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Fixed-Point Theorems for Discontinuous Mappings on Topological Vector Spaces

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**Fixed-Point Theorems for Discontinuous Mappings
on Topological Vector Spaces**

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Contents

1	Introduction	3
2	Preliminaries	3
2.1	Topology	3
2.2	Convex analysis	6
2.3	Topological vector spaces	8
2.4	Topologies on dual	11
2.5	Correspondences	12
3	Main Results	13
4	Fixed-Point of Discontinuous Mappings	21
5	Generalized half-continuity	28
6	Generalization of Herings et al. theorem	31
7	Existence of approximate fixed-points	32
8	Applications to game theory	35

1 Introduction

This is the Master 2 thesis of the author, completed under the guidance of Professor Philippe Bich.

In this master thesis, we present, and sometimes generalize, some recent fixed-point theorems. Most of them are generalizations of Brouwer fixed-point theorem allowing discontinuities. Many existence results in economic (e.g. general equilibrium) or in game theory (e.g. Nash equilibrium) rely upon fixed-point theorems. Allowing discontinuities is important since, for example, many payoff functions are discontinuous in game theory.

We establish an existence theorem of Yannelis-Prabhakar type (Theorem 22) for maximal element. From this result, we can easily derive several existing theorems in the literature. Inspired by [28] and [5], we introduce a new relaxed definition of continuity (Definition 17) which guarantees the existence of a fixed-point on Hausdorff linear spaces (Theorem 29). In Section 6, we generalize a theorem of Herings et al. [25]: the authors proved a fixed-point theorem for certain discontinuous self mappings on polytopes in \mathbb{R}^n . Then [6] generalized it to compact convex subsets of \mathbb{R}^n , and we extend it further to arbitrary inner product space (Theorem 38). In Section 7 we discuss approximate fixed-points. And we end with some applications to game theory.

2 Preliminaries

In this chapter, we recall some definitions and results useful for the subsequent sections. By vector space we mean a real one. In a vector space, we write $\text{co}(C)$ for the convex hull of a subset C , and $\text{ri}(A)$ for the relative interior of a convex subset A . When X is a topological space and $f : X \rightarrow \mathbb{R}$ is a function, we denote by $\text{Supp}(f)$ the closure of $\{x \in X : f(x) \neq 0\}$.

2.1 Topology

Definition 1. *Let X be a topological space. We call X Hausdorff if for every distinct points $a, b \in X$, there exist two disjoint open sets $U, V \subset X$ such that $a \in U, b \in V$. We call X normal if for every disjoint closed subsets $A, B \subset X$, there exist two disjoint open sets $U, V \subset X$ such that $A \subset U, B \subset V$.*

Lemma 1 (Urysohn's Lemma). *A topological space X is normal if and only if for every two disjoint closed subsets $A, B \subset X$, there exists a continuous function $f : X \rightarrow [0, 1]$ such that $A \subset f^{-1}(0), B \subset f^{-1}(1)$.*

Theorem 1 (Tietze Extension Theorem). *A topological space X is normal if and only if for every closed set $A \subset X$ and every continuous function $f : A \rightarrow [0, 1]$, there exists a continuous function $\bar{f} : X \rightarrow [0, 1]$ such that $\bar{f}|_A = f$.*

Definition 2 (Paracompact). *A topological space X is called paracompact if every open cover $\{U_\alpha\}_{\alpha \in A}$ of X admits a locally finite refinement, that is an open cover $\{V_\beta\}_{\beta \in B}$ such that*

- *refinement* : for every $\beta \in B$, there is $\alpha \in A$ such that $V_\beta \subset U_\alpha$;
- *locally finite* : for every $x \in X$, there is an open neighborhood $W \subset X$ of x such that W intersects with only finitely many V_β .

Every compact space is paracompact.

Theorem 2 (Stone). *A metric space is paracompact.*

Theorem 3 (Jean Dieudonné [17]). *A Hausdorff paracompact space is normal.*

Lemma 2 (Partition of Unity). *Let X be a Hausdorff paracompact space with an open cover $\{U_\alpha\}_{\alpha \in A}$, then there exists a collection of continuous functions $\{f_\alpha : X \rightarrow [0, 1]\}_{\alpha \in A}$ such that:*

- for every α , we have $\text{Supp}(f_\alpha) \subset U_\alpha$;
- for every $x \in X$, there is an open neighborhood $V \subset X$ of x such that the set $\{\alpha \in A : V \cap \text{Supp}(f_\alpha) \neq \emptyset\}$ is finite;
- $\sum_{\alpha \in A} f_\alpha = 1$ on X . (By the previous point, the sum is finite everywhere and hence well-defined.)

Lemma 3 (Partition of Unity). *Let X be a normal topological space with a finite open cover $\{U_i\}_{i=1}^n$. Then there exist continuous functions $h_1, \dots, h_n : X \rightarrow [0, 1]$ with the following properties:*

- $\sum_{i=1}^n h_i = 1$ on X ;
- for each $1 \leq i \leq n$, $\text{Supp}(h_i) \subset U_i$.

Although this result is classical, we recall the well-known proof for the convenience of readers. Below are two lemmas needed.

Lemma 4. *If X is a normal topological space with an open cover $\{U_1, U_2\}$, then there is a closed cover $\{F_1, F_2\}$ of X with $F_1 \subset U_1, F_2 \subset U_2$.*

Proof. In X , the two subsets U_1^c and U_2^c are disjoint and closed. By normality, there exist two disjoint open subsets V_1 and V_2 of X with $U_i^c \subset V_i$ for $i = 1, 2$. Now take $F_i = V_i^c$. For each i , F_i is closed in X and contained in U_i . As V_1 and V_2 are disjoint, we have $X = F_1 \cup F_2$. \square

Lemma 5 (bump function). *Let X be a normal topological space, $F \subset X$ closed and $U \subset X$ open. If $F \subset U$, then there is a continuous function $h : X \rightarrow [0, 1]$ such that $h = 1$ on F and $\text{Supp}(h) \subset U$.*

Proof. In X , the two subsets F and U^c are disjoint and closed. By normality, there exist two disjoint open subsets V and W of X with $F \subset V$ and $U^c \subset W$. Applying Lemma 4 to the disjoint closed subsets F and W^c , we find a continuous function $h : X \rightarrow [0, 1]$ such that $h = 1$ on F and $h = 0$ on W^c . The latter shows $\{h \neq 0\} \subset V \subset W^c$, so $\text{Supp}(h) \subset W^c \subset U$. \square

Then we prove Lemma 3.

Proof. We prove it by induction on n . The case $n = 1$ being trivial, we start with $n = 2$. We take a closed cover F_1, F_2 given by Lemma 4 and take a function $h_1 : X \rightarrow [0, 1]$ by applying Lemma 5 to $F_1 \subset U_1$, so $h_1 = 1$ on F_1 and $\text{Supp}(h_1) \subset U_1$. Then we define $h_2 = 1 - h_1$. Since $\{h_2 \neq 0\} = \{h_1 \neq 1\} \subset F_1^c \subset F_2$, we have $\text{Supp}(h_2) \subset F_2 \subset U_2$.

Now suppose that the case with at most n ($n \geq 2$) open subsets is proved, we proceed to the case with $n + 1$ open subsets U_1, \dots, U_{n+1} . By inductive hypothesis, for the cover with two open subsets $V = \cup_{i=1}^n U_i$ and U_{n+1} , there exist two continuous functions $f, h_{n+1} : X \rightarrow [0, 1]$ with $f + h_{n+1} = 1$ on X such that $C := \text{Supp}(f)$ is contained in V and $\text{Supp}(h_{n+1}) \subset U_{n+1}$. As a closed subset of a normal space, C is normal with a cover by the n open subsets $C \cap U_1, \dots, C \cap U_n$. By induction hypothesis again, there exist continuous functions $g_1, \dots, g_n : C \rightarrow [0, 1]$ with

$$\sum_{i=1}^n g_i = 1 \quad (1)$$

on C and $\text{Supp}_C(g_i) \subset C \cap U_i$ for each $1 \leq i \leq n$. By Tietze extension theorem, each g_i has a continuous extension $\bar{g}_i : X \rightarrow [0, 1]$. Define continuous functions $h_i = f\bar{g}_i : X \rightarrow [0, 1]$ for $1 \leq i \leq n$. By (1) and that $f|_{C^c} = 0$, we have $\sum_{i=1}^n h_i = f$ on X and hence $\sum_{i=1}^{n+1} h_i = 1$. For $i \leq n$, we have

$$\{h_i \neq 0\} = \{f \neq 0\} \cap \{\bar{g}_i \neq 0\} \subset C \cap \{\bar{g}_i \neq 0\} \subset \text{Supp}_C(g_i).$$

So, $\text{Supp}(h_i) \subset \text{Supp}_C(g_i) \subset U_i$. Thus, the case with $n + 1$ open subsets is proved. The induction is completed. \square

We state a corollary that we use repeatedly.

Corollary 1. [23, Exercise 5, Ch.6, p.81][37, Theorem 2.13] *If X is compact Hausdorff, then the result of Lemma 3 still holds.*

Proof. In fact, by Theorem 3, a compact Hausdorff space is normal. Then we apply Lemma 3. \square

Corollary 2 (Shrinking Lemma). *If X is compact Hausdorff with a finite open cover U_1, \dots, U_n , then there exists a finite closed cover F_1, \dots, F_n with $F_i \subset U_i$ for each i .*

Proof. Take h_i given by Corollary 1. Then each $F_i := \text{Supp}(h_i)$ is contained in U_i . Since $\sum_i h_i = 1$, we have $X = \cup_i F_i$. \square

Lemma 6 (Tube Lemma). *Suppose X, Y are two topological spaces, $A \subset X$ and $B \subset Y$ are compact subsets. If N is an open subset of $X \times Y$ containing $A \times B$, then there exist an open subset U of X containing A and an open subset V of Y containing B such that $U \times V \subset N$.*

Definition 3. A topological space X is called separable if there exists a countable subset $A \subset X$ such that $\bar{A} = X$.

Lemma 7. Every compact metric space is separable. A subspace of a separable metric space is separable.

Then we recall Baire's category theorem.

Definition 4. In a topological space X , a subset N is called nowhere dense if its closure \bar{N} has an empty interior. A countable union of nowhere dense subsets of X is called meagre (also called first category) in X .

Exercise 1. Let X be a topological space, $S \subset X$ a subset, then S is nowhere dense in X if and only if \bar{S}^c is dense in X .

Proof. Indeed, S is nowhere dense \iff Interior $\bar{S} = \emptyset \iff$ (Interior \bar{S}) $^c = X \iff \bar{S}^c = X \iff \bar{S}^c$ is dense of X . \square

Theorem 4 (Baire's Category Theorem). If A is a meagre subset of a Hausdorff compact space X , then A^c is dense in X .

Exercise 2. Let X be a topological space, then the following properties are equivalent:

- For every sequence of dense open subsets U_1, U_2, \dots in X , the intersection $\bigcap_{i \geq 1} U_i$ is dense in X .
- If $A \subset X$ is meagre, then A^c is dense in X .

Proof. Assume the first. Since A is meagre, then $A = \bigcup_{i \geq 1} F_i$, where each F_i is nowhere dense in X . By Exercise 1, each \bar{F}_i^c is open and dense in X . So $\bigcap_{i \geq 1} \bar{F}_i^c$ is contained in A^c and dense by assumption. Thus A^c is dense in X .

Assume the second. Let $A = \bigcup_{i \geq 1} U_i^c$. By Exercise 1, each U_i^c is nowhere dense in X , so A is meagre in X . By assumption, $\bigcap_i U_i = A^c$ is dense in X . \square

2.2 Convex analysis

Below, we remind the readers about two classical concepts: quasiconvexity and strict quasiconvexity. At the same time, we introduce a new one, which we call strong quasiconvexity serving as an intermediate between the two.

Definition 5 (Quasiconvex Function). Let S be a convex subset of a vector space and $f : S \rightarrow \mathbb{R}$ a function. The function f is called quasiconvex if for every $x, y \in S$, every $\lambda \in (0, 1)$, we have

$$f(\lambda x + (1 - \lambda)y) \leq \max(f(x), f(y)).$$

The function f is called strongly quasiconvex if it is quasiconvex and for every $x, y \in S$ with $f(x) \neq f(y)$, every $\lambda \in (0, 1)$, we have

$$f(\lambda x + (1 - \lambda)y) < \max(f(x), f(y)).$$

The function f is called strictly quasiconvex if for every $x, y \in S$ with $x \neq y$, every $\lambda \in (0, 1)$, we have

$$f(\lambda x + (1 - \lambda)y) < \max(f(x), f(y)).$$

Note that strictly quasiconvex functions are strongly quasiconvex, and strongly quasiconvex functions are quasiconvex. If f is quasiconvex, then for each $a \in \mathbb{R}$, the sublevel set $\{x : f(x) < a\}$ is convex.

For instance, convex functions, hence linear functions are strongly quasiconvex. Constant functions are strongly quasiconvex but not strictly quasiconvex. Strictly convex functions are strictly quasiconvex. We can similarly define quasiconcave functions used in Section 8.

Exercise 3. If $f, g : S \rightarrow \mathbb{R}$ are two quasiconvex (resp. strongly quasiconvex) functions, then $h = \max(f, g)$ is also quasiconvex (resp. strongly quasiconvex).

Proof. Fix $x, y \in S$, $\lambda \in (0, 1)$ and put $z = \lambda x + (1 - \lambda)y$.

Case 1: If f, g are quasiconvex: then

$$f(z) \leq \max(f(x), f(y)) \leq \max(h(x), h(y)).$$

Likewise, we have $g(z) \leq \max(h(x), h(y))$, so $h(z) \leq \max(h(x), h(y))$. Thus h is quasiconvex.

Case 2: If f, g are strongly quasiconvex: we suppose $h(x) > h(y)$. By symmetry, we may assume $f(x) \geq g(x)$, then $h(x) = f(x) > h(y) \geq f(y)$. As f is strongly quasiconvex, we have $f(z) < f(x) = h(x)$.

If $g(x) = g(y)$, then $g(z) \leq \max(g(x), g(y)) \leq h(y) < h(x)$; if $g(x) \neq g(y)$, then $g(z) < \max(g(x), g(y)) \leq \max(h(x), h(y)) = h(x)$. In both cases, $g(z) < h(x)$ and hence $h(z) < h(x)$. Thus h is strongly quasiconvex. \square

Theorem 5 (Carathéodory). Let A be a subset of \mathbb{R}^n . Fix $a \in A$ and $x \in \text{co}(A)$, then there is a subset $B \subset A$ with $\#B \leq n + 1$ and $a \in B$ such that $x \in \text{co}(B)$.

Corollary 3. If $A \subset \mathbb{R}^n$ is a compact subset, then $\text{co}(A)$ is compact.

Proof. Define $\Delta = \{(\lambda_1, \dots, \lambda_{n+1}) \mid \lambda_i \geq 0, \sum_i \lambda_i = 1\}$, then Δ is a compact subset of \mathbb{R}^{n+1} . Define $\phi : A^{n+1} \times \Delta \rightarrow \text{co}(A)$ by $((a_1, \dots, a_{n+1}), (\lambda_1, \dots, \lambda_{n+1})) \mapsto \sum_i \lambda_i a_i$, then ϕ is continuous. By Theorem 5, ϕ is surjective, so $\text{co}(A)$ is compact. \square

Definition 6 (Inward/Outward Set). Let C be a subset of a vector space E . For $x \in E$, the inward set of C at x is $I_C(x) = \cup_{\lambda > 0} \lambda C + (1 - \lambda)x$. Dually, the outward set of C at x is $O_C(x) = \cup_{\lambda < 0} \lambda C + (1 - \lambda)x$.

