



HAL
open science

On the representation of fourth and higher order anisotropic elasticity tensors in generalized continuum models

Marco Valerio d'Agostino, Robert J Martin, Peter Lewintan, Davide Bernardini, Alexandre Danescu, Patrizio Neff

► To cite this version:

Marco Valerio d'Agostino, Robert J Martin, Peter Lewintan, Davide Bernardini, Alexandre Danescu, et al.. On the representation of fourth and higher order anisotropic elasticity tensors in generalized continuum models. *Mathematics and Mechanics of Solids*, In press. hal-04391796

HAL Id: hal-04391796

<https://hal.science/hal-04391796v1>

Submitted on 12 Jan 2024

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.



Distributed under a Creative Commons Attribution - NonCommercial - NoDerivatives 4.0 International License

On the representation of fourth and higher order anisotropic elasticity tensors in generalized continuum models

Marco Valerio d'Agostino¹ Robert J. Martin² Peter Lewintan³ Davide Bernardini⁴
Alexandre Danescu⁵ and Patrizio Neff⁶

January 12, 2024

Abstract

The classification of all fourth order anisotropic tensor classes for classical linear elasticity is well known. We review the mathematical procedure behind this classification and extend it to fourth order elasticity tensors acting on non-symmetric matrices. These tensors naturally appear in generalised continuum models. Based on tensor-symmetrization we provide the most general form of these tensors for the case of orthotropic, transversely isotropic, cubic and isotropic materials. We give a self-contained presentation and provide a detailed calculation for simple examples.

Keywords: representation theory, group action, character, elasticity tensor, material symmetries, relaxed micro-morphic model, generalized continua.

AMS 2010 subject classification: 20C (representation theory of groups), 58D19 (group actions and symmetry properties), 20C15 (ordinary representations and characters), 74A30 (nonsimple materials), 74A60 (micromechanical theories), 74B05 (classical linear elasticity), 74M25 (micromechanics), 74Q15 (effective constitutive equations).

¹Marco Valerio d'Agostino, corresponding author, marco-valerio.dagostino@insa-lyon.fr, GEOMAS, INSA-Lyon, Université de Lyon, 20 avenue Albert Einstein, 69621, Villeurbanne cedex, France

²Robert J. Martin, robert.martin@uni-due.de, Chair for Nonlinear Analysis and Modelling, Fakultät für Mathematik, Universität Duisburg-Essen, Thea-Leymann-Straße 9, 45127 Essen, Germany.

³Peter Lewintan, peter.lewintan@uni-due.de, Chair for Nonlinear Analysis and Modelling, Fakultät für Mathematik, Universität Duisburg-Essen, Thea-Leymann-Straße 9, 45127 Essen, Germany

⁴Davide Bernardini, dbernardini.didattica@uniroma1.it, Department of Structural and Geotechnical Engineering, Sapienza University of Rome, Rome, Italy

⁵Alexandre Danescu, alexandre.danescu@ec-lyon.fr, Lyon Institute of Nanotechnology, École Centrale de Lyon, 69131 Ecully, France

⁶Patrizio Neff, patrizio.neff@uni-due.de, Head of Chair for Nonlinear Analysis and Modelling, Fakultät für Mathematik, Universität Duisburg-Essen, Thea-Leymann-Straße 9, 45127 Essen, Germany

Contents

1	Notation	3
2	Preliminaries	4
2.1	The trace formula	10
2.2	Explicit expression for the Hooke tensor in classical elasticity	15
3	Extended continuum models	16
3.1	Non-symmetric theories	16
3.2	Second gradient theory	19
3.2.1	Extended Voigt isomorphisms and structure of the symmetrized tensors	23
3.2.2	Cubic case	24
3.2.3	Transversal hemitropic and transversal isotropic case	25
3.3	Indeterminate couple stress model	27
3.4	Micromorphic type models	28
3.4.1	3D-relaxed micromorphic model	28
3.4.2	2D-relaxed micromorphic model, plane strain	29
3.4.3	Symmetrization of \mathbb{L}	30
3.4.4	Symmetrization of \mathbb{C}_e and $\mathbb{C}_{\text{micro}}$	32
3.5	2D-micromorphic model	33
A	Haar integration	35
A.1	Haar measure induced by a local chart	38
A.1.1	Cardan angles	40
B	Useful isomorphisms between vector spaces	41
B.1	Voigt isomorphism	41
B.2	axl isomorphism	42
B.3	Computation of characters	43
B.3.1	Classical elasticity	44
B.3.2	Second gradient elasticity	44
B.3.3	Non symmetric theories	46

Introduction

Starting from the very early years of the linear theory of elasticity the question concerning the number of material parameters involved in the general theory and various particular symmetric situations was a matter of active scientific debate. A historical overview of the field is presented in the Historical Introduction chapter of Love's monograph [36] and the most fundamental debate concerned the opposition between the supporters and the opponents of the *rari-constant* and *multi-constant* elasticity theories. Introduced in the isotropic setting by Navier in 1821 [44] and augmented in the general *æolotropic* (anisotropic) setting by Cauchy in 1828 [11–13], the *rari-constant* elasticity theory is based upon several assumptions, including central forces with arbitrary range of action (but small with respect to the macroscopic size of the body) and no residual stress. It predicts, in the general anisotropic case, only 15 elastic constants and in the particular case of isotropic materials $\nu = 1/4$ (Poisson ratio). For a detailed historical account of the controversy involving the *rari-constant* and the *multi-constant* elasticity theories, we refer to [65].

From a different phenomenological perspective, Green (and later Stokes) introduced what is called today the *strain-energy density* W and assumed that it depends on the infinitesimal strain tensor ε . Neglecting higher order terms and assuming that the initial stress vanishes, the first non-trivial contribution comes from the second order polynomial terms which can be expressed as

$$W(\varepsilon) = \frac{1}{2} \langle \mathbb{C} \varepsilon, \varepsilon \rangle.$$

In this form and without additional assumptions, taking into account the symmetry of the strains one is left with 21 elastic coefficients. In the particular case of anisotropic materials, once the symmetry class of the body is established, the demand for the elastic constitutive law to remain invariant with respect to the elements of the

accounted symmetry group turns into a large system of identities that must be satisfied by the components \mathbb{C}_{ijkl} of the 4th order Hooke tensor \mathbb{C} . An illustration of this straightforward but very technical method is presented in the monograph by Love [36]. The monograph of Gurtin [32] attributes the results to Voigt [66] and mentions that they can be obtained by using the results of Smith and Rivlin [63] and Sirotnin [62]. Ting’s monograph [64] is dedicated entirely to anisotropic bodies and introduces an abstract formalism (see also [10]) that associates to each transformation in the symmetry group of the material a second-order non-symmetric tensor K . In this framework, the invariance of the Hooke tensor with respect to the symmetry group is rewritten as the invariance of the fourth-order tensor \mathbb{C} with respect to the transformation $\mathbb{C} \rightarrow K\mathbb{C}K^T$. The Ting formalism is merely a condensed form of the classical invariance relation since now K is, as expected, quadratic with respect to the element of the symmetry group. As noticed by Bower [10], the Ting formalism is rather *convenient for computer applications*.

To the best of our knowledge, the first attempt to unify the restrictions imposed by the symmetry group on the general form of the Hooke law and to extend them to a larger framework (piezo-elastic materials, higher gradients, higher-order polynomial constitutive relations) with group theoretical methods including representation theory results, the trace theorem and Haar integration were presented in [17]. The basic idea is to notice that *the invariance of the constitutive relation can be regarded as a question concerning the invariant vector sub-space of a general vector space with respect to a suitable defined action of the symmetry group*. In the classical case of linear elasticity theory the vector space is that of the Hooke tensors, the invariant vector sub-space is that of Hooke tensors satisfying the invariance with respect to the (fixed) symmetry group \mathcal{G} which is a closed subgroup of $\text{SO}(3)$, while the action of the symmetry group associates to each symmetry element $Q \in \mathcal{G}$ the eighth-order linear operator acting on Hooke tensors as

$$\mathbb{C}_{ijkl} \rightarrow Q_{ia}Q_{jb}Q_{kc}Q_{ld}\mathbb{C}_{abcd}.$$

From classical group representation theory *the trace formula* provides the dimension of the invariant vector-subspace of Hooke tensors *as a function of the group only* and the explicit expression of an invariant Hooke tensor is obtained by symmetrization. Obviously, by definition,

$$\hat{\mathbb{C}}_{ijkl} = \frac{1}{\text{card}(\mathcal{G})} \sum_{Q \in \mathcal{G}} Q_{ia}Q_{jb}Q_{kc}Q_{ld}\mathbb{C}_{abcd}$$

is an invariant Hooke tensor with respect to the symmetry group \mathcal{G} .

From this perspective, the result in [17] provides a unified setting, independent of the symmetry group, that both (i) determine the complexity of the theory (i.e. the number of elastic moduli) through the dimension of the invariant vector subspace of Hooke tensors and (ii) provide an effective method to find them. Obviously, the method works not only for quadratic forms with respect to strains but also for arbitrary homogeneous higher-order polynomials and, as a consequence, for arbitrary polynomial constitutive relations [17]. The results in [17] were subsequently extended to electro-elastic materials in [19] and [18]. In the context of piezoelectric materials, results concerning material invariance were also obtained in [9, 68]. Recently, a series of papers (cf. [3, 5] and the author’s references within) rediscovered the symmetrization procedure in relation to the classical phenomenological higher gradient elasticity model of Mindlin. As noticed in [19], the symmetry properties of a classical (or generalized) constitutive relation depend strongly on the objects involved in its description so that as a consequence, higher-order tensors (involved in strain-gradient or second-gradient of strain) theories lead to more symmetry classes and, as expected, to a multitude of material parameters [2, 4, 22, 28, 52–55]. Along this path, the trace formula and its use in relation to the disjoint union decomposition of symmetry groups in [17] provide a way to explore the complexity of a theory without the explicit determination of the invariant forms of constitutive relation.

In this paper, building upon the results initially presented in [17], we explicitly calculate the expressions of the elasticity tensors used in enriched continuum models with respect to various symmetry classes. These models are, for example, currently employed to describe the mechanical properties of exotic metamaterials. Notably, in a series of papers [6, 15, 16, 21, 45, 56–60, 67], it has been convincingly demonstrated how the relaxed micromorphic model effectively describes the band-gap phenomena occurring in the dynamic behavior of metamaterials. Our primary focus is on this generalized model.

1 Notation

Throughout this paper the Einstein convention of summation over repeated indices is used if not differently specified. We denote by $\mathbb{R}^{3 \times 3}$ the set of real 3×3 second order tensors and by $\mathbb{R}^{3 \times 3 \times 3}$ the set of real $3 \times 3 \times 3$ third order tensors. The standard Euclidean scalar product on $\mathbb{R}^{3 \times 3}$ is given by $\langle X, Y \rangle_{\mathbb{R}^{3 \times 3}} = \text{tr}(X Y^T)$ and, thus, the Frobenius tensor norm is given by $\|X\|^2 = \langle X, X \rangle_{\mathbb{R}^{3 \times 3}}$. Moreover, the identity tensor on $\mathbb{R}^{3 \times 3}$ will be denoted by $\mathbb{1}$, so that $\text{tr}(X) = \langle X, \mathbb{1} \rangle$. We adopt the usual abbreviations of Lie-algebra theory, i.e.

- $\text{Sym}(3) := \{X \in \mathbb{R}^{3 \times 3} \mid X^T = X\}$ denotes the vector-space of all symmetric 3×3 matrices,
- $\mathfrak{so}(3) := \{X \in \mathbb{R}^{3 \times 3} \mid X^T = -X\}$ is the Lie-algebra of skew symmetric tensors,
- $\mathfrak{sl}(3) := \{X \in \mathbb{R}^{3 \times 3} \mid \text{tr}(X) = 0\}$ is the Lie-algebra of traceless tensors,
- $\mathbb{R}^{3 \times 3} \simeq \mathfrak{gl}(3) = \{\mathfrak{sl}(3) \cap \text{Sym}(3)\} \oplus \mathfrak{so}(3) \oplus \mathbb{R} \mathbf{1}$ is the *orthogonal Cartan-decomposition of the Lie-algebra*.

In other words, for all $X \in \mathbb{R}^{3 \times 3}$, we consider the decomposition

$$X = \text{dev sym } X + \text{skew } X + \frac{1}{3} \text{tr}(X) \mathbf{1}, \quad (1)$$

where

- $\text{sym } X = \frac{1}{2}(X^T + X) \in \text{Sym}(3)$ is the symmetric part of X ,
- $\text{skew } X = \frac{1}{2}(X - X^T) \in \mathfrak{so}(3)$ is the skew-symmetric part of X ,
- $\text{dev } X = X - \frac{1}{3} \text{tr}(X) \mathbf{1} \in \mathfrak{sl}(3)$ is the deviatoric part of X .

Throughout all the paper we indicate

- with an overline, e.g. $\overline{\mathbb{C}}$, the general fourth order tensors $\overline{\mathbb{C}} : \mathbb{R}^{3 \times 3} \rightarrow \mathbb{R}^{3 \times 3}$,
- without superscripts, e.g. \mathbb{C} , classical fourth order tensors acting only on symmetric matrices ($\mathbb{C} : \text{Sym}(3) \rightarrow \text{Sym}(3)$) or skew-symmetric ones ($\mathbb{C}_c : \mathfrak{so}(3) \rightarrow \mathfrak{so}(3)$).

We also define

- $\text{GL}^+(3) := \{X \in \mathbb{R}^{3 \times 3} \mid \det X > 0\}$,
- $\mathbf{1}_2 \in \mathbb{R}^{3 \times 3}$ the identity matrix, where the subscript 2 here pertains to the degree of the tensor (having 2 indices) and not to the dimension of \mathbb{R}^3 ,
- $\text{Sym}^+(3) := \{X \in \text{Sym}(3) \mid \langle Xv, v \rangle > 0 \quad \forall v \in \mathbb{R}^3, v \neq 0\}$,
- $\text{O}(3) := \{X \in \mathbb{R}^{3 \times 3} \mid \det X = \pm 1\}$,
- $\text{SO}(3) := \{X \in \mathbb{R}^{3 \times 3} \mid \det X = 1\}$.

We indicate by $\overline{\mathbb{C}}X$ the linear application of a tensor of 4th order to a tensor of 2nd order, i.e.

$$(\overline{\mathbb{C}}X)_{ij} = \overline{\mathbb{C}}_{ijhk} X_{hk}. \quad (2)$$

The operation of simple contraction between tensors of suitable order is also denoted by

$$(Xv)_i = X_{ij} v_j, \quad (ZX)_{ij} = Z_{ih} X_{hj}. \quad (3)$$

Typical conventions for differential operations are employed, such as a comma followed by a subscript to denote the partial derivative with respect to the corresponding Cartesian coordinate: $(\cdot)_{,j} = \frac{\partial(\cdot)}{\partial x_j}$.

2 Preliminaries

First, we illustrate in a very simple and understandable way the ideas of the procedure (proposed in [17]) to find the symmetrized structure of a tensor if we are considering a particularly symmetry class. The technical tools we need are rigorously presented in

- [20] for the Haar measure and invariant integration on locally compact topological groups like $\text{SO}(n)$,
- [33] for a general introduction to representation theory,
- [30] for the classification of the irreducible representations of $\text{SO}(3)$ and the classification of its closed subgroups (which gives us the set of possible symmetries of a considered material), and the trace formula for finite-dimensional representations.

In linearised elasticity, the elasticity tensor \mathbb{C} is a linear map

$$\mathbb{C} : \text{Sym}(3) \rightarrow \text{Sym}(3),$$

which gives the relation between the strain-tensor $\varepsilon := \text{sym} Du$ (the local deformation of the body) and the symmetric Cauchy stress-tensor σ via

$$\sigma = \mathbb{C} \varepsilon. \quad (4)$$

Thus, the elasticity tensor is an element of the vector space $\text{Lin}(\text{Sym}(3), \text{Sym}(3))$, whose dimension is 36. Moreover, according to the fact that we derive the relation (4) from a variational principle, i.e. that the equilibrium configuration of the system is characterized as the stationary point of the functional

$$\frac{1}{2} \int_{\Omega} \langle \mathbb{C} \varepsilon, \varepsilon \rangle_{\mathbb{R}^{3 \times 3}} dx,$$

the elasticity tensor has to be symmetric (and this symmetry is known as **major symmetry**) also with respect to the scalar product $\langle \mathbb{C} \varepsilon, \varepsilon \rangle_{\mathbb{R}^{3 \times 3}}$ which implies that \mathbb{C} lives in the smaller space $\text{Sym}(\text{Sym}(3), \text{Sym}(3))$ of the symmetric applications from $\text{Sym}(3)$ to $\text{Sym}(3)$. The dimension of this space is 21 and so the full anisotropic elasticity tensor has at most 21 independent components instead of $9 \times 9 = 81$. According to the map $\underline{\mathfrak{M}}$ introduced in (110) (see the Appendix), we can represent \mathbb{C} as a 6×6 symmetric matrix $\tilde{\mathbb{C}}$ with

$$\tilde{\mathbb{C}} = \begin{pmatrix} \tilde{\mathbb{C}}_{11} & \tilde{\mathbb{C}}_{12} & \tilde{\mathbb{C}}_{13} & \tilde{\mathbb{C}}_{14} & \tilde{\mathbb{C}}_{15} & \tilde{\mathbb{C}}_{16} \\ & \tilde{\mathbb{C}}_{22} & \tilde{\mathbb{C}}_{23} & \tilde{\mathbb{C}}_{24} & \tilde{\mathbb{C}}_{25} & \tilde{\mathbb{C}}_{26} \\ & & \tilde{\mathbb{C}}_{33} & \tilde{\mathbb{C}}_{34} & \tilde{\mathbb{C}}_{35} & \tilde{\mathbb{C}}_{36} \\ & & & \tilde{\mathbb{C}}_{44} & \tilde{\mathbb{C}}_{45} & \tilde{\mathbb{C}}_{46} \\ & & & & \tilde{\mathbb{C}}_{55} & \tilde{\mathbb{C}}_{56} \\ & \text{sym} & & & & \tilde{\mathbb{C}}_{66} \end{pmatrix} = \begin{pmatrix} \mathbb{C}_{1111} & \mathbb{C}_{2211} & \mathbb{C}_{3311} & \mathbb{C}_{3211} & \mathbb{C}_{3111} & \mathbb{C}_{2111} \\ & \mathbb{C}_{2222} & \mathbb{C}_{3322} & \mathbb{C}_{3222} & \mathbb{C}_{3122} & \mathbb{C}_{2122} \\ & & \mathbb{C}_{3333} & \mathbb{C}_{3332} & \mathbb{C}_{3331} & \mathbb{C}_{3321} \\ & & & \mathbb{C}_{3232} & \mathbb{C}_{3231} & \mathbb{C}_{3221} \\ & & & & \mathbb{C}_{3131} & \mathbb{C}_{3121} \\ & \text{sym} & & & & \mathbb{C}_{2121} \end{pmatrix}. \quad (5)$$

The question at hand is as follows: When we contemplate a material with a specific symmetry, how does this symmetry affect the structure of the elasticity tensor? To address this question, we must define the material invariance and use it to derive conditions on \mathbb{C} , which are then interpreted within the framework of representation theory.

According to [14], considering a Lagrangian energy density $W : \text{GL}^+(3) \rightarrow \mathbb{R}$ and a closed subgroup \mathcal{G} of $\text{SO}(3)$, we say that the material is \mathcal{G} -invariant if

$$W(FQ) = W(F) \quad \forall Q \in \mathcal{G} \quad \text{and} \quad \forall F \in \text{GL}^+(3). \quad (6)$$

In continuum mechanics the matrix F is the gradient of a deformation field $\varphi : \Omega \subseteq \mathbb{R}^3 \rightarrow \mathbb{R}^3$. Under the frame-invariance requirement for the energy density, setting $C = F^T F$, the invariance condition (6) can be expressed in terms of the auxiliary function¹

$$\widehat{W} : \text{Sym}^+(3) \rightarrow \mathbb{R}, \quad \widehat{W}(F^T F) = W(F) \quad \forall F \in \text{GL}^+(3),$$

as

$$\widehat{W}(Q^T C Q) = \widehat{W}(C) \quad \forall Q \in \mathcal{G} \quad \text{and} \quad \forall C \in \text{Sym}^+(3). \quad (7)$$

Defining the displacement field $u : \Omega \rightarrow \mathbb{R}^3$ as $u = \varphi - \mathbf{1}$ (i.e. $u(x) = \varphi(x) - x$ for every $x \in \Omega$), and introducing the Green-St.Venant strain tensor

$$E := \frac{1}{2}(F^T F - \mathbf{1}) = \frac{1}{2}(Du^T + Du) + \frac{1}{2} Du^T Du, \quad (8)$$

relation (7) reads

$$\widehat{W}(Q^T (\mathbf{1} + 2E) Q) = \widehat{W}(\mathbf{1} + 2E) \quad \forall Q \in \mathcal{G} \quad \text{and} \quad \forall E \in \text{Sym}(3). \quad (9)$$

Expanding the energy density in a neighbourhood of the origin by

$$\widehat{W}(\mathbf{1} + 2E) = \widehat{W}(\mathbf{1}) + 2 \langle D\widehat{W}(\mathbf{1}), E \rangle + 2 \langle D^2 \widehat{W}(\mathbf{1}) E, E \rangle + o(\|E\|^2),$$

¹The polar factorization of invertible matrices theorem [14, Thm.3.2-2, pag.95] establishes that $\text{GL}^+(3) \simeq \text{SO}(3) \times \text{Sym}^+(3)$. For a matrix $F \in \text{GL}^+(3)$, the unique pair $(R, U) \in \text{SO}(3) \times \text{Sym}^+(3)$ that yields $F = RU$ is determined by $U = \sqrt{F^T F}$ and $R = F U^{-1}$ (see also [27, 49]).

if we choose $\widehat{W}(\mathbf{1}) = 0$ and the reference configuration is a natural state (i.e. $D\widehat{W}(\mathbf{1}) = 0$), then taking only the linear part $\varepsilon = \frac{1}{2}(Du^T + Du) = \text{sym} Du$ of E (geometric linearity) and considering a homogeneous material, setting $\mathbb{C} = 4D^2\widehat{W}(\mathbf{1})$ the linearized energy can be expressed as a quadratic form in ε , i.e. $2\overline{W}(\varepsilon) = \langle \mathbb{C} \varepsilon, \varepsilon \rangle$. The invariance condition (9) then reads

$$\langle \mathbb{C} (Q^T \varepsilon Q), Q^T \varepsilon Q \rangle = \langle \mathbb{C} \varepsilon, \varepsilon \rangle \quad \forall Q \in \mathcal{G} \quad \text{and} \quad \forall \varepsilon \in \text{Sym}(3). \quad (10)$$

Writing the relation (10) component-wise, we obtain

$$\mathbb{C}_{abcd} Q_{ch}^T \varepsilon_{hk} Q_{kd} Q_{ai}^T \varepsilon_{ij} Q_{jb} = \mathbb{C}_{ijhk} \varepsilon_{hk} \varepsilon_{ij}, \quad (11)$$

which is satisfied if and only if

$$\mathbb{C}_{ijhk} = Q_{ia} Q_{jb} Q_{hc} Q_{kd} \mathbb{C}_{abcd} \quad \forall Q \in \mathcal{G}. \quad (12)$$

Therefore, the requirement of material invariance for the quadratic stored energy density translates into relation (12) for the elasticity tensor which gives us relations between the components of \mathbb{C} . In this way, a part of the components of \mathbb{C} can be expressed as linear combinations of a subset of its components.

We now ask for an algorithmic procedure which allows us to understand, once we fix the subgroup \mathcal{G} , the number of independent components of a tensor \mathbb{C} which satisfy the relation (12) and their position in its matrix representation $\widehat{\mathbb{C}}$.

In order to do this, the idea is to interpret the relation (12) in terms of an action of \mathcal{G} over the vector space $\text{Sym}(\text{Sym}(3), \text{Sym}(3))$ and characterize the tensors which respect the relation (12) as those ones which are left fixed by the action. We give the formal definition of a linear action of a group over a vector space and we illustrate why we need this mathematical tool.

Definition 1 (Linear action). A **linear action** is a triple $(V, \mathcal{G}, \varphi)$ where V is a finite-dimensional real vector space, \mathcal{G} a topological group with unit element e and $\varphi : \mathcal{G} \times V \rightarrow V$ is a map such that

1. φ is a continuous map from $\mathcal{G} \times V$ to V , i.e. $\varphi \in \mathcal{C}^0(\mathcal{G} \times V, V)$,
2. $\varphi(e, v) = v$ for every $v \in V$,
3. for fixed $g \in \mathcal{G}$ the application $\varphi_g : V \rightarrow V$, $\varphi_g(v) := \varphi(g, v)$ is linear,
4. $\varphi(g_1, \varphi(g_2, v)) = \varphi(g_1 g_2, v)$ for all $g_1, g_2 \in \mathcal{G}$ and $v \in V$.

An action of a group on a vector space is therefore a way to move the elements inside the vector space along trajectories, called **orbits**, which have to respect the structure of the group. In the case of classical linear elasticity, the action we account for is given by the triple $(\mathfrak{Ela}(3), \mathcal{G}, \varphi)$ where

- $\mathfrak{Ela}(3)$ is the space of the elasticity tensors, defined as

$$\begin{aligned} \mathfrak{Ela}(3) &:= \text{Sym}(\text{Sym}(3), \text{Sym}(3)) \\ &= \{ \mathbb{C} \in \text{Lin}(\text{Sym}(3), \text{Sym}(3)) \mid \langle \mathbb{C} A, B \rangle_{\mathbb{R}^{3 \times 3}} = \langle A, \mathbb{C} B \rangle_{\mathbb{R}^{3 \times 3}} \quad \forall A, B \in \text{Sym}(3) \}, \end{aligned}$$

- \mathcal{G} is a proper subgroup of $\text{SO}(3)$ (and we will denote the property of being a closed subgroup by $\mathcal{G} \leq \text{SO}(3)$),
- $\varphi \in \mathcal{C}^0(\mathcal{G} \times \mathfrak{Ela}(3), \mathfrak{Ela}(3))$ is the map

$$\varphi : \mathcal{G} \times \mathfrak{Ela}(3) \rightarrow \mathfrak{Ela}(3) \quad (Q, \mathbb{C}) \longmapsto \varphi(Q, \mathbb{C}) =: \widehat{\mathbb{C}}, \quad \widehat{\mathbb{C}}_{ijkl} = Q_{ia} Q_{jb} Q_{kc} Q_{ld} \mathbb{C}_{abcd}.$$

The invariance condition established in (12) reads in this new language as follows:

$$\mathbb{C} \text{ satisfies (12) if and only if } \varphi(Q, \mathbb{C}) = \mathbb{C} \quad \forall Q \in \mathcal{G}.$$

Remark 1. The map φ is well-defined; more specifically,

$$\varphi(Q, \mathbb{C}) \in \mathfrak{Ela}(3) \quad \forall \mathbb{C} \in \mathfrak{Ela}(3).$$

Indeed, let us consider $\mathbb{C} \in \mathfrak{Ela}(3)$. The minor symmetry $1 \leftrightarrow 2$ allows us to establish

$$\widehat{\mathbb{C}}_{jikl} = Q_{ja} Q_{ib} Q_{kc} Q_{ld} \mathbb{C}_{abcd} = Q_{ja} Q_{ib} Q_{kc} Q_{ld} \mathbb{C}_{bacd} = Q_{ib} Q_{ja} Q_{kc} Q_{ld} \mathbb{C}_{bacd} = \widehat{\mathbb{C}}_{ijkl}.$$

It holds the same for the second minor symmetry $3 \leftrightarrow 4$ and the major symmetry (12) \leftrightarrow (34).

