diagonal entry at  $(i_1, \dots, i_k; j_1, \dots, j_k)$ , say with  $l \geq 1$  different indices. Then the integral is like  $\int_B |z|^{2l} e^{-a|z|^2} \mu_0 = O(k^{-2})$ , where  $a \geq (k-l)\lambda$ .

Therefore

$$\int_B H_{2,P}^{\otimes k} \mu_0 = \pi D^{-1} + O(k^{-2}) = \pi D^{-1} (Id_{s^k} + O(k^{-1})).$$

By Corollary, we get the upper bound

$$B_k(p) \leq \frac{D \otimes Id_r}{2\pi} (Id_{s^k \otimes r} + O(k^{-1/2} \log^3 k)).$$

We turn to the lower bound. For multi-indices  $I = (i_1, \dots, i_k; \alpha)$ ,  $J = (j_1, \dots, j_k; \beta) \in [s]^k \times [r]$ ,

$$G'_{I,J} = \int_B H_{i_1, j_1}^P \cdots H_{i_k, j_k}^P H_{\alpha, \beta}^E \mu. \quad (5.18)$$

On the diagonal, i.e.  $I = J$ ,

$$\begin{aligned} G'_{II} &= 2 \int_B e^{-\sum_{j=1}^k F_{i_j, i_j} |z|^2} (1 + kO(|z|^3))(1 + O(|z|^2)) \mu_0 \\ &= \left( \int_B e^{-\sum F |z|^2} \right) (1 + O(k^{-1/2})) = \frac{2\pi}{\sum_{j=1}^k F_{i_j, i_j}} (1 + O(k^{-1/2})) \end{aligned}$$

If  $I \neq J$ , the integral is at most  $\int_B e^{-a|z|^2} O(|z|^2) \mu = O(k^{-2})$  where  $a \geq k\lambda$ . Thus by (5.16) and (5.17), we find

$$B_k(p) \geq \frac{D \otimes Id_r}{2\pi} (Id_{s^k \otimes r} + O(k^{-1/2})).$$

We conclude by combining the upper and the lower bound.  $\square$

**Corollary 6** (Asymptotic Riemann-Roch). *We have asymptotic expansion*

$$\dim H^0(M, P^k \otimes E) = s^{k-1} k r \int_M c_1(P) + O(s^k k^{-1/2} \log^3 k).$$

*Proof.* Recall that  $z$  is K-coordinate,  $\mu = idz \wedge d\bar{z}$  by Exercise 4, so

$$c_1(P, h^P) = \frac{i}{2\pi} \text{Tr}(R^P) = \frac{i}{2\pi} \text{Tr}(\dot{R}^P) dz \wedge d\bar{z} = \frac{1}{2\pi} \text{Tr}(F) \mu.$$

We conclude by Proposition 4 and Theorem 17.  $\square$

Some questions that we don't know: does Catlin-Zelditch expansion (Theorem 8) hold in this high rank case? Do we have vanishing theorem of the form  $H^{>1}(M, P^k \otimes E) = 0$ ?

## Chapter 6

# An application to Riemann surfaces

We apply the same strategy to generalize [13, Theorem 1.1]. Recall that every compact Riemann surface admits a hermitian metric of constant scalar curvature.

**Theorem 18.** *In Theorem 7, if  $n = 1$ ,  $E$  is trivial and  $(M, \omega)$  is of constant scalar curvature  $\rho$ , then*

$$B_k(p) = \frac{2k + \rho}{4\pi} (1 + O(k^{-(\log k)/4})).$$

*Proof.* Again, let  $B = B(p, \frac{\log k}{\sqrt{k}})$  in a fixed holomorphic chart centered in  $p \in M$ .

Upper bound: As in Lemma 3,  $s = f\sigma^k$ ,  $|s|^2 = |f|^2 e^{-k\varphi}$ . Because  $\varphi$  is rotationally invariant, we argue as in (3.10) to get

$$\int_B |s|^2 \mu = \int_B e^{-k\varphi} |f|^2 \mu \geq |f(p)|^2 \int_B e^{-k\varphi} \mu = |s(p)|^2 \int_B e^{-k\varphi} \mu.$$

By Lemma 2,

$$B_k(p) \leq \left( \int_B e^{-k\varphi} \mu \right)^{-1}. \quad (6.1)$$

Lower bound: As in Section 5.2, let  $s_k \in H^0(M, L^k)$  be the peak section constructed from  $\sigma^k$ . Then

$$B_k(p) \geq A, \quad (6.2)$$

where  $A^{-1} = \langle s_k, s_k \rangle = G$ . By (3.31),  $G = (1 + O(k^{-(\log k)/4})) \int_B e^{-k\varphi} \mu$ .