Note that $C \subset I_C(x)$ and if $x \in \text{Interior}C$, then $I_C(x) = E$.

Exercise 4. If C is convex in a vector space E and $x \in E$, then $I_C(x)$ is also convex.

Proof. For every $z_1, z_2 \in I_C(x)$, we can write

$$\begin{aligned} z_1 &= \lambda_1 c_1 + (1 - \lambda_1)x; \\ z_2 &= \lambda_2 c_2 + (1 - \lambda_2)x \end{aligned}$$

for some $c_1, c_2 \in C$, $\lambda_1, \lambda_2 > 0$. For every $\rho \in (0, 1)$, we put $\mu := \rho\lambda_1 + (1 - \rho)\lambda_2 > 0$. Since C is convex, $c := \frac{\rho\lambda_1}{\mu}c_1 + \frac{(1-\rho)\lambda_2}{\mu}c_2 \in C$. Therefore, we have $\rho z_1 + (1 - \rho)z_2 = \mu c + (1 - \mu)x \in I_C(x)$. This completes the proof. \square

2.3 Topological vector spaces

Definition 7 (Inner Product Space). *An inner product space is a vector space E together with an inner product, that is a function $\langle \cdot, \cdot \rangle : E \times E \rightarrow \mathbb{R}$ satisfying the following properties : for every $x, y, z \in E$ and $a, b \in \mathbb{R}$, we have*

- $\langle x, y \rangle = \langle y, x \rangle$;
- $\langle ax + by, z \rangle = a\langle x, z \rangle + b\langle y, z \rangle$;
- if $x \neq 0$, then $\langle x, x \rangle > 0$.

A complete inner product space is called a Hilbert space.

Definition 8 (Normed Space). *On a vector space E , a seminorm is a function $\|\cdot\| : E \rightarrow [0, +\infty)$ which satisfies*

- For every $\lambda \in \mathbb{R}, v \in E$, we have $\|\lambda v\| = |\lambda| \cdot \|v\|$;
- For every $u, v \in E$, we have $\|u + v\| \leq \|u\| + \|v\|$.

A seminorm $\|\cdot\|$ is called a norm if for every $v \in E$ with $\|v\| = 0$, we have $v = 0$. In that case, the pair $(E, \|\cdot\|)$ is called a normed space.

An inner product space has a natural induced norm. A complete normed space is called a Banach space.

Definition 9 (TVS). [38, Section 1.6] *Suppose τ is a topology on a vector space E such that*

- $\{0\}$ is a closed set;
- the addition $E \times E \rightarrow E$ and the scalar multiplication $\mathbb{R} \times E \rightarrow E$ are continuous.

Then we call (E, τ) a topological vector space (henceforth TVS).

For example, a normed space $(E, \|\cdot\|)$ with the topology induced by the norm is a TVS.

We recall some basic facts about TVS.

Theorem 6. [38, Theorem 1.12] *Every TVS is Hausdorff.*

Theorem 7. [12, Ch.IV, §1, Exercise 18] Let (E, τ) be a finite dimensional TVS, then τ is the standard Euclidean topology.

Theorem 8. [38, Theorem 1.42] Let E' be a finite dimensional subspace of a TVS E , then $E' \subset E$ is closed.

Theorem 9. [38, Theorem 1.24] If (E, τ) is a first-countable TVS, then there is a metric d on E such that d is compatible with the topology τ and d is invariant: $d(x, y) = d(x - y, 0)$ for every $x, y \in X$.

Lemma 8 (Minkowski functional). If V is a convex neighborhood of 0 in a TVS E and $q : E \rightarrow [0, +\infty)$ is the associated Minkowski functional:

$$q(x) = \inf\{t > 0 : x \in tV\}.$$

Then q is a continuous, subadditive, positively homogeneous and convex function. If V is also closed, then $\{x \in E : q(x) \leq 1\} = V$ and for every $x \notin V$, we have $\frac{x}{q(x)} \in \partial V$.

Definition 10 (Locally Convex). [38, Definition 1.8 (a)] A TVS E is called locally convex if for every open neighborhood $U \subset E$ of 0, there is a convex open neighborhood $V \subset U$ of 0.

For a TVS E , we write E^* for its (topological) dual, that is the vector space of all continuous linear functionals on E . If for every $v(\neq 0) \in E$, there is $p \in E^*$ with $(p, v) := p(v) \neq 0$, then we say that E^* separates E .

Definition 11. A subset $B \subset E$ is called (τ) -bounded if for every neighborhood V of 0 in E , there is $s > 0$ such that $B \subset tV$ for every $t > s$.

Theorem 10. [38, Theorem 1.15 (b)] In a TVS, every compact subset is bounded.

Definition 12 (Weak Topology). Given a TVS (E, τ) , let τ_w be the weakest topology on E such that for every $p \in E^*$, $p : (E, \tau_w) \rightarrow \mathbb{R}$ is continuous. This τ_w is called the weak topology of (E, τ) . (The adjective "weak" is justified since $\tau_w \subset \tau$.)

Self-explanatory expressions such as weakly bounded, original closure, etc., will be used to make it clear with respect to which topology these terms are to be understood.

Theorem 11. Assume that E^* separates points on a TVS (E, τ) , then (E, τ_w) is a locally convex TVS whose dual is still E^* . If $K \subset E$ is τ -compact, then the two topologies τ, τ_w coincide on K .

Proof. The first part is proved in [38, Theorem 3.10], so we only prove the second part. Since the identity mapping $\iota : (K, \tau) \rightarrow (K, \tau_w)$ is a continuous bijection from a compact to a Hausdorff space, it is a homeomorphism. \square

Theorem 12. [20, Theorem 8.2.1, 8.2.2] *Let E be a locally convex TVS with a subset $A \subset E$. Then A is bounded if and only if it is weakly bounded. If A is convex, then A is closed if and only if it is weakly closed.*

Theorem 13 (Hahn-Banach). [38, Theorem 3.4, Theorem 3.21] *Suppose A, B are disjoint nonempty convex sets in a TVS E .*

- *If A is open, then there exist $p \in E^*$ and $r \in \mathbb{R}$ such that $(p, a) < r \leq (p, b)$ for every $a \in A, b \in B$.*
- *If E is locally convex, A is compact, B is closed, then there exists $p \in E^*, r_1, r_2 \in \mathbb{R}$ such that $(p, a) < r_1 < r_2 < (p, b)$ for every $a \in A, b \in B$.*
- *If E^* separates points on E and A is weakly compact, B is weakly closed, then there exists $p \in E^*, r_1, r_2 \in \mathbb{R}$ such that $(p, a) < r_1 < r_2 < (p, b)$ for every $a \in A, b \in B$.*

Corollary 4. [38, Corollary of Theorem 3.4] *If a TVS E is locally convex, then E^* separates E .*

Remark 1. *As shown in [38, Exercise 5 (d), Ch.3], there exists a TVS which is not locally convex, but whose dual separates points.*

Exercise 5. *Let E be a TVS such that for every $x \neq 0$, there is a continuous quasiconvex function $p : E \rightarrow \mathbb{R}$ such that $p(0) < p(x)$, then E^* separates E .*

Proof. Fix $x \in E, x \neq 0$. By assumption, there is a continuous quasiconvex function $p : E \rightarrow \mathbb{R}$ with $p(0) < p(x)$. Then $A := \{v \in E : p(v) < p(x)\}$ is an open convex subset. Applying Theorem 13 to x and A we get $q \in E^*$ such that $q(a) < q(x)$ for every $a \in A$. Since $0 \in A$, we have $q(0) < q(x)$. This completes the proof. \square

Exercise 6. *Let E be a TVS, $A \subset C \subset E$ be subsets where $A = \{c_1, c_2, \dots\}$ is countable and dense in C , then $\text{Span}(C)$ is separable with a countable dense subset $S = \text{Span}_{\mathbb{Q}}(A)$.*

Proof. We write $S = \cup_{A' \subset A} \text{Span}_{\mathbb{Q}}(A')$, where A' runs through all finite subset of A . Each $\text{Span}_{\mathbb{Q}}(A')$ is countable and A has countably many finite subsets. Therefore S is indeed countable.

We now prove that $\text{Span}_{\mathbb{Q}}(C)$ is in the closure of S inside $\text{Span}(C)$. In fact, for every $x \in \text{Span}_{\mathbb{Q}}(C)$, there is an expression $x = \sum_{i=1}^N b_i x_i$, where $N \in \mathbb{N}^+, b_i \in \mathbb{Q}, x_i \in C$. For every $U \subset \text{Span}(C)$ open neighborhood of 0, there exists an open subset $V \subset E$ such that $V \cap \text{Span}(C) = U$ and $V_0 \subset V$ open such that $\sum_{i=1}^N b_i V_0 \subset V$. For each i , since $A \subset C$ is dense, there exists an index $j(i)$ such that $c_{j(i)} \in x_i + V_0$. Consider $x' = \sum_{i=1}^N b_i c_{j(i)} \in S$, then $x' - x \in V \cap \text{Span}(C) = U$.

But $\text{Span}_{\mathbb{Q}}(C)$ is dense in $\text{Span}(C)$, so S is also dense in $\text{Span}(C)$. \square

The following is taken from [1, Theorem 1]. We reproduce their proof here for the sake of completeness.

Proposition 1. *Let E be a first-countable locally convex TVS, $C \subset E$ be a weakly compact subset. If a mapping $f : C \rightarrow E$ is weakly sequentially continuous, then f is weakly continuous.*

Proof. By Theorem 9, E is metrizable. For every weakly closed subset $F \subset E$, $f^{-1}(F)$ is weakly relative compact. For every x_0 in the weak closure of $f^{-1}(F)$ inside E , by Eberlein–Šmulyan theorem [20, Theorem 8.12.4 (b)], there is a sequence $\{x_n\}$ in $f^{-1}(F)$ that converges weakly to x_0 . As f is weakly sequentially continuous, we have $f(x_n) \rightarrow f(x_0)$ weakly, so $f(x_0) \in F$, i.e., $x_0 \in f^{-1}(F)$. We see that $f^{-1}(F)$ is weakly closed in C , from which we deduce that f is weakly continuous. \square

2.4 Topologies on dual

Let E be a TVS. For every $x \in E$, let \tilde{x} be the linear form $p \mapsto (p, x)$ on E^* .

Definition 13. *The weakest topology on E^* making each \tilde{x} continuous is called the weak*-topology.*

Theorem 14. [38, §3.14] *The weak*-topology makes E^* a locally convex TVS whose dual set is E .*

We introduce a second topology on the dual.

Definition 14 (Compact Convergence Topology). *For every $p \in E^*$, every nonempty compact subset $C \subset E$ and every $r > 0$, we put*

$$B(p, C, r) := \{q \in E^* : |(q, x) - (p, x)| < r, \forall x \in C\}.$$

The topology on E^ generated by the base $\{B(p, C, r)\}$ is called the compact convergence topology (or compact-open topology).*

Note that each $B(p, C, r)$ is convex, so E^* with compact convergence topology is a locally convex TVS.

Exercise 7. *If $C \subset E$ is compact and E^* is endowed with compact convergence topology, then the canonical mapping $C \times E^* \rightarrow \mathbb{R}$ defined by $(x, p) \mapsto p(x)$ is continuous. In particular on E^* , the weak*-topology is weaker than the compact convergence topology.*

Proof. Fix $x \in C$, $p \in E^*$ and $\epsilon > 0$. Since $p : C \rightarrow \mathbb{R}$ is continuous, $U := p^{-1}(B(p(x), \epsilon/2)) \subset C$ is an open neighborhood of x . Moreover, $S = \{x' \in C : |p(x') - p(x)| \leq \epsilon/2\}$ is closed in C and hence compact. By definition, the set $B(p, S, \epsilon/2)$ is an open neighborhood of $p \in E^*$. For every $p' \in B(p, S, \epsilon/2)$, $x' \in U$, since $U \subset C$ we have

$$|p'(x') - p(x)| \leq |p'(x') - p(x')| + |p(x') - p(x)| < \epsilon/2 + \epsilon/2 = \epsilon.$$

This shows the joint continuity. \square

Now we introduce the strong topology on E^* . For each nonempty bounded subset $B \subset E$, each $p \in E^*$, the image $p(B)$ is bounded in \mathbb{R} . Define $q_B : E^* \rightarrow [0, +\infty)$ by $p \mapsto \sup_{x \in B} |(p, x)|$, then q_B is a seminorm on E^* . The family of seminorms $\{q_B\}$ is separating, that is to each $x(\neq 0) \in E$ corresponds at least one member q_B of this family with $q_B(x) \neq 0$. By [38, Theorem 1.37], this family induces a topology τ_p on E^* making (E^*, τ_p) a locally convex TVS. We call τ_p the *strong topology* on E^* and (E^*, τ_p) the strong dual of (E, τ) . For instance, if E is a normed (or seminormable) space, then τ_p is the natural norm topology on E^* .

One can see that the family $\{q_C\}$ where C runs through all nonempty compact subsets in E induces the compact convergence topology. The family $\{q_{\{x\}}\}_{x \in E}$ induces the weak*-topology. This point of view makes the following result obvious.

Exercise 8. *On E^* , the compact-convergence topology is weaker than the strong topology.*

Proof. Every nonempty compact subset $C \subset E$ is bounded by Theorem 10, then q_C is defined above and continuous about τ_p . Now for every $r > 0$,

$$B(0, C, r) = \{p \in E^* : \forall x \in C, |p(x)| < r\} = q_C^{-1}[0, r) \in \tau_p,$$

which allows us to conclude. □

Now we introduce reflexivity following [20, §8.4]. The dual of (E^*, τ_p) is called the bidual of E and is denoted by E^{**} . For each $x \in E$ we have $\tilde{x} \in E^{**}$.

Definition 15 (Semireflexive). *A locally convex TVS E is called semi-reflexive (resp. reflexive) if the canonical mapping $c_E : E \rightarrow E^{**}$ by $x \mapsto \tilde{x}$ is bijective (resp. is a TVS isomorphism).*

Theorem 15. [20, Theorem 8.4.2] *A locally convex TVS E is semireflexive if and only if each bounded weakly closed subset of E is weakly compact.*

2.5 Correspondences

Exercise 9. *Let S be a topological space with a finite closed cover C_1, \dots, C_n and V be a TVS. For each $1 \leq j \leq n$, take $v_j \in V$. Define a correspondence $F : S \rightarrow V$ by $F(x) = \text{co}(v_j | x \in C_j)$. Then F has a closed graph.*

This result is used implicitly in proof of [30, Lemma 7.1]. In [2, p.823], the authors need a similar result. However, they erroneously cite [14, Theorem 17.27] to show their Φ is upper hemicontinuous, which is not sufficient for that purpose.

Proof. In fact, take any net $(x_\alpha, z_\alpha) \rightarrow (x, z)$ in $S \times V$, where $z_\alpha \in F(x_\alpha)$ for each α . We want to show $z \in F(x)$.

We can write $z_\alpha = \sum_{i=1}^n a_\alpha^i v_i$, where each $a_\alpha^i \geq 0$, $\sum_{i=1}^n a_\alpha^i = 1$ and $a_\alpha^i = 0$ when $x_\alpha \notin C_i$.

Since $[0, 1]$ is compact, we may find a subnet (still denoted by the original one) such that for each i , $a_\alpha^i \rightarrow a^i \in [0, 1]$. Then $\sum_i a^i = 1$ and $z = \sum_i a^i v_i$. If j is an index such that $x \in C_j^c$, then there is α_0 such that for every $\alpha \geq \alpha_0$ we have $x_\alpha \in C_j^c$ and thus $a_\alpha^j = 0$. Taking limit, we get $a^j = 0$. Therefore, $z \in \text{co}(v_i | x \in C_i) = F(x)$.

