



**HAL**  
open science

# A Nonlinear Elliptic Equation with a Degenerate Diffusion and a Source Term in $L^1$

Guillaume Leloup, Roger Lewandowski

► **To cite this version:**

Guillaume Leloup, Roger Lewandowski. A Nonlinear Elliptic Equation with a Degenerate Diffusion and a Source Term in  $L^1$ . Applied Mathematics Letters, 2024, 153, pp.109077. hal-04392244v2

**HAL Id: hal-04392244**

**<https://hal.science/hal-04392244v2>**

Submitted on 22 Jan 2024

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.



Distributed under a Creative Commons Attribution 4.0 International License

# A Nonlinear Elliptic Equation with a Degenerate Diffusion and a Source Term in $L^1$

Guillaume Leloup<sup>1</sup> and Roger Lewandowski<sup>2</sup>

<sup>1,2</sup>Univ Rennes, IRMAR, UMR CNRS 6625, and Odyssey Team, INRIA Rennes, France, E-mail: Guillaume.Leloup@univ-rennes.fr, Roger.Lewandowski@univ-rennes1.fr

## Abstract

We study a simplified equation governing turbulent kinetic energy  $k$  in a bounded domain, arising from turbulence modeling where the eddy diffusion is given by  $\varrho(x) + \varepsilon$ , with  $\varrho$  representing the Prandtl mixing length of the order of the distance to the boundary, and a right-hand side in  $L^1$ . We obtain estimates of  $\sqrt{\varrho}\nabla k$  in  $L^q$  spaces and we establish the convergence toward the formal limit equation in the sense of the distributions as  $\varepsilon$  goes to 0.

**Key words** : Fluid mechanics, Turbulence models, degenerate operators, Navier-Stokes Equations, Turbulent Kinetic Energy.

**2010 MSC**: 76D05, 35Q30, 76F65, 76D03, 35Q30.

## 1 Introduction

In this paper we consider the non linear elliptic equation

$$(1.1) \quad \begin{cases} -\operatorname{div}((\varrho(x) + \varepsilon)\nabla k) + \frac{k|k|^{\alpha-1}}{\varrho + \varepsilon} = \mathbb{D}(x) & \text{in } \Omega, \\ k = 0 & \text{at } \Gamma, \end{cases}$$

where  $\Omega \subset \mathbb{R}^N$  ( $N = 2, 3$ ) is a  $\mathcal{C}^2$  bounded domain,  $\Gamma = \partial\Omega$ ,  $\alpha > 0$ ,  $\mathbb{D} \geq 0 \in L^1(\Omega)$ ,  $\varrho : \Omega \rightarrow \mathbb{R}^+$  is of  $\mathcal{C}^2$  class behaving like the distance to  $\Gamma$  (see (2.1)), and  $\varepsilon > 0$ .

Equation (1.1) is a by-product of the equation for the turbulent kinetic energy used in turbulence models (see [3, 12, 14]), in which  $\nu_{\text{turb}} = \varrho + \varepsilon$  represents an eddy diffusion coefficient in a simplified manner. Here  $\varepsilon$  is a molecular diffusion coefficient,  $\mathbb{D}$  is the production of energy,  $\mathcal{E} = k|k|^{\alpha-1}(\varrho + \varepsilon)^{-1}$  is the dissipation of energy. In physical models  $\alpha = 3/2$ , the function  $\varrho$  represents the Prandtl Mixing length.

Elliptic equations with r.h.s in  $L^1$  have been intensively studied since the seminal work of Boccardo-Gallouët [2], in the context of turbulent and/or thermodynamical coupling [3, 4, 6, 10, 11]. Therefore, we know the existence of a non negative distributional solution to (1.1), in  $W_0^{1,p}(\Omega)$ , for all  $p < N'$ . Moreover, due to the fact that the term  $\mathcal{E}$  is monotonic, there is a unique renormalized solution to this equation [5, 13].

We address in this paper the problem of letting  $\varepsilon$  go to zero in equation (1.1), which is related to the common question of determining whether molecular diffusion can be neglected in comparison to the eddy diffusion terms in the models (see, for instance, [7]). The challenge here arises from the fact that  $\varrho$  vanishes at  $\Gamma$ .