Combine (6.1) and (6.2) we find that

$$B_k(p) = \left( \int_B e^{-k\varphi} \mu \right)^{-1} (1 + O(k^{-\frac{\log k}{4}})).$$



**Case**  $\rho = 0$ , locally we may assume that  $\varphi(z) = |z|^2$ , then  $\mu = 2\mu_0$ .

$$\int_B e^{-k\varphi} \mu = 2 \int_B e^{-k|z|^2} \mu_0 = \frac{2\pi}{k} (1 - e^{-\log^2 k}). \quad (6.3)$$

We can also get (6.3) by formally letting  $\rho \rightarrow 0$  in result of (6.4).

**Case**  $\rho \neq 0$ , by [13, (3.5)], locally we may assume that

$$\varphi(z) = \frac{2}{\rho} \log(1 + \frac{\rho}{2}|z|^2).$$

Then

$$g = \frac{\partial^2 \varphi}{\partial z \partial \bar{z}} = (1 + \frac{\rho}{2}|z|^2)^{-2}$$

and

$$\mu = 2g\mu_0 = \frac{2}{(1 + \frac{\rho}{2}|z|^2)^2} \mu_0.$$

We calculate

$$\begin{aligned} \int_B e^{-k\varphi} \mu &= \int_B e^{-\frac{2k}{\rho} \log(1 + \frac{\rho}{2}|z|^2)} \frac{2}{(1 + \frac{\rho}{2}|z|^2)^2} \mu_0 \\ &= 2 \int_B (1 + \frac{\rho}{2}r^2)^{-2 - \frac{2k}{\rho}} r dr d\theta \\ &= 4\pi \int_1^{1 + \frac{\rho}{2} \frac{\log^2 k}{k}} t^{-2 - \frac{2k}{\rho}} \frac{dt}{\rho} \\ &= \frac{4\pi}{2k + \rho} \left[ 1 - \frac{1}{(1 + \frac{\rho}{2} \frac{\log^2 k}{k})^{1 + \frac{2k}{\rho}}} \right], \end{aligned} \quad (6.4)$$

where we used the change of variable  $z = re^{i\theta}$  and  $t = 1 + \frac{\rho}{2}r^2$ .  $\square$

In fact, everything involved is of local nature, therefore the same result holds for any (open) Riemann surfaces (with locally uniform convergence), which is a particular case of [1, Corollary 2.4] (when  $m = 0$ ).

**Remark 5.** *It seems possible to improve Theorem 18 with an error term of size  $O(e^{-\delta k})$  for certain constant  $\delta > 0$  (on Riemann surface). See for instance [8, Theorem A]. The case of hermitian symmetric manifold could also be treated, cf.[16, Theorem 1]. We are grateful to Professor X.Ma for providing us these references. One can ask if a similar result holds for any dimensional manifold endowed with a constant scalar curvature Kähler metric.*

# Appendix A

## Appendix

### A.1 An alternative proof of Corollary 2

Following [15, Remark 4.1.4], we give another proof of Corollary 2.

**Exercise 12.** [15, (4.1.5)] Let  $(M^n, \omega)$  be a hermitian manifold,  $E \rightarrow M$  a hermitian holomorphic vector bundle of Chern curvature  $R^E$ , then

$$niR^E \wedge \omega^{n-1} = (i\Lambda R^E)\omega^n. \quad (\text{A.1})$$

Taking trace,

$$\text{Tr}(i\Lambda R^E)\omega^n = 2\pi n c_1(E)\omega^{n-1}.$$

In particular, when  $E = T_h M$  with  $h^E$  the hermitian metric of  $M$ ,

$$\text{scal} \cdot \omega^n = 2\pi n c_1(T_h M)\omega^{n-1}.$$

*Proof.* We check the identity pointwisely. Fix  $p \in M$  and a holomorphic chart centered in  $p$  and orthonormal at  $p$ . Put  $\eta_l = dz_l \wedge d\bar{z}_l$  for short. At  $p$ ,

$$g_{i\bar{j}} = \delta_{ij}; \quad \omega = i \sum_{l=1}^n \eta_l; \quad i\Lambda R^E = \sum_{j=1}^n R_{j\bar{j}}^E.$$

