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# A Nonlinear Elliptic Equation with a Degenerate Diffusion and a Source Term in $L^1$

Guillaume Leloup<sup>1</sup> and Roger Lewandowski<sup>2</sup>

<sup>1,2</sup>Univ Rennes, IRMAR, UMR CNRS 6625, and Odyssey Team, INRIA Rennes, France, E-mail: Guillaume.Leloup@univ-rennes.fr, Roger.Lewandowski@univ-rennes1.fr

## Abstract

We study a simplified equation governing turbulent kinetic energy in a bounded domain, arising from turbulence modeling where the eddy diffusion is given by  $\rho(x) + \varepsilon$ , with  $\rho$  representing the Prandtl mixing length of the order of the distance to the boundary, and a right-hand side in  $L^1$ . We establish the convergence toward the formal limit equation as  $\varepsilon$  approaches 0, within fractional Sobolev spaces  $W^{1/2,q}$ .

**Key words** : Fluid mechanics, Turbulence models, degenerate operators, Navier-Stokes Equations, Turbulent Kinetic Energy.

**2010 MSC**: 76D05, 35Q30, 76F65, 76D03, 35Q30.

## 1 Introduction

In this paper we consider the non linear elliptic equation

$$(1.1) \quad \begin{cases} -\operatorname{div}((\varrho(x) + \varepsilon)\nabla k) + \frac{k|k|^{\alpha-1}}{\varrho + \varepsilon} = \mathbb{D}(x) & \text{in } \Omega, \\ k = 0 & \text{at } \Gamma, \end{cases}$$

where  $\Omega \subset \mathbb{R}^N$  ( $N = 2, 3$ ) is a  $\mathcal{C}^2$  bounded domain,  $\Gamma = \partial\Omega$ ,  $\alpha > 0$ ,  $\mathbb{D} \geq 0 \in L^1(\Omega)$ ,  $\varrho : \Omega \rightarrow \mathbb{R}^+$  is of  $\mathcal{C}^2$  class behaving like the distance to  $\Gamma$  (see (2.1)), and  $\varepsilon > 0$ .

Equation (1.1) is a by-product of the equation for the turbulent kinetic energy used in turbulence models (see [4, 15, 17]), in which  $\nu_{\text{turb}} = \varrho + \varepsilon$  represents an eddy diffusion coefficient in a simplified manner. Here  $\varepsilon$  is a molecular diffusion coefficient,  $\mathbb{D}$  is the production of energy,  $\mathcal{E} = k|k|^{\alpha-1}(\varrho + \varepsilon)^{-1}$  is the dissipation of energy. In physical models  $\alpha = 3/2$ , the function  $\rho$  represents the Prandtl Mixing length.

Elliptic equations with r.h.s in  $L^1$  have been intensively studied since the seminal work of Boccardo-Gallouët [3], in the context of turbulent and/or thermodynamical coupling [4, 5, 7, 13, 14]. Therefore, we know the existence of a non negative distributional solution to (1.1), in  $W_0^{1,p}(\Omega)$ , for all  $p < N'$ . Moreover, due to the fact that the term  $\mathcal{E}$  is monotonic, there is a unique renormalized solution to this equation [6, 16].

We address in this paper the problem of letting  $\varepsilon$  go to zero in equation (1.1), which is related to the common question of determining whether molecular diffusion can be neglected in comparison to the eddy diffusion terms in the models (see, for instance, [9]). The challenge here arises from the fact that  $\rho$  vanishes at  $\Gamma$ .

This question was previously examined in the relaxed case in [1, section 3], where we set  $\nu_{\text{turb}} = \varrho^\beta + \varepsilon$  for  $\beta < 1$ . We then demonstrated the convergence of the corresponding solutions to the limit problem in  $W_0^{1,q}(\Omega)$  when  $\beta < \frac{1}{2N-1}$  and for any  $q < \frac{N'(1-\beta)}{1+\beta}$ . However, in most physical models,  $\beta = 1$ , and the situation becomes more intricate. Our main result is the Theorem (3.1) which states that when  $1 < \alpha < N'$ , the family  $(k_\varepsilon)_{\varepsilon>0}$  converges, up to a subsequence, to a distributional solution  $k \in W^{1/2,q}(\Omega)$  for  $q < \frac{2\alpha}{1+\alpha} = q_c$ , of the limit equation given by (3.1) below.