This question was previously examined in the relaxed case in [1, section 3], where we set  $\nu_{\text{turb}} = \varrho^\beta + \varepsilon$  for  $\beta < 1$ . We then demonstrated the convergence of the corresponding solutions to the limit problem in  $W_0^{1,q}(\Omega)$  when  $\beta < \frac{1}{2N-1}$  and for any  $q < \frac{N'(1-\beta)}{1+\beta}$ . However, in most physical models,  $\beta = 1$ , and the situation becomes more intricate. Our main result is the Theorem (3.1) which states that when  $1 < \alpha < \frac{N}{N-2}$ , the family  $(k_\varepsilon)_{\varepsilon>0}$  converges in a certain sense, up to a subsequence, to a distributional solution  $k \in W_{\text{loc}}^{1,q}(\Omega) \cap L^\alpha(\frac{dx}{\rho})$  for  $q < \frac{2\alpha}{1+\alpha} = q_c$ , of the limit equation given by (3.1) below. Notice that the case  $\alpha = 3/2$  is achieved whether  $N = 2$  or  $N = 3$ . If we cannot say that  $k = 0$  at the boundary at this stage, this will be possible in a suitable sense because  $k \in L^\alpha(\frac{dx}{\rho})$ , which will be discussed in a next paper. Finally, the case  $\frac{1}{2N-1} \leq \beta < 1$  is in progress.

The paper is organized as follows. In the first section, we derive *a priori* estimates of  $\varrho^{1/2}\nabla k$  in  $L^q$  spaces, following the approach of Boccardo-Gallouët [2] (also discussed in [8]), adapted to the present case. In the second section, we pass to the limit in the equations in the sense of the distributions.

## 2 Estimates in weighted spaces

### 2.1 Framework and $L^\alpha$ estimates

We assume that  $\varrho \in \mathcal{C}^2(\overline{\Omega})$  satisfies:

$$(2.1) \quad \lim_{\substack{d(x,\Gamma) \rightarrow 0 \\ x \in \Omega}} \frac{\varrho(x)}{d(x,\Gamma)} = 1, \quad \forall n > 0, \quad \varrho_n = \inf_{\substack{d(x,\Gamma) \geq \frac{1}{n} \\ x \in \Omega}} \varrho(x) > 0.$$

Let  $T_n$  be the truncation at height  $n$ , namely, the odd function defined by  $\forall x \geq 0$ ,  $T_n(x) = \min(n, x)$ , and we set  $G_n = T_{n+1} - T_n$ . Finally, let  $\eta > 0$  and  $H_\eta$  be the  $\mathcal{C}^{0,1}$  approximation of the Heaviside function, namely, the odd function given by,  $\forall 0 \leq x \leq \eta$ ,  $H_\eta(x) = \frac{x}{\eta}$ ,  $\forall x \geq \eta$ ,  $H_\eta(x) = 1$ . These three functions are Lipschitz continuous, the derivatives of which have a finite number of discontinuities. Moreover, they vanish at 0. Therefore, according to Stampacchia [15], if  $F$  is one of these three functions and  $v \in H_0^1(\Omega)$ , then  $F(v) \in H_0^1(\Omega)$  and  $\nabla(F(v)) = F'(v)\nabla v$ , which will be used all along the following proof. In the following, given  $s > 0$  and  $p > 1$ , we set

$$(2.2) \quad W^{s,p^-} = \bigcap_{1 \leq q < p} W^{s,q}.$$

We consider the renormalized solution  $k$  of (1.1). We do not give the full definition of the renormalized solution, which is rather intricate, referring to [5, 13]. We retain that this solution is essentially a distributional solution that satisfies several additional properties, such as  $\forall n \in \mathbb{N}$ ,  $T_n(k) \in H_0^1(\Omega)$ ,  $k \in W_0^{1,(N')^-}(\Omega)$ , and mainly for any  $F \in W^{1,\infty}(\mathbb{R})$  entering in Stampacchia's framework, such as  $T_n$ ,  $G_n$  or  $H_\eta$ ,

$$(2.3) \quad \int_{\Omega} (\varrho + \varepsilon) F'(k) |\nabla k|^2 + \int_{\Omega} \frac{k|k|^{\alpha-1}}{\varrho} F(k) = \int_{\Omega} \mathbb{D}F(k).$$

Moreover, we know from [9] that  $k \geq 0$  *a.e.* in  $\Omega$ , so that  $k|k|^{\alpha-1} = k^\alpha$ . The following estimate is essential in our proof, telling among other that  $k \in L^\alpha(\Omega)$ .