We give another proof. Fix $x_0 \in S$, then $U = \bigcap_{i: C_i^c \ni x_0} C_i^c$ is an open neighborhood of x_0 and for every $x \in U$, we have $F(x) \subset F(x_0)$. Therefore, the correspondence F is upper semicontinuous with closed values, and thus has a closed graph. \square

Product of topological spaces are always endowed with the product topology.

Exercise 10. Let S be a topological space, $\{T_i\}_{i \in N}$ be a family of topological spaces. For each i , let $H_i : S \rightrightarrows T_i$ be a closed graph correspondence. Then $H : S \rightrightarrows \prod_i T_i$ defined by $H = \prod_i H_i$ has a closed graph.

We don't think the same holds for upper semicontinuity.

Proof. Take any net $(x(\alpha), y(\alpha))$ converging in $S \times \prod_i T_i$ to (x, y) with $y(\alpha) \in H(x(\alpha))$ for every α . For every $i \in N$, $y(\alpha)_i \in H_i(x(\alpha))$ and $(x, y(\alpha)_i) \rightarrow (x, y_i)$ in $S \times T_i$. Since H_i has a closed graph, we find $y_i \in H_i(x)$. Thus $y \in H(x)$. \square

Exercise 11. Let C be a compact subset of a TVS. Let S be a topological space, $f : S \rightarrow \mathbb{R}$ be a continuous function and $F : S \rightrightarrows C$ be a closed graph correspondence, then the correspondence $S \rightrightarrows C$ defined by $x \mapsto f(x)F(x)$ has a closed graph.

This result is contained in the proof of [3, Lemma A.2] with a minor problem. They write "and a fortiori $z^\alpha \rightarrow z$ with $y = cz$ " without justification.

Proof. Take any net (x_α, y_α) converging in $S \times C$ to (x, y) with $y_\alpha \in f(x_\alpha)F(x_\alpha)$ for each α . That means for each α , there is $z_\alpha \in F(x_\alpha)$ with $f(x_\alpha)z_\alpha = y_\alpha$. Then $x_\alpha \rightarrow x$ in S . Since f is continuous, we have $f(x_\alpha) \rightarrow f(x)$. Since C is compact, there is a subnet (still denoted by the original one) $z_\alpha \rightarrow z$ for some $z \in C$. Because F has a closed graph, we find $z \in F(x)$ and $y = f(x)z \in f(x)F(x)$. \square

3 Main Results

We fix notations. From now on, let (E, τ) be a TVS, $K \subset E$ be a nonempty τ -compact convex subset and $C \subset E$ be a nonempty subset. The weak topology of (E, τ) is denoted by τ_w . A correspondence $F : C \rightrightarrows E$ is called a convex-valued correspondence if for every $x \in C$, the set $F(x)$ is convex. An element $x_0 \in C$ is called a maximal element of F if $F(x_0)$ is empty. Let $\text{Fix}(F) := \{x \in C : x \in F(x)\}$ be the set of fixed-points of F and $M(F) := \{x \in C : F(x) = \emptyset\}$ be the set of maximal elements of F . For two correspondences $F, G : C \rightrightarrows E$, by $F \subset_C G$ we mean $F(x) \subset G(x)$ for every $x \in C$.

Theorem 16 (Schauder, Singbal). [7, Appendix] *If E is locally convex, A is a nonempty convex subset of E and B is a compact subset of A , then every continuous mapping $f : A \rightarrow B$ has a fixed-point.*

In [7], the above theorem requires A to be also closed, which is not used in the proof. To convince the readers, we repeat detailedly the argument.

Proof. Since B is compact and $f(A) \subset B$, we know that the closure $\overline{f(A)}$ of $f(A)$ in E is contained in B and compact.

For every convex open neighborhood V of $0 \in E$, the family $\{f(x)+V\}_{x \in A}$ is an open cover of $\overline{f(A)}$. In fact, for every $y \in \overline{f(A)}$, its open neighborhood $y - V$ contains $f(x)$ for some $x \in A$ and thus $y \in f(x)+V$. As $\overline{f(A)}$ is compact, we may find a finite subcover $y_1 + V, \dots, y_n + V$, where each $y_i \in \overline{f(A)}$. Take functions $h_1, \dots, h_n : \overline{f(A)} \rightarrow [0, 1]$ given by Corollary 1. We put $K_V := \text{co}(y_1, \dots, y_n)$, which is a polytope. Since each $y_i \in A$ and A is convex, we find $K_V \subset A$.

We define a mapping $f_V : A \rightarrow K_V$ by

$$f_V(x) = \sum_{i=1}^n h_i(f(x))y_i.$$

For every $x \in A$, every i with $h_i(f(x)) > 0$, we have $f(x) \in y_i + V$. Because V is convex, we find

$$f(x) - f_V(x) = \sum_{i=1}^n h_i(f(x))(f(x) - y_i) \in V, \forall x \in A. \quad (2)$$

As $f_V : K_V \rightarrow K_V$ is continuous, by Brouwer theorem there is $x_V \in K_V$ with $f_V(x_V) = x_V$. When V runs through all convex open neighborhoods of $0 \in E$, we get a net $\{f(x_V)\}$ in B . Since B is compact, we find a subnet (still denoted by the original one) $f(x_V) \rightarrow x_0$ for some $x_0 \in B$. By (2), we have $f(x_V) - x_V \in V$, so $x_V \rightarrow x_0$ in A . By continuity of f , we have $f(x_V) \rightarrow f(x_0)$. Since E is Hausdorff by Theorem 6, the two limits x_0 and $f(x_0)$ of the same net $\{f(x_V)\}$ coincide. Or equivalently, x_0 is a fixed-point of f . \square

We apply it to get a generalization of Rothe's fixed-point theorem [36, Fixpunktsatz, p.186], where the ambient space is a Banach space and C is the closed unit ball of E . Rothe proves it using degree argument.

Theorem 17 (Rothe). *Assume E is locally convex and $C \subset E$ is a convex closed subset with nonempty interior $\text{Interior}C$. Let $f : C \rightarrow E$ be a continuous mapping such that $f(\partial C) \subset C$ and $A := \overline{f(C)}$ is compact, then f has a fixed-point.*

Proof. By translation, we may assume that $0 \in \text{Interior}C$. By Theorem 10, there is $a > 1$ such that $A \subset aC$. Let $q : E \rightarrow [0, +\infty)$ be the Minkowski functional associated to C . Define $r : E \rightarrow C$ by

$$r(x) = \frac{x}{\max(1, q(x))}.$$

By Lemma 8, r is continuous, $r|_C = Id_C$ and for every $x \notin C$ we have $r(x) \in \partial C$. (The construction of r is inspired by [48, Lemma 2.4].)

Define $g : aC \rightarrow A$ by $g(x) = f \circ r(x)$. By Theorem 16, there is $x_0 \in aC$ such that $g(x_0) = x_0$. Suppose that $x_0 \notin C$, then $r(x_0) \in \partial C$ and $x_0 = g(x_0) = f \circ r(x_0) \in C$ (by assumption), a contradiction. Thus $x_0 \in C$. Then $f(x_0) = g(x_0) = x_0$, which shows that x_0 is a fixed-point of f . \square

Conjecture 1 (Schauder). [29, Problem 54] *A continuous mapping $f : K \rightarrow K$ admits fixed-point.*

R. Cauty in [11] proposed an answer to the Schauder's Conjecture. In the international conference of Fixed-Point Theory and its Applications in 2005, T. Dobrowolski remarked that there is a gap in the proof. After proving Theorem 17, we found a very similar result [26, Theroem 2] in the literature. However, his proof uses Cauty's doubtful result.

Another proof of Schauder's conjecture given in [21] relies on [48, Theorem 2.13], which seems questionable for the following reasons.

First of all, the first line of the proof of Theorem 2.13 given in [48] says that by their Lemma 2.3, we can assume 0 is in the relative interior of C . But in fact, their Lemma 2.3 is not sufficient for this purpose. For example, take $s = p = 1$ and $0 \neq x_0 \in C$ not in the relative interior of C in their Lemma 2.3, then 0 is not in the relative interior of their $D = C - x_0$.

Secondly, Lemma 2.12 also plays an important role in the proof of Theorem 2.13 given in [48] but it seems unreliable. In fact, they said "Without loss of generality we suppose that $\{x_n\}_{n=1}^{\infty}$ is linearly independent." However, consider an one dimensional example: $s = p = 1$, $X = \mathbb{R}^1$, $D = [-1, 1]$, then there are at most one vector linearly independent in \mathbb{R}^1 , say x_1 . But then $D \subset \overline{\text{co}}(0, x_1)$ is impossible! The existence of their M_n^* is also doubtful. Moreover, as $X_0 \subset X$ is not closed in general, $D \cap X_0$ need not to be compact and their F_0^{-1} may not be continuous on $F_0(D \cap X_0)$.

Besides, a result similar to [48, Theorem 2.13] is [4, Theorem 34, p.64], but unfortunately its proof seems false, too. In fact, the $d(x, \lambda f(x))$ on the last line of the proof can be 0, making his V empty!

The following variant of [47, Theorem 1] can be proved using the same (classical) argument: gluing local sections to be a global one via Lemma 2. It also generalizes [50, Theorem 3.1] where they assume existence of local constant selections.

Theorem 18 (Selection). *Let X be a Hausdorff paracompact space and Y a subset of a TVS E . Suppose that $F : X \rightrightarrows E$ is a convex valued correspondence such that for every $x \in X$, there exist an open neighborhood $U \subset X$ of x and a continuous (local) selection $s : U \rightarrow Y$ of F . If either Y is convex or F takes value in Y , then F has a continuous (global) selection $f : X \rightarrow Y$.*

Theorem 19 (Max-Fix Alternative). *Assume Schauder's conjecture for E . Let $F : K \rightrightarrows E$ be a convex-valued correspondence. If for every $x \in K \setminus (\text{Fix}(F) \cup M(F))$, there exist $U \subset K$ open neighborhood of x , and a continuous*

mapping $s : U \rightarrow K$ such that for every $x' \in U \setminus (\text{Fix}(F) \cup M(F))$ we have $s(x') \in F(x')$, then F has a maximal element or a fixed-point.

For example, when E^* separates E , then Schauder's conjecture is known for E and hence Theorem 19 contains [46, Corollary 3.2] as a corollary.

Proof. Assume the opposite. By Theorem 18, there exists a continuous selection $f : K \rightarrow K$ of F . If Schauder's conjecture is true, then f and hence F admits a fixed-point, a contradiction! □

For readers who prefer to avoiding Schauder's conjecture, we list a weaker version of last result whose proof is also a direct adaption. In the sequel, **we will only use this weaker version.**

Theorem 20. *Let $F : K \rightrightarrows E$ be a convex-valued correspondence. If for every $x \in K \setminus (\text{Fix}(F) \cup M(F))$, there exist $U \subset K$ an open neighborhood of x and $y \in K$ such that for every $x' \in U \setminus (\text{Fix}(F) \cup M(F))$, we have $y \in F(x')$. Then F has a maximal element or a fixed-point.*

Proof. Assume the contrary. For every $x \in K$, there exist $U_x \subset K$ an open neighborhood of x and $y_x \in K$ such that $y_x \in F(x')$ for every $x' \in U_x$. Since K is compact, the open cover $\{U_x\}_{x \in K}$ admits a finite subcover U_1, \dots, U_n , with corresponding points x_1, \dots, x_n and elements y_1, \dots, y_n .

Take functions h_i given by Lemma 1. Denote $K_0 = \text{co}(y_1, \dots, y_n)$, then K_0 is a finite dimensional convex compact contained in K . We define a continuous mapping $f : K \rightarrow K_0$ by

$$f(x) = \sum_{i=1}^n h_i(x) y_i.$$

For every $x \in K$, every index i with $h_i(x) > 0$, we have $x \in U_i$, so $y_i \in F(x)$. Since $F(x)$ is convex, we have $f(x) \in F(x)$, i.e., f is a selection of F . By Brouwer's theorem, $f|_{K_0}$, and hence F , admits a fixed-point, a contradiction. □

Corollary 5 (Urai). [43, Theorem 1] *Let $G : K \rightrightarrows E$ be a correspondence. If G satisfies the following condition (K^*) , then G has a fixed-point :*

- (K^*) *There is a convex valued correspondence $F : K \rightrightarrows E$ such that for each $x \in K \setminus \text{Fix}(G)$, there exist a neighborhood $U \subset K$ of x and $y \in K$ such that $x' \notin F(x')$ and $y \in F(x')$ for each $x' \in U$.*

Proof. Otherwise, for each $x \in K$, $x \notin F(x)$ and there exists some $y \in F(x)$, or equivalently F has neither fixed-point nor maximal element. But then condition (K^*) contradicts Theorem 20. □

Remark 2. *Actually, Urai added an unnecessary condition in (K^*) , namely $G \subset_U F$.*

Moreover, [43, Theorem 2, (NK*)] is certainly false. For example, take $I = \{1, 2\}$, $X^1 = X^2 = [0, 1] \subset E = \mathbb{R}$ and let $\phi^1 = \Phi^1 : X = [0, 1]^2 \rightarrow X^1$ be constant mapping 0, $\phi^2 = \Phi^2 : [0, 1]^2 \rightarrow X^2$ given by

$$\phi^2(a, b) = \begin{cases} 1 & b < 1; \\ 0 & b = 1. \end{cases}$$

Clearly, Urai's (NK*) condition holds because for each $z \in X$, $0 \in \Phi^1(z)$. However, contrary to his statement, $\phi = \phi^1 \times \phi^2$ has no fixed-point.

The following result generalizes simultaneously Theorem 16, [47, Corollay 3] and [50, Theorem 3.2], who assume in addition that for every $x \in X$, there exist $U \subset X$ open neighborhood of x and $y \in Y$ such that for every $x' \in U$, we have $y \in F(x')$.

Theorem 21 (Yannelis-Prabhkar, Wu-Shen). *Assume E is locally convex, $X \subset E$ is convex and $Y \subset X$ is compact. If $F : X \rightrightarrows Y$ is a convex-valued correspondence such that for every $x \in X \setminus (\text{Fix}(F) \cup M(F))$, there exist $U \subset X$ an open neighborhood of x and a continuous mapping $s : U \rightarrow Y$ such that for every $x' \in U \setminus (\text{Fix}(F) \cup M(F))$, we have $s(x') \in F(x')$. Then F has a maximal element or a fixed-point.*

Proof. Assume the contrary. Denote $A = \text{co}(Y)$. As X is convex, we have $A \subset X$. By [18, Lemma 1], A is paracompact. By Theorem 18, $F|_A : A \rightrightarrows Y$ admits a continuous selection $f : A \rightarrow Y$. By Theorem 16, f hence F admits a fixed-point on A , which is a contradiction. \square

We derive quickly the main result of [22].

Corollary 6. [22, Theorem 1, p.557] *Assume $K \subset C$ and $F : C \rightrightarrows E$ is a correspondence. Put $S = C \setminus \text{Fix}(F)$ and suppose that $G : S \rightrightarrows K$ is another correspondence with nonempty convex values satisfying that for every $x \in S$, there exist $U \subset S$ an open neighborhood of x and $y \in E$ such that for every $x' \in U$, we have $y \in G(x')$ and $x' \notin G(x')$. Then F admits a fixed-point.*

Proof. Otherwise, $S = C$ and $G : C \rightrightarrows K$ has neither fixed-point nor maximal element. But in that case, $G|_K$ contradicts Theorem 20! \square

Theorem 22 (Majorization). *Let $F : K \rightrightarrows E$ be a correspondence and $f : K \setminus M(F) \rightarrow K$ be a mapping. If for every $x \in K \setminus M(F)$, there exist $U \subset K$ an open neighborhood of x and $G : U \rightrightarrows E$ a correspondence*

- *with convex values;*
- *without fixed-points ;*
- *for every $x' \in U \setminus M(F)$, we have that $G^{-1}(f(x'))$ is a neighborhood of x' in U and $F(x') \subset G(x')$.*

Then F admits a maximal element.