Therefore,

$$(i\Lambda R^E)\omega^n = i^n \left( \sum_j R_{j\bar{j}}^E \right) n! \eta_1 \wedge \cdots \wedge \eta_n;$$

$$niR^E \wedge \omega^{n-1} = ni^n \left( \sum_{l=1}^n \eta_l \right)^{n-1} \wedge \left( \sum_{j,k} R_{j\bar{k}}^E dz_j \wedge d\bar{z}_k \right) = i^n \left( \sum_j R_{j\bar{j}}^E \right) n! \eta_1 \wedge \cdots \wedge \eta_n.$$

We proved (A.1). For the last equation, apply Exercise 6.  $\square$

We recall two characteristic classes.

**Definition 17** (Todd class). *If a complex vector bundle  $E$  on a topological space  $S$  has  $\alpha_i$  as its Chern roots, the Todd class  $Td(E) = \prod_i Q(\alpha_i)$ , where  $Q(x) = \frac{x}{1-e^{-x}}$ . In particular,  $Td(E) = 1 + \frac{c_1(E)}{2} + \dots$ . For a complex manifold  $X$ , define  $Td(X) := Td(T_h X)$ .*

**Definition 18** (Chern character). *[15, (1.3.45)] For a holomorphic vector bundle  $P \rightarrow X$ , define Chern character*

$$ch(E) = \text{Tr} \exp\left(\frac{iR^E}{2\pi}\right),$$

where  $R^E$  is Chern curvature of  $E$ . We define the  $2j$  forms  $c_j(E)$  by

$$ch(E) = \sum_{j \geq 0} c_j(E).$$

(In particular  $c_1(E) = \frac{i}{2\pi} \text{Tr}(R^E)$ , which agrees with Definition 12. And  $c_0(E) = rk(E)$ .)

We state the following celebrated theorem.

**Theorem 19** (Hirzebruch-Riemann-Roch). *[15, Theorem 1.4.6] Let  $P$  be a holomorphic vector bundle over a compact complex manifold  $X$ , then  $\chi(X, P) = \int_X ch(P)Td(X)$ .*

Now we derive the desired result as a corollary.

A second proof of Corollary 2. Note that

$$ch(L^k \otimes E) = \text{Tr} \exp\left(\frac{iR^{L^k \otimes E}}{2\pi}\right) = \exp\left(\frac{ikR^L}{2\pi}\right) \text{Tr} \exp\left(\frac{iR^E}{2\pi}\right) = e^{kc_1(L)} ch(E).$$

By Theorem 19,  $\chi(M, L^k \otimes E)$  expands into a degree  $n$  polynomial in  $k$ .

In this asymptotic expansion, the coefficient of  $k^n$  is  $rk(E) \int_M \frac{c_1^n(L)}{n!} = \frac{rk(E)}{(2\pi)^n} \int_M \mu$ .

By Exercise 12,

$$\int_M \text{Tr}(i\Lambda R^E) c_1^n(L) = n \int_M c_1(E) c_1^{n-1}(L); \quad (\text{A.2})$$

$$\int_M \text{scal} c_1^n(L) = n \int_M c_1(T_h M) c_1^{n-1}(L). \quad (\text{A.3})$$

The coefficient of  $k^{n-1}$  is

$$\begin{aligned} & \int_M \frac{c_1^{n-1}(L)}{(n-1)!} \left( rk(E) \frac{c_1(T_h M)}{2} + c_1(E) \right) \\ &= \frac{1}{n!} \left[ \frac{rk(E)}{2} \int_M \text{scal} c_1^n(L) + \int_M \text{Tr}(i\Lambda R^E) c_1^n(L) \right] \\ &= \frac{1}{(2\pi)^n} \left[ \frac{rk(E)}{2} \int_M \text{scal} \mu + \int_M \text{Tr}(i\Lambda R^E) \mu \right]. \end{aligned}$$

Thus we get the said asymptotic expansion. □

## A.2 Controlling supreme norm

The following Theorem 20 is a start point of complex analysis.

**Theorem 20** (Cauchy integral formula). *Let  $D$  be an open subset of  $\mathbb{C}^n$  with  $0 \in D$ , for function  $f \in C_c^\infty(D)$ ,*

$$f(0) = \left(\frac{-1}{\pi}\right)^n \int_D \frac{\partial^n f}{\partial \bar{z}_1 \dots \partial \bar{z}_n} \frac{\mu_0}{z_1 \dots z_n}.$$

We give another theorem of same flavor.