**Lemma 2.1.** *Let  $k$  be the renormalized solution to problem (1.1). Then:*

$$(2.4) \quad 0 \leq \int_{\Omega} \frac{k^\alpha}{\varrho + \varepsilon} \leq \|\mathbb{D}\|_{0,1}.$$

*Proof.* Take  $H_\eta(k)$  as test in (1.1). Then by (2.3), as  $k \geq 0$ ,  $|H_\eta| \leq 1$ :

$$\int_{\Omega} \varrho \tilde{\mu}(k) H'_\eta(k) |\nabla k|^2 + \int_{\Omega} H_\eta(k) \frac{k^\alpha}{\varrho + \varepsilon} = \int_{\Omega} \mathbb{D}H_\eta(k) \leq \|\mathbb{D}\|_{0,1,\Omega}.$$

Since  $\varrho \tilde{\mu}(k) H'_\eta(k) |\nabla k|^2 \geq 0$  and  $H_\eta(k) \xrightarrow{\eta \rightarrow 0} 1$  a.e in  $\Omega$ , by Fatou's lemma

$$0 \leq \int_{\Omega} \frac{k^\alpha}{\varrho + \varepsilon} \leq \liminf_{\eta \rightarrow 0} \int_{\Omega} H_\eta(k) \frac{k^\alpha}{\varrho + \varepsilon} \leq \|\mathbb{D}\|_{0,1},$$

which proves (2.4). □

## 2.2 Ladder process

**Lemma 2.2.** *let  $q < N'$  and*

$$(2.5) \quad r > \frac{q}{2 - q}.$$

*Then  $\forall n_0 \in \mathbb{N}$ ,  $\exists \lambda_1(n_0), \lambda_2(n_0) > 0$  such that  $\lim_{n_0 \rightarrow \infty} \lambda_2(n_0) = 0$  and such that*

$$(2.6) \quad \int_{\Omega} \varrho^{\frac{q}{2}} |\nabla k|^q \leq \lambda_1(n_0) + \lambda_2(n_0) \|k\|_{0,r,\Omega}^{\frac{r(2-q)}{2}}.$$

*Proof.* Let  $B_n = \{x \in \Omega; n \leq k \leq n+1\}$ . We take  $G_n(k) = T_{n+1}(k) - T_n(k)$  as test in (1.1) and use (2.3). As  $G'_n(k)$  vanishes outside  $B_n$  and  $0 \leq G_n(k) \leq 1$ , we obtain

$$(2.7) \quad \varepsilon \int_{B_n} |\nabla k|^2 + \int_{B_n} \varrho |\nabla k|^2 + \int_{\Omega} \frac{k^\alpha}{\varrho + \varepsilon} G_n(k) = \int_{\Omega} \mathbb{D}G_n(k) \leq \|\mathbb{D}\|_{0,1,\Omega}.$$

Then by (2.7) we get:

$$(2.8) \quad \int_{B_n} \varrho |\nabla k|^2 \leq \|\mathbb{D}\|_{0,1,\Omega} =: M.$$

Let us write the following decomposition “into slices”:

$$(2.9) \quad \int_{\Omega} \varrho^{q/2} |\nabla k|^q = \sum_{n=0}^{+\infty} \int_{B_n} \varrho^{q/2} |\nabla k|^q.$$

Therefore, combining Hölder inequality and (2.8) yields

$$(2.10) \quad \forall n \in \mathbb{N}, \quad \int_{B_n} \varrho^{q/2} |\nabla k|^q \leq \left( \int_{B_n} \varrho |\nabla k|^2 \right)^{q/2} |B_n|^{\frac{2-q}{2}} \leq M^{q/2} |B_n|^{\frac{2-q}{2}}.$$