Proof. The proof is essentially that of [8, Corollary 1]. Assume the contrary. For every $x \in K$, there exist U_x and G_x as in the assumption. Take finite subcover U_1, \dots, U_n , points x_1, \dots, x_n and correspondences G_1, \dots, G_n . Take closed subsets $F_1 = \text{Supp}(h_1), \dots, F_n = \text{Supp}(h_n)$ given by Lemma 1.

For each i , define $H_i : K \rightarrow K$ by

$$H_i(x) = \begin{cases} G_i(x), & x \in F_i; \\ E, & x \notin F_i. \end{cases}$$

Then $F \subset_K H_i$ and H_i is of convex value. For every $x \in K$ and every i , $H_i^{-1}(f(x))$ is a neighborhood of x in K . In fact, if $x \in F_i$, then $G_i^{-1}(f(x)) \subset H_i^{-1}(f(x))$ and the former is a neighborhood by our assumption; if $x \notin F_i$, then $F_i^c \subset H_i^{-1}(f(x))$ and again the former is a neighborhood.

Define $H = \cap H_i$ another correspondence $K \rightarrow E$. Then H has convex values and $F \subset_K H$. For every $x \in K$, $H^{-1}(f(x)) = \cap_{i=1}^n H_i^{-1}(f(x))$ is a neighborhood of x in K . Therefore, there is $U \subset H^{-1}(f(x))$ open neighborhood of x in K , such that every $x' \in U$, $f(x) \in H(x')$.

For every $x \in K$, there is i with $x \in F_i$. As G_i has no fixed-point on U_i , we have $x \notin G_i(x) = H_i(x)$ and hence $x \notin H(x)$. That means H admits no fixed-point on K . By (weak version of) Theorem 20, there is $x_0 \in K$ with $H(x_0)$ empty, so $F(x_0)$ empty. \square

Definition 16 (Class \mathcal{L}). [50, Definition 5.1] A correspondence $F : C \rightarrow C$ is called of class \mathcal{L} if is

- *irreflexive*: for every $x \in C$, $x \notin \text{co}(F(x))$;
- *of open preimage*: for every $y \in C$, $F^{-1}(y) := \{x \in C : y \in F(x)\}$ is open in C .

A correspondence $G : C \rightarrow C$ is called \mathcal{L} -majorized if for every $x \in C \setminus M(G)$, there exist an open neighborhood $U \subset C$ of x and $H : C \rightarrow C$ a correspondence of class \mathcal{L} with $G \subset_U H$.

Obviously, a correspondence of class \mathcal{L} is \mathcal{L} -majorized. The following result, due to Yannelis and Prabhakar, is very handy in practice. Now we show that Theorem 22 is a natural extension of it.

Theorem 23 (Yannelis-Prabhakar). [50, Corollary 5.1] If $F : K \rightarrow K$ is an \mathcal{L} -majorized correspondence, then F admits maximal element.

Proof. Otherwise, for every $x \in K$, $F(x)$ is nonempty, so there exists a selection f of F . Fix $x \in K$, there exist $U \subset K$ open neighborhood of x and a correspondence $G : K \rightarrow K$ of class \mathcal{L} with $F \subset_U G$. For every $x' \in U$, since $f(x') \in F(x') \subset G(x')$ and G has open preimages, we find $G^{-1}(f(x'))$ is an open neighborhood of x' in K . Define $H : K \rightarrow K$ by $H = \text{co}(G)$. Then H is convex-valued and $F \subset_U H$. For every $x' \in U$, $H^{-1}(f(x')) \supset G^{-1}(f(x'))$ is a neighborhood of x' in K . By irreflexivity of G , H admits no fixed-point. Now we can conclude by Theorem 22. \square

To illustrate the power of Theorem 22, we give two applications. This first is inspired by [28, Theorem 4.1].

Theorem 24 (Li). *If a mapping $f : K \rightarrow K$ satisfies: for every $x \in K \setminus \text{Fix}(f)$, there exist an open neighborhood $U_x \subset K$ of x and a continuous quasiconvex function $p : E \rightarrow \mathbb{R}$ such that $p(0) < p(x' - f(x'))$ for all $x' \in U_x \setminus \text{Fix}(f)$, then f admits fixed-point.*

Proof. Otherwise, fix any $x \in K$, there is such p with $p(0) < p(x - f(x))$. By continuity of p and f , there is an open neighborhood $U \subset K$ of x such that for every $x' \in U$ we have $p(f(x') - f(x)) < p(x' - f(x))$.

Define $G : K \rightarrow K$ by $G(x') = \{y \in K : p(y - f(x)) < p(x' - f(x))\}$, then $f \subset U G$ and G has no fixed-point. Since p is quasiconvex, G is of convex value. Since p is continuous, G is of open preimage. Therefore, G is of class \mathcal{L} and f is \mathcal{L} -majorized which contradicts Theorem 23! \square

As a second application, we give an alternative proof of the main result of [45].

Theorem 25. [45, Theorem 1] *Assume E is locally convex and $F : K \rightarrow K$ is a correspondence without maximal element. Endow E^* with compact convergence topology. Assume for every $x \in K \setminus \text{Fix}(F)$, there exist an open neighborhood $U \subset K$ of x and a compact-valued upper semi-continuous correspondence $P : U \rightarrow E^*$ without maximal element such that for every $x' \in U, y' \in F(x'), p \in P(x')$, we have $(p, x' - y') < 0$. Then F has fixed-point.*

Proof. Otherwise, fix any $x \in K$, there exists U and P as in the above theorem. Define $G : U \rightarrow K$ by $G_x(x') = \{y \in K : (p, x' - y) < 0, \forall p \in P(x')\}$. Then G is convex valued without fixed-point and $F \subset U G$.

We show that G has open preimages. In fact, fix $y_0 \in K$ and $x_0 \in G^{-1}(y_0)$, then every $p \in P(x_0)$, we have $(p, x_0 - y_0) < 0$. By Exercise 7, the mapping $\phi : K \times E^* \rightarrow \mathbb{R}$ defined by $(x, p) \mapsto p(x - y_0)$ is continuous. So, $\phi^{-1}(-\infty, 0) \subset K \times E^*$ is open and contains $\{x_0\} \times P(x_0)$. By Lemma 6 there exist open neighborhoods $V \subset K$ of x_0 and $U \subset E^*$ of $P(x_0)$ such that $V \times U \subset \phi^{-1}(-\infty, 0)$. Or equivalently, for every $x' \in V, q \in U$, we have $(q, x' - y_0) < 0$. (These details are omitted in [45].)

As P is upper semi-continuous, there exists a neighborhood $V_1 \subset V$ of x_0 such that for every $x_1 \in V_1$, $P(x_1) \subset U$. Thus, $(q, x_1 - y_0) < 0$ for all $q \in P(x_1)$, which means $V_1 \subset G^{-1}(y_0)$. Hence $G^{-1}(y_0)$ is open.

By Theorem 22, F has a maximal element, a contradiction. \square

For our purpose, we need the following.

Theorem 26. *If we have a family of quasiconvex functions $\{T_x\}_{x \in K}$ on K such that for every $v \in K$, the function $K \rightarrow \mathbb{R}$ defined by*

$$u \mapsto T_u(u) - T_u(v)$$

is continuous, then there is $x_0 \in K$ such that $T_{x_0}(x_0) \leq T_{x_0}(x)$ for every $x \in K$.

Proof. Define a correspondence $F : K \rightrightarrows K$ by

$$F(x) := \{y \in K : T_x(x) - T_x(y) > 0\}.$$

For every $x \in K$, since T_x is quasiconvex, the set $F(x)$ is convex. Clearly, F has no fixed-points. For every $y \in K$, by our continuity assumption, $F^{-1}(y) = \{x \in K : T_x(x) - T_x(y) > 0\}$ is open in K . We find that F is of class \mathcal{L} . By Theorem 23, there is a maximal element $x_0 \in M(F)$, i.e., for every $x \in K$ we have $T_{x_0}(x_0) \leq T_{x_0}(x)$. \square

Corollary 7 (Browder). *Let $T : K \rightarrow E^*$ be a mapping such that for every $v \in K$, the function $K \rightarrow \mathbb{R}$ defined by $u \mapsto (T(u), u - v)$ is continuous. Then there is $x_0 \in K$ such that for every $x \in K$, $(T(x_0), x - x_0) \geq 0$.*

This result is similar to [9, Theorem 2], where Browder confusingly talks about the continuity of $T : K \rightarrow E^*$ without specifying the topology on E^* .

Proof. Define a family of linear functions by $T_x = T(x)|_K$ for $x \in K$. For every $v \in K$, the function $K \rightarrow \mathbb{R}$ defined by $u \mapsto T_u(v) - T_u(u) = (T(u), v - u)$ is continuous. We conclude by Theorem 26. \square

Example 1. *If $(E; \langle, \rangle)$ is a Hilbert space, and fix $v \in E$, we define $T : K \rightarrow E^*$ by $T(x) = \langle v - x, * \rangle$. Then the x_0 given by Corollary 7 is unique. In fact, it is the projection of v into K , i.e., $|v - x_0| = \min_{x \in K} |v - x|$.*

The following two results generalize Lemma 1 and Theorem 6 of [9] respectively, in the sense that E may not be locally convex.

Lemma 9. *Endow E^* with compact convergence topology. Let $A \subset E$ be convex and $B \subset E^*$ be convex compact. If for every $p \in B$, there is $a \in A$ with $(p, a) < 0$, then there is $a_0 \in A$ such that $(p, a_0) < 0$ for every $p \in B$.*

Proof. For every $a \in A$, let $U_a = \{p \in B : (p, a) < 0\}$. Then $U_a \subset B$ is weak*-open hence open in compact convergence topology by Exercise 7. By our assumption, $\{U_a\}_{a \in A}$ is an open cover of B . Since B is compact, it admits a finite subcover U_1, \dots, U_n corresponding to points $a_1, \dots, a_n \in A$. Take functions h_i given by Corollary 1. Define a continuous mapping $f : B \rightarrow A$ by $f(b) = \sum_{i=1}^n h_i(b)a_i$.

For every $p \in B$, every i with $h_i(p) > 0$, we have $p \in U_i$ and hence $(p, a_i) < 0$. Therefore, $(p, f(p)) < 0$.

Define a correspondence $F : B \rightrightarrows B$ by $F(p) = \{b \in B : (b, f(p)) \geq 0\}$. By last paragraph, F has no fixed-point. F has convex values. By Exercise 7, the canonical pairing $f(B) \times E^* \rightarrow \mathbb{R}$ is continuous, so is $\phi : B \times B \rightarrow \mathbb{R}$ by $(p, b) \mapsto b(f(p))$. The graph of F is $\phi^{-1}[0, +\infty)$ hence closed in $B \times B$. Recall that E^* is a locally convex TVS. By Kakutani's theorem, there is $b_0 \in B$ such that $F(b_0)$ is empty. Then $a_0 = f(b_0)$ is what we seek. \square

The following is based on the above lemma.

Theorem 27 (Browder). *Endow E^* with compact convergence topology. $T : K \rightarrow E^*$ an upper semicontinuous correspondence with nonempty compact convex values. Then there is $x_0 \in K$, $p_0 \in T(x_0)$ such that $(p_0, x_0 - u) \geq 0$ for all $u \in K$.*

Proof. Define a correspondence $S : K \rightarrow K$ by

$$S(x) := \{u \in K : \forall p \in T(x), (p, u - x) > 0\}.$$

S is of convex value. For each $x \in K$, $T(x)$ is nonempty, so $x \notin S(x)$. For each $u \in K$, $S^{-1}(u) = \{x \in K : \forall p \in T(x), (p, u - x) > 0\}$ is open.

Indeed, fix any $x_0 \in S^{-1}(u)$. By Exercise 7, the mapping $\phi : K \times E^* \rightarrow \mathbb{R}$ by $(x, p) \mapsto (p, u - x)$ is continuous, so $\phi^{-1}(0, +\infty)$ is open in $K \times E^*$ containing $\{x_0\} \times T(x_0)$. By Lemma 6, there exist opens $U \subset K$ containing x_0 and $V \subset E^*$ containing $T(x_0)$ such that $U \times V \subset \phi^{-1}(0, +\infty)$. T is upper semicontinuous, so $U' = \{x \in K : T(x) \subset V\}$ is open neighborhood of x_0 in K . For every $x \in U \cap U'$, we have $T(x) \subset V$, then $\{x\} \times T(x) \subset \phi^{-1}(0, +\infty)$, i.e., $x \in S^{-1}(u)$. Thus $U \cap U' \subset S^{-1}(u)$.

We have verified that S is of class \mathcal{L} . By Theorem 23, there is $x_0 \in K$ such that $S(x_0)$ is empty. Or equivalently, for every $u \in K$, there is $p \in T(x_0)$ with $(p, u - x_0) \leq 0$. By Lemma 9, there is $p_0 \in T(x_0)$ such that $(p_0, x_0 - u) \geq 0$ for every $u \in K$. \square

4 Fixed-Point of Discontinuous Mappings

In this section, the results from the previous one are applied to strength some earlier fixed-point theorems for discontinuous mappings. We start by generalizing the finite dimensional result [44, Theorem 2] to arbitrary inner product space.

Theorem 28 (Urai). *Let $(E; \langle \cdot, \cdot \rangle)$ be an inner product space. If $F : K \rightarrow K$ is a correspondence without maximal element and satisfying the following condition:*

- (LDV2) *for every $x \in K \setminus \text{Fix}(F)$, there exist $y \in E$ and an open neighborhood $U \subset K$ of x such that $\langle y - x', y' - x' \rangle > 0$ for every $x' \in U \setminus \text{Fix}(F)$, and every $y' \in F(x')$.*

Then F admits a fixed-point.

Proof. Otherwise, for every $x \in K$, consider $y \in E$ and some neighborhood U given by the hypothesis (LDV2). Define $G_x : K \rightarrow K$ by $G_x(z) = \{w \in K : \langle y - z, w - z \rangle > 0\}$. Then $F \subset_U G_x$ and G_x is convex-valued and without fixed-point. For each $w \in K$, $G_x^{-1}(w) \subset K$ is open. So, G_x is of class \mathcal{L} and hence F is \mathcal{L} -majorized, which contradicts Theorem 23. \square

Then we introduce some weak continuity definitions.

Definition 17 (Quasi-continuity). *Assume C is convex. A mapping $f : C \rightarrow C$ is called quasi-continuous if for every $x \in C \setminus \text{Fix}(f)$, there exist a continuous quasiconvex function $p : C \rightarrow \mathbb{R}$ and an open neighborhood $U \subset C$ of x such that*

$$p(f(x')) < p(x')$$

for every $x' \in U \setminus \text{Fix}(f)$.

This definition is inspired by a similar one :

Definition 18 (Half-continuity). *[5, Definition 2.1] A mapping $f : C \rightarrow E$ is called half-continuous if for every $x \in C \setminus \text{Fix}(f)$, there exist $p \in E^*$ and an open neighborhood $U \subset C$ of x such that*

$$(p, f(x') - x') < 0$$

for every $x' \in U \setminus \text{Fix}(f)$.

For an example of half-continuous mappings that are not continuous, see [40, Remark 3.5]. Clearly, a half-continuous mapping $f : C \rightarrow C$ is quasi-continuous. We don't know whether quasi-continuous mappings are always half-continuous.