Remark 2. In order to guarantee the well-posedness of the linearized elasticity problem, it is necessary to impose that the bilinear form $\langle \mathbb{C} X, X \rangle_{\mathbb{R}^{3 \times 3}}$ is positive-definite, i.e., we need to require that

$$\mathbb{C} \in \mathfrak{Ela}^+(3) := \{ \mathbb{D} \in \mathfrak{Ela}^+(3) \mid \langle \mathbb{D} S, S \rangle_{\mathbb{R}^{3 \times 3}} > 0 \quad \forall S \in \text{Sym}(3) \setminus \{0\} \}.$$

Note that $\mathfrak{Ela}^+(3)$ is not a vector subspace of $\mathfrak{Ela}(3)$; instead, it is an open half-cone. Nevertheless, the action φ will be defined on $\mathfrak{Ela}(3)$ rather than $\mathfrak{Ela}^+(3)$ because this allows us to handle linear actions. While it is conceivable to consider the action directly on $\mathfrak{Ela}^+(3)$, doing so would complicate the general framework (by involving actions on smooth manifolds). Due to the technical constraint of ensuring $\mathbb{C} \in \mathfrak{Ela}^+(3)$, it becomes necessary to demonstrate that $\mathfrak{Ela}^+(3)$ is an invariant subset of φ . In other words,

$$\varphi(Q, \mathfrak{Ela}^+(3)) \subseteq \mathfrak{Ela}^+(3) \quad Q \in \text{SO}(3).$$

To show that, it suffices to note that for all $S \in \text{Sym}(3) \setminus \{0\}$ we have

$$\begin{aligned} \langle \varphi(Q, \mathbb{C}) S, S \rangle_{\mathbb{R}^{3 \times 3}} &= Q_{ia} Q_{jb} Q_{kc} Q_{ld} \mathbb{C}_{abcd} S_{ij} S_{kl} = \mathbb{C}_{abcd} (Q_{ia} S_{ij} Q_{jb}) (Q_{kc} S_{kl} Q_{ld}) \\ &= \mathbb{C}_{abcd} (Q_{ai}^T S_{ij} Q_{jb}) (Q_{ck}^T S_{kl} Q_{ld}) = \langle \mathbb{C} (Q^T S Q), \underbrace{Q^T S Q}_{\in \text{Sym}(3) \setminus \{0\}} \rangle_{\mathbb{R}^{3 \times 3}} > 0 \end{aligned} \quad (13)$$

for all $(Q, \mathbb{C}) \in \text{SO}(3) \times \mathfrak{Ela}^+(3)$.

Thus, we can say that an elasticity tensor \mathbb{C} respects the considered symmetry if it is left fixed by the action of the group on the vector space $V = \mathfrak{Ela}(3)$. Therefore, the set of the tensors which verify (12) is the subset of $\mathfrak{Ela}(3)$ of tensors which do not move under the action of φ . This subset is a vector subspace² of V and it is called the **fixed-point subspace**

$$\text{Fix}_{\mathcal{G}}^{\varphi} V := \{ \mathbb{C} \in V \mid \varphi(Q, \mathbb{C}) = \mathbb{C} \text{ for all } Q \in \mathcal{G} \} = \bigcap_{Q \in \mathcal{G}} \underbrace{\text{Ker}(\mathbb{1}_V - \varphi_Q)}_{\in \text{Lin}(V, V)}. \quad (14)$$

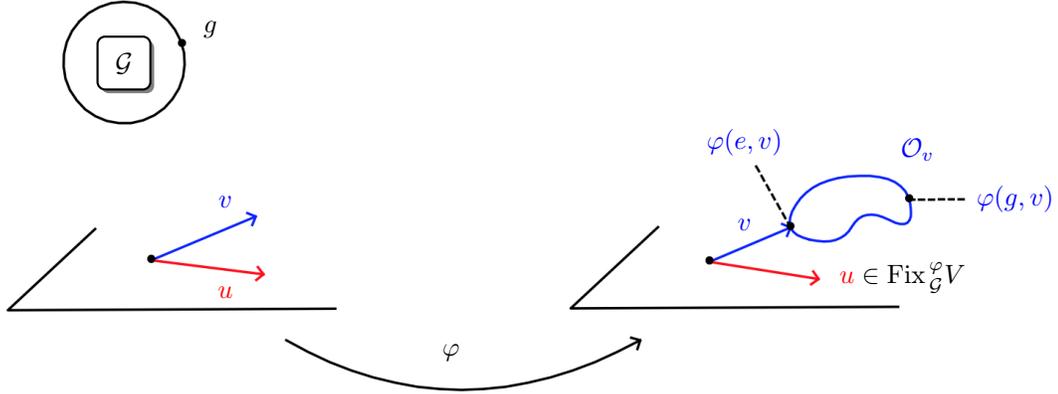


Figure 1: Action, orbit and fixed point subspace.

The dimension of the fixed-point subspace provides the count of independent components of the invariant tensors. By applying the map \mathfrak{M} (see formula (110)) to the elements of $\text{Fix}_{\mathcal{G}}^{\varphi} V$, we can derive the corresponding matrix representation. Therefore, the crucial point lies in comprehending how to determine the set $\text{Fix}_{\mathcal{G}}^{\varphi} V$. In elementary linear algebra, it is well-established that, in the context of a finite-dimensional vector space with a subspace, one can define a projection operator on it. This operator enables us to isolate, from any arbitrary element in the vector space, only the component that belongs to the subspace. Consequently, the subspace can be obtained by projecting the entire vector space using the corresponding projection operator. In this context, working with a subspace is entirely equivalent to working with a projection operator.

²Indeed, as showed in formula (14), $\text{Fix}_{\mathcal{G}}^{\varphi} V$ is the intersection of the vector subspaces $\{\text{Ker}(\mathbb{1}_V - \varphi_Q)\}_{Q \in \mathcal{G}}$ of V .

In our specific scenario, we lack the direct capability to define $\text{Fix}_{\mathcal{G}}^{\mathcal{C}}V$. However, we can introduce a projection operator, and through this, we can ascertain that the resultant subspace precisely coincides with the set of fixed elements.

The concept is as follows: Let us take a generic element denoted as \mathbb{C} belonging to the vector space $\mathfrak{Ela}(3)$. We then track its trajectory (its orbit) determined by the action of a discrete group \mathcal{G} (we are making the hypothesis for the moment that \mathcal{G} has a finite number of 6 elements). Now, if we perform the summation $\widehat{\mathbb{C}} = \frac{1}{6} \sum_{i=1}^6 \varphi(Q_i, \mathbb{C})$, $Q_i \in \mathcal{G}$, over all the elements in the orbit $\mathcal{O}_{\mathbb{C}}$ of \mathbb{C} , where

$$\mathcal{O}_{\mathbb{C}} := \{\mathbb{D} \in V \mid \exists Q \in \mathcal{G} \text{ such that } \mathbb{D} = \varphi(Q, \mathbb{C})\}, \quad (15)$$

we obtain a tensor. This tensor automatically retains its invariance under the action of φ . As illustrated in Figure 2, this invariance can be interpreted as φ simultaneously affecting all the terms in the sum, which are in turn interrelated such that each maps to another summand within the overall summation.

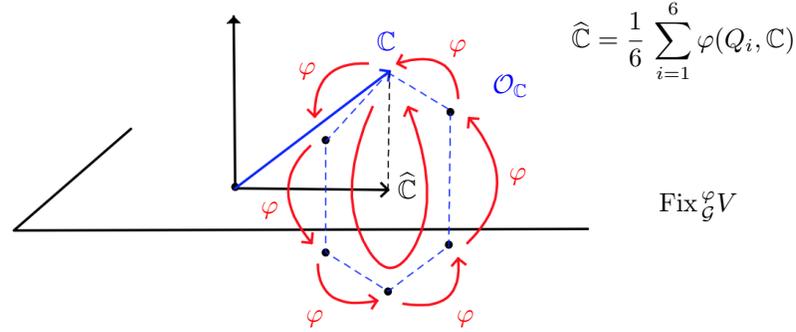


Figure 2: Given an element $\mathbb{C} \in V$, its symmetrized $\widehat{\mathbb{C}}$ is obtained averaging over its orbit.

Indeed, denoting with n the cardinality of the considered group, the tensor

$$\widehat{\mathbb{C}} = \frac{1}{n} \sum_{i=1}^n \varphi(Q_i, \mathbb{C}), \quad (16)$$

i.e. the average of the elements of the orbit $\mathcal{O}_{\mathbb{C}}$ of \mathbb{C} , is invariant under the action: for all $\Lambda \in \mathcal{G}$ we have

$$\varphi(\Lambda, \widehat{\mathbb{C}}) = \varphi\left(\Lambda, \frac{1}{n} \sum_{i=1}^n \varphi(Q_i, \mathbb{C})\right) = \frac{1}{n} \sum_{i=1}^n \varphi(\Lambda, \varphi(Q_i, \mathbb{C})) = \frac{1}{n} \sum_{i=1}^n \varphi(\Lambda Q_i, \mathbb{C}) \stackrel{(*)}{=} \frac{1}{n} \sum_{i=1}^n \varphi(Q_i, \mathbb{C}) = \widehat{\mathbb{C}}. \quad (17)$$

Another way to say this is: averaging the elements of the vector space with respect to the action of the group we obtain the elements of the fixed-point subspace. The projection we are looking for has to do exactly this, i.e. it has to average the elements of the space of symmetric fourth order tensors.

From a technical point of view, there are a series of difficulties we have to deal with. Indeed, not all the closed subgroups of $\text{SO}(3)$ are discrete and this means that we have to extend the sum on continuous group in a way in which we can still guarantee the validity of identities like $(*)$ in equation (17) (which is obvious in the case of a discrete sum but it demands the notions of an invariant measure over the group \mathcal{G} if we need to perform an integral).

The technical tool we need here is the Haar measure over topological groups. Roughly speaking, the Haar measure has, in the context of topological groups, the same role which the Lebesgue measure plays in \mathbb{R}^n . One of the fundamental properties of the Lebesgue measure is its invariance with respect to rigid transformations, i.e. roto-translation maps of the form $f(x) = Qx + b$ with $Q \in \text{O}(n)$ and $b \in \mathbb{R}^n$. This property guarantees that the Lebesgue measure is a good notion to measure the size of subsets of \mathbb{R}^n because if we move a subset with a rigid map its size does not change. On topological groups we do not have rotations and translations.

However, we still have something which works in an analogous way: left- and right- translations. The **left-translation** is defined as follows: for any $h \in \mathcal{G}$, we set

$$L_h: \mathcal{G} \rightarrow \mathcal{G}, \quad L_h(g) := hg. \quad (18)$$

In an analogous way we define **right-translations** by

$$R_h: \mathcal{G} \rightarrow \mathcal{G}, \quad R_h(g) := gh. \quad (19)$$

On a topological group, these movements precisely represent the transformations we wish to preserve. Luckily, as shown in [20], every compact topological group (which are the groups of interest, like $SO(3)$) has a measure μ which preserves both left- and right-translations such that the size of the group is 1 (i.e. $\mu(\mathcal{G}) = 1$); this is called the **normalized Haar measure** on \mathcal{G} . This means that if we are considering a measurable subset \mathcal{A} of \mathcal{G} , and we move this set according to L_h or R_h , then $\mu(L_h(\mathcal{A})) = \mu(R_h(\mathcal{A})) = \mu(\mathcal{A})$. This property translates for the derived notion of integration (the **normalized Haar integral**) as follows: for any integrable function³ $f: \mathcal{G} \rightarrow \mathbb{R}$,

$$\int_{\mathcal{G}} f(g) d\mu = \int_{\mathcal{G}} f(L_h(g)) d\mu = \int_{\mathcal{G}} f(hg) d\mu = \int_{\mathcal{G}} f(gh) d\mu = \int_{\mathcal{G}} f(R_h(g)) d\mu \quad \forall h \in \mathcal{G}. \quad (20)$$

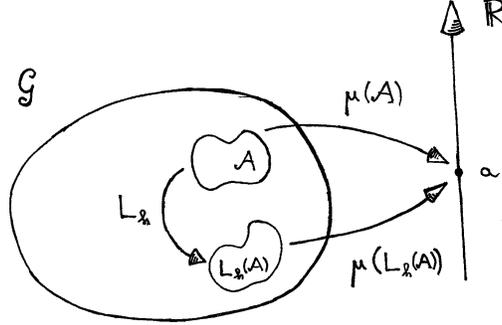


Figure 3: Left-invariance of a measure.

Therefore, the average in (16) can be expressed as follows for continuous groups:

$$\widehat{\mathbb{C}} = \int_{\mathcal{G}} \varphi(Q, \mathbb{C}) d\mu, \quad (21)$$

where μ is the normalized Haar measure on \mathcal{G} . It is the invariance property of the Haar measure with respect to both left- and right-translations which guarantees also in the continuous case that the averaged tensor $\widehat{\mathbb{C}}$ is left fixed by the action. Indeed, exactly as in (17), for $\Lambda \in SO(3)$ we have

$$\varphi(\Lambda, \widehat{\mathbb{C}}) = \varphi\left(\Lambda, \int_{SO(3)} \varphi(Q, \mathbb{C}) d\mu\right) = \int_{SO(3)} \varphi(\Lambda, \varphi(Q, \mathbb{C})) d\mu = \int_{SO(3)} \varphi(\Lambda Q, \mathbb{C}) d\mu = \int_{SO(3)} \varphi(Q, \mathbb{C}) d\mu = \widehat{\mathbb{C}}. \quad (22)$$

This motivates to define the projection

$$\mathcal{P}: \mathfrak{E}ta(3) \rightarrow \text{Fix}_{\mathcal{G}}^{\varphi} \mathfrak{E}ta(3),$$

where

$$\begin{aligned} \mathcal{P}(\mathbb{C}) &:= \int_{\mathcal{G}} \varphi(Q, \mathbb{C}) d\mu = \int_{\mathcal{G}} \varphi_Q(\mathbb{C}) d\mu && \text{for the continuous case,} \\ \mathcal{P}(\mathbb{C}) &:= \frac{1}{\text{card } \mathcal{G}} \sum_{Q \in \mathcal{G}} \varphi(Q, \mathbb{C}) = \frac{1}{\text{card } \mathcal{G}} \sum_{Q \in \mathcal{G}} \varphi_Q(\mathbb{C}) && \text{for the discrete case.} \end{aligned} \quad (23)$$

Remark 3. For a discrete group \mathcal{G} , the normalized Haar measure is simply a weighted counting measure; more specifically,

$$\mu(\mathcal{A}) = \frac{\text{card } \mathcal{A}}{\text{card } \mathcal{G}}$$

³The same invariance holds for both left and right-translations simultaneously. For instance, consider $h_1, h_2 \in \mathcal{G}$ and a function $f: \mathcal{G} \rightarrow \mathbb{R}$. We introduce the auxiliary function $\tilde{f}(g) = f(gh_2)$. Then

$$\int_{\mathcal{G}} f(h_1gh_2) d\mu = \int_{\mathcal{G}} f(h_1(gh_2)) d\mu = \int_{\mathcal{G}} \tilde{f}(h_1g) d\mu = \int_{\mathcal{G}} \tilde{f}(g) d\mu = \int_{\mathcal{G}} f(gh_2) d\mu = \int_{\mathcal{G}} f(g) d\mu.$$

for all measurable $\mathcal{A} \subset \mathcal{G}$. In this case,

$$\int_{\mathcal{G}} \varphi_Q(\mathbb{C}) \, d\mu = \frac{1}{\text{card } \mathcal{G}} \sum_{Q \in \mathcal{G}} \varphi_Q(\mathbb{C}),$$

thus the discrete case in (23) can be considered a special case of the general Haar-measure based formula.

Remark 4. Note that \mathcal{P} is surjective, i.e. $\mathcal{P}: \mathfrak{Ela}(3) \rightarrow \text{Fix}_{\mathcal{G}}^{\varphi} \mathfrak{Ela}(3)$. Indeed, if $\mathbb{C} \in \text{Fix}_{\mathcal{G}}^{\varphi} \mathfrak{Ela}(3)$, then by definition of the fixed point subspace, $\varphi(Q, \mathbb{C}) = \mathbb{C}$ for every $Q \in \mathcal{G}$ and thus

$$\mathcal{P}(\mathbb{C}) = \int_{\mathcal{G}} \underbrace{\varphi(Q, \mathbb{C})}_{=\mathbb{C} \, \forall Q \in \mathcal{G}} \, d\mu = \int_{\mathcal{G}} \mathbb{C} \, d\mu = \mathbb{C} \underbrace{\int_{\mathcal{G}} d\mu}_{=1} = \mathbb{C},$$

i.e., we obtain the inclusion $\text{Fix}_{\mathcal{G}}^{\varphi} \mathfrak{Ela}(3) \subseteq \text{Im}(\mathcal{P})$. Analogously for the discrete case,

$$\mathcal{P}(\mathbb{C}) = \frac{1}{\text{card } \mathcal{G}} \sum_{\mathcal{G}} \underbrace{\varphi(Q, \mathbb{C})}_{=\mathbb{C} \, \forall Q \in \mathcal{G}} = \frac{1}{\text{card } \mathcal{G}} \sum_{\mathcal{G}} \mathbb{C} = \frac{1}{\text{card } \mathcal{G}} (\text{card } \mathcal{G}) \mathbb{C} = \mathbb{C}.$$

2.1 The trace formula

We have successfully acquired the skills to compute the structure of symmetrized tensors. However, there remains the task of determining the dimension of $\text{Fix}_{\mathcal{G}}^{\varphi} V$. To confront this secondary challenge, we introduce a pivotal tool into our arsenal: the **trace formula**. Indeed, the general result establishes the following (see [30, Thm.2.3, p.76]).

Theorem 1. *Consider a linear action $(V, \mathcal{H}, \varphi)$ where \mathcal{H} is a compact Lie group. Then for a Lie subgroup $\mathcal{G} \leq \mathcal{H}$, we have*

$$\dim \text{Fix}_{\mathcal{G}}^{\varphi} V = \int_{g \in \mathcal{G}} \text{tr } \varphi_g \, d\mu,$$

where the measure $d\mu$ is the normalized Haar measure on \mathcal{G} and the maps $\{\varphi_g\}$ are the linear maps as defined in Definition 1.

Proof. Let us consider the projection operator \mathcal{P} associated to $\text{Fix}_{\mathcal{G}}^{\varphi} V$. Then

$$\text{tr } \mathcal{P} = \dim \text{Fix}_{\mathcal{G}}^{\varphi} V.$$

This is an elementary consequence of the fact that it is possible to find an isomorphism⁴ $\Upsilon: V \rightarrow \mathbb{R}^m$, for a suitable $m \in \mathbb{N}$, such that the projection \mathcal{P} can be represented as a matrix operator $A_{\mathcal{P}}$ with

$$\begin{array}{ccc} V & \xrightarrow{\mathcal{P}} & \text{Im}(\mathcal{P}) = \text{Fix}_{\mathcal{G}}^{\varphi} V \subseteq V \\ \Upsilon \downarrow & & \downarrow \Upsilon \\ \mathbb{R}^m & \xrightarrow{A_{\mathcal{P}}} & \mathbb{R}^p \subseteq \mathbb{R}^m \end{array} \quad A_{\mathcal{P}} = \left(\begin{array}{c|c} \mathbf{1}_{\mathbb{R}^p} & 0 \\ \hline 0 & 0 \end{array} \right). \quad (24)$$

The construction of \mathcal{P} we derived in eq.(23) can be straightforwardly generalized to any linear action $(V, \mathcal{H}, \varphi)$, where \mathcal{H} is a compact Lie group and \mathcal{G} is a Lie subgroup. Hence we obtain

$$\dim \text{Fix}_{\mathcal{G}}^{\varphi} V = \text{tr } \mathcal{P} \stackrel{(23)}{=} \text{tr} \int_{\mathcal{G}} \varphi_g \, d\mu = \int_{\mathcal{G}} \text{tr } \varphi_g \, d\mu. \quad \square$$

Therefore, from the knowledge of the traces $\{\text{tr } \varphi_Q\}_{Q \in \mathcal{G}}$, simply computing an integral we can finally obtain $\dim \text{Fix}_{\mathcal{G}}^{\varphi} V$. The quantities $\{\text{tr } \varphi_Q\}_{Q \in \mathcal{G}}$ are known in the literature as **characters** of $\{\varphi_Q\}_{Q \in \mathcal{G}}$ and they will be denoted by $\{\chi(Q)\}_{Q \in \mathcal{G}}$, $\chi(Q) \in \mathbb{R}$.

⁴Indeed, it is possible to choose a basis $\{v_i\}_{i=1}^m$ of V ($m = \dim V$) such that, denoting by d the dimension of the subspace $\text{Im}(\mathcal{P})$, v_1, \dots, v_d is a basis of $\text{Im}(\mathcal{P})$. Hence the isomorphism Υ is obtained via the identifications $v_i \mapsto e_i$ for each i , where $\{e_i\}_i^m$ is the canonical basis of \mathbb{R}^m .

Example 1. Let us illustrate the content of Theorem 1 with an example. Consider $V = \text{Sym}(2)$ and \mathcal{G}, φ and \mathcal{P} as in Example 2 below. Then $\dim V = 3$ and $\dim \text{Fix}_{\mathcal{G}}^{\varphi} V = 1$. Hence, after identifying V with \mathbb{R}^3 and $\text{Fix}_{\mathcal{G}}^{\varphi} V$ with the subspace of V generated by $(1, 0, 0)$ (denoted by $\langle(1, 0, 0)\rangle$), $A_{\mathcal{P}}: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ is such that $\text{Im}(A_{\mathcal{P}}) = \langle(1, 0, 0)\rangle$ and $A_{\mathcal{P}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$.

Equipping the space $\text{Lin}(V, V)$ with a scalar product $\langle \cdot, \cdot \rangle_{\text{Lin}(V, V)}$, the trace of a linear map $\varphi: V \rightarrow V$ can be introduced as

$$\text{tr } \varphi := \langle \varphi, \mathbf{1}_{\text{Lin}(V, V)} \rangle_{\text{Lin}(V, V)}.$$

From the definition of the action, it follows that the identity of $\text{Lin}(\mathfrak{E}\mathfrak{a}(3), \mathfrak{E}\mathfrak{a}(3))$ is $\varphi_{\mathbf{1}_2}$ which acts as

$$(\varphi(\mathbf{1}_2, \mathbb{C}))_{ijkl} = (\otimes^4 \mathbf{1}_2)_{iajbkcl} \mathbb{C}_{abcd} = \delta_{ia} \delta_{jb} \delta_{kc} \delta_{ld} \mathbb{C}_{abcd} = \mathbb{C}_{ijkl} \quad \forall i, j, k, l \in \{1, 2, 3\}.$$

Then it would be natural to set

$$\text{tr } \varphi_Q = \chi(Q) = \langle \varphi_Q, \otimes^4 \mathbf{1}_2 \rangle_{\otimes^4 \mathbb{R}^3}. \quad (25)$$

Nevertheless this is incorrect and the reason is at the same time very subtle and very simple. The crucial remark here is the fact that both $\otimes^4 \mathbf{1}_2$ and φ_Q can act as linear maps over the bigger vector space $\text{Lin}(\mathbb{R}^{3 \times 3}, \mathbb{R}^{3 \times 3})$ and formally these maps have the same expressions. Hence, in eq.(25) we need to impose that we are considering φ_Q as a map restricted to the subspace $\mathfrak{E}\mathfrak{a}(3) \subset \text{Lin}(\mathbb{R}^{3 \times 3}, \mathbb{R}^{3 \times 3})$. The simplest way to achieve this is considering the projection operator associated to $\mathfrak{E}\mathfrak{a}(3)$. This projection ‘‘has to symmetrize’’ elements belonging to $\text{Lin}(\mathbb{R}^{3 \times 3}, \mathbb{R}^{3 \times 3})$. For this reason it is denoted as **symmetrization identity**. In order to get an element of $\mathfrak{E}\mathfrak{a}(3)$ from an element of $\text{Lin}(\mathbb{R}^{3 \times 3}, \mathbb{R}^{3 \times 3})$ we need to symmetrize it w.r.t. the symmetries possessed by the elements of $\mathfrak{E}\mathfrak{a}(3)$. Let us introduce the symmetrization identity

$$\Pi^{\text{Sym}(3)} \in \text{Lin}(\mathbb{R}^{3 \times 3}, \text{Sym}(3)), \quad \text{where} \quad \Pi_{iajb}^{\text{Sym}(3)} = \frac{1}{2}(\delta_{ia} \delta_{jb} + \delta_{ib} \delta_{ja}), \quad (26)$$

which acts as

$$\Pi^{\text{Sym}(3)}: \mathbb{R}^{3 \times 3} \longrightarrow \mathbb{R}^{3 \times 3}, \quad S \longmapsto \Pi^{\text{Sym}(3)} S, \quad \text{where} \quad (\Pi^{\text{Sym}(3)} S)_{ij} = \frac{1}{2}(\delta_{ia} \delta_{jb} + \delta_{ib} \delta_{ja}) S_{ab}.$$

Clearly, $\Pi^{\text{Sym}(3)} S \in \text{Sym}(3)$ because

$$\begin{aligned} (\Pi^{\text{Sym}(3)} S)_{ji} &= \frac{1}{2}(\delta_{ja} \delta_{ib} + \delta_{jb} \delta_{ia}) S_{ab} = \frac{1}{2}(\delta_{ja} \delta_{ib} S_{ab} + \delta_{jb} \delta_{ia} S_{ab}) = \frac{1}{2}(\delta_{ja} \delta_{ib} S_{ab} + \delta_{ia} \delta_{jb} S_{ab}) \\ &= \frac{1}{2}(\delta_{ia} \delta_{jb} S_{ab} + \delta_{ja} \delta_{ib} S_{ab}) = (\Pi^{\text{Sym}(3)} S)_{ij}. \end{aligned}$$

Hence, to build up the projection from $\text{Lin}(\mathbb{R}^{3 \times 3}, \mathbb{R}^{3 \times 3})$ to $\mathfrak{E}\mathfrak{a}(3)$ we need to symmetrize w.r.t. both the arguments via $\Pi^{\text{Sym}(3)}$ and w.r.t. the exchange of the arguments to obtain the major symmetry. This means that the projection from $\text{Lin}(\mathbb{R}^{3 \times 3}, \mathbb{R}^{3 \times 3})$ to $\mathfrak{E}\mathfrak{a}(3)$ will be

$$\Pi^{\mathfrak{E}\mathfrak{a}(3)}: \text{Lin}(\mathbb{R}^{3 \times 3}, \mathbb{R}^{3 \times 3}) \longrightarrow \mathfrak{E}\mathfrak{a}(3), \quad \Pi^{\mathfrak{E}\mathfrak{a}(3)} := \text{sym}(\Pi^{\text{Sym}(3)} \otimes \Pi^{\text{Sym}(3)}),$$

where the symmetrization operation sym in the previous definition is defined as usual, i.e.,

$$\text{sym}(\Pi^{\text{Sym}(3)} \otimes \Pi^{\text{Sym}(3)})_{iajbkcl} = \frac{1}{2} \left((\Pi^{\text{Sym}(3)} \otimes \Pi^{\text{Sym}(3)})_{iajbkcl} + (\Pi^{\text{Sym}(3)} \otimes \Pi^{\text{Sym}(3)})_{kcl diajb} \right)$$

for every $i, a, j, b, k, c, l, d \in \{1, 2, 3\}$. Thus, finally, the component-wise expression of $\Pi^{\mathfrak{E}\mathfrak{a}(3)}$ is

$$\begin{aligned} \Pi_{iajbkcl}^{\mathfrak{E}\mathfrak{a}(3)} &= \frac{1}{8} \left(\delta_{ia} \delta_{jb} \delta_{kc} \delta_{ld} + \delta_{ia} \delta_{jb} \delta_{lc} \delta_{kd} + \delta_{ja} \delta_{ib} \delta_{lc} \delta_{kd} + \delta_{ja} \delta_{ib} \delta_{kc} \delta_{ld} \right. \\ &\quad \left. + \delta_{ka} \delta_{lb} \delta_{ic} \delta_{jd} + \delta_{ka} \delta_{lb} \delta_{jc} \delta_{id} + \delta_{la} \delta_{kb} \delta_{jc} \delta_{id} + \delta_{la} \delta_{kb} \delta_{ic} \delta_{jd} \right). \end{aligned} \quad (27)$$

Now we can compute the trace of the linear maps $\{\varphi_Q\}_{Q \in \mathcal{G}}$ as maps from $\mathfrak{E}\mathfrak{a}(3)$ to $\mathfrak{E}\mathfrak{a}(3)$ considering the composition maps $\{\varphi_Q \circ \Pi^{\mathfrak{E}\mathfrak{a}(3)}\}_{Q \in \mathcal{G}}$, obtaining

$$\chi(Q) = \text{tr}(\varphi_Q \circ \Pi^{\mathfrak{E}\mathfrak{a}(3)}) = \left\langle \varphi_Q \circ \Pi^{\mathfrak{E}\mathfrak{a}(3)}, \otimes^4 \mathbf{1}_2 \right\rangle_{\mathfrak{E}\mathfrak{a}(3)} = Q_{ia} Q_{jb} Q_{kc} Q_{ld} \Pi_{\alpha\alpha\beta\beta\gamma\gamma d\delta}^{\mathfrak{E}\mathfrak{a}(3)} \delta_{i\alpha} \delta_{j\beta} \delta_{k\gamma} \delta_{l\delta}$$

$$= Q_{ia}Q_{jb}Q_{kc}Q_{ld}\Pi_{aibjckdl} = \Pi_{iajbkcl d}Q_{ia}Q_{jb}Q_{kc}Q_{ld}.$$