**Theorem 21** (Bochner-Martinelli formula). *For  $\zeta, z \in \mathbb{C}^n$  the Bochner-Martinelli kernel  $\omega(\zeta, z)$  is a differential  $(n, n-1)$ -form in  $\zeta$  defined by*

$$\omega(\zeta, z) = \frac{(n-1)!}{(2\pi i)^n} \frac{1}{|z - \zeta|^{2n}} \sum_{1 \leq j \leq n} (\bar{\zeta}_j - \bar{z}_j) d\bar{\zeta}_1 \wedge d\zeta_1 \wedge \dots \wedge d\zeta_j \wedge \dots \wedge d\bar{\zeta}_n \wedge d\zeta_n$$

(where the term  $d\bar{\zeta}_j$  is omitted). *Let  $D \subset \mathbb{C}^n$  be a domain and  $f \in C_c^\infty(D)$ . Then for  $z \in D$ , we have*

$$f(z) = - \int_D \bar{\partial} f(\zeta) \wedge \omega(\zeta, z).$$

**Exercise 13.** *For  $n \in \mathbb{N}^+$ , there is a constant  $C = C(n) > 0$  such that for any  $f \in C^\infty(\mathbb{C}^n)$ ,*

1.

$$\frac{|f(0)|}{C} \leq |f|_{L^1(\Delta^n(1), \mu_0)} + \sup_{z \in \Delta^n(1), I(\neq \emptyset) \subset [n]} \left| \frac{\partial^{|I|} f}{\partial \bar{z}^I}(z) \right|$$

2. For any  $r > 0$ ,

$$\frac{|f(0)|}{C} \leq r^{-2n} |f|_{L^1(\Delta^n(r), \mu_0)} + \sup_{z \in \Delta^n(r), I(\neq \emptyset) \subset [n]} r^{|I|} \left| \frac{\partial^{|I|} f}{\partial \bar{z}^I}(z) \right|$$

*Proof.* By abuse of notation, we use  $C$  to denote various positive constant depending only on  $n$ .

1. Fix a smooth function  $\rho : \mathbb{C} \rightarrow [0, 1]$  such that  $\rho(z) = 1$  when  $|z| < 1/2$  and  $\rho$  compactly supported on  $\Delta^1(1)$ . Set  $\psi(z_1, \dots, z_n) := \prod_{i=1}^n \rho(z_i)$ . Apply Theorem 20 to  $f\psi$ ,

$$f(0) = f(0)\psi(0) = C \int_{\Delta^n(1)} \frac{\partial^n (f\psi)}{\partial z^{[n]}} \frac{\mu_0}{z_1 \dots z_n}$$

By Leibnitz formula,  $\frac{\partial^n (f\psi)}{\partial z^{[n]}} = \sum_{I \subset [n]} \frac{\partial^{|I|} f}{\partial \bar{z}^I} \frac{\partial^{n-|I|} \psi}{\partial z^{[n] \setminus I}}$ .

(A). When  $I = \emptyset$ ,  $\frac{\partial^n \psi}{\partial z^{[n]}} = \prod_{i=1}^n \frac{\partial \rho}{\partial \bar{z}}(z_i)$  is non-zero only when  $|z_i| \geq 1/2$  for all  $1 \leq i \leq n$ . Thus  $|\frac{\partial^n \psi}{\partial z^{[n]}} \frac{1}{z_1 \dots z_n}| \leq C$  over whole  $\mathbb{C}^n$ . so

$$\left| \int_{\Delta^n(1)} f \frac{\partial^n \psi}{\partial z^{[n]}} \frac{\mu_0}{z_1 \dots z_n} \right| = \left| \int_{1/2 \leq |z_i| < 1, \forall i} f \frac{\partial^n \psi}{\partial z^{[n]}} \frac{\mu_0}{z_1 \dots z_n} \right| \leq C |f|_{L^1(\Delta^n(1), \mu_0)}.$$