We now provide a trivial generalization of [40, Proposition 3.2], relaxing continuity to weaker continuity.

Proposition 2. *If E^* separates E , then every continuous mapping $f : (C, \tau) \rightarrow (E, \tau_w)$ is half-continuous. Conversely, if the zero mapping $f_0 : E \rightarrow E$ is half-continuous, then E^* separates E .*

Proof. For every $x \in C$ with $x \neq f(x)$, there is $p \in E^*$ such that $(p, f(x)) < (p, x)$. Now that the function $C \rightarrow \mathbb{R}$ by $z \mapsto (p, z) - (p, f(z))$ is continuous, thus there exists an open neighborhood $U \subset C$ of x such that for every $x' \in U$, $(p, f(x')) < (p, x')$.

We now show the second part. For every non zero $v \in E$, since $f_0(v) \neq v$ and f_0 is half-continuous, there is $p \in E^*$ such that $(p, f_0(v) - v) < 0$. In particular, $(p, v) \neq 0$. \square

Theorem 29. *If a mapping $f : K \rightarrow K$ is quasi-continuous, then there is $x_0 \in K$ such that $f(x_0) = x_0$.*

Proof. (We will give another proof, cf. Proof of Theorem 31.) Suppose f has no fixed point. By definition, there exist an open neighborhood $U_x \subset K$ of x and a continuous quasiconvex function $p_x : K \rightarrow \mathbb{R}$ such that $p_x(f(x')) < p_x(x')$ for every $x' \in U_x$. We can assume $p_x > 0$ by adding a constant. By compactness of K , the open cover $\{U_x\}_{x \in K}$ admits a finite subcover, say U_1, \dots, U_n with corresponding points $x_1, \dots, x_n \in K$ and functions p_1, \dots, p_n . Take functions h_i given by Corollary 1. For each $x \in K$, define a function $T_x : K \rightarrow [0, +\infty)$ by

$$T_x = \max_{i=1}^n h_i(x)p_i.$$

In fact, for every $x, y \in K$, since $\sum_{i=1}^n h_i(x) = 1$, we have $T_x(y) > 0$. By Exercise 3, T_x is quasiconvex.

For every $x \in K$, there is an index i_0 such that

$$T_x(f(x)) = h_{i_0}(x)p_{i_0}(f(x)).$$

Then $h_{i_0}(x) > 0$ and hence $x \in U_{i_0}$, so $p_{i_0}(f(x)) < p_{i_0}(x)$ by our choice of p_{i_0} . Therefore, we get

$$T_x(f(x)) < h_{i_0}(x)p_{i_0}(x) \leq T_x(x). \quad (3)$$

Fix $v \in E$, the function $K \rightarrow \mathbb{R}$ defined by

$$u \mapsto T_u(u) - T_u(v) = \max_{i=1}^n h_i(u)p_i(u) - \max_{j=1}^n h_j(u)p_j(v)$$

is continuous. By Theorem 26, there is $x_0 \in K$ such that for every $x \in K$, we have $T_{x_0}(x_0) \leq T_{x_0}(x)$. In particular, $T_{x_0}(x_0) \leq T_{x_0}(f(x_0))$, which contradicts (3). \square

Corollary 8 (Brouwer-Schauder-Tychonoff). *If E^* separates E , a subset $C \subset E$ is convex weakly compact, and a mapping $f : C \rightarrow C$ is τ_w -continuous, then there is $x_0 \in C$ with $f(x_0) = x_0$.*

Proof. By Theorem 11, (E, τ_w) is a locally convex TVS with dual E^* . We conclude by combining Proposition 2 and Theorem 29. Alternatively, we can apply Tychonoff's theorem to (E, τ_w) . \square

Remark 3. *The special case where E is locally convex, C is compact and f is continuous is a celebrated result of Tychonoff in [42, Satz, §2].*

Corollary 8 also covers Li's main result in [28, Theorem 4.1]. In fact, what Li views as a new concept is actually classical. If a TVS E is quasi-point-separable in Li's terminology [28, Definition 3.6 (ii)], then by Exercise 5 E^ separates E , or rather E is point-separable following Li.*

A weaker extension of Tychonoff's theorem is given by Urai.

Corollary 9 (Urai). [46, Corollary 3.1] *If E is such that*

- (T) *for every $x \neq y \in E$, there exist two disjoint open neighborhoods $U_x \ni x$, $U_y \ni y$, at least one being convex.*

Then every continuous function $f : K \rightarrow K$ has a fixed-point.

Proof. In fact, this condition (T) is stronger than the condition that E^* separates E , so we conclude by applying Corollary 8. Indeed, assume (T), assume for example U_x is convex, then by Theorem 13, there is $p \in E^*$ such that $p(x') < p(y)$ for every $x' \in U_x$. In particular, $p(x) \neq p(y)$ and hence E^* separates points. \square

As another application, we correct a mistake (see Remark 4 for explanation) in [43].

Theorem 30. *If E^* separates E and $F : K \rightarrow K$ is a convex valued correspondence such that for every $x \in K \setminus \text{Fix}(F)$, there exist an open neighborhood $U \subset K$ of x , a vector $v \in E$ and a function $\lambda : U \setminus \text{Fix}(F) \rightarrow (0, +\infty)$ (not necessarily continuous) such that $z + \lambda(z)v \in F(z)$ for every $z \in U \setminus \text{Fix}(F)$, then F has a fixed-point.*

Proof. Otherwise, for every $x \in K$, there is an open neighborhood $U_x \subset K$ of x , a vector v_x and a function $\lambda_x : U_x \rightarrow (0, +\infty)$ given by the above assumption. Since K is compact, the open cover $\{U_x\}_{x \in K}$ admits a finite subcover U_1, \dots, U_n with corresponding vectors v_1, \dots, v_n and functions $\lambda_1, \dots, \lambda_n$. Take functions h_i given by Corollary 1. Define a mapping $f : K \rightarrow E$ by

$$f(x) = x + \sum_{i=1}^n h_i(x) \lambda_i(x) v_i.$$

Fix any $x \in K$ and consider the nonempty index set $I = \{1 \leq i \leq n \mid h_i(x) > 0\}$. For every $i \in I$, we have $x \in U_i$, λ_i is defined at x and $x + \lambda_i(x)v_i \in F(x)$. As $F(x)$ is convex, we find $f(x) = \sum_{i \in I} h_i(x)(x + \lambda_i(x)v_i) \in F(x)$. In other words, f is a selection of F .

We first show that $0 \notin C$, where $C = \text{co}(v_i \mid i \in I)$. Otherwise we can write

$$0 = \sum_{i \in I} a_i v_i,$$

where each $a_i \geq 0$ and $\sum_{i \in I} a_i = 1$. Put $S = \sum_{i \in I} \frac{a_i}{\lambda_i(x)}$, then $S > 0$. For $j \in I$, define $b_j = \frac{a_j}{S \lambda_j(x)}$. Then each $b_j \geq 0$ and $\sum_{j \in I} b_j = 1$. As $F(x)$ is convex, we find $x = \sum_{j \in I} b_j(x + \lambda_j(x)v_j) \in F(x)$ which is a contradiction.

We check that f is half-continuous. Applying Theorem 13 to two disjoint convex weakly compact subsets $\{0\}$ and C , we find $p \in E^*$ such that $p(w) > 0$ for every $w \in C$. Then for every $i \in I$, $p(v_i) > 0$ and $\sum_{i=1}^n h_i(x)p(v_i) = \sum_{i \in I} h_i(x)p(v_i) > 0$. By continuity of the functions h_i , there is an open neighborhood $V \subset K$ of x such that for every $x' \in V$ we have $\sum_{i=1}^n h_i(x')p(v_i) > 0$ and hence

$$p(f(x') - x') = \sum_{i=1}^n h_i(x') \lambda_i(x') p(v_i) > 0.$$

So f is half-continuous at x .

By Theorem 29, f and hence F admits a fixed-point, a contradiction. \square

Corollary 10 (Urai). *Assume E^* separates E , and $F, G : K \rightarrow K$ are two correspondences, where G is convex-valued. If for every $x \in K \setminus \text{Fix}(F)$, there exists an open neighborhood $U \subset K$ of x , a vector $v \in E$ and a (possibly non continuous) function $\lambda : U \rightarrow (0, +\infty)$ such that for every $x' \in U \setminus \text{Fix}(F)$, we have $x' \notin G(x')$ and $x' + \lambda(x')v \in G(x')$, then F has a fixed-point.*

Proof. Assume the contrary. By our assumption, G satisfies the assumption of Theorem 30 but has no fixed-point, a contradiction. \square

Remark 4. Corollary 10 is stated in [43, Theorem 1, (K#)] for general TVS. However, his proof is wrong. Contrary to the claim he made during the proof, his mapping f may not be continuous nor taking values in his \hat{E} . We don't know whether his original statement remains true.

Now we extend Theorem 29 to correspondences.

Definition 19 (Quasi-continuous Correspondence). *A correspondence $F : C \rightrightarrows C$ is called quasi-continuous if for every $x \in C$ with $x \notin F(x)$, there exist an open neighborhood $U \subset C$ of x and a continuous quasiconvex function $p : C \rightarrow \mathbb{R}$ such that for every $x' \in U \setminus \text{Fix}(F)$, every $y' \in F(x')$, we have*

$$p(y') < p(x').$$

Half-continuous correspondence is similarly defined, see [40, Definition 4.1].

Lemma 10. *If $F : C \rightrightarrows C$ is quasi-continuous without maximal element, then it admits a quasi-continuous selection.*

Proof. Adopt the construction of [40, Lemma 4.5]. □

The following generalizes [43, Theorem 1. (K1)], relaxing his half-continuity property to our quasi-continuity property.

Theorem 31. *If $F : K \rightrightarrows K$ is a quasi-continuous correspondence without maximal element, then it admits a fixed-point.*

Proof. We conclude by combining Theorem 29 and Lemma 10. Still, we present another proof. Assume the contrary. For every $x \in K$, there exist an open neighborhood $U \subset K$ of x and a continuous quasiconvex function $p_x : K \rightarrow \mathbb{R}$ such that for every $x' \in U$, every $y' \in F(x')$, we have $p_x(y') < p_x(x')$. Define $G_x : K \rightarrow K$ by $G_x(x') = \{y \in K : p_x(y) < p_x(x')\}$, then $F \subset \bigcup_x G_x$.

Check that G_x is of class \mathcal{L} . Since p_x is quasiconvex, G_x is convex-valued. Clearly, G_x admits no fixed-point. For every $y \in K$, since p_x is continuous, $G_x^{-1}(y) = \{x' \in K : p_x(y) < p_x(x')\}$ is open in K .

Therefore, F is \mathcal{L} -majorized. But then F contradicts Theorem 23. □

Example 2. *Let $E = \mathbb{R}$, $K = [0, 1]$, $F : K \rightrightarrows K$ by $F(x) = K \setminus \{x\}$. Then for every $x \in K$, there exist an open neighborhood $U \subset K$ of x and $p \in E^*$ such that for every $x' \in U$, there exists $y' \in F(x')$ with $p(y') < p(x')$. Clearly, F admits no fixed-point. That is why in Definition 19 we require that the whole set $F(x')$ is separated from x' .*

As an example, we simplify the proof of another main result in [22].

Theorem 32. [22, Theorem 2, p.559] *If $K \subset C$ and $F : C \rightrightarrows E$ is a correspondence. Put $S = C \setminus \text{Fix}(F)$. Let $G : S \rightrightarrows K$ be a correspondence without maximal element. If for every $x \in S$, there exist open neighborhood $U \subset S$ of x and $p \in E^*$ such that for every $x' \in U$, every $y' \in G(x')$ we have $(p, y' - x') < 0$ (in particular $x' \notin \text{Fix}(G)$), then F admits a fixed-point.*

Proof. Otherwise, $S = C$ and $G|_K$ is half-continuous without fixed-point, which contradicts Theorem 31. \square

We generalize [40, Proposition 4.4] by relaxing continuity and weakening compactness to weakly closedness.

Proposition 3. *If E^* separates E and $F : (C, \tau) \rightarrow (E, \tau_w)$ is upper semi-continuous with convex weakly closed values, then F is half-continuous.*

Proof. Fix any $x \in C$ with $x \notin F(x)$. Since $\{x\}$ is weakly compact and $F(x)$ is weakly closed, by Theorem 13 there exist $p \in E^*$ and two reals $a < b$ such that $p(x) > b$ and for every $y \in F(x)$, $p(y) < a$. As F is upper semi-continuous, $U_1 = \{x' \in C : F(x') \subset p^{-1}(-\infty, a)\}$ is open in C . Also, by continuity of p , $U = U_1 \cap p^{-1}(b, +\infty)$ is an open neighborhood of x in C . For every $x' \in U$, every $y' \in F(x')$, we have $p(y') < a < b < p(x')$. Therefore F is half-continuous at x . \square

Theorem 33 (Kakutani-Fan-Glicksberg). *Assume E^* separates E and $C \subset E$ is a weakly compact convex subset. Let $F : C \rightarrow C$ be a correspondence with nonempty convex values, whose graph $Gr(F) \subset (C, \tau_w) \times (C, \tau_w)$ is closed. Then F admits a fixed-point.*

Proof. This proof is parallel to that of Corollary 8. Use the weak topology τ_w on E . Then we conclude by Theorem 31 and Proposition 3 or Kakutani's fixed-point theorem with Theorem 11. \square

The following result generalizes [40, Theorem 5.1]. The "locally convex" restriction is removed, continuity is relaxed, and single valued mapping is generalized to correspondence.

Theorem 34. *Let $G : K \rightarrow E$ be a correspondence such that for every $x \in K \setminus \text{Fix}(G)$, the set*

$$(x, G(x)) := \{\lambda x + (1 - \lambda)y : \lambda \in [0, 1), y \in G(x)\}$$

intersects K and there exist a continuous strongly quasiconvex function $p : E \rightarrow \mathbb{R}$ and an open neighborhood $U \subset K$ of x such that $p(y') < p(x')$ for every $x' \in U, y' \in G(x')$. Then G admits a fixed-point.

Proof. Assume the contrary. Define a correspondence $F : K \rightarrow K$ by $F(x) = (x, G(x)) \cap K$. Clearly, F has no fixed-point.

This F is quasi-continuous, which contradicts Theorem 31. In fact, for every $x \in K$, by assumption on G , there exist an open neighborhood $U \subset K$ of x and a continuous strongly quasiconvex function $p : K \rightarrow \mathbb{R}$ such that for every $x' \in U$, every $y' \in G(x')$, we have $p(y') < p(x')$. As p is strongly quasiconvex, for every $y'' \in F(x') \subset (x', G(x'))$, we have $p(y'') < p(x')$. Thus, F is quasi-continuous at x . \square

Corollary 11 (Fan-Kaczynski). *If E^* separates E , $C \subset E$ is weakly compact convex, and $g : C \rightarrow E$ is a τ_w -continuous mapping such that for every $x \in K$ with $x \neq g(x)$, the set*

$$(x, g(x)) := \{\lambda x + (1 - \lambda)y : \lambda \in [0, 1), y \in g(x)\}$$

intersects C , then g admits a fixed-point.

This corollary generalizes [40, Corollary 5.3], where the ambient space E is locally convex, C is compact, g is τ -continuous and hence τ_w -continuous.

Proof. Use topology τ_w and we conclude by Theorem 34. \square

Definition 20 (Weakly Inward/Outward). [33, p.3] *A correspondence $F : C \rightarrow E$ is called weakly inward (resp. weakly outward) if for each $x \in C \setminus \text{Fix}(F)$, $F(x)$ intersects $\bar{I}_C(x)$ (resp. $\bar{O}_C(x)$).*

Obviously, a weakly inward correspondence has no maximal element.