Notice that the case  $\alpha = 3/2$  is achieved when  $N = 2$ , but for  $N = 3$ , we must maintain  $\alpha < 3/2$ . Additionally, we must stress that we cannot impose  $k = 0$  at the boundary, since  $q_c < 2$ , and functions in  $W^{1/2,q}(\Omega)$  may have no trace at  $\Gamma$ . Finally, the case  $\frac{1}{2N-1} \leq \beta < 1$  is in progress.

The paper is organized as follows. In the first section, we derive *a priori* estimates of  $\rho^{1/2}\nabla k$  in  $L^p$  spaces, following the approach of Boccardo-Gallouët [3] (also discussed in [11]), adapted to the present case. In the second section, we derive estimates in  $W^{1/2,q}(\Omega)$  for  $q < q_c$  from a Hardy-like inequality. These estimates enable us to rigorously pass to the limit in the equations.

## 2 Estimates in weighted spaces

### 2.1 Framework and $L^\alpha$ estimates

We assume that  $\rho \in \mathcal{C}^2(\overline{\Omega})$  satisfies:

$$(2.1) \quad \lim_{\substack{d(x,\Gamma) \rightarrow 0 \\ x \in \Omega}} \frac{\varrho(x)}{d(x,\Gamma)} = 1, \quad \forall n > 0, \quad \varrho_n = \inf_{\substack{d(x,\Gamma) \geq \frac{1}{n} \\ x \in \Omega}} \varrho(x) > 0.$$

Let  $T_n$  be the truncation at height  $n$ , namely, the odd function defined by  $\forall x \geq 0$ ,  $T_n(x) = \min(n, x)$ , and we set  $G_n = T_{n+1} - T_n$ . Finally, let  $\eta > 0$  and  $H_\eta$  be the  $\mathcal{C}^{0,1}$  approximation of the Heaviside function, namely, the odd function given by,  $\forall 0 \leq x \leq \eta$ ,  $H_\eta(x) = \frac{x}{\eta}$ ,  $\forall x \geq \eta$ ,  $H_\eta(x) = 1$ . These three functions are Lipschitz continuous, the derivatives of which have a finite number of discontinuities. Moreover, they vanish at 0. Therefore, according to Stampacchia [18], if  $F$  is one of these three functions and  $v \in H_0^1(\Omega)$ , then  $F(v) \in H_0^1(\Omega)$  and  $\nabla(F(v)) = F'(v)\nabla v$ , which will be used all along the following proof. In the following, given  $s > 0$  and  $p > 1$ , we set

$$(2.2) \quad W^{s,p^-} = \bigcap_{1 \leq q < p} W^{s,q}.$$

We consider the renormalized solution  $k$  of (1.1). We do not give the full definition of the renormalized solution, which is rather intricate, referring to [6, 16]. We retain that this solution is essentially a distributional solution that satisfies several additional properties, such as  $T_n(k) \in H_0^1(\Omega)$ , and mainly for any  $F \in W^{1,\infty}(\mathbb{R})$  entering in Stampacchia's framework,

$$(2.3) \quad \int_{\Omega} (\varepsilon + \rho) F'(k) |\nabla k|^2 + \int_{\Omega} \frac{k|k|^{\alpha-1}}{\varrho} F(k) = \int_{\Omega} \mathbb{D}F(k).$$

Moreover, we know from [12] that  $k \geq 0$  *a.e.* in  $\Omega$ , so that  $k|k|^{\alpha-1} = k^\alpha$ . The following estimate is essential in our proof, telling among other that  $k \in L^\alpha(\Omega)$ .

**Lemma 2.1.** *Let  $k$  be the renormalized solution to problem (1.1). Then:*

$$(2.4) \quad 0 \leq \int_{\Omega} \frac{k^{\alpha}}{\varrho + \varepsilon} dx \leq \|\mathbb{D}\|_{0,1}.$$

*Proof.* Take  $H_{\eta}(k)$  as test in (1.1). Then by (2.3), as  $k \geq 0$ ,  $|H_{\eta}| \leq 1$ :

$$\int_{\Omega} \varrho \tilde{\mu}(k) H'_{\eta}(k) |\nabla k|^2 dx + \int_{\Omega} H_{\eta}(k) \frac{k^{\alpha}}{\varrho + \varepsilon} dx = \int_{\Omega} \mathbb{D} H_{\eta}(k) dx \leq \|\mathbb{D}\|_{0,1,\Omega}.$$