Expanding, we find

$$\begin{aligned} \chi(Q) &= \Pi_{iajbkcl d}Q_{ia}Q_{jb}Q_{kc}Q_{ld} \\ &= \frac{1}{8}(\delta_{ia}\delta_{jb}\delta_{kc}\delta_{ld} + \delta_{ia}\delta_{jb}\delta_{lc}\delta_{kd} + \delta_{ja}\delta_{ib}\delta_{lc}\delta_{kd} + \delta_{ja}\delta_{ib}\delta_{kc}\delta_{ld} \\ &\quad + \delta_{ka}\delta_{lb}\delta_{ic}\delta_{jd} + \delta_{ka}\delta_{lb}\delta_{jc}\delta_{id} + \delta_{la}\delta_{kb}\delta_{jc}\delta_{id} + \delta_{la}\delta_{kb}\delta_{ic}\delta_{jd})Q_{ia}Q_{jb}Q_{kc}Q_{ld}, \\ &= \frac{1}{8}(\delta_{ia}\delta_{jb}\delta_{kc}\delta_{ld}Q_{ia}Q_{jb}Q_{kc}Q_{ld} + \delta_{ia}\delta_{jb}\delta_{lc}\delta_{kd}Q_{ia}Q_{jb}Q_{kc}Q_{ld} \\ &\quad + \delta_{ja}\delta_{ib}\delta_{lc}\delta_{kd}Q_{ia}Q_{jb}Q_{kc}Q_{ld} + \delta_{ja}\delta_{ib}\delta_{kc}\delta_{ld}Q_{ia}Q_{jb}Q_{kc}Q_{ld} \\ &\quad + \delta_{ka}\delta_{lb}\delta_{ic}\delta_{jd}Q_{ia}Q_{jb}Q_{kc}Q_{ld} + \delta_{ka}\delta_{lb}\delta_{jc}\delta_{id}Q_{ia}Q_{jb}Q_{kc}Q_{ld} \\ &\quad + \delta_{la}\delta_{kb}\delta_{jc}\delta_{id}Q_{ia}Q_{jb}Q_{kc}Q_{ld} + \delta_{la}\delta_{kb}\delta_{ic}\delta_{jd}Q_{ia}Q_{jb}Q_{kc}Q_{ld}) \\ &= \frac{1}{8}\left(\underbrace{\delta_{ia}Q_{ia}}_{\text{tr}Q}\underbrace{\delta_{jb}Q_{jb}}_{\text{tr}Q}\underbrace{\delta_{kc}Q_{kc}}_{\text{tr}Q}\underbrace{\delta_{ld}Q_{ld}}_{\text{tr}Q} + \underbrace{\delta_{ia}Q_{ia}}_{\text{tr}Q}\underbrace{\delta_{jb}Q_{jb}}_{\text{tr}Q}\underbrace{\delta_{cl}Q_{ld}\delta_{dk}Q_{kc}}_{Q_{cd}Q_{dc}=\text{tr}Q^2}\right. \\ &\quad + \underbrace{\delta_{bi}Q_{ia}\delta_{aj}Q_{jb}}_{\text{tr}Q^2}\underbrace{\delta_{kd}Q_{kc}\delta_{cl}Q_{ld}}_{\text{tr}Q^2} + \underbrace{\delta_{aj}Q_{jb}\delta_{bi}Q_{ia}}_{\text{tr}Q^2}\underbrace{\delta_{kc}Q_{kc}}_{\text{tr}Q}\underbrace{\delta_{ld}Q_{ld}}_{\text{tr}Q} \\ &\quad + \underbrace{\delta_{ak}Q_{kc}\delta_{ci}Q_{ia}}_{\text{tr}Q^2}\underbrace{\delta_{dj}Q_{jb}\delta_{bl}Q_{ld}}_{\text{tr}Q^2} + \underbrace{\delta_{ak}Q_{kc}\delta_{cj}Q_{jb}\delta_{bl}Q_{ld}\delta_{di}Q_{ia}}_{\text{tr}Q^4} \\ &\quad \left. + \underbrace{\delta_{al}Q_{ld}\delta_{di}Q_{ia}}_{\text{tr}Q^2}\underbrace{\delta_{cj}Q_{jb}\delta_{bk}Q_{kc}}_{\text{tr}Q^2} + \underbrace{\delta_{al}Q_{ld}\delta_{dj}Q_{jb}\delta_{bk}Q_{kc}\delta_{ci}Q_{ia}}_{\text{tr}Q^4}\right) \\ &= \frac{1}{8}\left((\text{tr}Q)^4 + (\text{tr}Q)^2\text{tr}Q^2 + (\text{tr}Q^2)^2 + (\text{tr}Q)^2\text{tr}Q^2 + (\text{tr}Q^2)^2 + \text{tr}Q^4 + (\text{tr}Q^2)^2 + \text{tr}Q^4\right) \end{aligned}$$

and thus

$$\chi(Q) = \frac{1}{8}\left((\text{tr}Q)^4 + 2(\text{tr}Q)^2\text{tr}Q^2 + 2\text{tr}Q^4 + 3(\text{tr}Q^2)^2\right). \quad (28)$$

Moreover, using the Cayley-Hamilton theorem (see the Appendix) for proper orthogonal rotations, the character χ can be also written as a polynomial in the first invariant of Q , i.e. $\text{tr}Q$, as⁵

$$\chi(Q) = (\text{tr}Q)^4 - 3(\text{tr}Q)^3 + 2(\text{tr}Q)^2 + \text{tr}Q. \quad (29)$$

Finally, we can summarize the proposed method according to the developed scheme:

- The material invariance condition requires the relations

$$\mathbb{C}_{ijhk} = Q_{ia}Q_{jb}Q_{hc}Q_{kd}\mathbb{C}_{abcd} \quad \forall Q \in \mathcal{G}$$

between the components of the elasticity tensor. We express this in terms of the action φ as

$$\varphi(Q, \mathbb{C}) = \mathbb{C} \quad \forall Q \in \mathcal{G},$$

which translates the problem of looking for constitutive tensors satisfying the material invariance condition to the search for a particular subspace of $V = \mathfrak{Ela}(3)$, the fixed point subspace $\text{Fix}_{\mathcal{G}}^{\varphi}V$.

- The fixed point subspace can be obtained computing the projection \mathcal{P} which acts by averaging a considered tensor over its orbit.

⁵In the following section, we will analyze generalized elasticity models that accommodate ambient space dimensions different from 3. This exploration delves into the spaces of elasticity tensors $\mathfrak{Ela}(n)$. It's noteworthy that the trace formula remains consistent in the n -dimensional case up to (28). However, the Cayley-Hamilton formula for $Q \in \text{SO}(n)$ is dimension-dependent, impacting the expression of the characters as polynomials in $\text{tr}Q$. Consequently, we will derive appropriate expressions based on n . For instance, when $n = 2$, (29) would undergo a transformation to

$$\chi(Q) = (\text{tr}Q)^4 - 3(\text{tr}Q)^2 + 2.$$

- The dimension of the fixed-point subspace, which gives the number of independent components of the invariant tensors, is calculated via the trace formula, i.e. by evaluating

$$\int_{\mathcal{G}} \text{tr } \varphi(Q, \mathbb{C}) \, d\mu \quad \text{or} \quad \frac{1}{\text{card } \mathcal{G}} \sum_{Q \in \mathcal{G}} \text{tr } \varphi(Q, \mathbb{C}),$$

where the trace of the linear applications is computed by virtue of the symmetrization identity $\Pi^{\mathfrak{e}\mathfrak{t}\mathfrak{a}(3)}$, i.e., the projection $\Pi^{\mathfrak{e}\mathfrak{t}\mathfrak{a}(3)} : \text{Lin}(\mathbb{R}^{3 \times 3}, \mathbb{R}^{3 \times 3}) \rightarrow \mathfrak{e}\mathfrak{t}\mathfrak{a}(3)$.

Example 2. We consider the illustrative variational problem

$$\mathcal{A}[f] = \int_{\Omega} W(Df(x)) \, dx,$$

where $\Omega \subseteq \mathbb{R}^2$ is a Lipschitz domain, $f: \Omega \subseteq \mathbb{R}^2 \rightarrow \mathbb{R}$ are the admissible functions and $W: \mathbb{R}^2 \rightarrow \mathbb{R}$ is the Lagrangian energy density. From a physical standpoint, we are dealing with a variational problem concerning a scalar quantity (such as the temperature) on a two-dimensional body. In this situation, we say that the Lagrangian energy density is invariant with respect to the action of a closed subgroup \mathcal{G} of $\text{SO}(2)$ if

$$W(Q\zeta) = W(\zeta) \quad \forall Q \in \mathcal{G} \text{ and } \zeta \in \mathbb{R}^2.$$

If we consider the simpler case in which W is a quadratic form in ζ , we have the linear problem

$$\mathcal{A}_{\text{lin}}[f] = \frac{1}{2} \int_{\Omega} \langle \mathbb{C} Df(x), Df(x) \rangle \, dx,$$

where $\mathbb{C} \in \text{Sym}(2)$, and the invariance condition becomes

$$\langle \mathbb{C}(Q\zeta), Q\zeta \rangle = \langle \mathbb{C}\zeta, \zeta \rangle \quad \forall Q \in \mathcal{G} \quad \text{and} \quad \forall \zeta \in \mathbb{R}^2$$

i.e.

$$\mathbb{C}_{ab} Q_{bj} \zeta_j Q_{ai} \zeta_i = \mathbb{C}_{ij} \zeta_j \zeta_i \quad \forall Q \in \mathcal{G} \quad \text{and} \quad \forall \zeta \in \mathbb{R}^2$$

which gives

$$\mathbb{C}_{ab} Q_{ai} Q_{bj} = \mathbb{C}_{ij} \quad \forall Q \in \mathcal{G} \quad \text{and} \quad \forall i, j.$$

Thus, in this case, the symmetry invariance condition reads as follows: let us consider the triple $(\text{Sym}(2), \mathcal{G}, \varphi)$, where the action φ is defined by

$$\varphi : \mathcal{G} \times \text{Sym}(2) \rightarrow \text{Sym}(2), \quad \varphi(Q, \mathbb{C}) = \widehat{\mathbb{C}} \quad \text{with} \quad \widehat{\mathbb{C}}_{ij} := \mathbb{C}_{ab} Q_{ai} Q_{bj}. \quad (30)$$

Then \mathbb{C} respects the symmetry if and only if $\mathbb{C} = \varphi(Q, \mathbb{C})$ for every $Q \in \mathcal{G}$. We are going now to study the class of tensors \mathbb{C} which are invariant with respect to the action of the full group $\text{SO}(2)$ following the stated scheme. Setting

$$\varphi_Q : \text{Sym}(2) \rightarrow \text{Sym}(2), \quad \varphi_Q(\mathbb{C})_{ij} = \mathbb{C}_{ab} Q_{ai} Q_{bj},$$

and denoting with μ the Haar measure on $\text{SO}(2)$, the projection \mathcal{P} is given by

$$\mathcal{P}(\mathbb{C}) = \left(\int_{Q \in \text{SO}(2)} \varphi_Q \, d\mu \right) \mathbb{C} = \int_{\text{SO}(2)} \varphi_Q \mathbb{C} \, d\mu,$$

or component-wise by

$$(\mathcal{P}(\mathbb{C}))_{ij} = \int_{Q \in \text{SO}(2)} \mathbb{C}_{ab} Q_{ai} Q_{bj} \, d\mu.$$

Setting

$$\psi_{\mathbb{C}}^{ij} : \text{SO}(2) \rightarrow \mathbb{R}, \quad \psi_{\mathbb{C}}^{ij}(Q) := \mathbb{C}_{ab} Q_{ai} Q_{bj}$$

we can use the formula (98) to calculate such integrals. Indeed, we have that

$$\int_{\text{SO}(2)} \psi_{\mathbb{C}}^{ij}(Q) \, d\mu = \frac{1}{2\pi} \int_0^{2\pi} \psi_{\mathbb{C}}^{ij}(\vartheta) \, d\vartheta,$$

where we use the parametrization

$$\vartheta \mapsto Q(\vartheta) = \begin{pmatrix} \cos \vartheta & -\sin \vartheta \\ \sin \vartheta & \cos \vartheta \end{pmatrix}, \quad \vartheta \in [0, 2\pi)$$

of $\text{SO}(2)$. Thanks to the considered parametrization, we obtain the following expressions for the functions $\psi_{\mathbb{C}\text{per}}^{ij}(\vartheta)$:

$$\psi_{\mathbb{C}\text{per}}^{ij}(\vartheta) = \mathbb{C}_{ab} Q_{ai}(\vartheta) Q_{bj}(\vartheta) = (Q^T(\vartheta) (\mathbb{C} Q(\vartheta)))_{ij} = \left(\begin{pmatrix} \cos \vartheta & \sin \vartheta \\ -\sin \vartheta & \cos \vartheta \end{pmatrix} \begin{bmatrix} \mathbb{C}_{11} & \mathbb{C}_{12} \\ \mathbb{C}_{12} & \mathbb{C}_{22} \end{bmatrix} \begin{pmatrix} \cos \vartheta & -\sin \vartheta \\ \sin \vartheta & \cos \vartheta \end{pmatrix} \right)_{ij},$$

and thus

$$\begin{cases} \psi_{\mathbb{C}\text{per}}^{11}(\vartheta) = \mathbb{C}_{11} \cos^2 \vartheta + \mathbb{C}_{22} \sin^2 \vartheta + \mathbb{C}_{12} \sin 2\vartheta, \\ \psi_{\mathbb{C}\text{per}}^{22}(\vartheta) = \mathbb{C}_{11} \sin^2 \vartheta + \mathbb{C}_{22} \cos^2 \vartheta - \mathbb{C}_{12} \sin 2\vartheta, \\ \psi_{\mathbb{C}\text{per}}^{12}(\vartheta) = \frac{1}{2} (\mathbb{C}_{22} - \mathbb{C}_{11}) \sin 2\vartheta + \mathbb{C}_{12} \cos 2\vartheta. \end{cases}$$

Integrating, we find

$$(\mathcal{P}(\mathbb{C}))_{11} = \frac{\mathbb{C}_{11} + \mathbb{C}_{22}}{2}, \quad (\mathcal{P}(\mathbb{C}))_{12} = 0, \quad (\mathcal{P}(\mathbb{C}))_{22} = \frac{\mathbb{C}_{11} + \mathbb{C}_{22}}{2}.$$

Thus, recalling that

$$\text{Fix}_{\text{SO}(2)}^{\varphi} \text{Sym}(2) = \{ \mathbb{C} \in \text{Sym}(2) \mid \mathcal{P}(\mathbb{C}) = \mathbb{C} \},$$

i.e. that

$$\mathcal{P}(\mathbb{C}) = \begin{pmatrix} \frac{\mathbb{C}_{11} + \mathbb{C}_{22}}{2} & 0 \\ 0 & \frac{\mathbb{C}_{11} + \mathbb{C}_{22}}{2} \end{pmatrix} = \begin{pmatrix} \mathbb{C}_{11} & \mathbb{C}_{12} \\ \mathbb{C}_{12} & \mathbb{C}_{22} \end{pmatrix} = \mathbb{C},$$

and therefore

$$\frac{\mathbb{C}_{11} + \mathbb{C}_{22}}{2} = \mathbb{C}_{11} \Leftrightarrow \mathbb{C}_{11} = \mathbb{C}_{22}, \quad \text{and} \quad \mathbb{C}_{12} = 0$$

for $\mathbb{C} \in \text{Fix}_{\text{SO}(2)}^{\varphi} \text{Sym}(2)$, we find

$$\text{Fix}_{\text{SO}(2)}^{\varphi} \text{Sym}(2) = \left\{ \mathbb{C} \in \text{Sym}(2) \mid \mathbb{C} = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}, a \in \mathbb{R} \right\}.$$

In this case, the dimension of the fixed subspace (namely 1) can be obtained directly, but let us calculate it with the trace formula in order to show how the theoretical machinery works. To do this (as shown in eq.(26)), we write down the symmetrization identity of $\text{Sym}(2)$,

$$\Pi^{\text{Sym}(2)} = \frac{1}{2} (\delta_{ia} \delta_{jb} + \delta_{ja} \delta_{ib}).$$

Thus, the character $\chi(Q)$ of $Q \in \text{SO}(2)$ is

$$\begin{aligned} \chi(Q) &= \left\langle \varphi_Q \circ \Pi^{\text{Sym}(2)}, \mathbf{1}_2 \otimes \mathbf{1}_2 \right\rangle_{\mathbb{R}^{2 \times 2}} = Q_{ia} Q_{jb} \frac{1}{2} (\delta_{ia} \delta_{jb} + \delta_{ja} \delta_{ib}) = \frac{1}{2} (Q_{ia} Q_{jb} \delta_{ia} \delta_{jb} + Q_{ia} Q_{jb} \delta_{ja} \delta_{ib}) \\ &= \frac{1}{2} ((\text{tr } Q)^2 + \text{tr } Q^2) = \frac{1}{2} [(2 \cos \vartheta)^2 + 2 \cos^2 \vartheta - 2 \sin^2 \vartheta] = 3 \cos^2 \vartheta - \sin^2 \vartheta \end{aligned}$$

and therefore

$$\dim \left(\text{Fix}_{\text{SO}(2)}^{\varphi} \text{Sym}(2) \right) = \int_{Q \in \text{SO}(2)} \chi(Q) dQ = \frac{1}{2\pi} \int_0^{2\pi} (3 \cos^2 \vartheta - \sin^2 \vartheta) d\vartheta = 1.$$

We can also visualize the fixed point subspace in \mathbb{R}^3 identifying it with $\text{Sym}(2)$ via the following isometry:

$$\Lambda : \text{Sym}(2) \rightarrow \mathbb{R}^3, \quad \begin{pmatrix} \mathbb{C}_{11} & \mathbb{C}_{12} \\ \mathbb{C}_{12} & \mathbb{C}_{22} \end{pmatrix} = \mathbb{C} \mapsto \xi_{\mathbb{C}} := \begin{pmatrix} \mathbb{C}_{11} \\ \mathbb{C}_{22} \\ \sqrt{2} \mathbb{C}_{12} \end{pmatrix}.$$

- **Transversely isotropic materials** (5 elastic constants) The symmetry group contains all rotations of a fixed given axis (here e_3 .) The general form of the Hooke tensor is

$$\begin{pmatrix} \tilde{\mathbb{C}}_{11} & \tilde{\mathbb{C}}_{12} & \tilde{\mathbb{C}}_{13} & 0 & 0 & 0 \\ & \tilde{\mathbb{C}}_{11} & \tilde{\mathbb{C}}_{13} & 0 & 0 & 0 \\ & & \tilde{\mathbb{C}}_{33} & 0 & 0 & 0 \\ & & & \tilde{\mathbb{C}}_{44} & 0 & 0 \\ \text{sym} & & & & \tilde{\mathbb{C}}_{44} & 0 \\ & & & & & \tilde{\mathbb{C}}_{66} \end{pmatrix} \quad (32)$$

with $\tilde{\mathbb{C}}_{11} = \tilde{\mathbb{C}}_{12} + 2\tilde{\mathbb{C}}_{66}$.

- **Cubic materials** (3 elastic constants) The symmetry group is that of the symmetries of the cube. The general form of the Hooke tensor is

$$\begin{pmatrix} \tilde{\mathbb{C}}_{11} & \tilde{\mathbb{C}}_{12} & \tilde{\mathbb{C}}_{12} & 0 & 0 & 0 \\ & \tilde{\mathbb{C}}_{11} & \tilde{\mathbb{C}}_{12} & 0 & 0 & 0 \\ & & \tilde{\mathbb{C}}_{11} & 0 & 0 & 0 \\ & & & \tilde{\mathbb{C}}_{44} & 0 & 0 \\ \text{sym} & & & & \tilde{\mathbb{C}}_{44} & 0 \\ & & & & & \tilde{\mathbb{C}}_{44} \end{pmatrix}. \quad (33)$$

- **Isotropic material** (2 elastic constants); the symmetry group contains all proper orthogonal rotations. Using the notation of Voigt, the Hooke tensor is that of a cubic material but the additional relation $\tilde{\mathbb{C}}_{11} = \tilde{\mathbb{C}}_{12} + 2\tilde{\mathbb{C}}_{44}$ holds.

3 Extended continuum models

We now extend the previous setting to more general situations which can occur in mathematical modeling of mechanical phenomena.

3.1 Non-symmetric theories

In a more general framework including additional degrees of freedom (see for example [15, 38, 47]), a constitutive relation between an arbitrary second-order tensor, denoted by P and a non-symmetric stress tensor will involve, in a general case, 45 elastic constants. Using the classical notation we have

$$\sigma_{ij} = \bar{\mathbb{C}}_{ijkl} P_{kl}, \quad (34)$$

and only the major symmetries $\bar{\mathbb{C}}_{ijkl} = \bar{\mathbb{C}}_{klij}$ are assumed. Hereafter, it will be more convenient to use an extended notation of Voigt

$$\begin{pmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{32} \\ \sigma_{13} \\ \sigma_{31} \\ \sigma_{12} \\ \sigma_{21} \end{pmatrix} = \begin{pmatrix} \tilde{\mathbb{C}}_{11} & \tilde{\mathbb{C}}_{12} & \tilde{\mathbb{C}}_{13} & \tilde{\mathbb{C}}_{14} & \tilde{\mathbb{C}}_{15} & \tilde{\mathbb{C}}_{16} & \tilde{\mathbb{C}}_{17} & \tilde{\mathbb{C}}_{18} & \tilde{\mathbb{C}}_{19} \\ & \tilde{\mathbb{C}}_{22} & \tilde{\mathbb{C}}_{23} & \tilde{\mathbb{C}}_{24} & \tilde{\mathbb{C}}_{25} & \tilde{\mathbb{C}}_{26} & \tilde{\mathbb{C}}_{27} & \tilde{\mathbb{C}}_{28} & \tilde{\mathbb{C}}_{29} \\ & & \tilde{\mathbb{C}}_{33} & \tilde{\mathbb{C}}_{34} & \tilde{\mathbb{C}}_{35} & \tilde{\mathbb{C}}_{36} & \tilde{\mathbb{C}}_{37} & \tilde{\mathbb{C}}_{38} & \tilde{\mathbb{C}}_{39} \\ & & & \tilde{\mathbb{C}}_{44} & \tilde{\mathbb{C}}_{45} & \tilde{\mathbb{C}}_{46} & \tilde{\mathbb{C}}_{47} & \tilde{\mathbb{C}}_{48} & \tilde{\mathbb{C}}_{49} \\ & & & & \tilde{\mathbb{C}}_{55} & \tilde{\mathbb{C}}_{56} & \tilde{\mathbb{C}}_{57} & \tilde{\mathbb{C}}_{58} & \tilde{\mathbb{C}}_{59} \\ & & & & & \tilde{\mathbb{C}}_{66} & \tilde{\mathbb{C}}_{67} & \tilde{\mathbb{C}}_{68} & \tilde{\mathbb{C}}_{69} \\ \text{sym} & & & & & & \tilde{\mathbb{C}}_{77} & \tilde{\mathbb{C}}_{78} & \tilde{\mathbb{C}}_{79} \\ & & & & & & & \tilde{\mathbb{C}}_{88} & \tilde{\mathbb{C}}_{89} \\ & & & & & & & & \tilde{\mathbb{C}}_{99} \end{pmatrix} \begin{pmatrix} P_{11} \\ P_{22} \\ P_{33} \\ P_{23} \\ P_{32} \\ P_{13} \\ P_{31} \\ P_{12} \\ P_{21} \end{pmatrix}. \quad (35)$$

Note that the physical meaning of $\tilde{\mathbb{C}}_{44}$ above is slightly different from that of $\tilde{\mathbb{C}}_{44}$ in (5).

The character of the representation used in the extended theory can again be computed using the symmetry properties of the Hooke tensor. In this case, since only the major symmetry of the Hooke tensor is involved, the symmetrization identity $\bar{\Pi}$ which projects $\text{Lin}(\mathbb{R}^{3 \times 3}, \mathbb{R}^{3 \times 3})$ to $\text{Sym}(\mathbb{R}^{3 \times 3}, \mathbb{R}^{3 \times 3})$ is

$$\bar{\Pi} = \frac{1}{2} (\delta_{ia} \delta_{jb} \delta_{kc} \delta_{ld} + \delta_{ka} \delta_{lb} \delta_{ic} \delta_{jd}), \quad (36)$$

so that the corresponding character is $\chi(Q) = \frac{1}{2} \left((\text{tr } Q)^4 + (\text{tr}(Q^2))^2 \right)$ (see the calculations in Appendix B.3.3). Notice that for $Q = \mathbb{1}_{\mathbb{R}^3}$ we have $\text{tr } \mathbb{1}_{\mathbb{R}^3} = 3$ so that $\chi(\mathbb{1}_{\mathbb{R}^3}) = \frac{1}{2} (3^4 + 3^2) = 45$ which is, as expected, the number of elastic constants without any additional symmetry.

In the following, we give the details of the computations that provide the general forms of the Hooke tensor for orthotropic, transversely isotropic, cubic and isotropic materials. Although the computations are performed using the fourth-order notations for the Hooke tensor, the results are presented in the following using the extended Voigt notation. For each of the symmetry classes above we start with a generic (45 independent elastic constants) Hooke tensor $\bar{\mathbb{C}}$ and use the following procedure.

Step 1: Using the trace formula and the normalized Haar integral (or average over the group in the discrete case) we compute the number of independent parameters after symmetrization as

$$\dim(\text{Fix}_{\mathcal{G}}^{\varphi} \text{Sym}(\mathbb{R}^{3 \times 3}, \mathbb{R}^{3 \times 3})) = \int_{\mathcal{G}} \chi(Q) d\mu. \quad (37)$$

Step 2: We compute the symmetrization $\mathcal{P}(\bar{\mathbb{C}})$ over the group \mathcal{G} . Notice that, by definition, the components of $\mathcal{P}(\bar{\mathbb{C}})$ are linear combinations of elements of $\bar{\mathbb{C}}$.

Step 3: By using standard linear algebra computations, we choose among the components of $\mathcal{P}(\bar{\mathbb{C}})$ a basis and we check that it contains exactly

$$\dim(\text{Fix}_{\mathcal{G}}^{\varphi} \text{Sym}(\mathbb{R}^{3 \times 3}, \mathbb{R}^{3 \times 3}))$$

elements.