Theorem 35. *Consider a weakly inward correspondence $F : K \rightarrow E$ satisfies : for every $x \in K \setminus \text{Fix}(F)$, there exist $U \subset K$ an open neighborhood of x and $p : E \rightarrow \mathbb{R}$ continuous strongly quasiconvex function such that every $x' \in U \setminus \text{Fix}(F)$, every $y' \in F(x')$, we have $p(y') < p(x')$. Then F has a fixed-point.*

Proof. Assume the contrary. We repeat the proof of Theorem 29 with necessary modifications. For every $x \in K$, there exist a continuous strongly quasiconvex function $T_x : E \rightarrow \mathbb{R}$ such that for every $y \in F(x)$, $T_x(y) < T_x(x)$. (Note that we used Exercise 3 implicitly in construction of T_x .) Moreover, there is $x_0 \in K$ such that

$$T_{x_0}(x_0) \leq T_{x_0}(x), \forall x \in K. \quad (4)$$

Take $y_0 \in F(x_0) \cap \bar{I}_K(x_0)$, then

$$T_{x_0}(y_0) < T_{x_0}(x_0) \quad (5)$$

by our assumption. Since T_{x_0} is continuous, there is an open neighborhood $U \subset E$ of 0 such that for every $y \in y_0 + U$, $T_{x_0}(y) < T_{x_0}(x_0)$. Choose $y \in (y_0 + U) \cap I_K(x_0)$, then there is $\lambda > 0$, $z \in K$ with $y = x_0 + \lambda(z - x_0)$. If $\lambda \geq 1$, then $z = \frac{1}{\lambda}y + (1 - \frac{1}{\lambda})x_0$. Since T_{x_0} is strongly quasiconvex, $T_{x_0}(z) < T_{x_0}(x_0)$ which contradicts (4). If $\lambda < 1$, then $y = (1 - \lambda)x_0 + \lambda z \in K$ and (4) contradicts (5). \square

Corollary 12 (Park). [33, Theorem 3.2, Theorem 5.7] *A half-continuous weakly inward/outward correspondence $F : K \rightarrow E$ has fixed-point.*

Proof. If F is weakly inward, then we can apply Theorem 35 directly. Now assume that F is weakly outward. Define another correspondence $G : K \rightarrow E$ by $G(x) = 2x - F(x)$, then G is half-continuous weakly inward. We conclude by applying Theorem 35. \square

Thus we can generalize the result due to Halpern and Bergman [24, Theorem 4.1], where they require that C is compact and f is continuous.

Corollary 13. *If E^* separates E , C convex weakly compact, $f : C \rightarrow E$ weakly continuous such that for every $x \in C$, $f(x)$ lies in the weak closure of $I_C(x)$, then f has a fixed-point.*

Proof. Apply Corollary 12 to (E, τ_w) . □

Remark 5. *This corollary differs a bit from [32, Corollary 1]. Park requires that $f(x)$ lies in the τ -closure of $I_C(x)$, which is stronger than $f(x)$ being in the weak closure of $I_C(x)$. In fact, we can replace the original closure by weak closure in all his three corollaries loc.cit. But the weak continuity of f assumed by us is stronger than his condition: for each $p \in E^*$, $\{x \in C : p(x) < p(f(x))\}$ is weakly open in C .*

The following corollary generalizes [16, Ch 2, §10, Exercise 4, p.93] and [32, Corollary 3], where in both cases E is restricted to a reflexive Banach space.

Corollary 14. *Let E be a locally convex semi-reflexive TVS with a nonempty closed bounded convex subset $C \subset E$. Assume a mapping $f : C \rightarrow E$ is such that for every $x \in C$, $f(x)$ lies in the weak closure of $I_C(x)$. If either*

- (1) *the mapping f weakly continuous or*
- (2) *the mapping f is weakly sequentially continuous and E is metrizable,*

then f has a fixed-point.

Proof. By Theorem 15, C is weakly compact. Under condition (1), we can conclude by Corollary 13. Condition (2) implies (1) by Proposition 1. □

5 Generalized half-continuity

The following result is of spirit similar to [34, Theorem, p.473].

Theorem 36 (Bich). *Let $K \subset \mathbb{R}^n$ be a convex compact with nonempty interior, $f : K \rightarrow K$ be a measurable function such that for every $x \in K \setminus \text{Fix}(f)$, there exist an open neighborhood $U \subset K$ of x and $p \in \mathbb{R}^n$, such that $(p, f(x')) < (p, x')$*

- *boundary condition: for all $x' \in (U \cap \partial K) \setminus \text{Fix}(f)$ and*
- *inner condition: for almost everywhere (with respect to Lebesgue measure) $x' \in U \setminus \text{Fix}(f)$,*

then f admits a fixed-point.

Proof. Assume that f has no fixed-point. For $n \geq 1$, define $r_n : K \rightarrow [0, 1]$ by $r_n(x) = \min(1/n, d(x, K^c))$. Then each r_n is continuous and $r_n > 0$ on $\text{Interior}K$. Notice that the open ball $B(x, r_n(x)) \subset K$.

For $x \in \text{Interior}K$, $r > 0$ with $B(x, r) \subset K$, recall that the integral average of f over the ball $B(x, r)$ is given by $I(f, x, r) = \frac{\int_{B(x, r)} f(y) dy}{\text{Vol}(B(x, r))}$. Clearly, $I(\text{Id}, x, r) = x$. Given $p \in \mathbb{R}^n$, we have that $(p, I(f, x, r)) = I(pf, x, r)$. Moreover, if $g_1 < g_2$ almost everywhere on $B(x, r)$, then

$$I(g_1, x, r) < I(g_2, x, r). \quad (6)$$

We have $I(f, x, r) \in K$. Otherwise by Theorem 13, there is $p \in \mathbb{R}^n$ such that $(p, c) < (p, I(f, x, r))$ for every $c \in K$. In particular, $(p, f(y)) < (p, I(f, x, r))$ for every $y \in B(x, r)$, then by (6), we have $p(I(f, x, r)) < p(I(f, x, r))$, a contradiction.

Now define $f_n : K \rightarrow K$ by

$$f_n(x) = \begin{cases} I(f, x, r_n(x)), & x \in \text{Interior}K; \\ f(x), & x \in \partial K. \end{cases}$$

By absolute continuity of integral, each f_n is continuous on $\text{Interior}K$. Then we check that f_n is half-continuous on K .

Indeed, fix $x \in \partial K$, recall that there is an open neighborhood $U \subset K$ of x , $p \in \mathbb{R}^n$ such that $(p, f(x')) < (p, x')$ for all $x' \in U \cap \partial K$ and for almost everywhere $x' \in U$. There is an open neighborhood $U' \subset U$ of x such that for every $y \in U'$, $B(y, d(y, K^c)) \subset U$. The existence of such U' relies on $x \in \partial K$. Fix $x' \in U'$, if $x' \in \partial K$, then $p(f_n(x')) = p(f(x')) < p(x')$; if $x' \in \text{Interior}K$, then $B(x', r_n(x')) \subset B(x', d(x', K^c)) \subset U$. So $y \in B(x', r_n(x'))$ for almost everywhere we have $(p, f(y)) < (p, y)$, then by (6),

$$(p, f_n(x')) = (p, I(f, x', r_n(x'))) < (p, I(\text{Id}, x', r_n(x'))) = p(x').$$

In both cases we have $p(f_n(x')) < p(x')$ which shows f_n is half-continuous.

By Theorem 29, there is $x_n \in K$ fixed-point of f_n . As K is compact, there is $x_0 \in K$ such that $x_n \rightarrow x_0$. Recall again that there is $U_0 \subset K$ open neighborhood of x_0 and $p_0 \in \mathbb{R}^n$ such that $(p_0, f(x')) < (p_0, x')$ for almost every $x' \in U_0$. When n is large enough, we have $B(x_n, 1/n) \subset U_0$. Fix one such n . Then $f_n(x_n) = x_n \neq f(x_n)$, so $x_n \in \text{Interior}K$ and $f(x_n) = I(f, x_n, r_n(x_n))$. For almost every $y \in B(x_n, r_n(x_n))$ we have $(p_0, f(y)) < (p_0, y)$, so $(p_0, f_n(x_n)) < (p_0, I(\text{Id}, x_n, r_n(x_n))) = (p_0, x_n)$ which contradicts $f_n(x_n) = x_n$. \square

Example 3. Recall Dirichlet function $g : (0, 1) \rightarrow [0, 1]$ is defined by

$$g(x) = \begin{cases} 1 & x \in \mathbb{Q}; \\ 0 & x \notin \mathbb{Q}. \end{cases}$$

Take $K = [0, 1]^n \subset \mathbb{R}^n$ and define $f : K \rightarrow K$ by

$$f(x) = \begin{cases} x, & x \in \partial K; \\ (g(x_1), \dots, g(x_n)), & x \in \text{Interior}K. \end{cases}$$

Then f satisfies the condition of Theorem 36 but not half-continuous.

Example 4. Consider $K = \overline{B(0,1)} \subset \mathbb{R}^2$ and the function $f : K \rightarrow K$, given by

$$f(x, y) = \begin{cases} (1, 0), & 0 < x \leq 1, y = 0; \\ (x/2, y/2), & \text{else.} \end{cases}$$

Then f satisfies the condition of Theorem 36 but it is not half-continuous.

In fact, f is continuous on the subset $\overline{B(0,1)} \setminus [0,1] \times \{0\}$. The two ends of the line segment $[0,1] \times \{0\}$ are fixed. Let $g : K \rightarrow K$ be $g(x, y) = (x/2, y/2)$. Then almost everywhere $f = g$. For any $x \in (0,1)$, f is not half-continuous but satisfies the condition of Theorem 36 at $(x, 0)$.

Example 5. Consider $f : [0,1] \rightarrow [0,1]$ defined by $f(x) = 1$ when $x < 1$ and $f(1) = 0$, then for all but one $x \in [0,1]$ we have $f(x) > x$. But f admits no fixed-point. This example illustrates that the boundary condition in Theorem 36 is necessary.

We would like to raise the following question: Let $K \subset \mathbb{R}^n$ be a convex compact of nonempty interior, $f : K \rightarrow K$ be a measurable function such that for every $x \in K \setminus \text{Fix}(f)$, there exist an open neighborhood $U \subset K$ of x and $p \in \mathbb{R}^n$, such that $(p, f(x')) < (p, x')$ for $x' = x$ and almost everywhere $x' \in U$. Does f necessarily admit a fixed-point?

We give another result of similar nature.

Theorem 37 (Bich). *If $K \subset E$, $f : K \rightarrow K$ a mapping such that for every $x \in K \setminus \text{Fix}(f)$, there exist an open neighborhood $U \subset K$ of x , a subset $N \subset U \setminus \partial K$ meagre in K and $p \in E^*$, such that $(p, f(x')) < (p, x')$ for **all** $x' \in U \setminus (N \cup \text{Fix}(f))$, then f admits a fixed-point.*

The following proof is due to Philippe Bich.

Proof. Assume that f has no fixed-point. Then for every $x \in K$, there exist an open neighborhood $U_x \subset K$ of x , a meagre subset $N_x \subset U_x \cap \text{Interior}K$ and $p_x \in E^*$ such that for all $x' \in U_x \setminus N_x$, we have $(p_x, f(x')) < (p_x, x')$. As K is compact, the open cover $\{U_x\}_{x \in K}$ admits a finite subcover U_1, \dots, U_n with corresponding points x_1, \dots, x_n , subsets N_1, \dots, N_n , functions p_1, \dots, p_n . Take h_i given by Corollary 1. Define $T : K \rightarrow E^*$ by

$$T(x) = \sum_{i=1}^n h_i(x) p_i.$$

By Corollary 7, there is $x_0 \in K$ such that

$$(T(x_0), x_0) \leq (T(x_0), x), \forall x \in K. \quad (7)$$

Put $I = \{1 \leq i \leq n : h_i(x_0) > 0\}$. Then $\cap_{i \in I} U_i$ is an open neighborhood of x_0 . On the other hand, $\cup_{i \in I} N_i$ is meagre in K . By Theorem 4, $K \setminus \cup_{i \in I} N_i$ is dense in K and thus intersects the nonempty open $\cap_{i \in I} U_i$. That is

$$\cap_{i \in I} U_i \setminus \cup_{i \in I} N_i \neq \emptyset. \quad (8)$$

Take any $x \in \bigcap_{i \in I} U_i \setminus \bigcup_{i \in I} N_i$. Then for each $i \in I$, $x \in U_i \setminus N_i$, $p_i(f(x)) < p_i(x)$ by our assumption. Sum up, we have $\sum_{i \in I} h_i(x_0)(p_i, f(x)) < \sum_{i \in I} h_i(x_0)(p_i, x)$. That is

$$(T(x_0), f(x)) < (T(x_0), x), \forall x \in \bigcap_{i \in I} U_i \setminus \bigcup_{i \in I} N_i. \quad (9)$$

In particular, $T(x_0) \neq 0$. By (7) we have $x_0 \in \partial K$. In particular, $x_0 \in \bigcap_{i \in I} U_i \setminus \bigcup_{i \in I} N_i$. By (9), we have $(T(x_0), f(x_0)) < (T(x_0), x_0)$ which contradicts (7). This contradiction finishes the proof. \square

Remark 6. *As one can remark, the only use of "meagre" assumption is to derive the nonemptiness in (8). We can replace the adjective "meagre" by other properties describing how "thin" a set is. For example, when $E = \text{Span}(K)$ is finite dimensional, we can use "of zero measure" at the place of "meagre", and this strategy provides an alternative proof of Theorem 36.*

6 Generalization of Herings et al. theorem

Herings et al. [25] proved a fixed-point theorem for polytopes in \mathbb{R}^n . Later, Philippe Bich generalized it to convex compact subsets of \mathbb{R}^n . In this section only, we fix $(E; \langle \cdot, \cdot \rangle)$ an inner product space.

Definition 21. [25, Definition 2.1] *A correspondence $F : C \rightrightarrows E$ is called locally gross direction preserving if for every $x \in C \setminus \text{Fix}(F)$, there exist $U \subset C$ open neighborhood of x such that for every $x_1, x_2 \in U \setminus \text{Fix}(F)$ and every $y_1 \in F(x_1), y_2 \in F(x_2)$, we have*

$$\langle y_1 - x_1, y_2 - x_2 \rangle \geq 0.$$

The following theorem generalizes the finite dimensional case in [5, Theorem 3.7] to arbitrary inner product spaces.

Theorem 38. *A locally gross direction preserving correspondence $F : K \rightrightarrows K$ admits either maximal element or fixed-point.*

Proof. Assume the contrary. Since K is a compact metric space, it is separable by Lemma 7. By Exercise 6, replacing E with $\text{Span}(K)$, we may assume that E is separable. Again, replacing E by its completion, we may assume E is further complete. (These replacements do not change the topology of K .)

Check that F is half-continuous. Fix $x \in K$, take U given by Definition 21. As a subset of E , by Lemma 7 $D := \{y' - x' : x' \in U, y' \in F(x')\}$ is separable, so it has a countable dense subset $A = \{y_1 - x_1, y_2 - x_2, \dots\}$. Define

$$p = \sum_{i=1}^{+\infty} \frac{1}{2^i} \frac{y_i - x_i}{|y_i - x_i|},$$

then $p \in E$ is a well-defined element since E is complete. Fix $x' \in U$ and $y' \in F(x')$, then $\langle p, y' - x' \rangle \geq 0$. If it is an equality, then for every $a \in A$,

$\langle a, y' - x' \rangle = 0$, so $y' - x' \in \text{Span}(A)^\perp$. By Exercise 6, $\text{Span}(A) \subset \text{Span}(D)$ is dense in D . Therefore $y' - x' \in \text{Span}(D)^\perp \cap \text{Span}(D)$, so $y' - x' = 0$, i.e., $x' \in \text{Fix}(F)$, contradiction. So, we have $\langle p, y' - x' \rangle > 0$ which means F is half-continuous at x .