(B). When  $I$  is nonempty,  $|\frac{\partial^{n-|I|}\psi}{\partial\bar{z}^{[n]\setminus I}}| \leq C$  over  $\mathbb{C}^n$ , so

$$\begin{aligned} & \left| \int_{\Delta^n(1)} \frac{\partial^{|I|}f}{\partial\bar{z}^I} \frac{\partial^{n-|I|}\psi}{\partial\bar{z}^{[n]\setminus I}} \frac{\mu_0}{z_1 \dots z_n} \right| \\ & \leq C \sup_{z \in \Delta^n(1)} \left| \frac{\partial^{|I|}f}{\partial\bar{z}^I} \right| \int_{\Delta^n(1)} \frac{\mu_0}{|z_1 \dots z_n|} \\ & = C \sup_{z \in \Delta^n(1)} \left| \frac{\partial^{|I|}f}{\partial\bar{z}^I} \right| \end{aligned} \quad (\text{A.4})$$

and we proved 1.

2. Let  $g(z) := f(rz)$  and apply 1. to  $g$ .  $\square$

**Remark 6.** *With similar proof but applying Theorem 21 instead of Theorem 20, we can get rid of higher derivatives. More precisely, there is  $C = C(n) > 0$  such that for any  $f \in C^\infty(\mathbb{C}^n)$  and  $r > 0$ ,*

$$\frac{|f(0)|}{C} \leq r^{-2n} |f|_{L^1(B(r), \mu_0)} + r \sup_{z \in B(r)} |\bar{\partial}f|.$$

For the sake of completeness, we state a generalization of [3, Theorem 2.1], which is used to control peak section constructed loc.cit. Note how we avoid this result in Section 3.2.2.

**Proposition 5.** *Let  $M^n$  be a compact hermitian manifold of volume form  $\mu$ ,  $L$  and  $E$  be hermitian holomorphic vector bundles with  $\text{rk}(L) = 1$ . Then there is  $C > 0$  such that for any  $s \in C^\infty(M, L^k \otimes E)$ , any  $p \in M$ , any  $(U, z_1, \dots, z_n)$  holomorphic chart centered at  $p$ , for large  $k \in \mathbb{N}^+$  we have*

$$\frac{1}{C} |s(p)| \leq k^n \|s\|_{L^1(\Delta^n(r_0), \mu)} + \sup_{z \in \Delta^n(r_0), I(\neq \emptyset) \subset [n]} r_0^{|I|} \left| \frac{\partial^{|I|}s}{\partial\bar{z}^I} \right|, \quad (\text{A.5})$$

where  $r_0 = 1/\sqrt{k}$ .

*Proof.* By Proposition 3, we can choose a frame  $\sigma$  for  $L$  in a neighborhood of  $p \in M$  such that  $\varphi(z) := -\log |\sigma|_{h^L}^2 = O(|z|^2)$ , and  $\{v_1, \dots, v_r\}$  as in the Proposition. Expand  $s = \sum_{\alpha=1}^r f_\alpha(\sigma^k \otimes v_\alpha)$ , where  $f_\alpha \in C^\infty$  near  $p$ .

$$|s(z)|_h^2 = e^{-k\varphi} (f_1, \dots, f_r) H^E (f_1, \dots, f_r)^H.$$

In particular,

$$|s(p)| = \left( \sum_{\alpha} |f_\alpha(p)|^2 \right)^{1/2} \leq \sum_{\alpha} |f_\alpha(p)|. \quad (\text{A.6})$$

By Exercise 13, (2).

$$\frac{1}{C} |f_\alpha(p)| \leq r_0^{-2n} |f_\alpha|_{L^1(\Delta^n(r_0), \mu_0)} + \sup_{z \in \Delta^n(r_0), I \neq \emptyset} r_0^{|I|} \left| \frac{\partial^{|I|}f_\alpha}{\partial\bar{z}^I}(z) \right|$$

There is  $A > 0$  such that in  $\Delta^n(r_0)$ ,  $|\varphi(z) - \varphi(0)| \leq 2Ar_0^2$ , so

$$e^{-k\varphi(z)} \geq e^{-2Akr_0^2} = e^{-2A}$$

There is  $a > 0$  such that  $H^E \geq a^2 Id_r$  and  $\mu \geq a\mu_0$  on  $\Delta^n(r_0)$ .

$$\begin{aligned} \|s\|_{L^1(\Delta^n(r_0), \mu)} &= \int_{\Delta^n(r_0)} |s| \mu \\ &= \int_{\Delta^n(r_0)} e^{-k\varphi/2} [(f_1, \dots, f_r) H^E (f_1, \dots, f_r)^H]^{1/2} \mu \\ &\geq \int_{\Delta^n(r_0)} e^{-A} a \left( \sum_{\alpha=1}^r |f_\alpha(z)|^2 \right)^{1/2} a \mu_0 \\ &\geq e^{-A} a^2 \int_{\Delta^n(r_0)} r^{-1/2} \sum_{\alpha} |f_\alpha| \mu_0 \\ &= e^{-A} a^2 r^{-1/2} \sum_{\alpha} \|f_\alpha\|_{L^1(\Delta^n(r_0), \mu_0)}, \end{aligned} \tag{A.7}$$

where the last inequality is Cauchy-Schwarz.