- **Orthotropic materials:** From the trace formula, the number of elastic constants is 15. The general form of the Hooke tensor is

$$\begin{pmatrix} \tilde{\mathbb{C}}_{11} & \tilde{\mathbb{C}}_{12} & \tilde{\mathbb{C}}_{13} & 0 & 0 & 0 & 0 & 0 & 0 \\ & \tilde{\mathbb{C}}_{22} & \tilde{\mathbb{C}}_{23} & 0 & 0 & 0 & 0 & 0 & 0 \\ & & \tilde{\mathbb{C}}_{33} & 0 & 0 & 0 & 0 & 0 & 0 \\ & & & \tilde{\mathbb{C}}_{44} & \tilde{\mathbb{C}}_{45} & 0 & 0 & 0 & 0 \\ & & & & \tilde{\mathbb{C}}_{55} & 0 & 0 & 0 & 0 \\ & & & & & \tilde{\mathbb{C}}_{66} & \tilde{\mathbb{C}}_{67} & 0 & 0 \\ & \text{sym} & & & & & \tilde{\mathbb{C}}_{77} & 0 & 0 \\ & & & & & & & \tilde{\mathbb{C}}_{88} & \tilde{\mathbb{C}}_{89} \\ & & & & & & & & \tilde{\mathbb{C}}_{99} \end{pmatrix}. \quad (38)$$

- **Transversely isotropic materials:** In contrast to the case of classical elasticity, when the involved 4-order tensor has only the major symmetry, the actions with respect to the two closed subgroups $\text{SO}(2; e_3)$ and $\text{O}(2; e_3)$ of $\text{SO}(3)$ are not equivalent. In other words, $\text{SO}(2; e_3)$ and $\text{O}(2; e_3)$ are different symmetry groups for the vector space $\text{Sym}(\mathbb{R}^{3 \times 3}, \mathbb{R}^{3 \times 3})$, see for example [52]. Recall the definitions of these two groups,

$$\text{SO}(2; e_3) := \left\{ \begin{pmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \mid \theta \in [0, 2\pi) \right\} \quad \text{transversal hemitropic}, \quad (39)$$

$$\text{O}(2; e_3) := \text{SO}(2; e_3) \cup \left\{ \begin{pmatrix} \cos \theta & \sin \theta & 0 \\ \sin \theta & -\cos \theta & 0 \\ 0 & 0 & -1 \end{pmatrix} \mid \theta \in [0, 2\pi) \right\} \quad \text{transversal isotropic}. \quad (40)$$

From a geometrical point of view (39) reflects the invariance with respect to the rotations keeping fixed the axis e_3 of \mathbb{R}^3 (proper rotations), while the group given in (40) accounts also for the inversions with respect to the

plane $\langle e_1, e_2 \rangle$ (improper rotations)⁶. Via the trace formula and the explicit expression for the Haar measure on $\text{SO}(2)$ and $\text{O}(2)$ derived in (98) and (99) respectively, we compute the following numbers of independent components for the two considered symmetries:

$$\dim \text{Fix}_{\text{SO}(2, e_3)}^{\varphi} \text{Sym}(\mathbb{R}^{3 \times 3}, \mathbb{R}^{3 \times 3}) = 11, \quad \dim \text{Fix}_{\text{O}(2, e_3)} \text{Sym}(\mathbb{R}^{3 \times 3}, \mathbb{R}^{3 \times 3}) = 8. \quad (41)$$

The symmetrization process, for the invariance with respect to $\text{SO}(2, e_3)$, gives

$$\left(\begin{array}{cccccccccc} \tilde{\mathbb{C}}_{11} & \tilde{\mathbb{C}}_{12} & \tilde{\mathbb{C}}_{13} & 0 & 0 & 0 & 0 & \tilde{\mathbb{C}}_{18} & -\tilde{\mathbb{C}}_{18} & \\ & \tilde{\mathbb{C}}_{11} & \tilde{\mathbb{C}}_{13} & 0 & 0 & 0 & 0 & \tilde{\mathbb{C}}_{18} & -\tilde{\mathbb{C}}_{18} & \\ & & \tilde{\mathbb{C}}_{33} & 0 & 0 & 0 & 0 & \tilde{\mathbb{C}}_{38} & -\tilde{\mathbb{C}}_{38} & \\ & & & \tilde{\mathbb{C}}_{44} & \tilde{\mathbb{C}}_{45} & 0 & -\tilde{\mathbb{C}}_{56} & 0 & 0 & \\ & & & & \tilde{\mathbb{C}}_{55} & \tilde{\mathbb{C}}_{56} & 0 & 0 & 0 & \\ & & & & & \tilde{\mathbb{C}}_{44} & \tilde{\mathbb{C}}_{45} & 0 & 0 & \\ \text{sym} & & & & & & \tilde{\mathbb{C}}_{55} & 0 & 0 & \\ & & & & & & & \tilde{\mathbb{C}}_{88} & \tilde{\mathbb{C}}_{89} & \\ & & & & & & & & \tilde{\mathbb{C}}_{88} & \end{array} \right), \quad (42)$$

where $\tilde{\mathbb{C}}_{11} = \tilde{\mathbb{C}}_{12} + \tilde{\mathbb{C}}_{88} + \tilde{\mathbb{C}}_{89}$, while for the invariance with respect to $\text{O}(2, e_3)$ we obtain

$$\left(\begin{array}{cccccccccc} \tilde{\mathbb{C}}_{11} & \tilde{\mathbb{C}}_{12} & \tilde{\mathbb{C}}_{13} & 0 & 0 & 0 & 0 & 0 & 0 & \\ & \tilde{\mathbb{C}}_{11} & \tilde{\mathbb{C}}_{13} & 0 & 0 & 0 & 0 & 0 & 0 & \\ & & \tilde{\mathbb{C}}_{33} & 0 & 0 & 0 & 0 & 0 & 0 & \\ & & & \tilde{\mathbb{C}}_{44} & \tilde{\mathbb{C}}_{45} & 0 & 0 & 0 & 0 & \\ & & & & \tilde{\mathbb{C}}_{55} & 0 & 0 & 0 & 0 & \\ & & & & & \tilde{\mathbb{C}}_{44} & \tilde{\mathbb{C}}_{45} & 0 & 0 & \\ \text{sym} & & & & & & \tilde{\mathbb{C}}_{55} & 0 & 0 & \\ & & & & & & & \tilde{\mathbb{C}}_{88} & \tilde{\mathbb{C}}_{89} & \\ & & & & & & & & \tilde{\mathbb{C}}_{88} & \end{array} \right), \quad (43)$$

again with $\tilde{\mathbb{C}}_{11} = \tilde{\mathbb{C}}_{12} + \tilde{\mathbb{C}}_{88} + \tilde{\mathbb{C}}_{89}$.

- **Cubic materials:** From the trace formula, the number of elastic constants in the extended theory is 4. Using the notation of Voigt, the general form of the Hooke tensor is

$$\left(\begin{array}{cccccccccc} \tilde{\mathbb{C}}_{11} & \tilde{\mathbb{C}}_{12} & \tilde{\mathbb{C}}_{12} & 0 & 0 & 0 & 0 & 0 & 0 & \\ & \tilde{\mathbb{C}}_{11} & \tilde{\mathbb{C}}_{12} & 0 & 0 & 0 & 0 & 0 & 0 & \\ & & \tilde{\mathbb{C}}_{11} & 0 & 0 & 0 & 0 & 0 & 0 & \\ & & & \tilde{\mathbb{C}}_{44} & \tilde{\mathbb{C}}_{45} & 0 & 0 & 0 & 0 & \\ & & & & \tilde{\mathbb{C}}_{44} & 0 & 0 & 0 & 0 & \\ & & & & & \tilde{\mathbb{C}}_{44} & \tilde{\mathbb{C}}_{45} & 0 & 0 & \\ \text{sym} & & & & & & \tilde{\mathbb{C}}_{44} & 0 & 0 & \\ & & & & & & & \tilde{\mathbb{C}}_{44} & \tilde{\mathbb{C}}_{45} & \\ & & & & & & & & \tilde{\mathbb{C}}_{44} & \end{array} \right) \quad (44)$$

- **Isotropic materials:** The number of elastic moduli is 3; the Hooke tensor using Voigt notation is

$$\left(\begin{array}{cccccccccc} \tilde{\mathbb{C}}_{11} & \tilde{\mathbb{C}}_{12} & \tilde{\mathbb{C}}_{12} & 0 & 0 & 0 & 0 & 0 & 0 & \\ & \tilde{\mathbb{C}}_{11} & \tilde{\mathbb{C}}_{12} & 0 & 0 & 0 & 0 & 0 & 0 & \\ & & \tilde{\mathbb{C}}_{11} & 0 & 0 & 0 & 0 & 0 & 0 & \\ & & & \tilde{\mathbb{C}}_{44} & \tilde{\mathbb{C}}_{45} & 0 & 0 & 0 & 0 & \\ & & & & \tilde{\mathbb{C}}_{44} & 0 & 0 & 0 & 0 & \\ & & & & & \tilde{\mathbb{C}}_{44} & \tilde{\mathbb{C}}_{45} & 0 & 0 & \\ \text{sym} & & & & & & \tilde{\mathbb{C}}_{44} & 0 & 0 & \\ & & & & & & & \tilde{\mathbb{C}}_{44} & \tilde{\mathbb{C}}_{45} & \\ & & & & & & & & \tilde{\mathbb{C}}_{44} & \end{array} \right) \quad (45)$$

⁶ e_1, e_2, e_3 are the elements of the canonical orthonormal basis of \mathbb{R}^3 .

with $\tilde{\mathbb{C}}_{11} = \tilde{\mathbb{C}}_{12} + \tilde{\mathbb{C}}_{44} + \tilde{\mathbb{C}}_{45}$. This relation generalizes the well-known formula $\tilde{\mathbb{C}}_{11} = \tilde{\mathbb{C}}_{12} + 2\tilde{\mathbb{C}}_{44}$ from classical elasticity. We can also write the constitutive relation as

$$\text{sym } \sigma = 2\mu \text{sym } P + \lambda(\text{tr } P)\mathbf{1}, \quad \text{skew } \sigma = 2\mu_c \text{skew } P, \quad (46)$$

where

$$\lambda = \tilde{\mathbb{C}}_{12}, \quad \mu = \tilde{\mathbb{C}}_{11} - \tilde{\mathbb{C}}_{12} - \frac{(\tilde{\mathbb{C}}_{44} + \tilde{\mathbb{C}}_{55})}{2} = \frac{(\tilde{\mathbb{C}}_{44} + \tilde{\mathbb{C}}_{45})}{2} \quad \text{and} \quad \mu_c = \frac{(\tilde{\mathbb{C}}_{44} - \tilde{\mathbb{C}}_{45})}{2}. \quad (47)$$

Indeed, thanks to the relations in (47), we have

$$\begin{pmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{32} \\ \sigma_{13} \\ \sigma_{31} \\ \sigma_{12} \\ \sigma_{21} \end{pmatrix} = \begin{pmatrix} 2\mu + \lambda & \lambda & \lambda & 0 & 0 & 0 & 0 & 0 & 0 \\ \lambda & 2\mu + \lambda & \lambda & 0 & 0 & 0 & 0 & 0 & 0 \\ \lambda & \lambda & 2\mu + \lambda & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \mu + \mu_c & \mu - \mu_c & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \mu - \mu_c & \mu + \mu_c & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \mu + \mu_c & \mu - \mu_c & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \mu - \mu_c & \mu + \mu_c & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mu + \mu_c & \mu - \mu_c \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mu - \mu_c & \mu + \mu_c \end{pmatrix} \begin{pmatrix} P_{11} \\ P_{22} \\ P_{33} \\ P_{23} \\ P_{32} \\ P_{13} \\ P_{31} \\ P_{12} \\ P_{21} \end{pmatrix}, \quad (48)$$

which gives

$$\begin{pmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{32} \\ \sigma_{13} \\ \sigma_{31} \\ \sigma_{12} \\ \sigma_{21} \end{pmatrix} = \begin{pmatrix} (2\mu + \lambda)P_{11} + \lambda P_{22} + \lambda P_{33} \\ \lambda P_{11} + (2\mu + \lambda)P_{22} + \lambda P_{33} \\ \lambda P_{11} + \lambda P_{22} + (2\mu + \lambda)P_{33} \\ \mu(P_{23} + P_{32}) + \mu_c(P_{23} - P_{32}) \\ \mu(P_{23} + P_{32}) + \mu_c(P_{23} - P_{32}) \\ \mu(P_{13} + P_{31}) + \mu_c(P_{13} - P_{31}) \\ \mu(P_{13} + P_{31}) + \mu_c(P_{13} - P_{31}) \\ \mu(P_{12} + P_{21}) + \mu_c(P_{12} - P_{21}) \\ \mu(P_{12} + P_{21}) + \mu_c(P_{12} - P_{21}) \end{pmatrix} = \begin{pmatrix} 2\mu P_{11} + \lambda \text{tr } P \\ 2\mu P_{22} + \lambda \text{tr } P \\ 2\mu P_{33} + \lambda \text{tr } P \\ \mu(P_{23} + P_{32}) + \mu_c(P_{23} - P_{32}) \\ \mu(P_{23} + P_{32}) + \mu_c(P_{23} - P_{32}) \\ \mu(P_{13} + P_{31}) + \mu_c(P_{13} - P_{31}) \\ \mu(P_{13} + P_{31}) + \mu_c(P_{13} - P_{31}) \\ \mu(P_{12} + P_{21}) + \mu_c(P_{12} - P_{21}) \\ \mu(P_{12} + P_{21}) + \mu_c(P_{12} - P_{21}) \end{pmatrix} \quad (49)$$

$$= 2\mu \begin{pmatrix} P_{11} \\ P_{22} \\ P_{33} \\ \frac{P_{23} + P_{32}}{2} \\ \frac{P_{23} - P_{32}}{2} \\ \frac{P_{13} + P_{31}}{2} \\ \frac{P_{13} - P_{31}}{2} \\ \frac{P_{12} + P_{21}}{2} \\ \frac{P_{12} - P_{21}}{2} \end{pmatrix} + \lambda \text{tr } P \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} + 2\mu_c \begin{pmatrix} 0 \\ 0 \\ 0 \\ \frac{P_{23} - P_{32}}{2} \\ \frac{P_{23} + P_{32}}{2} \\ \frac{P_{13} - P_{31}}{2} \\ \frac{P_{13} + P_{31}}{2} \\ \frac{P_{12} - P_{21}}{2} \\ \frac{P_{12} + P_{21}}{2} \end{pmatrix}, \quad (50)$$

i.e.

$$\sigma = 2\mu \text{sym } P + 2\mu_c \text{skew } P + \lambda \text{tr}(P)\mathbf{1}. \quad (51)$$

Thus, in this case, the elastic energy can be expressed as

$$\begin{aligned} \frac{1}{2} \langle \bar{\mathbb{C}} P, P \rangle &= \frac{1}{2} \langle \sigma, P \rangle = \frac{1}{2} \langle 2\mu \text{sym } P + \lambda \text{tr}(P)\mathbf{1} + 2\mu_c \text{skew } P, P \rangle \\ &= \mu \langle \text{sym } P, P \rangle + \mu_c \langle \text{skew } P, P \rangle + \frac{\lambda}{2} \text{tr}(P) \langle \mathbf{1}, P \rangle = \mu \|\text{sym } P\|^2 + \mu_c \|\text{skew } P\|^2 + \frac{\lambda}{2} \text{tr}(P)^2. \end{aligned} \quad (52)$$

3.2 Second gradient theory

In the following, we want to show how the general framework proposed in this paper can be easily applied to the second gradient theories which, in recent years, have found wide use in modeling physical phenomena (see [7, 26, 34, 37, 68], for example). First, however, we would like to dedicate some words to a more detailed discussion on how the invariance conditions are derived from the Lagrangian energy density in this case. Indeed, concerning the second gradient theory, some authors derived an incorrect invariance law [25, 43] and, to our knowledge, the

correct answer is given in [41]. Thus, we follow [41] to explain the situation: let us consider a (non-linear) energy density

$$W(D\varphi, D^2\varphi)$$

depending also on the second gradient of the displacement field $\varphi : \Omega \subseteq \mathbb{R}^n \rightarrow \mathbb{R}^n$. Considering a generic orientation preserving diffeomorphism $\zeta : \mathbb{R}^n \rightarrow \mathbb{R}^n$, $\xi \mapsto \zeta(\xi)$, we say that the considered body is invariant with respect to ζ if the total energy is preserved under a diffeomorphic change of variables, i.e.

$$\int_{\xi \in \zeta^{-1}(\Omega)} W(D_\xi \varphi(\zeta(\xi)), D_\xi^2 \varphi(\zeta(\xi))) |\det D_\xi \zeta(\xi)| d\xi = \int_{x \in \Omega} W(D_x \varphi(x), D_x^2 \varphi(x)) dx. \quad (53)$$

Since material invariance means invariance of the elastic response with respect to rotations of the specimen with respect to the machine test, we need to ask for the following constraint for the change of variables:

$$D_\xi \zeta(\xi) \in \text{SO}(3) \quad \forall \xi \in \zeta^{-1}(\Omega).$$

Under this hypothesis, we can firstly remark that $\det D_\xi \zeta(\xi) = 1$ for all $\xi \in \zeta^{-1}(\Omega)$. Secondly, setting $\varphi^b(\xi) := \varphi(\zeta(\xi))$ for all $\xi \in \zeta^{-1}(\Omega)$, and observing that from the identity $x = \zeta(\zeta^{-1}(x))$ we obtain the identity

$$\mathbf{1} = D_\xi [\underbrace{\zeta(\zeta^{-1}(x))}_{=\xi}] D_x \zeta^{-1}(x) \iff D_x \zeta^{-1}(x) = [D_\xi \zeta(\xi)]^{-1},$$

expanding the expressions $D_\xi \varphi(\zeta(\xi))$ and $D_\xi^2 \varphi(\zeta(\xi))$ we arrive at $D_\xi \varphi(\zeta(\xi)) = D_x \varphi(\zeta(\xi)) D_\xi \zeta(\xi) = D_x \varphi(x) D_\xi \zeta(\xi)$ and

$$\begin{aligned} D_\xi^2 \varphi(\zeta(\xi)) &= D_\xi [D_x \varphi(\zeta(\xi)) D_\xi \zeta(\xi)] = [D_\xi D_x \varphi(\zeta(\xi))] D_\xi \zeta(\xi) + D_x \varphi(\zeta(\xi)) D_\xi^2 \zeta(\xi) \\ &= [D_x^2 \varphi(\zeta(\xi)) D_\xi \zeta(\xi)] \odot D_\xi \zeta(\xi) + D_x \varphi(x) D_\xi^2 \zeta(\xi) \\ &= [D_x^2 \varphi(x) D_\xi \zeta(\xi)] \odot D_\xi \zeta(\xi) + D_x \varphi(x) D_\xi^2 \zeta(\xi), \end{aligned}$$

where the contraction \odot is defined by $(A \odot B)_{ijk} = A_{imk} B_{mj}$. for $A \in \mathbb{R}^{3 \times 3 \times 3}$ and $B \in \mathbb{R}^{3 \times 3}$. Thus in our case, we have

$$([D_x^2 \varphi(x) D_\xi \zeta(\xi)] \odot D_\xi \zeta(\xi))_{ijk} = \frac{\partial^2 \varphi_i}{\partial x_n \partial x_m}(\zeta(\xi)) \frac{\partial \zeta_n}{\partial \xi_k}(\xi) \frac{\partial \zeta_m}{\partial \xi_j}(\xi) \quad \forall \xi \in \zeta^{-1}(\Omega).$$

Due to a rigidity result (see [35] and, for a modern version, [50]) asserting that if ζ is sufficiently regular (for example in $W^{1,2}(\zeta^{-1}(\Omega))$) and such that $D_\xi \zeta(\xi) \in \text{SO}(3)$ for every ξ , then there exists one constant $\bar{Q} \in \text{SO}(3)$ such that $D_\xi \zeta(\xi) = \bar{Q}$ for all ξ . This means that the term $D_\xi^2 \zeta(\xi)$ is 0 and the invariance condition (53) reduces to

$$\int_{x \in \Omega} W(D_x \varphi(x), D_x^2 \varphi(x)) dx = \int_{\xi \in \zeta^{-1}(\Omega)} W(D_x \varphi(\zeta(\xi)) \bar{Q}, [D_x^2 \varphi(\zeta(\xi)) \bar{Q}] \odot \bar{Q}) d\xi. \quad (54)$$

In the linearized boundary value homogeneous problem, the internal energy is given as the functional

$$\mathcal{A}[u] := \frac{1}{2} \int_{\Omega} (\langle \mathbb{C} \varepsilon, \varepsilon \rangle + 2 \langle \mathbb{H} \varepsilon, D^2 u \rangle + \langle \mathbb{G} D^2 u, D^2 u \rangle) dx, \quad (55)$$

where $\varepsilon := \text{sym } Du$ and

$$\mathbb{C} \in \text{Sym}(\text{Sym}(3), \text{Sym}(3)), \quad \mathbb{H} \in \text{Lin}(\text{Sym}(3), \mathbb{R}^3 \otimes \text{Sym}(3)), \quad \mathbb{G} \in \text{Sym}(\mathbb{R}^3 \otimes \text{Sym}(3), \mathbb{R}^3 \otimes \text{Sym}(3)).$$

We shall specify the invariance condition (54) to the linearized case in (55). Denoting with $\omega : \Omega \rightarrow \mathfrak{so}(3)$ the skew-symmetric part of Du , we find that

$$D^2 u : \Omega \rightarrow \mathbb{R}^3 \otimes \text{Sym}(3) \subseteq \mathbb{R}^{3 \times 3 \times 3}$$

can be represented as the sum

$$D^2 u = D(Du) = D(\varepsilon + \omega) = D\varepsilon + D\omega, \quad (56)$$

with

$$D\varepsilon : \Omega \rightarrow \text{Sym}(3) \otimes \mathbb{R}^3 \subseteq \mathbb{R}^{3 \times 3 \times 3}, \quad D\omega : \Omega \rightarrow \mathfrak{so}(3) \otimes \mathbb{R}^3 \subseteq \mathbb{R}^{3 \times 3 \times 3}, \quad \mathbb{R}^{3 \times 3 \times 3} \simeq (\text{Sym}(3) \otimes \mathbb{R}^3) \oplus (\mathfrak{so}(3) \otimes \mathbb{R}^3),$$

and

$$\dim(\text{Sym}(3) \otimes \mathbb{R}^3) = 18, \quad \dim(\mathfrak{so}(3) \otimes \mathbb{R}^3) = 9.$$

Remark 5. There exists a linear operator L such that $D^2u = L(D\varepsilon)$. Indeed, $u_{k,ij} = \varepsilon_{ik,j} + \varepsilon_{kj,i} - \varepsilon_{ji,k}$ for every $i, j, k \in \{1, 2, 3\}$.

Remark 6. The tensor product \otimes is not symmetric, so the two spaces $\text{Sym}(3) \otimes \mathbb{R}^3$ and $\mathbb{R}^3 \otimes \text{Sym}(3)$ are not the same space (but they are isomorphic having the same dimension). Denoting with $\text{Sym}(3, 3)$ the space of fully symmetric third-order tensors, i.e.

$$\text{Sym}(3, 3) := \{T \in \mathbb{R}^{3 \times 3 \times 3} \mid T_{ijh} = T_{jih} = T_{ihj} \ \forall i, j, h \in \{1, 2, 3\}\},$$

which is a 10-dimensional vector space, we have that $\text{Sym}(3, 3) = (\text{Sym}(3) \otimes \mathbb{R}^3) \cap (\mathbb{R}^3 \otimes \text{Sym}(3))$. Decomposing the identity of $\mathbb{R}^{3 \times 3 \times 3}$ using the two projections $\text{pr}_{\text{Sym}(3) \otimes \mathbb{R}^3}$ and $\text{pr}_{\mathfrak{so}(3) \otimes \mathbb{R}^3}$ associated respectively to the two orthogonal subspaces $\text{Sym}(3) \otimes \mathbb{R}^3$ and $\mathfrak{so}(3) \otimes \mathbb{R}^3$, i.e.

$$\mathbb{1}_{\mathbb{R}^{3 \times 3 \times 3}} = \text{pr}_{\text{Sym}(3) \otimes \mathbb{R}^3} + \text{pr}_{\mathfrak{so}(3) \otimes \mathbb{R}^3},$$

we have that

$$D^2u = D\varepsilon + D\omega = \text{pr}_{\text{Sym}(3) \otimes \mathbb{R}^3}(D^2u) + \text{pr}_{\mathfrak{so}(3) \otimes \mathbb{R}^3}(D^2u).$$

The second gradient model can be also formulated considering the gradient of ε (strain gradient theories, see Remark 5 (cf. [48])). In this case the action functional defining the problem is given by

$$\mathcal{A}[u] := \frac{1}{2} \int_{\Omega} (\langle \mathbb{C} \varepsilon, \varepsilon \rangle + 2 \langle \mathbb{H} \varepsilon, D\varepsilon \rangle + \langle \mathbb{G} D\varepsilon, D\varepsilon \rangle) dx, \quad (57)$$

with

$$\mathbb{H} : \text{Sym}(3) \rightarrow \text{Sym}(3) \otimes \mathbb{R}^3, \quad \mathbb{G} : \text{Sym}(3) \otimes \mathbb{R}^3 \rightarrow \text{Sym}(3) \otimes \mathbb{R}^3.$$

We will study the symmetrizations of the tensors $\mathbb{H}, \mathbb{G}, \hat{\mathbb{H}}, \hat{\mathbb{G}}$. In order to apply the mathematical tools introduced in Section 2, we have to identify, in this setting, the elements of the triple $(V, \mathcal{G}, \varphi)$. Let us introduce

$$\begin{aligned} V_0 &:= \text{Sym}(\text{Sym}(3), \text{Sym}(3)) & V_1 &:= \text{Lin}(\text{Sym}(3), \mathbb{R}^3 \otimes \text{Sym}(3)) & V_2 &:= \text{Sym}(\mathbb{R}^3 \otimes \text{Sym}(3), \mathbb{R}^3 \otimes \text{Sym}(3)) \\ \underline{V}_1 &:= \text{Lin}(\text{Sym}(3), \text{Sym}(3) \otimes \mathbb{R}^3) & \underline{V}_2 &:= \text{Sym}(\text{Sym}(3) \otimes \mathbb{R}^3, \text{Sym}(3) \otimes \mathbb{R}^3), \end{aligned}$$

and define $V := V_0 \times V_1 \times V_2$ and $\underline{V} := V_0 \times \underline{V}_1 \times \underline{V}_2$. Introducing the actions

$$\begin{aligned} \varphi_0 : \mathcal{G} \times V_0 &\rightarrow V_0, & \varphi_0(Q, \mathbb{C}) &= \hat{\mathbb{C}}, & \hat{\mathbb{C}}_{abcd} &= Q_{ai} Q_{bj} Q_{ck} Q_{dl} \mathbb{C}_{ijkl}, \\ \varphi_1 : \mathcal{G} \times V_1 &\rightarrow V_1, & \varphi_1(Q, \mathbb{H}) &= \hat{\mathbb{H}}, & \hat{\mathbb{H}}_{abcde} &= Q_{ai} Q_{bj} Q_{ck} Q_{dl} Q_{em} \mathbb{H}_{ijklm}, \\ \varphi_2 : \mathcal{G} \times V_2 &\rightarrow V_2, & \varphi_2(Q, \mathbb{G}) &= \hat{\mathbb{G}}, & \hat{\mathbb{G}}_{abcdef} &= Q_{ai} Q_{bj} Q_{ck} Q_{dl} Q_{em} Q_{fn} \mathbb{G}_{ijklmn} \\ \underline{\varphi}_1 : \mathcal{G} \times \underline{V}_1 &\rightarrow \underline{V}_1, & \underline{\varphi}_1(Q, \mathbb{H}) &= \hat{\mathbb{H}}, & \hat{\mathbb{H}}_{abcde} &= Q_{ai} Q_{bj} Q_{ck} Q_{dl} Q_{em} \mathbb{H}_{ijklm}, \\ \underline{\varphi}_2 : \mathcal{G} \times \underline{V}_2 &\rightarrow \underline{V}_2, & \underline{\varphi}_2(Q, \mathbb{G}) &= \hat{\mathbb{G}}, & \hat{\mathbb{G}}_{abcdef} &= Q_{ai} Q_{bj} Q_{ck} Q_{dl} Q_{em} Q_{fn} \mathbb{G}_{ijklmn}, \end{aligned} \quad (58)$$

and setting

$$\begin{aligned} \varphi &:= \varphi_0 \times \varphi_1 \times \varphi_2, & \varphi : \mathcal{G} \times V &\rightarrow V, & \varphi(Q, (\mathbb{C}, \mathbb{H}, \mathbb{G})) &:= (\varphi_0(Q, \mathbb{C}), \varphi_1(Q, \mathbb{H}), \varphi_2(Q, \mathbb{G})) = (\hat{\mathbb{C}}, \hat{\mathbb{H}}, \hat{\mathbb{G}}), \\ \underline{\varphi} &:= \varphi_0 \times \underline{\varphi}_1 \times \underline{\varphi}_2, & \underline{\varphi} : \mathcal{G} \times \underline{V} &\rightarrow \underline{V}, & \underline{\varphi}(Q, (\mathbb{C}, \mathbb{H}, \mathbb{G})) &:= (\varphi_0(Q, \mathbb{C}), \varphi_1(Q, \mathbb{H}), \varphi_2(Q, \mathbb{G})) = (\hat{\mathbb{C}}, \hat{\mathbb{H}}, \hat{\mathbb{G}}), \end{aligned}$$

we consider the two following triples for problems (55) and (57) respectively: $(V, \mathcal{G}, \varphi)$ and $(\underline{V}, \mathcal{G}, \varphi)$. Initially, we will ascertain the number of independent components for the tensors in question, namely, $\mathbb{H}, \mathbb{G}, \hat{\mathbb{H}}, \hat{\mathbb{G}}$, when subjected to their respective actions $\varphi_1, \varphi_2, \underline{\varphi}_1, \underline{\varphi}_2$. Subsequently, we will determine the matrix representations of these symmetrized tensors, focusing on the cubic symmetry, extending the classical Voigt notation to the case of third order tensors.