Since  $\varrho \tilde{\mu}(k) H'_{\eta}(k) |\nabla k|^2 \geq 0$ ,  $H_{\eta}(k) \xrightarrow{\eta \rightarrow 0} k$  a.e in  $\Omega$ , by Fatou's lemma

$$0 \leq \int_{\Omega} \frac{k^{\alpha}}{\varrho + \varepsilon} dx \leq \liminf_{\eta \rightarrow 0} \int_{\Omega} H_{\eta}(k) \frac{k^{\alpha}}{\varrho + \varepsilon} dx \leq \|\mathbb{D}\|_{0,1},$$

which proves (2.4). □

## 2.2 Ladder process

**Lemma 2.2.** *let  $q < N'$  and*

$$(2.5) \quad r > \frac{q}{2-q}.$$

*Then  $\forall n_0 \in \mathbb{N}$ ,  $\exists \lambda_1(n_0), \lambda_2(n_0) > 0$  such that  $\lim_{n_0 \rightarrow \infty} \lambda_2(n_0) = +\infty$  and such that*

$$(2.6) \quad \int_{\Omega} \varrho^{\frac{q}{2}} |\nabla k|^q dx \leq \lambda_1(n_0) + \lambda_2(n_0) \|k\|_{0,r,\Omega}^{\frac{r(2-q)}{2}},$$

*Proof.* Let  $B_n = \{x \in \Omega; n \leq k \leq n+1\}$ . We take  $G_n(k) = T_{n+1}(k) - T_n(k)$  as test in (1.1) and use (2.3). As  $G'_n(k)$  vanishes outside  $B_n$  and  $0 \leq G_n(k) \leq 1$ , we obtain

$$(2.7) \quad \varepsilon \int_{B_n} |\nabla k|^2 + \int_{B_n} \varrho |\nabla k|^2 dx + \int_{\Omega} \frac{k^{3/2}}{\varrho + \varepsilon} G_n(k) dx = \int_{\Omega} \mathbb{D} G_n(k) dx \leq \|\mathbb{D}\|_{0,1,\Omega}.$$

$$(2.8) \quad \text{Then by (2.7):} \quad \int_{B_n} \varrho(k) |\nabla k|^2 dx \leq \|\mathbb{D}\|_{0,1,\Omega} =: M.$$

$$(2.9) \quad \text{We write:} \quad \int_{\Omega} \varrho^{q/2} |\nabla k|^q dx = \sum_{n=0}^{+\infty} \int_{B_n} \varrho^{q/2} |\nabla k|^q dx.$$

Therefore, combining Hölder inequality and (2.8) yields

$$(2.10) \quad \forall n \in \mathbb{N}, \quad \int_{B_n} \varrho^{q/2} |\nabla k|^q dx \leq \left( \int_{B_n} \varrho |\nabla k|^2 dx \right)^{q/2} \left( \int_{B_n} dx \right)^{\frac{2-q}{2}} \leq M^{q/2} |B_n|^{\frac{2-q}{2}}.$$

Let  $n_0 \in \mathbb{N}$ . On one hand,

$$(2.11) \quad \sum_{n=0}^{n_0} \int_{B_n} \varrho^{q/2} |\nabla k|^q dx \leq (n_0 + 1) \max(M, |\Omega|).$$

On the other hand, let  $r$  that satisfies (2.5). Then,  $\int_{B_n} k^r dx \geq \int_{B_n} n^r dx = n^r |B_n|$ , giving

$$(2.12) \quad M^{q/2} |B_n|^{\frac{2-q}{2}} \leq \frac{M^{q/2}}{n^{\frac{r(2-q)}{2}}} \left( \int_{B_n} k^r dx \right)^{\frac{2-q}{2}}.$$

Combining (2.10) and (2.12), adding the terms for  $n > n_0$ , by Hölder inequality,

$$(2.13) \quad \begin{aligned} \sum_{n=n_0+1}^{+\infty} \int_{B_n} \varrho^{q/2} |\nabla k|^q dx &\leq M^{q/2} \sum_{n=n_0+1}^{+\infty} \frac{1}{n^{\frac{r(2-q)}{2}}} \left( \int_{B_n} k^r dx \right)^{\frac{2-q}{2}} \\ &\leq M^{q/2} \left[ \sum_{n=n_0+1}^{+\infty} \left( \int_{B_n} k^r dx \right) \right]^{\frac{2-q}{2}} \left[ \sum_{n=n_0+1}^{+\infty} \frac{1}{n^{\frac{r(2-q)}{q}}} \right]^{q/2}. \end{aligned}$$