Let  $n_0 \in \mathbb{N}$ . On one hand,

$$(2.11) \quad \sum_{n=0}^{n_0} \int_{B_n} \varrho^{q/2} |\nabla k|^q \leq (n_0 + 1) \max(M, |\Omega|).$$

On the other hand, let  $r$  that satisfies (2.5). Then,  $\int_{B_n} k^r dx \geq \int_{B_n} n^r = n^r |B_n|$ , giving

$$(2.12) \quad M^{q/2} |B_n|^{\frac{2-q}{2}} \leq \frac{M^{q/2}}{n^{\frac{r(2-q)}{2}}} \left( \int_{B_n} k^r \right)^{\frac{2-q}{2}}.$$

Combining (2.10) and (2.12), adding the terms for  $n > n_0$ , by Hölder inequality,

$$(2.13) \quad \begin{aligned} \sum_{n=n_0+1}^{+\infty} \int_{B_n} \varrho^{q/2} |\nabla k|^q &\leq M^{q/2} \sum_{n=n_0+1}^{+\infty} \frac{1}{n^{\frac{r(2-q)}{2}}} \left( \int_{B_n} k^r \right)^{\frac{2-q}{2}} \\ &\leq M^{q/2} \left[ \sum_{n=n_0+1}^{+\infty} \left( \int_{B_n} k^r \right) \right]^{\frac{2-q}{2}} \left[ \sum_{n=n_0+1}^{+\infty} \frac{1}{n^{\frac{r(2-q)}{q}}} \right]^{q/2}. \end{aligned}$$

The term in the last bracket is a convergent series as a consequence of (2.5). Then let

$$\lambda_1(n_0) = (n_0 + 1) \max(M, |\Omega|) \quad \text{and} \quad \lambda_2(n_0) = M^{q/2} \left[ \sum_{n=n_0+1}^{+\infty} \frac{1}{n^{\frac{r(2-q)}{q}}} \right]^{q/2}.$$

Inequalities (2.11) and (2.13) become

$$(2.14) \quad \int_{\Omega} \varrho^{q/2} |\nabla k|^q \leq \lambda_1(n_0) + \lambda_2(n_0) \left( \int_{\Omega} k^r \right)^{\frac{2-q}{2}} = \lambda_1(n_0) + \lambda_2(n_0) \|k\|_{0,r,\Omega}^{\frac{r(2-q)}{2}},$$

hence (2.6).  $\square$

### 3 Passing to the limit

**Theorem 3.1.** *Assume  $1 < \alpha < \infty$  if  $N = 2$ ,  $1 < \alpha < 3$  if  $N = 3$ . Let  $k_\varepsilon$  be the renormalized solution to equation (1.1). Then there exists  $(\varepsilon_n)_{n \in \mathbb{N}}$  with  $\lim_{n \rightarrow \infty} \varepsilon_n = 0$  and such that for any  $q < \frac{2\alpha}{1+\alpha} = q_c$ , for any  $\omega \subset\subset \Omega$  the sequence  $(k_{\varepsilon_n})_{n \in \mathbb{N}}$  converges weakly in  $W^{1,q}(\omega)$  to  $k \in W_{loc}^{1,q_c}(\Omega) \cap L^\alpha(\Omega, \varrho^{-1} dx)$ , solution in the sense of the distributions of the equation*

$$(3.1) \quad -\operatorname{div}(\varrho(x) \nabla k) + \frac{k|k|^{\alpha-1}}{\varrho} = \mathbb{D}(x).$$

*Proof. Step 1 : extracting subsequences.* Notice first that because  $\varrho \in \mathcal{C}^2(\overline{\Omega})$ , we deduce from (2.4) that  $(k_\varepsilon)_{\varepsilon>0}$  is bounded in  $L^\alpha(\Omega)$ . By (2.6) with  $\alpha = r$ , which imposes  $\frac{q}{2-q} < \alpha$  because of (2.5), or equivalently