Case $(V_1, \mathcal{G}, \varphi)$: The number of independent components of D^2u is 18, the symmetries of which are

$$u_{i,jk} = u_{i,kj}, \quad \text{with } i, j, k = 1, \dots, 3.$$

In this way the tensor \mathbb{H} has the symmetries

$$\mathbb{H}_{abcde} = \mathbb{H}_{acbde} = \mathbb{H}_{abced}, \quad (59)$$

and the set of index permutations which leaves \mathbb{H} invariant is equivalent to $S_2 \times S_2$ (S_2 is the group of the permutations of two indices), so it has 4 different elements. In order to determine the character $\chi(Q)$ of an element $Q \in \mathcal{G}$ we have to specify the identity tensor $\mathbb{1}_{V_1}$ of V_1 . Due to the symmetries (59), the symmetrization identity Π^{V_1} is

$$\Pi_{iajkbkclmdme}^{V_1} = \frac{1}{4}(\delta_{ia}\delta_{jb}\delta_{kc}\delta_{ld}\delta_{me} + \delta_{ia}\delta_{kb}\delta_{jc}\delta_{ld}\delta_{me} + \delta_{ia}\delta_{jb}\delta_{kc}\delta_{md}\delta_{le} + \delta_{ia}\delta_{kb}\delta_{jc}\delta_{md}\delta_{le}).$$

The character $\chi(Q)$, being defined as $\langle \varphi_Q \circ \Pi^{V_1}, \otimes^5 \mathbb{1}_2 \rangle$ with $\varphi_Q = \otimes^5 Q$ is (see the Appendix for detailed calculations)

$$\langle \varphi_Q \circ \Pi^{V_1}, \otimes^5 \mathbb{1}_2 \rangle = \frac{1}{4} \left((\text{tr } Q)^5 + 2(\text{tr } Q)^3 \text{tr } Q^2 + \text{tr } Q (\text{tr } Q^2)^2 \right), \quad (60)$$

which, due to the Hamilton-Cayley transformation, becomes

$$\chi(Q) = (\text{tr } Q)^5 - 2(\text{tr } Q)^4 + (\text{tr } Q)^3. \quad (61)$$

A very useful check to verify the accuracy of the performed calculations consists in computing the character of the matrix $Q = \mathbb{1}$. In this case, if the derived formula (61) is correct, we have to find the dimension of the vector space V_1 . In our case, having that $\dim \text{Sym}(3) \otimes \mathbb{R}^3 = 18$, the dimension of V_1 is $18 \times 6 = 108$, and indeed

$$\chi(\mathbb{1}) = (\text{tr } \mathbb{1})^5 - 2(\text{tr } \mathbb{1})^4 + (\text{tr } \mathbb{1})^3 = 108.$$

Case $(\underline{V}_1, \mathcal{G}, \underline{\varphi})$: The number of independent components of $D\varepsilon$ is 18, the symmetries of which are

$$\varepsilon_{ij,k} = \varepsilon_{ji,k}, \quad \text{with } i, j, k = 1, \dots, 3.$$

The tensor $\underline{\mathbb{H}}$ has the symmetries

$$\underline{\mathbb{H}}_{abcde} = \underline{\mathbb{H}}_{bacde} = \underline{\mathbb{H}}_{abced},$$

and the set of index permutations which leaves $\underline{\mathbb{H}}$ invariant is still equivalent to $S_2 \times S_2$. The symmetrized identity tensor $\Pi^{\underline{V}_1}$ of \underline{V}_1 is

$$\Pi_{iajkbkclmdme}^{\underline{V}_1} = \frac{1}{4}(\delta_{ia}\delta_{jb}\delta_{kc}\delta_{ld}\delta_{me} + \delta_{ja}\delta_{ib}\delta_{kc}\delta_{ld}\delta_{me} + \delta_{ia}\delta_{jb}\delta_{kc}\delta_{md}\delta_{le} + \delta_{ja}\delta_{ib}\delta_{kc}\delta_{md}\delta_{le}).$$

The character $\chi(Q)$ is

$$\chi_Q = \langle \varphi_Q \circ \Pi^{\underline{V}_1}, \otimes^5 \mathbb{1}_2 \rangle = \frac{1}{4} \left((\text{tr } Q)^5 + 2(\text{tr } Q)^3 \text{tr } Q^2 + \text{tr } Q (\text{tr } Q^2)^2 \right), \quad (62)$$

which, using to the Hamilton-Cayley transformation, can be expressed as

$$\chi(Q) = (\text{tr } Q)^5 - 2(\text{tr } Q)^4 + (\text{tr } Q)^3. \quad (63)$$

Case $(V_2, \mathcal{G}, \varphi_2)$: The tensor \mathbb{G} has the symmetries

$$\mathbb{G}_{abcdef} = \mathbb{G}_{acbdef} = \mathbb{G}_{abcdfe} = \mathbb{G}_{defabc}, \quad (64)$$

and the set of index permutations which leaves \mathbb{G} invariant is equivalent to $S_2 \times S_2 \times S_2$ and has 8 different elements. The symmetrization identity tensor Π^{V_2} is

$$\Pi_{iajkbkclmdmenf}^{V_2} = \frac{1}{8}(\delta_{ia}\delta_{jb}\delta_{kc}\delta_{ld}\delta_{me}\delta_{nf} + \delta_{ia}\delta_{kb}\delta_{jc}\delta_{ld}\delta_{me}\delta_{nf} + \delta_{ia}\delta_{jb}\delta_{kc}\delta_{ld}\delta_{ne}\delta_{mf} + \delta_{ia}\delta_{kb}\delta_{jc}\delta_{ld}\delta_{ne}\delta_{mf}$$

$$+ \delta_{la}\delta_{mb}\delta_{nc}\delta_{id}\delta_{je}\delta_{kf} + \delta_{la}\delta_{nb}\delta_{mc}\delta_{id}\delta_{je}\delta_{kf} + \delta_{la}\delta_{mb}\delta_{nc}\delta_{id}\delta_{ke}\delta_{jf} + \delta_{la}\delta_{nb}\delta_{mc}\delta_{id}\delta_{ke}\delta_{jf}).$$

The character $\chi(Q)$, defined as $\langle \varphi_Q \circ \Pi^{V_2}, \otimes^6 \mathbb{1}_2 \rangle$ with $\varphi_Q = \otimes^6 Q$, is

$$\langle \varphi_Q \circ \Pi^{V_2}, \otimes^6 \mathbb{1}_2 \rangle = \frac{1}{8} \left((\text{tr } Q)^6 + 2(\text{tr } Q)^4 \text{tr } Q^2 + (\text{tr } Q)^2 (\text{tr } Q^2)^2 + 2 \text{tr } Q^4 \text{tr } Q^2 + 2 (\text{tr } Q^2)^3 \right) \quad (65)$$

which, with to the Hamilton-Cayley transformation, becomes

$$\chi(Q) = (\text{tr } Q)^6 - 4(\text{tr } Q)^5 + 6(\text{tr } Q)^4 - 2(\text{tr } Q)^3 - 2(\text{tr } Q)^2. \quad (66)$$

A quick check yields

$$\chi(\mathbb{1}) = (\text{tr } \mathbb{1})^6 - 4(\text{tr } \mathbb{1})^5 + 6(\text{tr } \mathbb{1})^4 - 2(\text{tr } \mathbb{1})^3 - 2(\text{tr } \mathbb{1})^2 = 171,$$

which is exactly the dimension of \underline{V}_2 .

Case $(\underline{V}_2, \mathcal{G}, \underline{\varphi}_2)$: The tensor $\underline{\mathbb{G}}$ has the symmetries

$$\underline{\mathbb{G}}_{abcdef} = \underline{\mathbb{G}}_{bacdef} = \underline{\mathbb{G}}_{abcdef} = \underline{\mathbb{G}}_{defabc}, \quad (67)$$

and the set of index permutations which leaves $\underline{\mathbb{G}}$ invariant is again equivalent to $S_2 \times S_2 \times S_2$ with 8 different elements. The symmetrization identity tensor Π^{V_2} is

$$\begin{aligned} \Pi_{iajbkclmdef}^{V_2} = & \frac{1}{8} (\delta_{ia}\delta_{jb}\delta_{kc}\delta_{ld}\delta_{me}\delta_{nf} + \delta_{ja}\delta_{ib}\delta_{kc}\delta_{ld}\delta_{me}\delta_{nf} + \delta_{ia}\delta_{jb}\delta_{kc}\delta_{md}\delta_{le}\delta_{nf} + \delta_{ja}\delta_{ib}\delta_{kc}\delta_{md}\delta_{le}\delta_{nf} \\ & + \delta_{la}\delta_{mb}\delta_{nc}\delta_{id}\delta_{je}\delta_{kf} + \delta_{ma}\delta_{lb}\delta_{nc}\delta_{id}\delta_{je}\delta_{kf} + \delta_{la}\delta_{mb}\delta_{nc}\delta_{jd}\delta_{ie}\delta_{kf} + \delta_{ma}\delta_{lb}\delta_{nc}\delta_{jd}\delta_{ie}\delta_{kf}). \end{aligned}$$

The character $\chi(Q)$, defined as $\langle \varphi_Q \circ \Pi^{V_2}, \otimes^6 \mathbb{1}_2 \rangle$ with $\varphi_Q = \otimes^6 Q$, is

$$\langle \varphi_Q \circ \Pi^{V_2}, \otimes^6 \mathbb{1}_2 \rangle = \frac{1}{8} \left((\text{tr } Q)^6 + 2(\text{tr } Q)^4 \text{tr } Q^2 + (\text{tr } Q)^2 (\text{tr } Q^2)^2 + 2 \text{tr } Q^4 \text{tr } Q^2 + 2 (\text{tr } Q^2)^3 \right) \quad (68)$$

which, again using the Hamilton-Cayley transformation, can be written as

$$\chi(Q) = (\text{tr } Q)^6 - 4(\text{tr } Q)^5 + 6(\text{tr } Q)^4 - 2(\text{tr } Q)^3 - 2(\text{tr } Q)^2. \quad (69)$$

Checking the character of the identity $\mathbb{1}$ results in

$$\chi(\mathbb{1}) = (\text{tr } \mathbb{1}_2)^6 - 4(\text{tr } \mathbb{1}_2)^5 + 6(\text{tr } \mathbb{1}_2)^4 - 2(\text{tr } \mathbb{1}_2)^3 - 2(\text{tr } \mathbb{1}_2)^2 = 171,$$

which is again equal to the dimension of \underline{V}_2 .

3.2.1 Extended Voigt isomorphisms and structure of the symmetrized tensors

As for classical elasticity with to the Voigt representation isomorphism, we now want to define an analogous isomorphism to represent the tensors $\underline{\mathbb{H}}$ and $\underline{\mathbb{G}}$ as matrices. In [5], following reasonable criteria (i.e. organizing the components in a matrix in order to group together the zero entries when accounting for material symmetries), the Voigt isomorphism is generalized as follows: denoting by $\{\tau_\alpha\}_{\alpha=1}^{18}$ the canonical basis of \mathbb{R}^{18} and considering the orthonormal basis

$$\left\{ \varsigma_{ijk} := \left(\frac{1 - \delta_{ij}}{\sqrt{2}} + \frac{\delta_{ij}}{2} \right) (e_i \otimes e_j + e_j \otimes e_i) \otimes e_k \right\}_{i,j,k=1}^3$$

of $\text{Sym}(3) \otimes \mathbb{R}^3$ we set

$$\mathfrak{N} : \text{Sym}(3) \otimes \mathbb{R}^3 \rightarrow \mathbb{R}^{18}$$

as follows:

$$\mathfrak{N}_{\varsigma_{111}} = \tau_1, \quad \mathfrak{N}_{\varsigma_{221}} = \tau_2, \quad \mathfrak{N}_{\varsigma_{122}} = \tau_3, \quad \mathfrak{N}_{\varsigma_{331}} = \tau_4, \quad \mathfrak{N}_{\varsigma_{133}} = \tau_5,$$

$$\begin{array}{lllll}
\mathfrak{N}_{\zeta_{222}} = \tau_6, & \mathfrak{N}_{\zeta_{112}} = \tau_7, & \mathfrak{N}_{\zeta_{121}} = \tau_8, & \mathfrak{N}_{\zeta_{332}} = \tau_9, & \mathfrak{N}_{\zeta_{233}} = \tau_{10}, \\
\mathfrak{N}_{\zeta_{333}} = \tau_{11}, & \mathfrak{N}_{\zeta_{113}} = \tau_{12}, & \mathfrak{N}_{\zeta_{131}} = \tau_{13}, & \mathfrak{N}_{\zeta_{223}} = \tau_{14}, & \mathfrak{N}_{\zeta_{232}} = \tau_{15}, \\
\mathfrak{N}_{\zeta_{123}} = \tau_{16}, & \mathfrak{N}_{\zeta_{132}} = \tau_{17}, & \mathfrak{N}_{\zeta_{231}} = \tau_{18}. & &
\end{array}$$

In this way we can define

$$\begin{aligned}
\mathfrak{M}\mathfrak{M} &: \text{Lin}(\text{Sym}(3), \text{Sym}(3) \otimes \mathbb{R}^3) \rightarrow \text{Lin}(\mathbb{R}^6, \mathbb{R}^{18}), \\
\mathfrak{N} &: \text{Sym}(\text{Sym}(3) \otimes \mathbb{R}^3, \text{Sym}(3) \otimes \mathbb{R}^3) \rightarrow \text{Sym}(\mathbb{R}^{18}, \mathbb{R}^{18})
\end{aligned}$$

by

$$\langle \mathfrak{M}\mathfrak{M}(\mathbb{H}) a, b \rangle_{\mathbb{R}^{18}} = \langle \mathbb{H} \mathfrak{M}^{-1} a, \mathfrak{N}^{-1} b \rangle_{\mathbb{R}^{3 \times 3}} \quad \forall a \in \mathbb{R}^6, b \in \mathbb{R}^{18}, \quad (70)$$

$$\langle \mathfrak{N}(\mathbb{G}) p, q \rangle_{\mathbb{R}^{18}} = \langle \mathbb{G} \mathfrak{N}^{-1} p, \mathfrak{N}^{-1} q \rangle_{\mathbb{R}^{3 \times 3 \times 3}} \quad \forall p, q \in \mathbb{R}^{18}. \quad (71)$$

3.2.2 Cubic case

Now, we want to study in detail the particular case in which the symmetry group is the cubic group \mathbb{O} which has 24 elements. We want to calculate the dimension of the fixed subspaces for $V_1, V_2, \underline{V}_1, \underline{V}_2$, and determine the structure of the symmetrized tensors. We have

$$\dim \text{Fix}_{\mathbb{O}}^{\varphi} V_j = \frac{1}{24} \sum_{Q \in \mathbb{O}} \chi_{\varphi_j}(Q), \quad \dim \text{Fix}_{\mathbb{O}}^{\varphi} \underline{V}_j = \frac{1}{24} \sum_{Q \in \mathbb{O}} \chi_{\varphi_j}(Q), \quad j \in \{1, 2\},$$

thus, thanks to the character formulas (63),(61),(66),(69) we find

- for V_1, \underline{V}_1 : $\dim \text{Fix}_{\mathbb{O}}^{\varphi} V_1 = \dim \text{Fix}_{\mathbb{O}}^{\varphi} \underline{V}_1 = 3$, which gives 3 independent constants for \mathbb{H} and $\underline{\mathbb{H}}$,
- for V_2, \underline{V}_2 : $\dim \text{Fix}_{\mathbb{O}}^{\varphi} V_2 = \dim \text{Fix}_{\mathbb{O}}^{\varphi} \underline{V}_2 = 11$ which gives 11 independent constants for \mathbb{G} and $\underline{\mathbb{G}}$.

Thanks to the transformations defined in (70) and (71), the matrix structures of the symmetrized tensors are (these results were already derived in [5])

$$\underline{\mathbb{G}} = \begin{pmatrix} \underline{G}_1 & 0 & 0 & 0 \\ 0 & \underline{G}_1 & 0 & 0 \\ 0 & 0 & \underline{G}_1 & 0 \\ 0 & 0 & 0 & \underline{G}_2 \end{pmatrix} \in \text{Sym}(18),$$

where

$$\underline{G}_1 = \begin{pmatrix} \eta_{11} & \eta_{12} & \eta_{13} & \eta_{12} & \eta_{13} \\ & \eta_{22} & \eta_{23} & \eta_{24} & \eta_{25} \\ & & \eta_{33} & \eta_{25} & \eta_{35} \\ \text{sym} & & & \eta_{22} & \eta_{23} \\ & & & & \eta_{33} \end{pmatrix} \in \text{Sym}(5), \quad \underline{G}_2 = \begin{pmatrix} \gamma_{11} & \gamma_{12} & \gamma_{12} \\ & \gamma_{11} & \gamma_{12} \\ \text{sym} & & \gamma_{11} \end{pmatrix} \in \text{Sym}(3),$$

with

$$\begin{aligned}
\eta_{11} &= \frac{1}{3} (\underline{\mathbb{G}}_{111111} + \underline{\mathbb{G}}_{222222} + \underline{\mathbb{G}}_{333333}), & \eta_{22} &= \frac{1}{6} (\underline{\mathbb{G}}_{112112} + \underline{\mathbb{G}}_{113113} + \underline{\mathbb{G}}_{221221} + \underline{\mathbb{G}}_{223223} + \underline{\mathbb{G}}_{331331} + \underline{\mathbb{G}}_{332332}), \\
\eta_{24} &= \frac{1}{3} (\underline{\mathbb{G}}_{112332} + \underline{\mathbb{G}}_{113223} + \underline{\mathbb{G}}_{221331}), & \eta_{33} &= \frac{1}{6} (\underline{\mathbb{G}}_{121121} + \underline{\mathbb{G}}_{122122} + \underline{\mathbb{G}}_{131131} + \underline{\mathbb{G}}_{133133} + \underline{\mathbb{G}}_{232232} + \underline{\mathbb{G}}_{233233}), \\
\eta_{35} &= \frac{1}{3} (\underline{\mathbb{G}}_{121233} + \underline{\mathbb{G}}_{122133} + \underline{\mathbb{G}}_{131232}), & \eta_{12} &= \frac{1}{6} (\underline{\mathbb{G}}_{111221} + \underline{\mathbb{G}}_{111331} + \underline{\mathbb{G}}_{112222} + \underline{\mathbb{G}}_{113333} + \underline{\mathbb{G}}_{222332} + \underline{\mathbb{G}}_{223333}), \\
\eta_{11} &= \frac{1}{3} (\underline{\mathbb{G}}_{123123} + \underline{\mathbb{G}}_{132132} + \underline{\mathbb{G}}_{231231}), & \eta_{13} &= \frac{1}{6} (\underline{\mathbb{G}}_{111122} + \underline{\mathbb{G}}_{111133} + \underline{\mathbb{G}}_{121222} + \underline{\mathbb{G}}_{131333} + \underline{\mathbb{G}}_{222233} + \underline{\mathbb{G}}_{232333}), \\
\eta_{12} &= \frac{1}{3} (\underline{\mathbb{G}}_{123132} + \underline{\mathbb{G}}_{123231} + \underline{\mathbb{G}}_{132231}), & \eta_{23} &= \frac{1}{6} (\underline{\mathbb{G}}_{112121} + \underline{\mathbb{G}}_{113131} + \underline{\mathbb{G}}_{122221} + \underline{\mathbb{G}}_{133331} + \underline{\mathbb{G}}_{223232} + \underline{\mathbb{G}}_{233332}), \\
& & \eta_{25} &= \frac{1}{6} (\underline{\mathbb{G}}_{112233} + \underline{\mathbb{G}}_{113232} + \underline{\mathbb{G}}_{121332} + \underline{\mathbb{G}}_{122331} + \underline{\mathbb{G}}_{131223} + \underline{\mathbb{G}}_{133221}),
\end{aligned}$$

and

$$\mathbb{H} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \underline{H}_2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \underline{H}_2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \underline{H}_2 \\ \underline{H}_1 & 0 & 0 & 0 \end{pmatrix} \in \mathbb{R}^{6 \times 18}, \quad \text{where } \underline{H}_1 = \begin{pmatrix} \zeta_1 & -\zeta_1 & 0 \\ -\zeta_1 & 0 & \zeta_1 \\ 0 & \zeta_1 & -\zeta_1 \end{pmatrix} \in \mathbb{R}^{3 \times 3}, \quad \underline{H}_2 = \begin{pmatrix} \zeta_2 \\ \zeta_3 \\ -\zeta_2 \\ -\zeta_3 \end{pmatrix} \in \mathbb{R}^{3 \times 3},$$

with

$$\begin{aligned} \zeta_1 &= \frac{1}{6} (-\mathbb{H}_{11213} + \mathbb{H}_{11312} + \mathbb{H}_{22123} - \mathbb{H}_{22312} - \mathbb{H}_{33123} + \mathbb{H}_{33213}), \\ \zeta_2 &= \frac{1}{6} (\mathbb{H}_{12311} - \mathbb{H}_{12322} - \mathbb{H}_{13211} + \mathbb{H}_{13233} + \mathbb{H}_{23122} - \mathbb{H}_{23133}), \\ \zeta_3 &= \frac{1}{6} (\mathbb{H}_{12113} - \mathbb{H}_{12223} - \mathbb{H}_{13112} + \mathbb{H}_{13323} + \mathbb{H}_{23212} - \mathbb{H}_{23313}). \end{aligned}$$

3.2.3 Transversal hemitropic and transversal isotropic case

Next, we study the two particular cases in which the invariance of the elasticity tensor $\underline{\mathbb{G}}$ is taken with respect to $\text{SO}(2; e_3)$ and $\text{O}(2; e_3)$. These results are not new; indeed, the structure for the symmetrized tensors was already obtained in [5]. Via the trace formula and the explicit expression for the Haar measure on $\text{SO}(2)$ and $\text{O}(2)$ derived in (98) and (99) respectively, we compute the following numbers of independent components for the two considered symmetries:

$$\dim \text{Fix}_{\text{SO}(2, e_3)}^{\varphi} \underline{V}_2 = 31, \quad \dim \text{Fix}_{\text{O}(2, e_3)}^{\varphi} \underline{V}_2 = 21. \quad (72)$$

The correct number of independent components for the transversal isotropic case was also obtained with another approach in an unpublished note [8]. Denoting with $\underline{\mathbb{G}}_{\text{SO}(2; e_3)}$ and $\underline{\mathbb{G}}_{\text{O}(2; e_3)}$ the symmetrized tensors with respect to the actions of $\text{SO}(2; e_3)$ and $\text{O}(2; e_3)$ respectively, their matrix representations via the isomorphism \mathfrak{N} are

$$\underline{\mathbb{G}}_{\text{SO}(2; e_3)} = \begin{pmatrix} \underline{G}_1 & \underline{G}_2 & 0 & 0 \\ \underline{G}_2 & \underline{G}_1 & 0 & 0 \\ 0 & 0 & \underline{G}_3 & \underline{G}_4 \\ 0 & 0 & \underline{G}_4 & \underline{G}_5 \end{pmatrix} \in \text{Sym}(18), \quad \underline{\mathbb{G}}_{\text{O}(2; e_3)} = \begin{pmatrix} \underline{G}_1 & 0 & 0 & 0 \\ 0 & \underline{G}_1 & 0 & 0 \\ 0 & 0 & \underline{G}_3 & 0 \\ 0 & 0 & 0 & \underline{G}_5 \end{pmatrix} \in \text{Sym}(18),$$

where

$$\underline{G}_1 = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} & \alpha_{14} & \alpha_{15} \\ & \alpha_{22} & -\alpha_{13} + \frac{\sqrt{2}(\alpha_{11} - \alpha_{22})}{2} & \alpha_{14} - \sqrt{2}\alpha_{34} & \alpha_{15} - \sqrt{2}\alpha_{35} \\ & & -\alpha_{12} + \frac{\alpha_{11} + \eta_{22}}{2} & \alpha_{34} & \alpha_{35} \\ \text{sym} & & & \alpha_{44} & \alpha_{45} \\ & & & & \alpha_{55} \end{pmatrix} \in \text{Sym}(5) \quad 11 \text{ independent components,}$$

$$\underline{G}_2 = \begin{pmatrix} 0 & \beta_{12} & -\frac{\sqrt{2}\beta_{12}}{2} & \beta_{24} + \sqrt{2}\beta_{34} & \beta_{25} + \sqrt{2}\beta_{35} \\ & 0 & -\frac{\sqrt{2}\beta_{12}}{2} & \beta_{24} & \beta_{25} \\ & & 0 & \beta_{34} & \beta_{35} \\ \text{skew} & & & 0 & \beta_{45} \\ & & & & 0 \end{pmatrix} \in \mathfrak{so}(5) \quad 6 \text{ independent components,}$$

$$\begin{aligned}
\underline{G}_3 &= \begin{pmatrix} \gamma_{11} & \gamma_{12} & \gamma_{13} & \gamma_{12} & \gamma_{13} \\ & \gamma_{22} & \gamma_{23} & \gamma_{22} & \gamma_{23} \\ & & \gamma_{33} & \gamma_{23} & \gamma_{33} \\ \text{sym} & & & \gamma_{22} & \gamma_{23} \\ & & & & \gamma_{33} \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ & 0 & 0 & -\eta_{11} & -\sqrt{2}\eta_{12} \\ & & 0 & -\sqrt{2}\eta_{12} & -(\eta_{22} + \eta_{33}) \\ \text{sym} & & & 0 & 0 \\ & & & & 0 \end{pmatrix} \in \text{Sym}(5) \quad 6 \text{ independent components,} \\
\underline{G}_4 &= \begin{pmatrix} 0 & \zeta_{12} & -\zeta_{12} \\ 0 & \zeta_{22} & -\zeta_{22} - \sqrt{2}\zeta_{31} \\ \zeta_{31} & \zeta_{32} & -\zeta_{32} \\ 0 & \zeta_{22} + \sqrt{2}\zeta_{31} & -\zeta_{22} \\ -\zeta_{31} & \zeta_{32} & -\zeta_{32} \end{pmatrix} \in \mathbb{R}^{5 \times 3} \quad 4 \text{ independent components,} \\
\underline{G}_5 &= \begin{pmatrix} \eta_{11} & \eta_{12} & \eta_{12} \\ & \eta_{22} & \eta_{23} \\ \text{sym} & & \eta_{22} \end{pmatrix} \in \text{Sym}(3) \quad 4 \text{ independent components,}
\end{aligned}$$

and

$$\begin{aligned}
\alpha_{11} &= \frac{1}{64} \left(5 \mathbb{G}_{111111} + 4 \mathbb{G}_{111122} + 2 \mathbb{G}_{111221} + \mathbb{G}_{112112} + 4 \mathbb{G}_{112121} + 2 \mathbb{G}_{112222} \right. \\
&\quad \left. + 4 \mathbb{G}_{121121} + 4 (\mathbb{G}_{121222} + \mathbb{G}_{122122} + \mathbb{G}_{122221}) + \mathbb{G}_{221221} + 5 \mathbb{G}_{222222} \right), \\
\alpha_{12} &= \frac{1}{64} \left(\mathbb{G}_{111111} + 6 \mathbb{G}_{111221} + \mathbb{G}_{112112} + 6 \mathbb{G}_{112222} - 4 \mathbb{G}_{121121} - 4 \mathbb{G}_{122122} + \mathbb{G}_{221221} + \mathbb{G}_{222222} \right), \\
\alpha_{13} &= \frac{1}{32\sqrt{2}} \left(\mathbb{G}_{111111} + 6 \mathbb{G}_{111122} - \mathbb{G}_{112112} - 2 \mathbb{G}_{112121} + 6 \mathbb{G}_{121222} - 2 \mathbb{G}_{122221} - \mathbb{G}_{221221} + \mathbb{G}_{222222} \right), \\
\alpha_{14} &= \frac{1}{32} \left(3 \mathbb{G}_{111331} + \mathbb{G}_{112332} + 2 \mathbb{G}_{121332} + 2 \mathbb{G}_{122331} + \mathbb{G}_{221331} + 3 \mathbb{G}_{222332} \right), \\
\alpha_{15} &= \frac{1}{16\sqrt{2}} \left(3 \mathbb{G}_{111133} + \mathbb{G}_{112233} + 2 \mathbb{G}_{121233} + 2 \mathbb{G}_{122133} + \mathbb{G}_{133221} + 3 \mathbb{G}_{222233} \right), \\
\alpha_{22} &= \frac{1}{64} \left(\mathbb{G}_{111111} - 4 \mathbb{G}_{111122} + 2 \mathbb{G}_{111221} + 5 \mathbb{G}_{112112} - 4 \mathbb{G}_{112121} + 2 \mathbb{G}_{112222} \right. \\
&\quad \left. + 4 \mathbb{G}_{121121} - 4 \mathbb{G}_{121222} + 4 \mathbb{G}_{122122} - 4 \mathbb{G}_{122221} + 5 \mathbb{G}_{221221} + \mathbb{G}_{222222} \right), \\
\alpha_{34} &= \frac{1}{16\sqrt{2}} \left(\mathbb{G}_{111331} - \mathbb{G}_{112332} + 2 \mathbb{G}_{121332} + 2 \mathbb{G}_{122331} - \mathbb{G}_{221331} + \mathbb{G}_{222332} \right), \\
\alpha_{35} &= \frac{1}{16} \left(\mathbb{G}_{111133} - \mathbb{G}_{112233} + 2 \mathbb{G}_{121233} + 2 \mathbb{G}_{122133} - \mathbb{G}_{133221} + \mathbb{G}_{222233} \right), \\
\alpha_{44} &= \frac{1}{8} \left(\mathbb{G}_{331331} + \mathbb{G}_{332332} \right), \quad \alpha_{45} = \frac{1}{4\sqrt{2}} \left(\mathbb{G}_{133331} + \mathbb{G}_{233332} \right), \quad \alpha_{55} = \frac{1}{4\sqrt{2}} \left(\mathbb{G}_{133133} + \mathbb{G}_{233233} \right), \\
\beta_{12} &= \frac{1}{16} \left(\mathbb{G}_{111112} - \mathbb{G}_{111121} + \mathbb{G}_{112122} - \mathbb{G}_{121221} + \mathbb{G}_{122222} - \mathbb{G}_{221222} \right), \\
\beta_{24} &= \frac{1}{32} \left(\mathbb{G}_{111332} - 3 \mathbb{G}_{112331} + 2 \mathbb{G}_{121331} - 2 \mathbb{G}_{122332} + 3 \mathbb{G}_{221332} - \mathbb{G}_{222331} \right), \\
\beta_{25} &= \frac{1}{16\sqrt{2}} \left(\mathbb{G}_{111233} - 3 \mathbb{G}_{112133} + 2 \mathbb{G}_{121133} - 2 \mathbb{G}_{122233} - \mathbb{G}_{133222} + 3 \mathbb{G}_{221233} \right), \\
\beta_{34} &= \frac{1}{16\sqrt{2}} \left(\mathbb{G}_{111332} + \mathbb{G}_{112331} - 2 \mathbb{G}_{121331} + 2 \mathbb{G}_{122332} - \mathbb{G}_{221332} - \mathbb{G}_{222331} \right), \\
\beta_{35} &= \frac{1}{16} \left(\mathbb{G}_{111233} + \mathbb{G}_{112133} - 2 \mathbb{G}_{121133} + 2 \mathbb{G}_{122233} - \mathbb{G}_{133222} - \mathbb{G}_{221233} \right), \\
\beta_{45} &= \frac{1}{4\sqrt{2}} \left(-\mathbb{G}_{133332} + \mathbb{G}_{233331} \right),
\end{aligned}$$