The term in the last bracket is a convergent series as a consequence of (2.5). Then let

$$\lambda_1(n_0) = (n_0 + 1) \max(M, |\Omega|) \quad \text{and} \quad \lambda_2(n_0) = M^{q/2} \left[ \sum_{n=n_0+1}^{+\infty} \frac{1}{n^{\frac{r(2-q)}{q}}} \right]^{q/2}.$$

Inequalities (2.11) and (2.13) become

$$(2.14) \quad \int_{\Omega} \varrho^{q/2} |\nabla k|^q dx \leq \lambda_1(n_0) + \lambda_2(n_0) \left( \int_{\Omega} k^r dx \right)^{\frac{2-q}{2}} = \lambda_1(n_0) + \lambda_2(n_0) \|k\|_{0,r,\Omega}^{\frac{r(2-q)}{2}},$$

hence (2.6). □

### 3 Passing to the limit

**Theorem 3.1.** *Assume  $1 < \alpha < N'$ . Let  $k_\varepsilon$  be the renormalized solution to equation (1.1). Then there exists  $(\varepsilon_n)_{n \in \mathbb{N}}$  with  $\lim_{n \rightarrow \infty} \varepsilon_n = 0$  and such that for any  $q < \frac{2\alpha}{1+\alpha} = q_c$ , the sequence  $(k_{\varepsilon_n})_{n \in \mathbb{N}}$  converges weakly in  $W^{\frac{1}{2},q}$  to  $k \in W^{\frac{1}{2},q_c^-} \cap L^\alpha(\rho^{-1})$ , solution in the sense of the distributions of the equation*

$$(3.1) \quad -\operatorname{div}(\varrho(x) \nabla k) + \frac{k|k|^{\alpha-1}}{\varrho} = \mathbb{D}(x).$$

*Proof.* We proceed in two steps. We first combine (2.4) and (2.6) to get a  $W^{1/2,q}$  estimate for  $q < q_c$ . Then we pass to the limit in the equation.

*Step 1.* Because  $\rho \in \mathcal{C}^2(\overline{\Omega})$ , we deduce from (2.4) that  $(k_\varepsilon)_{\varepsilon > 0}$  is bounded in  $L^\alpha(\Omega)$ . By (2.6) with  $\alpha = r$ , which imposes  $\frac{q}{2-q} < \alpha$  because of (2.5), or equivalently

$$(3.2) \quad q < \frac{2\alpha}{1+\alpha} = q_c,$$

we get, where  $C$  denotes any generic constant,

$$(3.3) \quad \int_{\Omega} \varrho^{q/2} |\nabla k_\varepsilon|^q dx \leq C.$$

Let  $1 < \alpha < N'$ ,  $1 < q < q_c < 2$ , and  $\mathbf{v} \in W_0^{1/2, q'}(\Omega)^N$ . By standard calculus we have

$$(3.4) \quad \left| \int_{\Omega} \nabla k_{\varepsilon} \cdot \mathbf{v} dx \right| = \left| \int_{\Omega} \varrho^{1/2} \nabla k_{\varepsilon} \cdot \varrho^{-1/2} \mathbf{v} dx \right| \leq \|\varrho^{1/2} \nabla k_{\varepsilon}\|_{0, q, \Omega} \|\varrho^{-1/2} \mathbf{v}\|_{0, q', \Omega},$$

which combined with (3.3) yields

$$(3.5) \quad \left| \int_{\Omega} \nabla k_{\varepsilon} \cdot \mathbf{v} dx \right| \leq C \|\varrho^{-1/2} \mathbf{v}\|_{0, q', \Omega}.$$

By Theorem 1.4.4.3 in [8, chapter 1] with  $s = 1/2$  and  $p = q' > 2$ , which derives from the Hardy inequality, we deduce from (3.5)

$$(3.6) \quad \forall \varepsilon > 0, \quad \left| \int_{\Omega} \nabla k_{\varepsilon} \cdot \mathbf{v} dx \right| \leq C \|\mathbf{v}\|_{1/2, q', \Omega}.$$

Therefore,  $\|\nabla k_{\varepsilon}\|_{-1/2, q} \leq C$  which yields (see in [1, 2])

$$(3.7) \quad \forall 1 < q < q_c, \quad \forall \varepsilon > 0, \quad \|k_{\varepsilon}\|_{1/2, q} \leq C.$$