Now that F is half-continuous, it contradicts Theorem 31. \square

Remark 7. *It is stated in [39, thm 4.5] that locally gross direction preserving is equivalent to some half-continuity, which seems false. In \mathbb{R}^2 , two vectors on the same side of a hyperplane may of negative inner product, so the last line of the proof therein is invalid.*

7 Existence of approximate fixed-points

In this section, we will demonstrate how to deduce the existence of various approximate fixed-point via half-continuity.

Let $V \subset E$ be an open neighborhood of 0. We recall some classical notions.

Definition 22 (*V-fixed-point*). *Fix a correspondence $F : C \rightrightarrows E$. For $U \subset E$ an open neighborhood of 0, if a point $x_U \in C$ is such that there exists $y_U \in F(x_U)$ with $x_U - y_U \in V + U$, then x_U is called fixed up to $V + U$ (with respect to F). If for every open neighborhood U of 0, there exists a point fixed up to $V + U$, then F is said to admit a V -fixed-point.*

Definition 23 (*V-continuity*). *A correspondence $F : C \rightrightarrows E$ is called V -continuous, if for every $U \subset E$ open neighborhood of 0, every $x \in C$ not fixed up to $V + U$, there exist W an open neighborhood of 0 and $y_0 \in F(x)$ such that for every $x' \in x + W$ not fixed up to $V + U$, we have $y' - y_0 \in V + U$ for every $y' \in F(x')$.*

We compare these to [5, Definition 4.1]. When E is a normed space, F is a mapping and $V = B(0, r)$ is the open ball of radius $r > 0$, then F admits V -fixed-point is exactly that F admits r -fixed-point. If F is r -continuous in the sense of [10, Definition 1.2], then it is V -continuous.

The following theorem generalizes [10, Theorem 3.1] and [27, Theorem 3]. They both are valid in normed spaces. Bula's result shows that r -continuity guarantees existence of a $2r$ -fixed-point and Klee requires strong r -continuity.

Theorem 39 (Klee). *If E is locally convex, $V \subset E$ is a convex open neighborhood of 0, and $F : K \rightrightarrows K$ is a V -continuous correspondence, then F has a V -fixed-point.*

Proof. Assume the contrary, i.e., assume there is $U \subset E$ an open neighborhood of 0, such that there is no point fixed up to $V + U$. So, for every $x \in K$, every $y \in F(x)$, we have $x - y \notin V + U$. Since E is locally convex, by shrinking U we may assume U is convex. In particular, $U/2 + U/2 \subset U$ and there is no point fixed up to $V + U/2$.

We check that F is half-continuous. Fix $x \in K$, since F is V -continuous and x is not fixed up to $V + U/2$, there exist an open neighborhood $W \subset E$ of 0

and $y_0 \in F(x)$ (in particular, F has no maximal element), such that for every $x' \in x + W$, we have $y' - y_0 \in V + U/2$ for every $y' \in F(x')$. Applying Theorem 13 to $x - y_0$ and $V + U$, there is $p \in E^*$ such that $p(x - y_0) = 1$ while for every $z \in V + U$, $p(z) < 1$.

For every $x' \in x + (-\frac{U}{2} \cap W)$ and every $y' \in f(x')$,

$$y' - y_0 + x - x' \in (V + U/2) + U/2 \subset V + U,$$

so $p(y' - y_0 + x - x') < 1$ and

$$p(x' - y') = p(x - y_0) - p(y' - y_0 + x - x') > 0,$$

which shows that F is half-continuous at x .

By Theorem 29, F admits a fixed-point, contradiction. \square

Theorem 40. *Let $K \subset E$ be an n -simplex of vertices $\{v_0, \dots, v_n\}$ and $f : K \rightarrow K$ be a mapping. Consider $g : K \rightarrow E$ given by $g(x) = f(x) - x$, then there is $x_0 \in K$ such that $0 \in \text{co}(g(U))$ for every neighborhood $U \subset K$ of x_0 .*

Proof. For each $N \geq 1$, we take a decomposition K_N of K into N^n smaller simplexes, each of diameter $\frac{\text{diam}(K)}{N}$. Define $f_N : K \rightarrow K$ as follows: for each vertex v of K_N we let $f_N(v) = f(v)$. And then extend f_N barycentrically to K . That is, if x lies in a small simplex with vertices u_0, \dots, u_n , we may write barycentric coordinate of x as $x = \sum_{i=0}^n \lambda_i(x)u_i$ where $\sum_{i=0}^n \lambda_i(x) = 1$ and each $\lambda_i(x) \geq 0$, then let

$$f_N(x) = \sum_{i=0}^n \lambda_i(x)f(u_i).$$

In fact, f_N is well-defined and continuous. (This construction is inspired by [19, §3, Example 1].)

By Brouwer's fixed-point theorem, there is a fixed-point $x_N \in K$ of f_N . As K is compact, a subsequence of $\{x_N\}_{N \geq 1}$ converges to some $x_0 \in K$. As the diameter of decomposition tends to 0, for every neighborhood $U \subset K$ of x_0 , there exist $N \geq 1$, a small simplex with vertices $\{u_0, \dots, u_n\}$ of K_N containing x_N and contained in U . By our construction of f_N , we have

$$\sum_{i=0}^n \lambda_i(x_N)u_i = x_N = f_N(x_N) = \sum_{i=0}^n \lambda_i(x_N)f(u_i).$$

Thus

$$0 = \sum_{i=0}^n \lambda_i(x_N)g(u_i).$$

Therefore, we find $0 \in \text{co}(g(U))$. \square

The following result is a generalization of Eaves' theorem in [31, §3], where they requires $x + v(x) \in \text{Interior}K$ for every $x \in \partial K$.

Corollary 15 (Eaves). *Let $K \subset E$ be an n -simplex of vertices $\{v_0, \dots, v_n\}$ and $v : K \rightarrow E$ be a mapping such that for every $x \in \partial K$, we have $x + v(x) \in K$, then there is $x_0 \in K$ such that $0 \in \text{co}(v(U))$ for every neighborhood $U \subset K$ of x_0 .*

Proof. Define $f : K \rightarrow K$ as follows. For $x \in \text{Interior}K$, there is $\lambda_x > 0$ such that $x + \lambda_x v(x) \in K$ and we let $f(x) = x + \lambda_x v(x)$. For $x \in \partial K$, we let $f(x) = x + v(x)$. We conclude by Theorem 40. \square

Urai in [43, Theorem 18] tried to generalize Eaves' theorem from simplex to convex compact subsets. The generalization is based on his Lemma 17, whose proof is false. The p^x in his proof may not exist. We don't know whether his statement is true. The following is a variant of [43, Lemma 17].

Lemma 11. *Assume E is locally convex. For every mapping $f : K \rightarrow K$, define $g(x) = f(x) - x$, then there is $x_0 \in K$ such that every neighborhood $U \subset K$ of x_0 , $0 \in \overline{\text{co}}g(U)$.*

Proof. Otherwise, for every $x \in K$, there is an open neighborhood $U_x \subset K$ of x such that $0 \notin \overline{\text{co}}g(U)$. By Theorem 13, there is $p_x \in E^*$ such that $p_x(z) < 0$ for every $z \in \overline{\text{co}}g(U)$. In particular, for every $x' \in U_x$, $p_x(f(x') - x') < 0$, so f is half-continuous. By Theorem 29, f admits a fixed-point $x_0 \in K$, then $0 = g(x_0)$, a contradiction. \square

Now let E be a normed space. Given a correspondence $F : C \rightrightarrows E$ without maximal element, we recall the measure of discontinuity introduced in [13, §2, Definition]:

$$\delta(F) = \sup_{x \in C} \limsup_{r \rightarrow 0} \sup_{x' \in B(x,r), y \in F(x), y' \in F(x')} |y' - y|.$$

If a mapping $f : C \rightarrow E$ is r -continuous in the sense of [10, Definition 1.2], then $\delta(f) \leq r$. We end up with a generalization of [13, Theorem 2.1 (i)], where the finite dimensional cases are treated. We give a proof similar to that of Theorem 39.

Theorem 41 (Cromme-Diener). *Let $F : K \rightrightarrows K$ be a correspondence without maximal element, then there exists $x_0 \in K, y_0 \in F(x_0)$ such that $|y_0 - x_0| \leq \delta(F)$.*

Proof. Assume the contrary. Choose any selection $f : K \rightarrow K$ of F . Clearly, $\delta(f) \leq \delta(F)$.

We check that f is half-continuous. Fix $x \in K$, then $|f(x) - x| > \delta(f)$. There is $a > 0$ such that $|f(x) - x| > \delta(f) + a$. By Theorem 13, there is $p \in E^*$ such that $p(x - f(x)) = 1$ and $p(z) < 1$ for every $z \in B(0, \delta(f) + a)$. By definition,

$$\delta(f) \geq \limsup_{r \rightarrow 0} \sup_{x' \in B(x,r)} |f(x') - f(x)|,$$

so there is $r_0 \in (0, a/2)$ such that

$$\sup_{x' \in B(x, r_0)} |f(x') - f(x)| < \delta(f) + a/2.$$

For every $x' \in B(x, r_0)$,

$$|x - x' + f(x') - f(x)| \leq |x - x'| + |f(x') - f(x)| < r_0 + \delta(f) + a/2 < \delta(f) + a.$$

Therefore $p(x - x' + f(x') - f(x)) < 1$ and

$$p(x' - f(x')) = p(x - f(x)) - p(x - x' + f(x') - f(x)) > 0,$$

which means f is half-continuous at x .

By Theorem 29, f and hence F admits a fixed-point, a contradiction. \square

8 Applications to game theory

Let N be an index set. For each index $i \in N$, let E_i be a TVS and $X_i \subset E_i$ be a nonempty convex compact subset. Put $E = \prod_{i \in N} E_i$ and $X = \prod_{i \in N} X_i$. As the product of a family of TVS E is a TVS. By Tychonoff's theorem, X is compact. We use X_{-i} to denote $\prod_{j \neq i} X_j$ with typical element x_{-i} . The setting has a natural interpretation in economy. Namely N is the set of players and X_i is the pure strategy set for each player $i \in N$. Then X is the set of joint strategy profiles.

Definition 24 (Qualitative Game). [41, p.101][3, §2] Assume for each player $i \in N$ there is a correspondence $P_i : X \rightarrow X_i$, called player i 's preference correspondence, such that $x_i \notin P_i(x)$ for every $x \in X$. Then we call $G = (X_i, P_i)_{i \in N}$ a qualitative game.

Definition 25 (Nash equilibrium). A (pure strategy) Nash equilibrium of a qualitative game $G = (X_i, P_i)_{i \in N}$ is a profile $\bar{x} \in X$ such that $P_i(\bar{x}) = \emptyset$ for all $i \in N$.

Definition 26 (Normal Form Game). If each player $i \in N$ has a payoff function $u_i : X \rightarrow \mathbb{R}$, then we call $G = (X_i, u_i)_{i \in N}$ a normal form game.

A normal form game is a special case of qualitative game since u_i induces a total preorder¹ \prec_i on X by defining $x \prec_i y$ if $u_i(x) < u_i(y)$. In that case, for each $i \in N$, the preference correspondence is $P_i(x) = \{y_i \in X_i | u_i(y_i, x_{-i}) > u_i(x)\}$. We define the best reply correspondence $R_i : X \rightarrow X_i$ of player i by

$$R_i(x) = \{y_i \in X_i : u_i(y_i, x_{-i}) = \sup_{z_i \in X_i} u_i(z_i, x_{-i})\}.$$

¹A preference relation (total preorder) on a set S is a binary relation \prec satisfies the following properties:

- Transitivity: for every x, y and z in S , if $x \prec y$ and $y \prec z$ then $x \prec z$;
- Strong connectedness: for every x and y in S , $x \prec y$ or $y \prec x$.

Define the joint best reply correspondence $R : X \rightarrow X$ by $R(x) = \prod_i R_i(x)$. Then a Nash equilibrium of G is exactly a fixed-point of R .

We introduce $B_i : X \times \mathbb{R} \rightarrow X_i$ given by

$$B_i(x, a) = \{y_i \in X_i : u_i(y_i, x_{-i}) \geq a\}.$$

Let $\pi_i : E \rightarrow E_i$ be the projection mapping. Given a correspondence F taking value in E , we let $F_i = \pi_i \circ F$ be the projection correspondence of F taking value in E_i . If F is convex valued, then so is F_i for each i . For brevity, we call a correspondence ‘‘Kakutani’’ if it has a closed graph with nonempty convex values.

The following theorem states the existence of an equilibrium without quasi-convexity of payoff functions. It generalizes the result of Nishimura and Friedman [31, Theorem 1], since the authors assume further that

- N is finite;
- each E_i is finite dimensional;
- each payoff function $u_i : X \rightarrow \mathbb{R}$ is continuous.

Theorem 42 (Generalization of Nishimura-Friedman). *Let $G = (X_i, u_i)$ be a normal form game such that the best reply correspondence R has no maximal element. If for every $x \in X$ which is not a Nash equilibrium, there exist an open neighborhood $U \subset X$ of x and $p \in E^*$ such that for every $x(1), x(2) \in U$ which are not equilibria, for every $y(1) \in R(x(1)), y(2) \in R(x(2))$ we have $(p, y(1) - x(1)) \cdot (p, y(2) - x(2)) > 0$, then G admits an equilibrium.*

Remark 8. *Actually, their proof of Lemma 4 in [31] is false because in the last paragraph, they only derived $\beta = 1$, which is not sufficient for concluding $\sum_{k=1}^l \alpha_{kr} = 1$ as stated in Lemma 4. Partial problem is that their selection f is discontinuous. This flaw renders their proof of Theorem 1 incomplete.*

Proof. Assume the contrary. For every $x \in K$, take p and U as in the assumption, then either $p(y' - x') > 0$ for all $x' \in U, y' \in R(x')$ or $-p(y' - x') > 0$ for all $x' \in U, y' \in R(x')$. We see that R is half-continuous. But then R contradicts Theorem 31. \square

We give some other existence results without continuity of payoff functions. First some necessary mathematical preparations.

Theorem 43. *Assume that for each $i \in N$, the dual E_i^* separates E_i . Let C be a subset of X . If for every $x \in X \setminus C$, there exist an open neighborhood $U_x \subset X$ of x and two correspondences $F_x, B_x : U_x \rightarrow X$ such that:*

- (1) F_x is a Kakutani correspondence;
- (2) B_x is convex valued;
- (3) for every $y \in U_x$, there is $i \in N$ with $y_i \notin B_{x,i}(y)$;

- (4) for every finitely many points $x(1), \dots, x(n) \in X \setminus C$, every $i \in N$, there is $1 \leq j_i \leq n$ satisfying $F_{x(j_i), i}(z) \subset B_{x(\alpha), i}(z)$ for every $1 \leq \alpha \leq n$, and $z \in \bigcap_{\alpha=1}^n U_{x(\alpha)}$.

Then C is nonempty.

Proof. Assume the contrary.

We have E^* separates E . In fact, for every non zero $v \in E$, there is $i \in N$ such that $v_i \neq 0$. By assumption, there is $p_i \in E_i^*$ such that $p_i(v_i) \neq 0$. Now $p = p_i \circ \pi_i \in E^*$ satisfies $p(v) \neq 0$.

For each $x \in X$, define $B'_x : U_x \rightarrow X$ by $B'_x = \prod_i B_{x, i}$. Note that $B_x \subset_{U_x} B'_x$. By (2) B'_x is convex valued. By (3) B'_x has no fixed-point. (This is stronger than B_x has no fixed-point.)