$$\begin{aligned}
\gamma_{11} &= \frac{\mathbb{G}_{333333}}{4}, & \gamma_{12} &= \frac{1}{8}(\mathbb{G}_{113333} + \mathbb{G}_{223333}), & \gamma_{13} &= \frac{1}{4\sqrt{2}}(\mathbb{G}_{131333} + \mathbb{G}_{232333}), \\
\gamma_{22} &= \frac{1}{32}(3\mathbb{G}_{113113} + 2\mathbb{G}_{113223} + 4\mathbb{G}_{123123} + 3\mathbb{G}_{223223}), \\
\gamma_{23} &= \frac{1}{16\sqrt{2}}(3\mathbb{G}_{113131} + \mathbb{G}_{113232} + 2\mathbb{G}_{123132} + 2\mathbb{G}_{123231} + \mathbb{G}_{131223} + 3\mathbb{G}_{223232}), \\
\gamma_{33} &= \frac{1}{16}(3\mathbb{G}_{131131} + 2\mathbb{G}_{131232} + \mathbb{G}_{132132} + 2\mathbb{G}_{132231} + \mathbb{G}_{231231} + 3\mathbb{G}_{232232}), \\
\zeta_{12} &= \frac{1}{4\sqrt{2}}(\mathbb{G}_{132333} - \mathbb{G}_{231333}), \\
\zeta_{22} &= \frac{1}{16\sqrt{2}}(3\mathbb{G}_{113132} - \mathbb{G}_{113231} - 2\mathbb{G}_{123131} + 2\mathbb{G}_{123232} + \mathbb{G}_{132223} - 3\mathbb{G}_{223231}), \\
\zeta_{31} &= \frac{1}{16}(-\mathbb{G}_{113132} - \mathbb{G}_{113231} + 2\mathbb{G}_{123131} - 2\mathbb{G}_{123232} + \mathbb{G}_{132223} + \mathbb{G}_{223231}), \\
\zeta_{32} &= \frac{1}{8}(\mathbb{G}_{131132} - \mathbb{G}_{131231} + \mathbb{G}_{132232} - \mathbb{G}_{231232}), \\
\eta_{11} &= \frac{1}{16}(\mathbb{G}_{113113} - 2\mathbb{G}_{113223} + 4\mathbb{G}_{123123} + \mathbb{G}_{223223}), \\
\eta_{12} &= \frac{1}{16}(\mathbb{G}_{113131} - \mathbb{G}_{113232} + 2\mathbb{G}_{123132} + 2\mathbb{G}_{123231} - \mathbb{G}_{131223} + \mathbb{G}_{223232}), \\
\eta_{22} &= \frac{1}{16}(\mathbb{G}_{131131} - 2\mathbb{G}_{131232} + 3\mathbb{G}_{132132} - 2\mathbb{G}_{132231} + 3\mathbb{G}_{231231} + \mathbb{G}_{232232}), \\
\eta_{23} &= \frac{1}{16}(\mathbb{G}_{131131} - 2\mathbb{G}_{131232} - \mathbb{G}_{132132} + 6\mathbb{G}_{132231} - \mathbb{G}_{231231} + \mathbb{G}_{232232}).
\end{aligned}$$

3.3 Indeterminate couple stress model

We now aim to apply the established invariance theory to a special sub-class of second gradient models: the indeterminate couple stress model (for a very good description of this model we refer to [29, 42, 46, 48, 51] and for a determination of the material invariance condition we refer to [41, 48]). The linearization of the indeterminate couple stress model is characterized via the action functional

$$\mathcal{A}[u] := \int_{\Omega} \frac{1}{2} \left(\langle \mathbb{C} \varepsilon, \varepsilon \rangle + \underbrace{\langle \mathbb{L} \operatorname{Dcurl} u, \operatorname{Dcurl} u \rangle}_{=: W_{\operatorname{curl}}(\operatorname{D}^2 u)} \right) dx, \quad (73)$$

i.e., in the indeterminate couple stress model, the non local term of the deformation energy is defined as a function of the gradient of only the rotational part of the gradient of u . Thus, we can immediately remark that

- $\operatorname{Dcurl} u$ is **not a symmetric** second order tensor and, as a consequence, the curvature elasticity tensor is a fourth order tensor involving only the major symmetry,
- considering the orthogonal split of the curvature term in the symmetric and skew symmetric parts

$$\langle \mathbb{L} \operatorname{Dcurl} u, \operatorname{Dcurl} u \rangle = \langle \mathbb{L}_s \operatorname{sym} \operatorname{Dcurl} u, \operatorname{sym} \operatorname{Dcurl} u \rangle + \langle \mathbb{L}_c \operatorname{skew} \operatorname{Dcurl} u, \operatorname{skew} \operatorname{Dcurl} u \rangle,$$

where

$$\mathbb{L}_s \in \operatorname{Sym}(\operatorname{Sym}(3), \operatorname{Sym}(3)) \quad \text{and} \quad \mathbb{L}_c \in \operatorname{Sym}(\mathfrak{so}(3), \mathfrak{so}(3)),$$

we can express the skew-symmetric term as a bilinear form acting on \mathbb{R}^3 via the axl operator (see the Appendix B.2). Indeed, in this case we have

$$\operatorname{axl} \operatorname{skew} \operatorname{Dcurl} u = \operatorname{curl} \operatorname{curl} u,$$

and we can introduce a matrix $\tilde{\mathbb{L}}_c \in \operatorname{Sym}(3)$ such that

$$\langle \mathbb{L}_c \operatorname{skew} \operatorname{Dcurl} u, \operatorname{skew} \operatorname{Dcurl} u \rangle_{\mathbb{R}^3 \times \mathbb{R}^3} = \langle \tilde{\mathbb{L}}_c \operatorname{curl} \operatorname{curl} u, \operatorname{curl} \operatorname{curl} u \rangle_{\mathbb{R}^3} \quad \forall \text{ admissible } u.$$

Applying the theoretical machinery developed in Section 3.1, we can obtain the structure of the elasticity tensor \mathbb{L} when we ask for the isotropic invariance of the indeterminate couple stress model. Indeed, having that

$$\mathbb{L} \in \text{Sym}(\mathbb{R}^{3 \times 3}, \mathbb{R}^{3 \times 3}),$$

the invariance with respect to the full group $\text{SO}(3)$ gives us only 3 independent material constants (see Eqs. (47), (48)). Thus, in this case, we have

$$\langle \mathbb{L} \text{Dcurl} u, \text{Dcurl} u \rangle = \alpha_1 \|\text{sym Dcurl} u\|^2 + \frac{\alpha_2}{2} (\text{tr Dcurl} u)^2 + \alpha_3 \|\text{skew Dcurl} u\|^2.$$

Moreover, remarking that $\text{tr D curl} u = \text{div curl} u = 0$, we can write the deformation energy in the case of isotropy as

$$\langle \mathbb{L} \text{Dcurl} u, \text{Dcurl} u \rangle = \alpha_1 \|\text{dev sym Dcurl} u\|^2 + \alpha_3 \|\text{skew Dcurl} u\|^2.$$

3.4 Micromorphic type models

Our next objective is to apply the well-established theory of invariance to a class of micromorphic-type generalized continua, which have garnered increasing attention in recent years for modeling the mechanical behavior of metamaterials. As we will discuss, the curvature term of the relaxed micromorphic model belongs to a different vector space than that of classical Cauchy continua. To address this discrepancy, we will leverage the findings from Section 3.1, which will be tailored to suit 2D models. These results have been utilized in [61] to model the size-effects of metamaterial beams under bending with the aid of the relaxed micromorphic continuum. We will also compare the symmetry classes of the relaxed micromorphic model with those of the classical micromorphic model.

3.4.1 3D-relaxed micromorphic model

We set

$$x = (x_1, x_2, x_3) = (\bar{x}, x_3), \quad \text{where} \quad \bar{x} = (x_1, x_2).$$

The involved kinematic fields are

$$\begin{aligned} u: \Omega \subset \mathbb{R}^3 &\rightarrow \mathbb{R}^3, & x \mapsto u(x) &= (u_1(x), u_2(x), u_3(x)), \\ P: \Omega \subset \mathbb{R}^3 &\rightarrow \mathbb{R}^{3 \times 3}, & x \mapsto P(x) &= \begin{pmatrix} P_{11}(x) & P_{12}(x) & P_{13}(x) \\ P_{21}(x) & P_{22}(x) & P_{23}(x) \\ P_{31}(x) & P_{32}(x) & P_{33}(x) \end{pmatrix}. \end{aligned}$$

The Curl operator is defined by

$$\text{Curl} P(x) = \text{Curl} \begin{pmatrix} P_{11}(x) & P_{12}(x) & P_{13}(x) \\ P_{21}(x) & P_{22}(x) & P_{23}(x) \\ P_{31}(x) & P_{32}(x) & P_{33}(x) \end{pmatrix} := \begin{pmatrix} \text{curl}(P_{11}(x), P_{12}(x), P_{13}(x)) \\ \text{curl}(P_{21}(x), P_{22}(x), P_{23}(x)) \\ \text{curl}(P_{31}(x), P_{32}(x), P_{33}(x)) \end{pmatrix}. \quad (74)$$

The full potential energy density for the linear model consists of the terms

$$W(u, Du, P, \text{Curl} P) = W_e(\text{sym} Du, \text{sym} P) + W_c(\text{skew} Du, \text{skew} P) + W_{\text{micro}}(\text{sym} P) + W_{\text{curv}}(\text{Curl} P),$$

where

$$\begin{aligned} W_e(\text{sym} Du, \text{sym} P) &:= \frac{1}{2} \langle \mathbb{C}_e \text{sym}(Du - P), \text{sym}(Du - P) \rangle_{\mathbb{R}^{3 \times 3}}, \\ W_c(\text{skew} Du, \text{skew} P) &:= \frac{1}{2} \langle \mathbb{C}_c \text{skew}(Du - P), \text{skew}(Du - P) \rangle_{\mathbb{R}^{3 \times 3}}, \\ W_{\text{micro}}(\text{sym} P) &:= \frac{1}{2} \langle \mathbb{C}_{\text{micro}} \text{sym} P, \text{sym} P \rangle_{\mathbb{R}^{3 \times 3}}, \\ W_{\text{curv}}(\text{Curl} P) &:= \frac{1}{2} \langle \mathbb{L} \text{Curl} P, \text{Curl} P \rangle_{\mathbb{R}^{3 \times 3}}, \end{aligned} \quad (75)$$

and the four involved tensors $\mathbb{C}_e, \mathbb{C}_c, \mathbb{C}_{\text{micro}}, \mathbb{L}$ are

$$\begin{aligned} \mathbb{C}_e &\in \mathfrak{Ela}^+(3) \subseteq \otimes^4 \mathbb{R}^3, & \mathbb{C}_c &\in \text{Sym}^+(\mathfrak{so}(3), \mathfrak{so}(3)), \\ \mathbb{C}_{\text{micro}} &\in \mathfrak{Ela}^+(3), & \mathbb{L} &\in \text{Sym}^+(\mathbb{R}^{3 \times 3}, \mathbb{R}^{3 \times 3}). \end{aligned}$$

3.4.2 2D-relaxed micromorphic model, plane strain

The involved kinematic fields are

$$\begin{aligned} u: \Omega \subset \mathbb{R}^2 &\rightarrow \mathbb{R}^2, & (x_1, x_2) &\mapsto u(x_1, x_2) = (u_1(x_1, x_2), u_2(x_1, x_2)), \\ P: \Omega \subset \mathbb{R}^2 &\rightarrow \mathbb{R}^{2 \times 2}, & (x_1, x_2) &\mapsto P(x_1, x_2) = \begin{pmatrix} P_{11}(x_1, x_2) & P_{12}(x_1, x_2) \\ P_{21}(x_1, x_2) & P_{22}(x_1, x_2) \end{pmatrix}. \end{aligned} \quad (76)$$

Given a vector field $v: \Omega \subset \mathbb{R}^2 \rightarrow \mathbb{R}^2$ we define the 2D-curl operator as

$$\text{curl}_{2\text{D}} v(x) := v_{2,1}(x) - v_{1,2}(x),$$

and for a matrix-valued field $P: \Omega \subset \mathbb{R}^2 \rightarrow \mathbb{R}^{2 \times 2}$,

$$\begin{aligned} \text{Curl}_{2\text{D}} P(x) &:= \text{Curl}_{2\text{D}} \begin{pmatrix} P_{11}(x_1, x_2) & P_{12}(x_1, x_2) \\ P_{21}(x_1, x_2) & P_{22}(x_1, x_2) \end{pmatrix} = \begin{pmatrix} \text{curl}_2(P_{11}(x_1, x_2), P_{12}(x_1, x_2)) \\ \text{curl}_2(P_{21}(x_1, x_2), P_{22}(x_1, x_2)) \end{pmatrix} \\ &= \begin{pmatrix} P_{12,1}(x_1, x_2) - P_{11,2}(x_1, x_2) \\ P_{22,1}(x_1, x_2) - P_{21,2}(x_1, x_2) \end{pmatrix}. \end{aligned} \quad (77)$$

The full potential energy density for the linear model consists of the following terms:

$$W(u, Du, P, \text{Curl}_{2\text{D}} P) = W_e(\text{sym } Du, \text{sym } P) + W_c(\text{skew } Du, \text{skew } P) + W_{\text{micro}}(\text{sym } P) + W_{\text{curv}}(\text{Curl}_{2\text{D}} P),$$

where

$$\begin{aligned} W_e(\text{sym } Du, \text{sym } P) &:= \frac{1}{2} \langle \mathbb{C}_e \text{sym } (Du - P), \text{sym } (Du - P) \rangle_{\mathbb{R}^{2 \times 2}}, \\ W_c(\text{skew } Du, \text{skew } P) &:= \frac{1}{2} \langle \mathbb{C}_c \text{skew } (Du - P), \text{skew } (Du - P) \rangle_{\mathbb{R}^{2 \times 2}}, \\ W_{\text{micro}}(\text{sym } P) &:= \frac{1}{2} \langle \mathbb{C}_{\text{micro}} \text{sym } P, \text{sym } P \rangle_{\mathbb{R}^{2 \times 2}}, \\ W_{\text{curv}}(\text{Curl}_{2\text{D}} P) &:= \frac{1}{2} \langle \mathbb{L} \text{Curl}_{2\text{D}} P, \text{Curl}_{2\text{D}} P \rangle_{\mathbb{R}^2}, \end{aligned}$$

and the four involved tensors $\mathbb{C}_e, \mathbb{C}_c, \mathbb{C}_{\text{micro}}, \mathbb{L}$ are

$$\begin{aligned} \mathbb{C}_e &\in \mathfrak{Ela}^+(2) \subseteq \otimes^4 \mathbb{R}^2, & \mathbb{C}_c &\in \text{Sym}(\mathfrak{so}(2), \mathfrak{so}(2)), \\ \mathbb{C}_{\text{micro}} &\in \mathfrak{Ela}^+(2), & \mathbb{L} &\in \text{Sym}^+(2) \subseteq \mathbb{R}^{2 \times 2}, \end{aligned}$$

where we set $\mathfrak{Ela}^+(2) := \text{Sym}^+(\text{Sym}(2), \text{Sym}(2))$.

2D-relaxed micromorphic model derived from the 3D model: Let us give the following definition.

Definition 2. Consider a matrix valued field $P: \Omega \rightarrow \mathbb{R}^{3 \times 3}$. We say that P is a **plane field** if it has the following structure:

$$P(x) = \begin{pmatrix} P_{11}(x) & P_{12}(x) & P_{13}(x) \\ P_{21}(x) & P_{22}(x) & P_{23}(x) \\ P_{31}(x) & P_{32}(x) & P_{33}(x) \end{pmatrix} = \begin{pmatrix} P_{11}(\bar{x}) & P_{12}(\bar{x}) & 0 \\ P_{21}(\bar{x}) & P_{22}(\bar{x}) & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

We can identify a plane field with ‘‘its restriction field’’

$$\bar{P}: \Omega \rightarrow \mathbb{R}^{2 \times 2},$$

and by an abuse of notation, whenever there is no confusion, we use the same symbol for both.

If a matrix-valued field P is a plane field we obtain

$$\begin{aligned} \text{Curl } P(x) &= \text{Curl} \begin{pmatrix} P_{11}(x) & P_{12}(x) & P_{13}(x) \\ P_{21}(x) & P_{22}(x) & P_{23}(x) \\ P_{31}(x) & P_{32}(x) & P_{33}(x) \end{pmatrix} = \begin{pmatrix} \text{curl}(P_{11}(x), P_{12}(x), P_{13}(x)) \\ \text{curl}(P_{21}(x), P_{22}(x), P_{23}(x)) \\ \text{curl}(P_{31}(x), P_{32}(x), P_{33}(x)) \end{pmatrix} \\ &= \begin{pmatrix} (0, 0, P_{12,1}(\bar{x}) - P_{11,2}(\bar{x})) \\ (0, 0, P_{22,1}(\bar{x}) - P_{21,2}(\bar{x})) \\ (0, 0, 0) \end{pmatrix} = \begin{pmatrix} 0 & 0 & P_{12,1}(\bar{x}) - P_{11,2}(\bar{x}) \\ 0 & 0 & P_{22,1}(\bar{x}) - P_{21,2}(\bar{x}) \\ 0 & 0 & 0 \end{pmatrix} = \left(\begin{array}{cc|c} 0 & 0 & \text{Curl}_{2\text{D}} \bar{P}(\bar{x}) \\ 0 & 0 & \\ 0 & 0 & 0 \end{array} \right). \end{aligned} \quad (78)$$

3.4.3 Symmetrization of \mathbb{L}

Introducing the symmetrization identity

$$\pi : \mathbb{R}^{2 \times 2} \rightarrow \text{Sym}(2), \quad \pi_{iajb} = \frac{1}{2}(\delta_{ia}\delta_{jb} + \delta_{ja}\delta_{ib}),$$

the characters can be computed via $\chi(Q) = \langle \varphi_Q, \pi \rangle = \langle Q^{\otimes 2}, \pi \rangle_{\mathbb{R}^{2 \times 2 \times 2 \times 2}}$ giving⁷

$$\begin{aligned} \chi(Q) &= \frac{1}{2}(\delta_{ia}\delta_{jb} + \delta_{ja}\delta_{ib})Q_{ia}Q_{jb} = \frac{1}{2}(\delta_{ia}\delta_{jb}Q_{ia}Q_{jb} + \delta_{ja}\delta_{ib}Q_{ia}Q_{jb}) \\ &= \frac{1}{2} \left(\underbrace{\delta_{ia}Q_{ia}}_{\text{tr } Q} \underbrace{\delta_{jb}Q_{jb}}_{\text{tr } Q} + \underbrace{\delta_{aj}Q_{jb}\delta_{bi}Q_{ia}}_{\text{tr } Q^2} \right) = \frac{1}{2} \left((\text{tr } Q)^2 + \text{tr } Q^2 \right). \end{aligned} \quad (80)$$

The Cayley-Hamilton theorem allows us to express $\chi(Q)$ as a polynomial in $\text{tr } Q$: since

$$p(Q) = Q^2 + \underbrace{c_1}_{-\text{tr } Q} Q + \underbrace{\det(Q)}_{=\pm 1} \mathbb{1} = 0, \quad (Q \in \text{O}(2))$$

we find $Q^2 = \text{tr}(Q)Q \mp \mathbb{1}$ and thus $\text{tr } Q^2 = \text{tr}(\text{tr}(Q)Q \mp \mathbb{1}) = (\text{tr } Q)^2 \mp 2$. Thus, finally, we have

$$\chi(Q) = (\text{tr } Q)^2 \mp 1, \quad (81)$$

where the sign choice is dictated by

$$\begin{cases} -1 & \text{if } Q \in \text{SO}(2), \\ 1 & \text{if } Q \in \text{O}^-(2). \end{cases}$$

The closed subgroups of $\text{O}(2)$ we account for are \mathbb{Z}_2, D_2, D_4 and $\text{O}(2)$.

O(2)-action: Consider

$$\text{O}(2) = \text{SO}(2) \cup \text{O}_-(2) \simeq \text{SO}(2) \times \underbrace{\{\mathbb{1}, R\}}_{\simeq \mathbb{Z}_2}$$

where

$$\text{SO}(2) = \left\{ \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \mid \theta \in [0, 2\pi) \right\} \simeq \text{U}(1) \simeq \mathbb{S}^1 \quad \text{and} \quad \text{O}_-(2) = \left\{ \begin{pmatrix} \cos \theta & \sin \theta \\ \sin \theta & -\cos \theta \end{pmatrix} \mid \theta \in [0, 2\pi) \right\}$$

and $R = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ is the reflection across the x_2 -axis. It acts on the space of the symmetric matrices as follows:

$$\varphi : \text{O}(2) \times \text{Sym}(2) \rightarrow \text{Sym}(2), \quad (Q, \mathbb{L}) \mapsto \varphi(Q, \mathbb{L}), \quad \text{where} \quad (\varphi(Q, \mathbb{L}))_{ij} = Q_{ia}Q_{jb}\mathbb{L}_{ab}.$$

To be invariant under the accounted action means that

$$\mathbb{L} \in \text{Fix}_{\text{O}(2)}^{\varphi} \text{Sym}(2), \quad \text{i.e.} \quad \varphi(Q, \mathbb{L}) = \mathbb{L} \quad \forall Q \in \text{O}(2).$$

In this case,

$$\begin{aligned} \dim \text{Fix}_{\text{O}(2)}^{\varphi} \text{Sym}(2) &= \int_{\text{O}(2)} \chi(Q) d\mu = \int_{\text{SO}(2) \cup \text{O}_-(2)} \chi(Q) d\mu \\ (\text{normalisation}) &= \frac{1}{2} \left(\int_{\text{SO}(2)} \chi(Q) d\mu + \int_{\text{SO}(2)} \underbrace{\chi(RQ)}_{\equiv 1} d\mu \right) = \frac{1}{2} \left(\frac{1}{2\pi} \int_0^{2\pi} (4 \cos^2 \theta - 1) d\theta + 1 \right) = 1. \end{aligned} \quad (82)$$

⁷Keep in mind that

$$\text{tr } Q^2 = \langle \mathbb{1}, Q^2 \rangle_{\mathbb{R}^2} = \delta_{ij}(Q^2)_{ij} = \delta_{ij}Q_{i\alpha}Q_{\alpha j} = \delta_{ij}Q_{i\alpha}\delta_{\alpha\beta}Q_{\beta j}. \quad (79)$$

Let us determine the structure of the admissible constitutive tensors. We have

$$\mathcal{P}_{\text{O}(2)}(\mathbb{L}) := \frac{1}{|\text{O}(2)|} \int_{\text{O}(2)} \varphi(Q, \mathbb{L}) \, d\mu = \frac{1}{2} (\mathbb{L}_{11} + \mathbb{L}_{22}) \mathbf{1}. \quad (83)$$

Thus, in the isotropic planar case, the curvature tensor reduces to $\mathbb{L} = l\mathbf{1} = l \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, with $l \in \mathbb{R}_*^+$ (since \mathbb{L} is positive-definite, i.e. $\mathbb{L} \in \text{Sym}^+(2)$, hence $\mathbb{L}_{11}, \mathbb{L}_{22} > 0$, thus $l = 1/2(\mathbb{L}_{11} + \mathbb{L}_{22}) > 0$).

N.B. In this case, to be able to integrate over the full group $\text{O}(2)$, which has two connected components, we use the following facts: since $\text{SO}(2)$ is a closed subgroup of $\text{O}(2)$ and $\text{SO}(2)$ is compact, $\Delta_{\text{O}(2)}|_{\text{SO}(2)} \equiv \Delta_{\text{SO}(2)}$ (modular functions) and hence, there exists an invariant Radon measure ν on the quotient $\text{O}(2)/\text{SO}(2)$ (in this case the normalized counting measure) such that

$$\begin{aligned} \int_{\text{O}(2)} f(A) \, d\mu &= \int_{\text{O}(2)/\text{SO}(2)} \int_{\text{SO}(2)} f(AQ) \, d\mu \, d\nu = \frac{1}{|\text{O}(2)/\text{SO}(2)|} \sum_{[A] \in \text{O}(2)/\text{SO}(2)} \int_{\text{SO}(2)} f(AQ) \, d\mu \\ &= \frac{1}{2} \left(\int_{\text{SO}(2)} f(\mathbf{1}Q) \, d\mu + \int_{\text{SO}(2)} f(RQ) \, d\mu \right). \end{aligned} \quad (84)$$

D_4 -action:  Consider the dihedral group⁸ of order four, D_4 whose cardinality is 8. We have

$$\dim \text{Fix}_{D_4}^\varphi \text{Sym}(2) = \frac{1}{|D_4|} \sum_{Q \in D_4} \chi(Q) = \frac{1}{|D_4|} \left[\sum_{Q \in \mathbb{Z}_4} \chi(Q) + \sum_{Q \in R\mathbb{Z}_4} \chi(Q) \right] = \frac{1}{8} \left[\sum_{k \in \{1,2,3,4\}} \left(4 \cos^2 \frac{k\pi}{2} - 1 \right) + 4 \right] = 1. \quad (85)$$

Hence, also in this case, we have the reduction $\mathbb{L} = l\mathbf{1} = l \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, with $l \in \mathbb{R}_*^+$.

D_2 -action:  Consider the dihedral group D_2 whose cardinality is 4. We have

$$\dim \text{Fix}_{D_2}^\varphi \text{Sym}(2) = \frac{1}{|D_2|} \sum_{Q \in D_2} \chi(Q) = \frac{1}{|D_2|} \left[\sum_{Q \in \mathbb{Z}_2} \chi(Q) + \sum_{Q \in R\mathbb{Z}_2} \chi(Q) \right] = \frac{1}{4} \left[2 + \sum_{k \in \{1,2\}} \left(4 \cos^2 k\pi - 1 \right) \right] = 2. \quad (86)$$

Therefore, we have the reduction $\mathbb{L} = \begin{pmatrix} l_1 & 0 \\ 0 & l_2 \end{pmatrix}$, $l_1, l_2 > 0$.