Note

$$\frac{\partial^{|I|} s}{\partial \bar{z}^I} = \sum_{\alpha} \frac{\partial^{|I|} f_{\alpha}}{\partial \bar{z}^I} (\sigma^k \otimes v_{\alpha}),$$

so

$$\begin{aligned} \left| \frac{\partial^{|I|} s}{\partial \bar{z}^I} \right| &= e^{-k\varphi/2} \left[ \left( \frac{\partial^{|I|} f_1}{\partial \bar{z}^I}, \dots, \frac{\partial^{|I|} f_r}{\partial \bar{z}^I} \right) H^E \left( \frac{\partial^{|I|} f_1}{\partial \bar{z}^I}, \dots, \frac{\partial^{|I|} f_r}{\partial \bar{z}^I} \right)^H \right]^{1/2} \\ &\geq e^{-A} a \left( \sum_{\alpha} \left| \frac{\partial^{|I|} f_{\alpha}}{\partial \bar{z}^I} \right|^2 \right)^{1/2} \\ &\geq e^{-A} a \left| \frac{\partial^{|I|} f_{\alpha_0}}{\partial \bar{z}^I} \right|, \end{aligned}$$

for any  $1 \leq \alpha_0 \leq r$ . Thus

$$\sup_z \left| \frac{\partial^{|I|} s}{\partial \bar{z}^I} \right| \geq e^{-A} a \sup_z \left| \frac{\partial^{|I|} f_{\alpha_0}}{\partial \bar{z}^I} \right|.$$

Taking sum we get

$$\sup_z \left| \frac{\partial^{|I|} s}{\partial \bar{z}^I} \right| \geq r^{-1} e^{-A} a \sum_{\alpha} \sup_z \left| \frac{\partial^{|I|} f_{\alpha}}{\partial \bar{z}^I} \right|. \tag{A.8}$$

Now

$$\begin{aligned}
\frac{1}{C}|s(p)| &\leq \sum_{\alpha} \frac{1}{C}|f_{\alpha}(p)| \\
&\leq \sum_{\alpha} (r_0^{-2n}|f_{\alpha}|_{L^1(\Delta^n(r_0),\mu_0)} + \sup_{z \in \Delta^n(r_0), I \neq \emptyset} r_0^{|I|} |\frac{\partial^{|I|} f_{\alpha}}{\partial \bar{z}^I}(z)|), \\
&\leq k^n \|s\|_{L^1(\Delta^n(r_0),\mu)} + \sup_{z \in \Delta^n(r_0), I(\neq \emptyset) \subset [n]} r_0^{|I|} |\frac{\partial^{|I|} s}{\partial \bar{z}^I}|
\end{aligned}$$

where we used (A.6),(A.7),(A.8). □

**Remark 7.** *By Remark 6, we can show*

$$\frac{|s(p)|}{C} \leq k^n \|s\|_{L^1(\Delta^n(r_0),\mu)} + k^{-1/2} \sup_{z \in \Delta^n(r_0)} |\bar{\partial} s|.$$

*By Cauchy-Schwarz inequality*

$$\|s\|_{L^1(\Delta^n(r_0),\mu)} \leq \text{vol}(\Delta^n(r_0),\mu)^{1/2} \|s\|_{L^2(\Delta^n(r_0),\mu)} \leq Ck^{-n/2} \|s\|_{L^2(\Delta^n(r_0),\mu)}.$$

*Therefore, we can deduce*

$$\frac{|s(p)|}{C} \leq k^{n/2} \|s\|_{L^2(\Delta^n(r_0),\mu)} + \sup_{z \in \Delta^n(r_0), I(\neq \emptyset) \subset [n]} r_0^{|I|} |\frac{\partial^{|I|} s}{\partial \bar{z}^I}|$$

*from (A.5).*

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