$$(3.2) \quad q < \frac{2\alpha}{1+\alpha} = q_c,$$

we get, where  $C$  denotes any generic constant,

$$(3.3) \quad \int_{\Omega} \varrho^{q/2} |\nabla k_\varepsilon|^q \leq C.$$

As  $(k_\varepsilon)_{\varepsilon>0}$  is bounded in  $L^\alpha(\Omega)$ ,  $1 < \alpha$ , there exists  $(\varepsilon_n)_{n \in \mathbb{N}}$ ,  $\varepsilon_n \rightarrow 0$ , and  $k \in L^\alpha(\Omega)$ , such that  $(k_{\varepsilon_n})_{n \in \mathbb{N}}$  converges weakly to  $k$  in  $L^\alpha(\Omega)$ . Let  $\omega \subset\subset \Omega$ . By (2.1) and (3.3),  $(\nabla k_{\varepsilon_n})_{n \in \mathbb{N}}$  is bounded in  $L^q(\omega)^N$  for any  $1 < q < q_c$ . From the  $L^\alpha$  bound, we deduce that  $(k_{\varepsilon_n})_{n \in \mathbb{N}}$  is bounded in  $W^{1,q}(\omega)$ . Therefore we can extract a subsequence that converges weakly in  $W^{1,q}(\omega)$  to a  $\tilde{k} \in W^{1,q}(\omega)$ . By the uniqueness of the limit,  $\tilde{k} = k|_\omega$ , and the whole sequence converges. In particular,  $k \in W_{loc}^{1,q}(\Omega)$ . Moreover, for any  $p < q_c^*$ , where

$$q_c^* = \frac{2\alpha N}{N + (N-2)\alpha},$$

$k_{\varepsilon_n} \rightarrow k$  in  $L^p(\omega)$  strongly, and a.e in  $\Omega$ , and  $k \geq 0$  a.e. Notice that if  $\omega \subset \tilde{\omega} \subset\subset \Omega$ , the sequence  $(k_{\varepsilon_n})_{n \in \mathbb{N}}$  converges weakly to  $k$  in  $W^{1,q}(\tilde{\omega})$ , strongly in  $L^p(\tilde{\omega})$  for  $p < q_c^*$ , by a standard uniqueness argument. In particular,  $k \in W_{loc}^{1,q}(\Omega)$ . Moreover, by (2.4) and Fatou's Lemma,  $k \in L^\alpha(\Omega, \frac{dx}{\rho})$ .

*Step 2. Passing to the limit.* Let  $\varphi \in \mathcal{D}(\Omega)$ , which we take as test in (1.1),  $\omega = \text{supp}\varphi$ . We have

$$-\int_{\Omega} \text{div}((\varrho + \varepsilon_n)\nabla k_{\varepsilon_n})\varphi = \int_{\Omega} (\varrho + \varepsilon_n)\nabla k_{\varepsilon_n} \cdot \nabla \varphi = -\int_{\Omega} \text{div}((\varrho + \varepsilon_n)\nabla \varphi)k_{\varepsilon_n}.$$

On one hand  $\text{div}((\varrho + \varepsilon_n)\nabla \varphi) \rightarrow \text{div}(\varrho\nabla \varphi)$  in  $L^{\alpha'}(\Omega)$ , while on the other hand  $k_{\varepsilon_n} \rightharpoonup k$  in  $L^\alpha(\Omega)$ , hence

$$(3.4) \quad \lim_{n \rightarrow \infty} \int_{\Omega} \text{div}((\varrho + \varepsilon_n)\nabla \varphi)k_{\varepsilon_n} = \int_{\Omega} \text{div}(\varrho\nabla \varphi)k =_{\mathcal{D}'} \langle \text{div}(\varrho\nabla k), \varphi \rangle_{\mathcal{D}}.$$

It remains to pass to the limit in the term  $\int_{\omega} \frac{k_{\varepsilon_n}^\alpha \varphi}{\varrho + \varepsilon_n}$ .