\mathbb{Z}_2 -action:  Consider the cyclic group \mathbb{Z}_2 whose cardinality is 2. We have

$$\dim \text{Fix}_{\mathbb{Z}_2}^\varphi \text{Sym}(2) = \frac{1}{|\mathbb{Z}_2|} \sum_{Q \in \mathbb{Z}_2} \chi(Q) = \frac{1}{2} \left[\sum_{k \in \{1,2\}} \left(4 \cos^2 k\pi - 1 \right) \right] = 3 = \dim \text{Sym}(2) \quad (87)$$

⁸A dihedral group of order n describes the $2n$ different symmetries of a regular polygon with n sides: n rotational symmetries and n reflection symmetries. The n rotational symmetries are the elements of the cyclic group

$$\mathbb{Z}_n = \left\{ \begin{pmatrix} \cos \frac{2\pi}{k} & -\sin \frac{2\pi}{k} \\ \sin \frac{2\pi}{k} & \cos \frac{2\pi}{k} \end{pmatrix} \mid k \in \{1, \dots, n\} \right\}$$

and

$$D_n \simeq \mathbb{Z}_n \times \underbrace{\{\mathbf{1}, R\}}_{\simeq \mathbb{Z}_2},$$

where R is the reflection $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$.

meaning that $\text{Sym}(2) = \text{Fix}_{\mathbb{Z}_2}^{\varphi} \text{Sym}(2)$, i.e., $\mathcal{P}_{\mathbb{Z}_2}(\mathbb{L}) = \mathbb{L} = \begin{pmatrix} l_1 & l_3 \\ l_3 & l_2 \end{pmatrix}$ for all $\mathbb{L} \in \text{Sym}(2)$.

3.4.4 Symmetrization of \mathbb{C}_e and $\mathbb{C}_{\text{micro}}$

The symmetrization identity in this case is

$$\Pi : \mathbb{R}^{2 \times 2 \times 2 \times 2} \rightarrow \mathfrak{gl}(2), \quad \Pi = \text{sym}(\pi \otimes \pi),$$

giving component-wise

$$\begin{aligned} \Pi_{iajblcdk} = & \frac{1}{8} (\delta_{ia}\delta_{jb}\delta_{kc}\delta_{ld} + \delta_{ia}\delta_{jb}\delta_{lc}\delta_{kd} + \delta_{ja}\delta_{ib}\delta_{lc}\delta_{kd} + \delta_{ja}\delta_{ib}\delta_{kc}\delta_{ld} \\ & + \delta_{ka}\delta_{lb}\delta_{ic}\delta_{jd} + \delta_{ka}\delta_{lb}\delta_{jc}\delta_{id} + \delta_{la}\delta_{kb}\delta_{jc}\delta_{id} + \delta_{la}\delta_{kb}\delta_{ic}\delta_{jd}). \end{aligned} \quad (88)$$

The characters can be computed via $\chi(Q) = \langle \varphi_Q, \pi \rangle = \langle Q^{\otimes 4}, \Pi \rangle_{\otimes^8 \mathbb{R}^2}$ giving

$$\chi(Q) = \frac{1}{8} [(\text{tr } Q)^4 + 2(\text{tr } Q)^2 \text{tr } Q^2 + 2 \text{tr } Q^4 + 3(\text{tr } Q^2)^2]. \quad (89)$$

The Cayley-Hamilton theorem allows us to express $\chi(Q)$ as a polynomial in $\text{tr } Q$.⁹ From eq. (81) we already know that $Q^2 = \text{tr}(Q)Q \mp \mathbb{1}$ and that $\text{tr } Q^2 = \text{tr}(\text{tr}(Q)Q \mp \mathbb{1}) = (\text{tr } Q)^2 \mp 2$. Hence we need to find the expression of $\text{tr } Q^4$ as a functions of the powers of $\text{tr } Q$:

$$\begin{aligned} Q^3 &= QQ^2 = Q(\text{tr}(Q)Q \mp \mathbb{1}) = \text{tr}(Q)Q^2 \mp Q, \\ \implies \text{tr } Q^3 &= \text{tr}(\text{tr}(Q)Q^2 \mp Q) = \text{tr } Q \text{tr } Q^2 \mp \text{tr } Q = \text{tr } Q ((\text{tr } Q)^2 \mp 2) \mp \text{tr } Q = \text{tr } Q ((\text{tr } Q)^2 \mp 3), \\ Q^4 &= QQ^3 = Q(\text{tr}(Q)Q^2 \mp Q) = \text{tr}(Q)Q^3 \mp Q^2, \\ \implies \text{tr } Q^4 &= \text{tr}(\text{tr}(Q)Q^3 \mp Q^2) = \text{tr } Q \text{tr } Q^3 \mp \text{tr } Q^2 = (\text{tr } Q)^2 ((\text{tr } Q)^2 \mp 3) \mp ((\text{tr } Q)^2 \mp 2) = (\text{tr } Q)^4 \mp 4(\text{tr } Q)^2 + 2. \end{aligned}$$

Hence we obtain

$$\begin{aligned} \chi(Q) &= \frac{1}{8} [(\text{tr } Q)^4 + 2(\text{tr } Q)^2 \text{tr } Q^2 + 2 \text{tr } Q^4 + 3(\text{tr } Q^2)^2] \\ &= \frac{1}{8} \left[(\text{tr } Q)^4 + 2(\text{tr } Q)^2 ((\text{tr } Q)^2 \mp 2) + 2 ((\text{tr } Q)^4 \mp 4(\text{tr } Q)^2 + 2) + 3 ((\text{tr } Q)^2 \mp 2)^2 \right] \\ &= \frac{1}{8} \left[(\text{tr } Q)^4 + 2(\text{tr } Q)^4 \mp 4(\text{tr } Q)^2 + 2(\text{tr } Q)^4 \mp 8(\text{tr } Q)^2 + 4 + 3 ((\text{tr } Q)^4 + 4 \mp 4(\text{tr } Q)^2) \right] \\ &= \frac{1}{8} \left[(\text{tr } Q)^4 + 2(\text{tr } Q)^4 \mp 4(\text{tr } Q)^2 + 2(\text{tr } Q)^4 \mp 8(\text{tr } Q)^2 + 4 + 3(\text{tr } Q)^4 + 12 \mp 12(\text{tr } Q)^2 \right] \\ &= \frac{1}{8} \left[8(\text{tr } Q)^4 \mp 24(\text{tr } Q)^2 + 16 \right] = (\text{tr } Q)^4 \mp 3(\text{tr } Q)^2 + 2, \end{aligned}$$

i.e.

$$\chi(Q) = \begin{cases} (\text{tr } Q)^4 - 3(\text{tr } Q)^2 + 2 & \text{if } Q \in SO(2), \\ (\text{tr } Q)^4 + 3(\text{tr } Q)^2 + 2 & \text{if } Q \in O^-(2). \end{cases}$$

D_4 -action: Consider the dihedral group D_4 whose cardinality is 8. Then

$$\begin{aligned} \dim \text{Fix}_{C_{4v}}^{\varphi} \mathfrak{gl}(2) &= \frac{1}{|C_{4v}|} \sum_{Q \in C_{4v}} \chi(Q) = \frac{1}{|C_{4v}|} \left[\sum_{Q \in C_4} \chi(Q) + \sum_{Q \in RC_4} \chi(Q) \right] \\ &= \frac{1}{8} \sum_{k \in \{1,2,3,4\}} \left[2(\cos(\pi k) + \cos(2\pi k) + 1) + 2 \right] = 3. \end{aligned}$$

⁹Avoiding explicit powers of Q reduces the computational cost of evaluating $\chi(Q)$.

The structure of the symmetrized tensors is obtained via

$$\mathcal{P}_{D_4}(\mathbb{C}) = \frac{1}{|D_4|} \sum_{Q \in D_4} \varphi(Q, \mathbb{C}) = \frac{1}{|D_4|} \sum_{Q \in \mathbb{Z}_4} \left(\varphi(Q, \mathbb{C}) + \varphi(QR, \mathbb{C}) \right), \quad (90)$$

giving

$$\begin{pmatrix} \tilde{\mathbb{C}}_{11} & \tilde{\mathbb{C}}_{12} & 0 \\ \tilde{\mathbb{C}}_{12} & \tilde{\mathbb{C}}_{11} & 0 \\ 0 & 0 & \tilde{\mathbb{C}}_{22} \end{pmatrix} \quad (91)$$

after considering a suitable Voigt isomorphism.

3.5 2D-micromorphic model

The major difference with the relaxed micromorphic model is the curvature term. Indeed, in the classical Eringen-Mindlin micromorphic model, we have that

$$W_{\text{curv}}(DP) := \frac{1}{2} \langle \mathfrak{L} DP, DP \rangle_{\mathbb{R}^{2 \times 2 \times 2}}, \quad (92)$$

where

$$\mathfrak{L} \in \text{Sym}^+(\otimes^3 \mathbb{R}^2, \otimes^3 \mathbb{R}^2) \subseteq \text{Sym}(\otimes^3 \mathbb{R}^2, \otimes^3 \mathbb{R}^2)$$

with the latter being a vector space of dimension 36. Moreover, the classical micromorphic model would allow for additional mixed terms as $\langle \text{sym}(Du - P), \text{sym} P \rangle$. We set

$$\mathfrak{H}\text{igh}(2) := \text{Sym}(\otimes^3 \mathbb{R}^2, \otimes^3 \mathbb{R}^2) \quad \text{and} \quad \mathfrak{H}\text{igh}^+(2) := \text{Sym}^+(\otimes^3 \mathbb{R}^2, \otimes^3 \mathbb{R}^2).$$

The only interesting term is the curvature term. In this case we remind that

$$P \in \mathbb{R}^{2 \times 2}, \quad DP \in \otimes^3 \mathbb{R}^2, \quad \mathfrak{L} \in \mathfrak{H}\text{igh}^+(2) := \text{Sym}^+(\otimes^3 \mathbb{R}^2, \otimes^3 \mathbb{R}^2).$$

The resultant action of $O(2)$ is

$$\varphi : O(2) \times \mathfrak{H}\text{igh}(2) \rightarrow \mathfrak{H}\text{igh}(2), \quad (Q, \mathfrak{L}) \mapsto \varphi(Q, \mathfrak{L}), \quad \text{where} \quad (\varphi(Q, \mathfrak{L}))_{ijklmn} = Q_{ia} Q_{jb} Q_{kc} Q_{ld} Q_{me} Q_{nf} \mathfrak{L}_{abcdef},$$

and the symmetrization identity is

$$\Pi^{\mathfrak{H}\text{igh}(2)} : \underbrace{(\otimes^3 \mathbb{R}^2) \otimes (\otimes^3 \mathbb{R}^2)}_{=: \otimes^6 \mathbb{R}^2} \rightarrow \underbrace{\text{Sym}(\otimes^3 \mathbb{R}^2, \otimes^3 \mathbb{R}^2)}_{=: \mathfrak{H}\text{igh}(2)}, \quad \Pi_{aijbchdkemfn}^{\mathfrak{H}\text{igh}(2)} = \frac{1}{2} (\delta_{ai} \delta_{bj} \delta_{ch} \delta_{dk} \delta_{em} \delta_{fn} + \delta_{ak} \delta_{bm} \delta_{cn} \delta_{di} \delta_{ej} \delta_{fh}).$$

The characters can be computed via $\chi(Q) = \text{tr}(\varphi_Q \circ \Pi^{\mathfrak{H}\text{igh}(2)}) = \langle \varphi_Q \circ \Pi^{\mathfrak{H}\text{igh}(2)}, \otimes^6 \mathbf{1}_2 \rangle_{\otimes^6 \mathbb{R}^2}$, giving

$$\begin{aligned} \chi(Q) &= (\delta_{ai} \delta_{bj} \delta_{ch} \delta_{dk} \delta_{em} \delta_{fn} + \delta_{ak} \delta_{bm} \delta_{cn} \delta_{di} \delta_{ej} \delta_{fh}) Q_{ia} Q_{jb} Q_{kc} Q_{kd} Q_{me} Q_{nf} \\ &= \frac{1}{2} \left(\delta_{ai} \delta_{bj} \delta_{ch} \delta_{dk} \delta_{em} \delta_{fn} Q_{ia} Q_{jb} Q_{kc} Q_{kd} Q_{me} Q_{nf} + \delta_{ak} \delta_{bm} \delta_{cn} \delta_{di} \delta_{ej} \delta_{fh} Q_{ia} Q_{jb} Q_{kc} Q_{kd} Q_{me} Q_{nf} \right) \\ &= \frac{1}{2} \left(\underbrace{\delta_{ia} Q_{ia}}_{\text{tr } Q} \underbrace{\delta_{jb} Q_{jb}}_{\text{tr } Q} \underbrace{\delta_{hc} Q_{hc}}_{\text{tr } Q} \underbrace{\delta_{kd} Q_{kd}}_{\text{tr } Q} \underbrace{\delta_{me} Q_{me}}_{\text{tr } Q} \underbrace{\delta_{nf} Q_{nf}}_{\text{tr } Q} + \underbrace{\delta_{ak} Q_{kd} \delta_{di} Q_{ia}}_{\text{tr } Q^2} \underbrace{\delta_{bm} Q_{me} \delta_{ej} Q_{jb}}_{\text{tr } Q^2} \underbrace{\delta_{cn} Q_{nf} \delta_{fh} Q_{hc}}_{\text{tr } Q^2} \right) \\ &= \frac{1}{2} \left((\text{tr } Q)^6 + (\text{tr } Q^2)^3 \right). \end{aligned} \quad (93)$$

Via the obtained formula $\text{tr } Q^2 = (\text{tr } Q)^2 \mp 2$ we find

$$(\text{tr } Q^2)^3 = ((\text{tr } Q)^2 \mp 2)^3 = (\text{tr } Q)^6 \mp 6(\text{tr } Q)^4 + 12(\text{tr } Q)^2 \mp 8 \quad (94)$$

and thus

$$\chi(Q) = \frac{1}{2} \left((\text{tr } Q)^6 + (\text{tr } Q)^6 \mp 6(\text{tr } Q)^4 + 12(\text{tr } Q)^2 \mp 8 \right) = (\text{tr } Q)^6 \mp 3(\text{tr } Q)^4 + 6(\text{tr } Q)^2 \mp 4. \quad (95)$$

D_4 -action: Concerning the number of independent components, we have

$$\begin{aligned} \dim \text{Fix}_{D_4}^{\varphi} \mathfrak{High}(2) &= \frac{1}{|D_4|} \sum_{Q \in D_4} \chi(Q) = \frac{1}{|D_4|} \left[\sum_{Q \in \mathbb{Z}_4} \chi(Q) + \sum_{Q \in R\mathbb{Z}_4} \chi(Q) \right] \\ &= \frac{1}{8} \sum_{k \in \{1,2,3,4\}} \left[2(6 + 9 \cos(k\pi) + 3 \cos(2k\pi) + \cos(3k\pi)) + 2 \right] = 10. \end{aligned}$$

The symmetrized tensor $\mathcal{P}_{D_4}(\mathfrak{L})$ has the structure

$$\left(\begin{array}{cccc|cccc} \tilde{\mathfrak{L}}_{11} & \tilde{\mathfrak{L}}_{12} & \tilde{\mathfrak{L}}_{13} & \tilde{\mathfrak{L}}_{14} & & & & & \\ & \tilde{\mathfrak{L}}_{22} & \tilde{\mathfrak{L}}_{23} & \tilde{\mathfrak{L}}_{24} & & & & & \\ \text{sym} & & \tilde{\mathfrak{L}}_{33} & \tilde{\mathfrak{L}}_{34} & & & & & \\ & & & \tilde{\mathfrak{L}}_{44} & & & & & \\ \hline & & & & \tilde{\mathfrak{L}}_{11} & \tilde{\mathfrak{L}}_{12} & \tilde{\mathfrak{L}}_{13} & \tilde{\mathfrak{L}}_{14} & \\ & & & & \tilde{\mathfrak{L}}_{22} & \tilde{\mathfrak{L}}_{23} & \tilde{\mathfrak{L}}_{24} & & \\ & 0 & & & \text{sym} & & \tilde{\mathfrak{L}}_{33} & \tilde{\mathfrak{L}}_{34} & \\ & & & & & & & \tilde{\mathfrak{L}}_{44} & \end{array} \right)$$

after considering a suitable Voigt isomorphism. By directly comparing the two curvature terms for the D_4 case, one for the relaxed micromorphic model and the one for the classical micromorphic model, we can observe that accounting for $\text{Curl } P$ as a curvature term significantly reduces the number of elastic moduli. This simplification is advantageous when attempting to characterize them. In this scenario, we also provide the explicit expression for the energy density:

$$W_{\text{Eringen}}(Du, P, DP) = \underbrace{\frac{1}{2} \langle \mathbb{C}_e \text{sym}(Du - P), \text{sym}(Du - P) \rangle}_{W_e(Du, P)} + \underbrace{\frac{1}{2} \langle \mathbb{C}_{\text{micro}} \text{sym } P, \text{sym } P \rangle}_{W_{\text{micro}}(P)} + \underbrace{\frac{1}{2} \mu L_c^2 \langle \mathfrak{L}DP, DP \rangle}_{W_{\text{curv}}(DP)}$$

where

$$\begin{aligned} W_{\text{curv}}(DP) &= \frac{1}{2} \mu L_c^2 \langle \mathfrak{L}DP, DP \rangle_{\mathbb{R}^2 \otimes \mathbb{R}^2 \otimes \mathbb{R}^2} \\ &= \frac{1}{2} \mu L_c^2 \left[(P_{11,1}^2 + P_{22,2}^2) \tilde{\mathfrak{L}}_{11} + 2(P_{11,1} P_{12,2} + P_{21,1} P_{22,2}) \tilde{\mathfrak{L}}_{12} + 2P_{11,1} P_{22,1} \tilde{\mathfrak{L}}_{13} \right. \\ &\quad + 2P_{11,2} P_{22,2} \tilde{\mathfrak{L}}_{13} + 2P_{11,1} P_{21,2} \tilde{\mathfrak{L}}_{14} + 2P_{12,1} P_{22,2} \tilde{\mathfrak{L}}_{14} + (P_{12,2}^2 + P_{21,1}^2) \tilde{\mathfrak{L}}_{22} \\ &\quad + 2P_{11,2} P_{21,1} \tilde{\mathfrak{L}}_{23} + 2P_{12,2} P_{22,1} \tilde{\mathfrak{L}}_{23} + 2P_{12,1} P_{21,1} \tilde{\mathfrak{L}}_{24} + 2P_{12,2} P_{21,2} \tilde{\mathfrak{L}}_{24} \\ &\quad \left. + P_{11,2}^2 \tilde{\mathfrak{L}}_{33} + P_{22,1}^2 \tilde{\mathfrak{L}}_{33} + 2P_{11,2} P_{12,1} \tilde{\mathfrak{L}}_{34} + 2P_{21,2} P_{22,1} \tilde{\mathfrak{L}}_{34} + (P_{12,1}^2 + P_{21,2}^2) \tilde{\mathfrak{L}}_{44} \right]. \end{aligned}$$

D_2 -action: Concerning the number of independent components, we have

$$\dim \text{Fix}_{D_2}^{\varphi} \mathfrak{High}(2) = \frac{1}{|D_2|} \sum_{Q \in D_2} \chi(Q) = \frac{1}{|D_2|} \left[\sum_{Q \in \mathbb{Z}_2} \chi(Q) + \sum_{Q \in R\mathbb{Z}_2} \chi(Q) \right] = 20.$$

The symmetrized tensor $\mathcal{P}_{D_2}(\mathcal{L})$ has the structure

$$\left(\begin{array}{cccc|cccc} \tilde{\mathcal{L}}_{11} & \tilde{\mathcal{L}}_{12} & \tilde{\mathcal{L}}_{13} & \tilde{\mathcal{L}}_{14} & 0 & 0 & 0 & 0 \\ & \tilde{\mathcal{L}}_{22} & \tilde{\mathcal{L}}_{23} & \tilde{\mathcal{L}}_{24} & 0 & 0 & 0 & 0 \\ \text{sym} & & \tilde{\mathcal{L}}_{33} & \tilde{\mathcal{L}}_{34} & 0 & 0 & 0 & 0 \\ & & & \tilde{\mathcal{L}}_{44} & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & \tilde{\mathcal{L}}_{55} & \tilde{\mathcal{L}}_{56} & \tilde{\mathcal{L}}_{57} & \tilde{\mathcal{L}}_{58} \\ 0 & 0 & 0 & 0 & & \tilde{\mathcal{L}}_{66} & \tilde{\mathcal{L}}_{67} & \tilde{\mathcal{L}}_{68} \\ 0 & 0 & 0 & 0 & \text{sym} & & \tilde{\mathcal{L}}_{77} & \tilde{\mathcal{L}}_{78} \\ 0 & 0 & 0 & 0 & & & & \tilde{\mathcal{L}}_{88} \end{array} \right)$$

after introducing a suitable Voigt isomorphism.

\mathbb{Z}_2 -**action:** Concerning the number of independent components, we have

$$\dim \text{Fix}_{\mathbb{Z}_2}^{\varphi} \text{Nigh}(2) = \frac{1}{|\mathbb{Z}_2|} \sum_{Q \in \mathbb{Z}_2} \chi(Q) = 36.$$

The symmetrized tensor $\mathcal{P}_{\mathbb{Z}_2}(\mathcal{L})$ has the structure

$$\left(\begin{array}{cccc|cccc} \tilde{\mathcal{L}}_{11} & \tilde{\mathcal{L}}_{12} & \tilde{\mathcal{L}}_{13} & \tilde{\mathcal{L}}_{14} & \tilde{\mathcal{L}}_{15} & \tilde{\mathcal{L}}_{16} & \tilde{\mathcal{L}}_{17} & \tilde{\mathcal{L}}_{18} \\ & \tilde{\mathcal{L}}_{22} & \tilde{\mathcal{L}}_{23} & \tilde{\mathcal{L}}_{24} & \tilde{\mathcal{L}}_{25} & \tilde{\mathcal{L}}_{26} & \tilde{\mathcal{L}}_{27} & \tilde{\mathcal{L}}_{28} \\ \text{sym} & & \tilde{\mathcal{L}}_{33} & \tilde{\mathcal{L}}_{34} & \tilde{\mathcal{L}}_{35} & \tilde{\mathcal{L}}_{36} & \tilde{\mathcal{L}}_{37} & \tilde{\mathcal{L}}_{38} \\ & & & \tilde{\mathcal{L}}_{44} & \tilde{\mathcal{L}}_{45} & \tilde{\mathcal{L}}_{46} & \tilde{\mathcal{L}}_{47} & \tilde{\mathcal{L}}_{48} \\ \hline & & & & \tilde{\mathcal{L}}_{55} & \tilde{\mathcal{L}}_{56} & \tilde{\mathcal{L}}_{57} & \tilde{\mathcal{L}}_{58} \\ & & & & & \tilde{\mathcal{L}}_{66} & \tilde{\mathcal{L}}_{67} & \tilde{\mathcal{L}}_{68} \\ & & & & \text{sym} & & \tilde{\mathcal{L}}_{77} & \tilde{\mathcal{L}}_{78} \\ & & & & & & & \tilde{\mathcal{L}}_{88} \end{array} \right)$$

after introducing a suitable Voigt isomorphism.

Conclusions

We applied the fundamental theoretical framework developed in the work by Danescu [17] to determine the number of independent components and the representations of tensor classes involved in generalized continuum models. This approach offers several advantages, primarily its simplicity and clarity in implementation.

It is worth noting that in recent years, various research directions [1, 4, 22, 23, 54, 55] have emerged with the goal of establishing a priori the permissible symmetry classes for a given category of tensors, building upon and extending the findings presented in previous works [28].

Appendix

A Haar integration

We want to show how it is possible to built up the Haar measure on the Lie group $\text{SO}(3)$ directly using the Cardan's angles parametrization. To facilitate a better understanding of how the Haar measure operates, we first provide

some classical examples. The general procedure, which we will present in Appendix A.1, will extend beyond the initial examples.

Haar measure on the multiplicative group of positive reals: Consider the multiplicative group (\mathbb{R}^+, \cdot) . The Lebesgue measure λ is not invariant with respect to the multiplication of real numbers but, instead of λ , we can define the measure

$$\lambda_*(A) := \int_{x \in A} \frac{1}{x} d\lambda(x), \quad \text{for all } A \in \mathfrak{B}(\mathbb{R}^+), \quad (96)$$

where $\mathfrak{B}(\mathbb{R}^+)$ denotes the σ -algebra of Borel sets. If $A = [a, b]$ with $a, b > 0$, we can explicitly calculate the introduced measure:

$$\lambda_*([a, b]) := \int_a^b \frac{1}{x} d\lambda(x) = \log b - \log a.$$

To verify that λ_* is indeed a Haar measure, we can simply show the invariance with respect to the left multiplication: using a simple change of variable ($y = x_0 x \rightsquigarrow d\lambda(y) = x_0 d\lambda(x)$), we find

$$\lambda_*(x_0 A) = \int_{y \in x_0 A} \frac{1}{y} d\lambda(y) = \int_{x \in A} \frac{x_0 d\lambda(x)}{x_0 x} = \lambda_*(A),$$

and, in the specific example in which $A = [a, b]$ with $a, b > 0$,

$$\lambda_*([x_0 a, x_0 b]) = \log(x_0 b) - \log(x_0 a) = \log x_0 + \log b - \log x_0 - \log a = \log b - \log a = \lambda_*([a, b]).$$

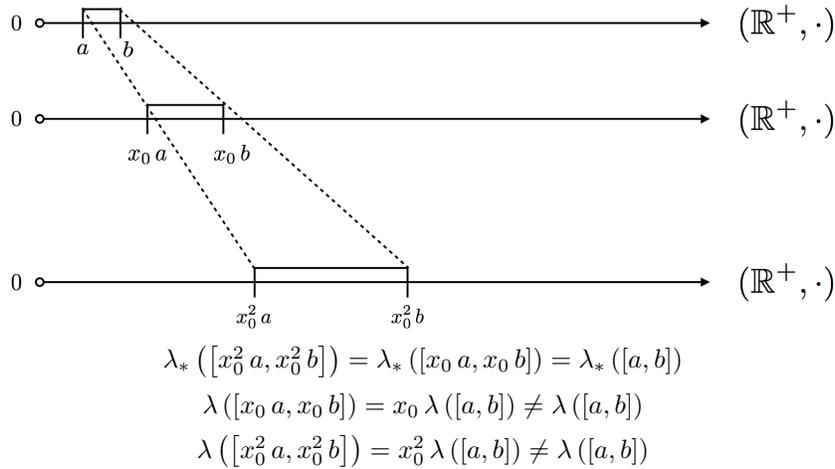


Figure 5: Difference between Lebesgue and Haar measures on (\mathbb{R}^+, \cdot)

Haar measure on the multiplicative group of the invertible matrices: Now, consider the multiplicative group $(\text{GL}^+(n), \cdot)$, where $\text{GL}^+(n)$ is the open subset of $\mathbb{R}^{n \times n}$ constituted by all matrices with positive determinant. The Lebesgue measure $\lambda^{n \times n}(M)$ on $\mathbb{R}^{n \times n}$ is not invariant with respect to the multiplication between matrices, but, similarly to \mathbb{R}^+ , we can define the measure

$$\lambda_*^{n \times n}(A) := \int_{M \in A} \frac{1}{\det^n M} d\lambda^{n \times n}(M) \quad \text{for } A \text{ measurable.} \quad (97)$$

Thanks to the change of variables formula, we can again verify that the introduced measure (97) is a Haar measure:

$$\lambda_*^{n \times n}(M_0 A) = \int_{N \in M_0 A} \frac{1}{(\det N)^n} d\lambda^{n \times n}(N) \quad (\text{Change of variables: } N = M_0 M \rightsquigarrow d\lambda^{n \times n}(N) = (\det M_0)^n d\lambda^{n \times n}(M))$$

$$= \int_{M \in \mathbf{A}} \frac{(\det M_0)^n \, d\lambda^{n \times n}(M)}{(\det M_0)^n (\det M)^n} = \lambda_*^{n \times n}(\mathbf{A}).$$

In order to best illustrate the meaning of the Haar measure, let us consider the simpler situation in which $n = 2$, \mathbf{A} is a four-dimensional rectangular set $[a, b] \times [a, b] \times [a, b] \times [a, b]$ and $\{\mathbf{C}_i\}_{i=1}^k$ is a partition in small 4-dimensional cubes of \mathbf{A} . Then

$$\begin{aligned} \lambda_*^{2 \times 2}(M_0 \mathbf{A}) &= \int_{N \in M_0 \mathbf{A}} \frac{1}{(\det N)^2} \, d\lambda^{2 \times 2}(N) \\ (M_i \text{ is the central point of } \mathbf{C}_i) &\approx \sum_{i=1}^k \frac{1}{(\det N_i)^2} \lambda^{2 \times 2}(M_0 \mathbf{C}_i) \stackrel{N_i = M_0 M_i}{=} \sum_{i=1}^k \frac{1}{(\det M_0)^2 (\det M_i)^2} (\det M_0)^2 \lambda^{2 \times 2}(\mathbf{C}_i) \\ &= \sum_{i=1}^k \frac{1}{(\det M_i)^2} \lambda^{2 \times 2}(\mathbf{C}_i) \approx \int_{M \in \mathbf{A}} \frac{1}{(\det M)^2} \, d\lambda^{2 \times 2}(M) = \lambda_*^{2 \times 2}(\mathbf{A}). \end{aligned}$$

Haar measure on $\mathbf{SO}(2)$: Recall that

$$\mathbf{SO}(2) = \left\{ R_\vartheta := \begin{pmatrix} \cos \vartheta & -\sin \vartheta \\ \sin \vartheta & \cos \vartheta \end{pmatrix} \mid \vartheta \in [0, 2\pi) \right\}.$$

This Lie group is compact and thus admits an unique left-invariant normalized Haar measure (see [20]). In order to explicitly obtain this measure, let us remark that every real-valued integrable function \widehat{f} on $\mathbf{SO}(2)$ can be identified with a 2π -periodic function $f_{\text{per}}: \mathbb{R} \rightarrow \mathbb{R}$ by setting

$$f_{\text{per}}(\vartheta) = \widehat{f}(R_\vartheta)$$

on $[0, 2\pi)$ and extending f_{per} to \mathbb{R} by periodicity. In this way, we can define the Haar measure μ on $\mathbf{SO}(2)$ by

$$\int_{\mathbf{SO}(2)} \widehat{f}(R_\vartheta) \, d\mu := \frac{1}{2\pi} \int_0^{2\pi} f_{\text{per}}(\vartheta) \, d\vartheta, \quad (98)$$

where ϑ is the Lebesgue measure on \mathbb{R} . In order to verify that the stated definition gives a left invariant measure, we have to show that

$$\int_{\mathbf{SO}(2)} \left(\widehat{f} \circ L_{R_{\vartheta_0}} \right) (R_\vartheta) \, d\mu = \int_{\mathbf{SO}(2)} \widehat{f}(R_\vartheta) \, d\mu \quad \forall \vartheta_0 \in \mathbb{R}.$$

Recalling that

$$R_{\vartheta_0} R_\vartheta = \begin{pmatrix} \cos \vartheta_0 & -\sin \vartheta_0 \\ \sin \vartheta_0 & \cos \vartheta_0 \end{pmatrix} \begin{pmatrix} \cos \vartheta & -\sin \vartheta \\ \sin \vartheta & \cos \vartheta \end{pmatrix} = \begin{pmatrix} \cos(\vartheta_0 + \vartheta) & -\sin(\vartheta_0 + \vartheta) \\ \sin(\vartheta_0 + \vartheta) & \cos(\vartheta_0 + \vartheta) \end{pmatrix} = R_{\vartheta_0 + \vartheta},$$

we find

$$\int_{\mathbf{SO}(2)} \left(\widehat{f} \circ L_{R_{\vartheta_0}} \right) (R_\vartheta) \, d\mu = \int_{\mathbf{SO}(2)} \widehat{f}(R_{\vartheta_0 + \vartheta}) \, d\mu = \frac{1}{2\pi} \int_0^{2\pi} f_{\text{per}}(\vartheta_0 + \vartheta) \, d\vartheta = \frac{1}{2\pi} \int_0^{2\pi} f_{\text{per}}(\vartheta) \, d\vartheta = \int_{\mathbf{SO}(2)} \widehat{f}(R_\vartheta) \, d\mu$$

for all $\vartheta_0 \in \mathbb{R}$.