The open cover $\{U_x\}_{x \in X}$ of X admits a finite subcover $\{U_\alpha\}$. By Corollary 2, there is a closed refinement $\{F_\alpha\}$. For every $x \in X$, the index set $\Lambda(x) := \{\alpha : x \in F_\alpha\}$ is nonempty. The sets $V_x = \bigcap_{\alpha: U_\alpha \ni x} U_\alpha$ and $W_x = V_x \cap \bigcap_{\alpha \notin \Lambda(x)} F_\alpha^c$ are open neighborhoods in X of x .

For every $x' \in W_x$, every $\alpha \in \Lambda(x')$, $x' \in W_x \cap F_\alpha$, so $\alpha \in \Lambda(x)$. We get

$$\Lambda(x') \subset \Lambda(x). \quad (10)$$

For every $i \in N, x \in X$, by (4) there is j_i among the finitely many indices α with $x \in U_\alpha$ such that

$$F_{x(j_i), i}(z) \subset B_{x(\alpha), i}(z) \quad (11)$$

for every α with $x \in U_\alpha$ and every $z \in \bigcap V_x$. Then we define $H_{x, i} : V_x \rightarrow X_i$ by $H_{x, i} = F_{x(j_i), i}$. By (1) $H_{x, i}$ is a Kakutani correspondence. We may rewrite (11) as

$$H_{x, i} \subset_{V_x} B_{x(\alpha), i} \quad (12)$$

for every α with $x \in U_\alpha$.

For every $x \in X$ we define $H_x : V_x \rightarrow X$ by $H_x = \prod_{i \in N} H_{x, i}$. Then (12) gives

$$H_x \subset_{V_x} B'_{x(\alpha)} \quad (13)$$

for every α with $x \in U_\alpha$. By Exercise 10, H_x is a Kakutani correspondence.

The open cover $\{W_x\}_{x \in X}$ of X admits a finite subcover $\{W_\beta\}$ with corresponding points $x(\beta)$ and correspondences $H_\beta : V_\beta \rightarrow X$. Take functions h_β given by Corollary 1. We define $H : X \rightarrow X$ by $H(x) = \sum_\beta h_\beta(x) H_\beta(x)$, then by Exercise 11 and [14, Theorem 17.32], H is a Kakutani correspondence.

For every $x \in X$, every $\alpha \in \Lambda(x)$, every β with $h_\beta(x) > 0$, we have $x \in W_\beta \subset V_\beta$, so by (10) $\alpha \in \Lambda(x) \subset \Lambda(x(\beta))$, then $x(\beta) \in F_\alpha \subset U_\alpha$. By (13) we have $H_\beta(x) \subset B'_{x(\alpha)}(x)$. The latter is convex, so

$$H(x) \subset B'_{x(\alpha)}(x). \quad (14)$$

Recall that E^* separates E . By Theorem 33, H has a fixed-point $x_0 \in X$. Recall that $\Lambda(x_0)$ is nonempty. Take every $\alpha \in \Lambda(x_0)$, then by (14) $x_0 \in H(x_0) \subset B'_\alpha(x_0)$, a contradiction. \square

The reader can see that the only use of the dual separation assumption is the application of Theorem 33. With this observation the following variant is immediate.

Theorem 44. *If for every $x \in X \setminus C$, there exist an open neighborhood $U_x \subset X$ of x , a finite dimensional subspace $E(x) \subset E$ and two correspondences $B_x : U_x \rightarrow X$ and $F_x : U_x \rightarrow X \cap E(x)$ such that:*

- (1) F_x is a Kakutani correspondence;
- (2) B_x is convex valued;
- (3) for every $y \in U_x$, there is $i \in N$ with $y(i) \notin B_{x,i}(y)$;
- (4) for every finitely many points $x(1), \dots, x(n) \in X \setminus C$, every $i \in N$, there is $1 \leq j_i \leq n$ satisfying $F_{x(j_i),i}(z) \subset B_{x(\alpha),i}(z)$ for every $1 \leq \alpha \leq n$, and $z \in \bigcap_{\alpha=1}^n U_{x(\alpha)}$.

Then C is nonempty.

Proof. Repeat last proof. We only indicate the difference. Note that $H : X \rightarrow X$ takes value in a finite dimensional subspace $E' \subset E$. Then by Theorem 33, $H : X \cap E' \rightarrow X \cap E'$ has a fixed-point. \square

We derive three results concerning existence of equilibria without continuity of payoff functions. The first is a generalization of [2, Theorem 2.2], in the sense that

- we allow infinitely many players;
- the ambient spaces are more general than locally convex ones;
- the payoff functions $u_i : X \rightarrow \mathbb{R}$ are not restricted to be bounded.

Corollary 16 (Generalization of Barelli-Meneghel). *Assume each E_i^* separates E_i . Let $G = (X_i, u_i)$ be a normal form game. If for every $x \in X$ not equilibrium, there exist $\alpha_x \in \mathbb{R}^N$, an open neighborhood $U_x \subset X$ of x and a Kakutani correspondence $F_x : U_x \rightarrow X$ such that*

- (a) $F_{x,i}(y) \subset B_i(y, \alpha_{x,i})$ for all $i \in N$, $y \in U_x$;
- (b) for each $y \in U_x$, there is $i \in N$ with $y(i) \notin coB_i(y, \alpha_{x,i})$.

Then G has pure strategy Nash equilibrium.

Proof. Let $C \subset X$ be the set of pure strategy Nash equilibrium. We are going to apply Theorem 43. Our assumption gives condition (1). Define $B_x : X \rightarrow X$ by $B_x(y) = \prod_i coB_i(y, \alpha_{x,i})$. Then (2) is verified and (b) becomes condition (3).

Check (4). For every $x(1), \dots, x(n) \in X \setminus C$, every $i \in N$, there is $1 \leq j_i \leq n$ with $\alpha_{x(j_i),i} = \max_{1 \leq j \leq n} \alpha_{x(j),i}$. Then for every $1 \leq j \leq N$ and every $z \in$

$U_{x(j_i)}$, we have $\alpha_{x(j),i} \leq \alpha_{x(j_i),i}$, so $B_{x(j),i}(z) \supset B_i(z, \alpha_{x(j),i}) \supset B_i(z, \alpha_{x(j_i),i}) \supset F_{x(j_i),i}(z)$, where the last inclusion is provided by (a).

Now that all conditions are verified, we conclude by applying Theorem 43. \square

Remark 9. *Although the authors don't require that F_x is convex valued and they emphasize this point after their Definition 2.3, their weak assumption makes their proof wrong. In fact, their Φ on [2, p.823] is not compact-valued in general, as shown in [14, Example 5.34].*

The following generalizes [30, Theorem 3.4]. Strictly speaking, the authors do not assume that each E_i is Hausdorff while we need it. Nevertheless, as Reny acknowledges, this mathematical refinement has not found any known economic applications yet.

We recall some terminology following [30, §3]. Fix a normal form game $G = (X_i, u_i)$. For each $i \in N$, fix a correspondence $\mathcal{X}_i : X \rightarrow X_i$, let $\mathcal{X} = \prod_i \mathcal{X}_i$ and call it a restriction operator. For each $i \in N$, we define $B_{\mathcal{X},i}, C_{\mathcal{X},i} : X \times \mathbb{R} \rightarrow X_i$ by

$$B_{\mathcal{X},i}(x, a) = \{y_i \in X_i(x) : u_i(y_i, x_{-i}) \geq a\} \quad (15)$$

$$C_{\mathcal{X},i}(x, a) = \text{co}(B_{\mathcal{X},i}(x, a)). \quad (16)$$

When we take $\mathcal{X} = \text{Id}$, then $B_{\mathcal{X},i}(x, a) = B_i(x, a)$.

Corollary 17 (Generalization of McLennan-Monteiro-Tourky). *Let $G = (X_i, u_i)$ be a normal form game. Assume for every $x \in X$ not equilibrium, there is open neighborhood $U_x \subset X$ of x , a finite dimensional subspace $E(x) \subset E$ and $\alpha_x \in \mathbb{R}^N$ such that*

- (a) *there is a Kakutani correspondence $F_x : U_x \rightarrow X \cap E(x)$;*
- (b) *$F_{x,i}(z) \subset C_{\mathcal{X},i}(z, \alpha_{x,i})$ for all $z \in U_x, i \in N$;*
- (c) *for every $z \in U_x$, there is $i \in N$ such that $z_i \notin C_{\mathcal{X},i}(z, \alpha_{x,i})$.*

Then G has a equilibrium.

Proof. The proof is parallel to that of Corollary 16 except that we apply Theorem 44 at the place of Theorem 43. \square

Remark 10. *There is a subtle minor point. In [30, Definition 3.1-3.3], at the place of our (a,b) their original assumption is the following:*

- (d) *there is a finite cover $C_1, \dots, C_{J(x)}$ of U_x , each closed in U_x such that for each $1 \leq j \leq J(x)$, there is $y(x, j) \in X$ such that $y(x, j)_i \in B_{\mathcal{X},i}(z, \alpha_{x,i})$ for all $z \in C_j$.*

Given (d), we define $F_x : U_x \rightarrow X$ by $F_x(z) = \text{co}(y(j) | z \in F_j)$ then F takes value in a finite dimensional subspace $E_x = \text{Span}(y(1), \dots, y(J(x)))$. For every $z \in U_x, i \in N$, we have $F_{x,i}(z) \subset C_{\mathcal{X},i}(z, \alpha_{x,i})$. By Exercise 9 $F_x : U_x \rightarrow X$ has closed graph so it is a Kakutani correspondence. Therefore their (d) implies our (a,b).

In [49] the following result is proved by reducing to games with finitely many players.

Corollary 18 (Debreu, Glicksberg, Fan). [49, Theorem 3.1][15, Theorem 1] *Let $G = (X_i, u_i)$ be a normal form game. Assume for every $i \in N$, u_i is continuous and $u_i(\cdot, x_{-i})$ is quasiconcave on X_i for every $x_{-i} \in X_{-i}$. Then there exists a Nash equilibrium.*

Proof. We are going to apply Corollary 17. Take the restriction operator $\mathcal{X} = \text{Id}$. For every $x \in X$ not equilibrium, there exist $i_x \in N$ and $y_{i_x} \in X_{i_x}$ such that $u_{i_x}(y_{i_x}, x_{-i_x}) > u_{i_x}(x)$. We choose $\alpha_{x, i_x} = \frac{1}{2}(u_{i_x}(y_{i_x}, x_{-i_x}) + u_{i_x}(x))$. For other index $i \neq i_x$, we choose

$$\alpha_{x, i} = \min_{z \in X} u_i(z). \quad (17)$$

Since u_{i_x} is continuous, there is an open neighborhood $U_x \subset X$ of x such that

$$u_{i_x}(y_{i_x}, z_{-i_x}) > \alpha_{x, i_x} > u_i(z). \quad (18)$$

Define mapping $F_x : U_x \rightarrow X$ of constant value (y_{i_x}, x_{-i_x}) .

Clearly F_x verifies (a). We check (b). If $i = i_x$, then $F_{i_x}(z) = y_{i_x} \in B_{i_x}(z, \alpha_{x, i_x})$ by (18). If $i \neq i_x$, then $F_{x, i}(z) = x_i \in B_i(z, \alpha_{x, i})$ by (17). Since $u_i(\cdot, z_{-i})$ is quasiconcave, each $B_i(z, a)$ is convex. By (18), we have $z_{i_x} \notin B_{i_x}(z, \alpha_{x, i_x})$ hence (c). Now we can conclude by Corollary 17. \square

We also generalize [3, Theorem 2.2].

Theorem 45 (Generalization of Barelli-Soza). *Assume each E_i^* separates E_i . Let $G = (X_i, P_i)$ be a qualitative game. For each $i \in N$, fix a convex valued correspondence $B_i : X \rightrightarrows X_i$. Assume that for every $x \in X$ not equilibrium, there exist an open neighborhood $U_x \subset X$ of x , a Kakutani correspondence $F_x : U_x \rightrightarrows X$ and a player $i_x \in N$ such that*

- (a) for every $z \in X$ with $x \in U_z$, we have $F_{z, i_x}(x) \subset B_{i_x}(x)$;
- (b) $x_{i_x} \notin B_{i_x}(x)$.

Then there exists an equilibrium.

Proof. Assume the contrary. The open cover $\{U_x\}_{x \in X}$ admits a finite subcover $\{U_\alpha\}$ with corresponding correspondence $F_\alpha : U_\alpha \rightrightarrows X$. Take functions h_α given by Corollary 1. Define $F : X \rightrightarrows X$ by

$$F(x) = \sum_{\alpha} h_{\alpha}(x) F_{\alpha}(x).$$

By Exercise 11 and [14, Theorem 17.32] F is a Kakutani correspondence.

By (a) for every $x \in X$, there is $i_x \in N$ such that for every α with $h_{\alpha}(x) > 0$, $x \in U_{\alpha}$, so $F_{\alpha, i_x}(x) \subset B_{i_x}(x)$. The latter is convex valued, so

$$F_{i_x}(x) \subset \sum_{\alpha} h_{\alpha}(x) F_{\alpha, i_x}(x) \subset B_{i_x}(x).$$

By (b), $x \notin F(x)$. But F has a fixed-point by Theorem 33, a contradiction. \square

Note that Corollary 16 and 17, Theorem 45 all three of them are generalizations of the well-known result due to Reny [35, Theorem 3.1]. We end up with another existence result in qualitative games.

Theorem 46 (Toussaint). *Let $G = (X_i, P_i)$ be a qualitative game such that for every $i \in N$, $\{x \in X | P_i(x) \neq \emptyset\}$ is open in X . Assume that for every $i \in N$ and every $x \in X$ not equilibrium, there exist $U \subset X$ open neighborhood of x and a correspondence of open preimages $Q_i : U \rightarrow X_i$ such that $P_i \subset_U Q_i$ and $x'_i \notin \text{co}(Q_i(x'))$ for every $x' \in U$. Then there exists an equilibrium.*

Proof. Assume the contrary. For every $x \in X$ let $I(x) = \{i \in N : P_i(x) \neq \emptyset\}$. By our assumption $I(x)$ is non empty. Define a correspondence $F : X \rightarrow X$ by

$$F(x) = \prod_{i \in I(x)} P_i(x) \prod_{j \notin I(x)} X_j.$$

Then F has no maximal element. For every $x \in X$, take one $i_x \in I(x)$. Then $V_x := \{y \in X | P_{i_x}(y) \neq \emptyset\}$ is an open neighborhood of x . By assumption, there is another open neighborhood U_x of x and a correspondence $Q_{i_x} : U_x \rightarrow X_{i_x}$ of open preimages such that $P_{i_x} \subset_{U_x} Q_{i_x}$ and $x'_{i_x} \notin \text{co}(Q_{i_x}(x'))$ for every $x' \in U_x$. Then $W_x = U_x \cap V_x$ is an open neighborhood of x . Define a correspondence $G_x : W_x \rightarrow X$ by $G_x(z) = \text{co}(Q_{i_x}(z)) \prod_{j \neq i_x} X_j$, then G_x is convex valued without fixed point and $F \subset_{W_x} G_x$. By Theorem 22, F admits a maximal element, a contradiction. \square

Remark 11. *If for every $i \in N$, P_i is of open preimages and for every $x \in X$, $P_i(x)$ is convex and $x_i \notin P_i(x)$, then for each x we may take $U = X$ and $Q_i = P_i$. Then Theorem 46 reduces to [49, Theorem 3.2]. If each P_i is KF majorized in the sense of [41], for each $x \in X$ we may take (U, Q_i) to be its local KF majorant and then we recover [41, Theorem 2.4].*

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