Notice that  $\alpha < \frac{N}{N-2}$  is equivalent to  $\alpha < q_c^*$ . Therefore, even if it means extracting another subsequence, by Lebesgue inverse Theorem, there exists  $K \in L^\alpha(\omega)$  s.t.  $\forall n \in \mathbb{N}$ ,  $0 \leq k_{\varepsilon_n} \leq K$  a.e in  $\omega$ . Therefore,

$$\forall n \in \mathbb{N}, \quad 0 \leq \frac{k_{\varepsilon_n}^\alpha \varphi}{\varrho + \varepsilon_n} \leq \frac{K^\alpha \|\varphi\|_\infty}{\inf_{\omega} \varrho} \in L^1(\omega).$$

Then as

$$\frac{k_{\varepsilon_n}^\alpha \varphi}{\varrho + \varepsilon_n} \rightarrow \frac{k^\alpha \varphi}{\varrho}$$

a.e. in  $\omega$ , we deduce from Lebesgue Theorem

$$(3.5) \quad \lim_{n \rightarrow \infty} \int_{\Omega} \frac{k_{\varepsilon_n}^\alpha \varphi}{\varrho + \varepsilon_n} = \lim_{n \rightarrow \infty} \int_{\omega} \frac{k_{\varepsilon_n}^\alpha \varphi}{\varrho + \varepsilon_n} = \int_{\omega} \frac{k^\alpha \varphi}{\varrho} = \int_{\Omega} \frac{k^\alpha \varphi}{\varrho}.$$

We conclude from (3.4) and (3.5) that  $k$  is a solution in  $\mathcal{D}'(\Omega)$  of the equation (3.1).  $\square$

## References

- [1] C. Amrouche, Leloup G., and R. Lewandowski. The model involving the distance to the wall. part 1: the relaxed case. *Preprint on Hal*, 2023.
- [2] L. Boccardo and T. Gallouët. Nonlinear elliptic and parabolic equations involving measure data. *J. Funct. Anal.*, 87(1):149–169, 1989.
- [3] T. Chacòn-Rebollo and R. Lewandowski. *Mathematical and Numerical Foundations of Turbulence Models and Applications*. Modeling and Simulation in Science, Engineering and Technology. Springer New York, 2014.
- [4] Stéphane Clain and Rachid Touzani. A two-dimensional stationary induction heating problem. *Math. Methods Appl. Sci.*, 20(9):759–766, 1997.
- [5] Gianni Dal Maso, François Murat, Luigi Orsina, and Alain Prignet. Renormalized solutions of elliptic equations with general measure data. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)*, 28(4):741–808, 1999.

- [6] T. Gallouët and R. Herbin. Existence of a solution to a coupled elliptic system. *Appl. Math. Lett.*, 7(2):49–55, 1994.
- [7] Kiera Kean, William Layton, and Michael Schneier. On the Prandtl-Kolmogorov 1-equation model of turbulence. *Philos. Trans. Roy. Soc. A*, 380(2226):Paper No. 20210054, 15, 2022.
- [8] G. Leloup. Coupled system involving eddy coefficients and a right hand side in L1: Review chapter. Introductory chapter to the PhD thesis, <https://hal.science/hal-04405798v1>, 2023.
- [9] R. Lewandowski. Modèles de turbulence et équations paraboliques. *C. R. Acad. Sci. Paris Sér. I Math.*, 317(9):835–840, 1993.
- [10] R. Lewandowski. *Analyse Mathématique et océanographie*. Elsevier-Masson, Paris, 1997.
- [11] R. Lewandowski. The mathematical analysis of the coupling of a turbulent kinetic energy equation to the Navier-Stokes equation with an eddy viscosity. *Nonlinear Anal.*, 28(2):393–417, 1997.
- [12] B. Mohammadi and O. Pironneau. *Analysis of the k-epsilon turbulence model*. RAM: Research in Applied Mathematics. Masson, Paris; John Wiley & Sons, Ltd., Chichester, 1994.
- [13] François Murat. Renormalized solutions of nonlinear elliptic equations with measure data. In *Journées “Équations aux Dérivées Partielles” (Saint-Jean-de-Monts, 1998)*, pages Exp. No. IX, 4. Univ. Nantes, Nantes, 1998.
- [14] S.-B. Pope. *Turbulent flows*. Cambridge University Press, Cambridge, 2000.
- [15] G. Stampacchia. *Équations elliptiques du second ordre à coefficients discontinus*. Séminaire de Mathématiques Supérieures, No. 16 (Été, 1965). Les Presses de l’Université de Montréal, Montreal, Que., 1966.