By the Sobolev embedding Theorem we have  $W^{1/2, q_c^-} \subset L^p(\Omega)$  for any  $p < q_c^*$ , where  $q_c^* = \frac{2\alpha N}{N + (N-1)\alpha}$ , with a compact injection for any  $q < \frac{2\alpha}{1+\alpha}$ . Notice that since  $\alpha > 1$ , then  $q^* > 1$ , and

$$(3.8) \quad q_c^* = \frac{4\alpha}{2+\alpha} \quad \text{when } N = 2, \quad q_c^* = \frac{6\alpha}{3+2\alpha} \quad \text{when } N = 3.$$

*Step 2. Passing to the limit.* We deduce from (3.7) that  $(k_{\varepsilon})_{\varepsilon > 0}$  is bounded in each  $W^{1/2, q}$ ,  $q < q_c$ . Hence by standards argument we can extract a sequence  $(k_{\varepsilon_n})_{n \in \mathbb{N}}$ , where  $\varepsilon_n \rightarrow 0$ , and there exists  $k \in W^{1/2, q_c^-}$  such that  $k_{\varepsilon_n} \rightarrow k$  weakly in each  $W^{1/2, q}$ , strongly in  $L^p(\Omega)$  for any  $p < q_c^*$ , a.e. in  $\Omega$ . In particular  $k \geq 0$  a.e. Let  $\varphi \in \mathcal{D}(\Omega)$ , which we take as test in (1.1). Let  $1 < p < q_c^*$  be fixed. We have

$$- \int_{\Omega} \operatorname{div}((\rho + \varepsilon_n) \nabla k_{\varepsilon_n}) \varphi = \int_{\Omega} (\rho + \varepsilon_n) \nabla k_{\varepsilon_n} \cdot \nabla \varphi = - \int_{\Omega} \operatorname{div}((\rho + \varepsilon_n) \nabla \varphi) k_{\varepsilon_n}.$$

On one hand  $\operatorname{div}((\rho + \varepsilon_n) \nabla \varphi) \rightarrow \operatorname{div}(\rho \nabla \varphi)$  in  $L^{p'}(\Omega)$ , while on the other hand  $k_{\varepsilon_n} \rightarrow k$  in  $L^p(\Omega)$ , hence

$$(3.9) \quad \lim_{n \rightarrow \infty} \int_{\Omega} \operatorname{div}((\rho + \varepsilon_n) \nabla \varphi) k_{\varepsilon_n} = \int_{\Omega} \operatorname{div}(\rho \nabla \varphi) k =_{\mathcal{D}'} (\operatorname{div}(\rho \nabla k), \varphi)_{\mathcal{D}}.$$

It remains to pass to the limit in the term  $\frac{k_{\varepsilon_n}^{\alpha} \varphi}{\varrho + \varepsilon_n}$ . Notice that  $\alpha < N'$  is equivalent to  $\alpha < q_c^*$ . Therefore, even if it means extracting another subsequence, by Lebesgue inverse Theorem, there exists  $K \in L^{\alpha}(\Omega)$  s.t.  $\forall n \in \mathbb{N}$ ,  $0 \leq k_{\varepsilon_n} \leq K$ . Consequently, by setting  $\omega = \operatorname{supp} \varphi$ , by (2.1)  $\inf_{\omega} \rho > 0$  and we deduce

$$\forall n \in \mathbb{N}, \quad 0 \leq \frac{k_{\varepsilon_n}^{\alpha} \varphi}{\varrho + \varepsilon_n} \leq \frac{K \alpha \|\varphi\|_{\infty}}{\inf_{\omega} \rho} \in L^1(\Omega).$$

Then as  $\frac{k_{\varepsilon_n}^{\alpha} \varphi}{\varrho + \varepsilon_n} \rightarrow \frac{k^{\alpha} \varphi}{\varrho}$  a.e. in  $\Omega$ , we deduce from Lebesgue Theorem

$$(3.10) \quad \lim_{n \rightarrow \infty} \int_{\Omega} \frac{k_{\varepsilon_n}^{\alpha} \varphi}{\varrho + \varepsilon_n} = \int_{\Omega} \frac{k^{\alpha} \varphi}{\varrho}.$$

We conclude from (3.9) and (3.10) that  $k$  is a solution in  $\mathcal{D}'(\Omega)$  of the equation (3.1).  $\square$

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