Haar measure on $\mathbf{O}(2)$: Setting

$$\mathbf{O}_-(2) := \left\{ \begin{pmatrix} \cos \vartheta & \sin \vartheta \\ \sin \vartheta & -\cos \vartheta \end{pmatrix} \mid \vartheta \in [0, 2\pi) \right\} \subseteq \mathbb{R}^{2 \times 2},$$

we observe that

$$\mathbf{O}(2) = \mathbf{SO}(2) \cup \mathbf{O}_-(2).$$

The group $\mathbf{O}(2)$ is compact as well, but unlike $\mathbf{SO}(2)$ it has two disconnected components. For this reason, the idea for defining the Haar measure consists in adapting the reasoning for $\mathbf{SO}(2)$ but considering that, in this case, the “size” of the involved group is doubled. We set

$$\int_{\mathbf{O}(2)} \widehat{f}(R_\vartheta) \, d\mu = \int_{\mathbf{SO}(2)} \widehat{f}(R_\vartheta^+) \, d\mu + \int_{\mathbf{O}_-(2)} \widehat{f}(R_\vartheta^-) \, d\mu = \frac{1}{4\pi} \left(\int_0^{2\pi} f_{\text{per}}^+(\vartheta) \, d\vartheta + \int_0^{2\pi} f_{\text{per}}^-(\vartheta) \, d\vartheta \right), \quad (99)$$

where

$$f_{\text{per}}^+(\vartheta) = \widehat{f}(R_\vartheta) \text{ for } R_\vartheta \in \text{SO}(2) \quad \text{and} \quad f_{\text{per}}^-(\vartheta) = \widehat{f}(R_\vartheta) \text{ for } R_\vartheta \in \text{O}_-(2)$$

and the factor $1/4\pi$ is taken to normalize the Haar measure.

A.1 Haar measure induced by a local chart

Let \mathcal{G} be a Lie group and (\mathcal{U}, φ) a local chart with \mathcal{U} open subset of \mathbb{R}^n such that $0 \in \mathcal{U}$ and $\varphi(0) = e$. In this Appendix, following [30, Exercice 1.8 pag.32], we show how it is possible to define a Haar measure on $\varphi(\mathcal{U})$ starting from the Lebesgue measure on \mathcal{U} .

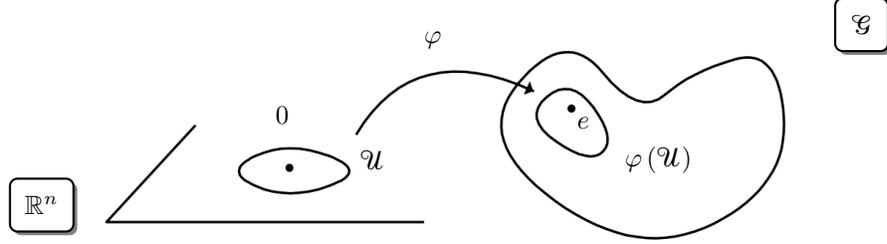


Figure 6: Local chart for a Lie group.

In order to define the Haar integral we need to introduce some auxiliary function. For every $h \in \varphi(\mathcal{U})$ we can find an open neighbourhood of the origin \mathcal{V}_h on which the function¹⁰(see Figures 7 and 8)

$$\psi_h: \mathcal{V}_h \subseteq \mathcal{U} \rightarrow \mathcal{U} \quad \psi_h := \varphi^{-1} \circ L_h \circ \varphi$$

is well defined.

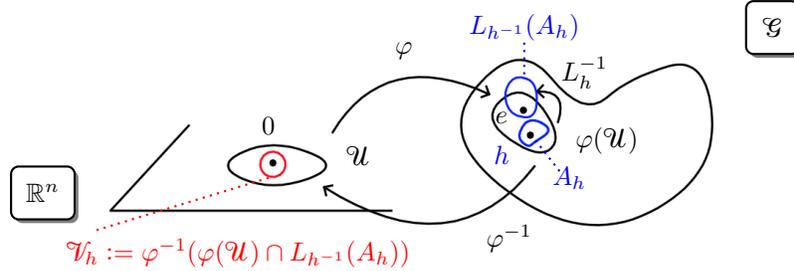


Figure 7: Construction of the neighbourhood \mathcal{V}_h (i.e. the domain of the functions ψ_h for $h \in \varphi(\mathcal{U})$) starting from an open neighbourhood A_h of the considered element h of \mathcal{G} .

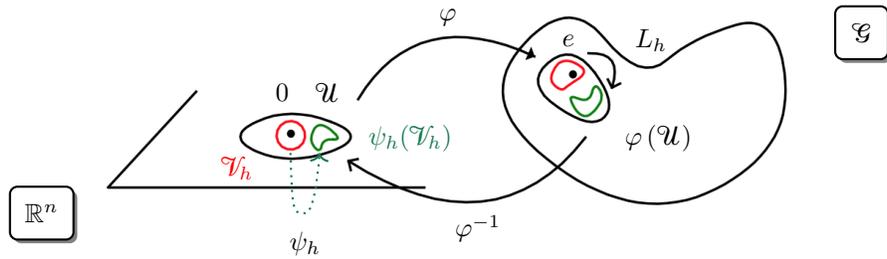


Figure 8: Construction of functions ψ_h for $h \in \varphi(\mathcal{U})$.

¹⁰Such an open neighbourhood \mathcal{V}_h of the origin always exists. Indeed, consider an open neighbourhood A_h of $h \in \varphi(\mathcal{U})$ in $\varphi(\mathcal{U})$. Then, $L_{h^{-1}}(A_h)$ is an open neighbourhood of the identity e and thus the intersection $\varphi(\mathcal{U}) \cap L_{h^{-1}}(A_h)$ is an open neighbourhood of e contained in $\varphi(\mathcal{U})$. Thus, considering $\varphi^{-1}(\varphi(\mathcal{U}) \cap L_{h^{-1}}(A_h))$, we obtain an open neighbourhood of the origin in \mathbb{R}^n on which the function ψ_h is well defined.

Setting

$$J_h := \det[\mathbf{D} \psi_h(0)] \quad \forall h \in \varphi(\mathcal{U}),$$

for every function f such that $\text{supp } f \subseteq \varphi(\mathcal{U})$ we define the desired integral in the following way:

$$\int_{\text{supp } f} f(g) \, d\mu := \int_{\varphi^{-1}(\text{supp } f)} f(\varphi(x)) \frac{1}{|\det [\mathbf{D} \psi_{\varphi(x)}(0)]|} \, dx.$$

Remark 7. The underlying concept behind this definition of the Haar integral is to adjust the Lebesgue measure on a point-by-point basis, resulting in a measure on $\varphi(\mathcal{U})$ that remains invariant under left multiplications. This pointwise correction at the position x is represented by the term $1/|J_{\varphi(x)}| = 1/|\det [\mathbf{D} \psi_{\varphi(x)}(0)]|$, which is always finite due to the fact that $\psi_{\varphi(x)}$ is a diffeomorphism, ensuring that $J_{\varphi(x)} = \det [\mathbf{D} \psi_{\varphi(x)}(0)] \neq 0$.

We want to show, with a direct calculation, that this definition is invariant with respect to left multiplications. Let us assume that we have a function f and an element $h \in \varphi(\mathcal{U})$ such that $L_{h^{-1}}(\text{supp}(f)) \subseteq \varphi(\mathcal{U})$. We have to show that

$$\int_{\text{supp } f} f(g) \, d\mu = \int_{\text{supp } f \circ L_h} (f \circ L_h)(g) \, d\mu,$$

i.e., denoting with x the points in $\varphi^{-1}(\text{supp } f)$ and with y the points in $\varphi^{-1}(\text{supp } f \circ L_h)$, that

$$\int_{\varphi^{-1}(\text{supp } f)} f(\varphi(x)) \frac{1}{|J_{\varphi(x)}|} \, dx = \int_{\varphi^{-1}(\text{supp } f \circ L_h)} (f \circ L_h)(\varphi(y)) \frac{1}{|J_{\varphi(y)}|} \, dy. \quad (100)$$

First of all, we can express y as a function of x . Indeed we have that

$$y = \psi_{h^{-1}}(x). \quad (101)$$

In this manner, we can effect a change of coordinates in the second integral of equation (100) with the hope that this transformation yields results akin to those in the first integral. Consequently, by virtue of the relation described in (101), we attain the following outcome:

- change of domain of integration: starting from the relation

$$\text{supp } (f \circ L_h) = L_{h^{-1}}(\text{supp } f), \quad (102)$$

and considering the change of variable $y = \psi_{h^{-1}}(x)$, we obtain

$$\psi_{h^{-1}}(\varphi^{-1}(\text{supp } f)) = \varphi^{-1} \circ L_{h^{-1}} \circ \varphi(\varphi^{-1}(\text{supp } f)) = (\varphi^{-1} \circ L_{h^{-1}})(\text{supp } f) = \varphi^{-1}(\text{supp } (f \circ L_h)), \quad (103)$$

- relation between differentials

$$dy = |\det [\mathbf{D} \psi_{h^{-1}}(x)]| \, dx, \quad (104)$$

- for the integrating function we have

$$f \circ L_h(\varphi(y)) = f \circ L_h(\varphi(\psi_{h^{-1}}(x))) = f \circ L_h(\varphi(\varphi^{-1} \circ L_{h^{-1}} \circ \varphi(x))) = f \circ L_h(L_{h^{-1}} \circ \varphi(x)) = f(\varphi(x)), \quad (105)$$

- and finally for the correction term

$$J_{\varphi(y)} = \det [\mathbf{D} \psi_{\varphi(y)}(0)] = \det [\mathbf{D} \psi_{\varphi(\psi_{h^{-1}}(x))}(0)] = \det [\mathbf{D} \psi_{\varphi(\varphi^{-1} \circ L_{h^{-1}} \circ \varphi(x))}(0)] = \det [\mathbf{D} \psi_{L_{h^{-1}}(\varphi(x))}(0)].$$

Remarking that

$$\begin{aligned} \psi_{L_{h^{-1}}(\varphi(x))}(0) &= (\varphi^{-1} \circ L_{L_{h^{-1}}(\varphi(x))} \circ \varphi)(0) = (\varphi^{-1} \circ L_{h^{-1}} \circ L_{\varphi(x)} \circ \varphi)(0) \\ &= (\varphi^{-1} \circ L_{h^{-1}} \circ \varphi \circ \varphi^{-1} \circ L_{\varphi(x)} \circ \varphi)(0) = (\psi_{h^{-1}} \circ \psi_{\varphi(x)})(0) \\ &= \psi_{h^{-1}}(\psi_{\varphi(x)}(0)), \end{aligned}$$

by the chain rule we have

$$\det [\mathbf{D} \psi_{L_{h^{-1}}(\varphi(x))}(0)] = \det [\mathbf{D} (\psi_{h^{-1}} \circ \psi_{\varphi(x)})(0)] = \det [\mathbf{D} \psi_{h^{-1}}(\psi_{\varphi(x)}(0)) \mathbf{D} \psi_{\varphi(x)}(0)]$$

$$= \det [\mathbf{D} \psi_{h^{-1}}(\psi_{\varphi(x)}(0))] \det [\mathbf{D} \psi_{\varphi(x)}(0)].$$

Moreover, we have also that

$$\psi_{\varphi(x)}(0) = (\varphi^{-1} \circ L_{\varphi(x)} \circ \varphi)(0) = (\varphi^{-1} \circ L_{\varphi(x)})(e) = \varphi^{-1}(\varphi(x)) = x,$$

thanks to which we finally find

$$\det [\mathbf{D} \psi_{\varphi(y)}(0)] = \det [\mathbf{D} \psi_{(L_{h^{-1}}(\varphi(x)))}(0)] = \det [\mathbf{D} \psi_{h^{-1}}(x)] \det [\mathbf{D} \psi_{\varphi(x)}(0)]. \quad (106)$$

Thus, inserting (103), (104), (105) and (106) into the second integral of (100), we obtain the desired identity. Indeed

$$\begin{aligned} & \int_{\varphi^{-1}(\text{supp } f \circ L_h)} (f \circ L_h)(\varphi(y)) \frac{1}{|\det [\mathbf{D} \psi_{\varphi(y)}(0)]|} dy \\ &= \int_{\varphi^{-1}(\text{supp } f)} f(\varphi(x)) \underbrace{\frac{1}{|\det [\mathbf{D} \psi_{h^{-1}}(x)]|} \frac{1}{|\det [\mathbf{D} \psi_{\varphi(x)}(0)]|}}_{\frac{1}{|\det [\mathbf{D} \psi_{\varphi(y)}(0)]|}} \underbrace{|\det [\mathbf{D} \psi_{h^{-1}}(x)]| dx}_{dy}. \end{aligned}$$

This technique proves especially advantageous when it is possible to cover a Lie group using a single chart, one that covers it entirely except for a subset of zero measure. In such instances, no compatibility condition arises between the support of a given function and the left translation by any arbitrary element within the group.

In the subsequent paragraph, it will be demonstrated that in the case of $\text{SO}(3)$, we precisely find ourselves in the aforementioned scenario: the local chart that covers $\text{SO}(3)$ but leaves out a subset of zero measure is defined by the Cardan angles.

A.1.1 Cardan angles

In the literature numerous local charts have been devised for the compact group¹¹ $\text{SO}(3)$. Among these, the Euler angles chart stands as one of the most prominent. However, for the specific objectives under consideration, the Euler angles chart is not the suitable choice. This is primarily due to the fact that it does not encompass the origin, as 0 lies outside its domain, and the identity element $\mathbf{1}$ is excluded from its range. Unfortunately, this incompatibility with our current approach presents a significant hurdle¹²: We are unable to directly apply our argument to derive the Haar measure representation within this chart. However, in lieu of considering this parametrization of $\text{SO}(3)$, we may opt for the one provided by the Cardan angles.

Consider the bijective map (see [24] for more details)

$$\mathcal{C} : [-\pi, \pi) \times \left[-\frac{\pi}{2}, \frac{\pi}{2}\right) \times [-\pi, \pi) \rightarrow \text{SO}(3), \quad \mathcal{C}(\phi, \theta, \psi) := R_\phi R_\theta R_\psi, \quad (107)$$

where

$$R_\phi = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \phi & \sin \phi \\ 0 & -\sin \phi & \cos \phi \end{pmatrix}, \quad R_\theta = \begin{pmatrix} \cos \theta & 0 & -\sin \theta \\ 0 & 1 & 0 \\ \sin \theta & 0 & \cos \theta \end{pmatrix}, \quad R_\psi = \begin{pmatrix} \cos \psi & -\sin \psi & 0 \\ \sin \psi & \cos \psi & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The three real numbers (ϕ, θ, ψ) are called **Cardan angles**. The map \mathcal{C} is not the desired local chart because its domain of definition is not an open subset of \mathbb{R}^3 . Its restriction to the open subset

$$\Delta := (-\pi, \pi) \times \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \times (-\pi, \pi),$$

¹¹It is known that to cover $\text{SO}(3)$ we need at least four local charts (see for example [31]).

¹²The Euler angles parametrization is provided by considering the mapping [40, pag. 497]

$$\mathcal{E} : (0, 2\pi) \times (0, \pi) \times (0, 2\pi) \rightarrow \text{SO}(3), \quad \mathcal{E}(\phi, \theta, \psi) := R_\psi R_\theta R_\phi,$$

where

$$R_\phi = \begin{pmatrix} \cos \phi & -\sin \phi & 0 \\ \sin \phi & \cos \phi & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad R_\theta = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \theta & -\sin \theta \\ 0 & \sin \theta & \cos \theta \end{pmatrix}, \quad R_\psi = \begin{pmatrix} \cos \psi & -\sin \psi & 0 \\ \sin \psi & \cos \psi & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The three real numbers (ϕ, θ, ψ) are called **Euler angles**.

however, is a chart (indeed $\mathcal{C}|_{\Delta}$ is injective and its Jacobian has always maximal rank) which covers $\text{SO}(3)$ except for a zero measure subset (the image of union of the three surfaces $\Sigma = (\{-\pi\} \times [-\frac{\pi}{2}, \frac{\pi}{2}] \times [-\pi, \pi]) \cup ([-\pi, \pi] \times \{-\frac{\pi}{2}\} \times [-\pi, \pi]) \cup ([-\pi, \pi] \times [-\frac{\pi}{2}, \frac{\pi}{2}] \times \{-\pi\})$).

This local chart possesses all the required properties to build the adapted Haar measure. In order to compute $J_{\mathcal{C}(\phi, \theta, \psi)}$, we need an explicit expression for the inverse map \mathcal{C}^{-1} : denoting by

$$\text{atan2}(y, x) = \begin{cases} \arctan \frac{y}{x} & \text{if } x > 0 \\ \arctan \frac{y}{x} + \pi & \text{if } x < 0 \text{ and } y \geq 0 \\ \arctan \frac{y}{x} - \pi & \text{if } x < 0 \text{ and } y < 0 \\ +\frac{\pi}{2} & \text{if } x = 0 \text{ and } y > 0 \\ -\frac{\pi}{2} & \text{if } x = 0 \text{ and } y < 0 \\ \text{not defined} & \text{if } x = 0 \text{ and } y = 0, \end{cases} \quad (108)$$

we have

$$\mathcal{C}^{-1}(Q) = \begin{pmatrix} \text{atan2}(Q_{23}, Q_{33}) \\ -\arcsin(Q_{13}) \\ \text{atan2}(Q_{12}, Q_{11}) \end{pmatrix} \quad \forall Q \in \mathcal{C}(\Delta).$$

In this way, the map

$$\psi_{\mathcal{C}(\phi, \theta, \psi)} : \mathbb{R}^3 \supseteq \mathcal{V} \rightarrow \mathbb{R}^3, \quad (\alpha, \beta, \gamma) \mapsto \psi_{\mathcal{C}(\phi, \theta, \psi)}(\alpha, \beta, \gamma) := (\mathcal{C}^{-1} \circ L_{\mathcal{C}(\phi, \theta, \psi)} \circ \mathcal{C})(\alpha, \beta, \gamma)$$

is given by

$$\mathcal{C}^{-1} \left[\underbrace{\mathcal{C}(\phi, \theta, \psi) \mathcal{C}(\alpha, \beta, \gamma)}_{\substack{\text{matrix product of} \\ \text{these two matrices}}} \right] = \begin{pmatrix} \text{atan2}((\mathcal{C}(\phi, \theta, \psi) \mathcal{C}(\alpha, \beta, \gamma))_{23}, (\mathcal{C}(\phi, \theta, \psi) \mathcal{C}(\alpha, \beta, \gamma))_{33}) \\ -\arcsin((\mathcal{C}(\phi, \theta, \psi) \mathcal{C}(\alpha, \beta, \gamma))_{13}) \\ \text{atan2}((\mathcal{C}(\phi, \theta, \psi) \mathcal{C}(\alpha, \beta, \gamma))_{12}, (\mathcal{C}(\phi, \theta, \psi) \mathcal{C}(\alpha, \beta, \gamma))_{11}) \end{pmatrix}.$$

Making the explicit calculation we find that

$$\frac{1}{|\det [\text{D} \psi_{\mathcal{C}(\phi, \theta, \psi)}(0, 0, 0)]|} = |\cos \theta|.$$

In this way

$$\begin{aligned} \int_{\text{SO}(3)} f(Q) d\mu &= \frac{1}{m(\Delta)} \int_{\Delta} f(\mathcal{C}(\phi, \theta, \psi)) \frac{1}{|\det [\text{D} \psi_{\mathcal{C}(\phi, \theta, \psi)}(0, 0, 0)]|} d\phi d\theta d\psi \\ &= \frac{1}{8\pi^2} \int_{-\pi}^{\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{-\pi}^{\pi} f(R_{\phi} R_{\theta} R_{\psi}) |\cos \theta| d\phi d\theta d\psi, \end{aligned}$$

and considering the linear change of variables

$$\phi' = \phi + \pi, \quad \theta' = \theta + \frac{\pi}{2}, \quad \psi' = \psi + \pi$$

we obtain the classical expression for the integration over $\text{SO}(3)$

$$\int_{\text{SO}(3)} f(Q) d\mu = \frac{1}{8\pi^2} \int_0^{2\pi} \int_0^{\pi} \int_0^{2\pi} f(R_{\phi'} R_{\theta'} R_{\psi'}) \sin \theta' d\phi' d\theta' d\psi'.$$

B Useful isomorphisms between vector spaces

B.1 Voigt isomorphism

In order to work with vectors and matrices, we define certain isomorphisms between finite-dimensional vector spaces. Let $\{e_1, e_2, e_3\}$ be the canonical basis¹³ of \mathbb{R}^3 and let $\{\epsilon_i\}_{i=1}^6$ be the canonical basis of \mathbb{R}^6 . We can consider the isomorphism $\mathfrak{M} : \text{Sym}(3) \rightarrow \mathbb{R}^6$ defined as follows:

$$\mathfrak{M}(e_1 \otimes e_1) = \epsilon_1, \quad \mathfrak{M}(e_2 \otimes e_2) = \epsilon_2, \quad \mathfrak{M}(e_3 \otimes e_3) = \epsilon_3,$$

¹³ $e_1 = (1, 0, 0)^T, e_2 = (0, 1, 0)^T, e_3 = (0, 0, 1)^T$

$$\mathfrak{M}(e_2 \otimes e_3 + e_3 \otimes e_2) = 2\epsilon_4, \quad \mathfrak{M}(e_3 \otimes e_1 + e_1 \otimes e_3) = 2\epsilon_5, \quad \mathfrak{M}(e_1 \otimes e_2 + e_2 \otimes e_1) = 2\epsilon_6.$$

Seen as a tensor $\mathfrak{M} \in \mathbb{R}^6 \otimes \mathbb{R}^{3 \times 3}$, its action on a matrix X is $(\mathfrak{M}X)_\alpha = \mathfrak{M}_{\alpha ij} X_{ij}$, where

$$\mathfrak{M}_{\alpha ij} = \tilde{\delta}_{\alpha 1} \delta_{i1} \delta_{j1} + \tilde{\delta}_{\alpha 2} \delta_{i2} \delta_{j2} + \tilde{\delta}_{\alpha 3} \delta_{i3} \delta_{j3} + \tilde{\delta}_{\alpha 4} (\delta_{i2} \delta_{j3} + \delta_{i3} \delta_{j2}) + \tilde{\delta}_{\alpha 5} (\delta_{i1} \delta_{j3} + \delta_{i3} \delta_{j1}) + \tilde{\delta}_{\alpha 6} (\delta_{i1} \delta_{j2} + \delta_{i2} \delta_{j1}),$$

and δ_{ij} is the usual Kronecker delta in \mathbb{R}^3 while $\tilde{\delta}_{\alpha\beta}$ is the Kronecker delta in \mathbb{R}^6 . Fixing the index α the components of \mathfrak{M} can be represented using the following matrices:

$$\begin{aligned} \mathfrak{M}_{1ij} &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, & \mathfrak{M}_{2ij} &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, & \mathfrak{M}_{3ij} &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \\ \mathfrak{M}_{4ij} &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, & \mathfrak{M}_{5ij} &= \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, & \mathfrak{M}_{6ij} &= \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \end{aligned}$$

In this way, given

$$X = \begin{pmatrix} X_{11} & X_{12} & X_{13} \\ X_{12} & X_{22} & X_{23} \\ X_{13} & X_{23} & X_{33} \end{pmatrix} \in \text{Sym}(3)$$

we have¹⁴

$$\mathfrak{M}X := (X_{11}, X_{22}, X_{33}, 2X_{23}, 2X_{13}, 2X_{12})^T \in \mathbb{R}^6. \quad (109)$$

The inverse map¹⁵ $\mathfrak{M}^{-1} : \mathbb{R}^6 \rightarrow \text{Sym}(3)$ is given in coordinates by

$$\mathfrak{M}_{ij\alpha}^{-1} = \tilde{\delta}_{\alpha 1} \delta_{i1} \delta_{j1} + \tilde{\delta}_{\alpha 2} \delta_{i2} \delta_{j2} + \tilde{\delta}_{\alpha 3} \delta_{i3} \delta_{j3} + \frac{1}{2} \left(\tilde{\delta}_{\alpha 4} (\delta_{i2} \delta_{j3} + \delta_{i3} \delta_{j2}) + \tilde{\delta}_{\alpha 5} (\delta_{i1} \delta_{j3} + \delta_{i3} \delta_{j1}) \right) + \frac{1}{2} \tilde{\delta}_{\alpha 6} (\delta_{i1} \delta_{j2} + \delta_{i2} \delta_{j1}),$$

and it acts by $(\mathfrak{M}^{-1}x)_{ij} = \mathfrak{M}_{ij\alpha}^{-1} x_\alpha$. From the definition of \mathfrak{M} , we can define

$$\underline{\mathfrak{M}} : \mathfrak{Ela}(3) \rightarrow \text{Sym}(\mathbb{R}^6, \mathbb{R}^6), \quad (110)$$

which associates to an elasticity tensor \mathbb{C} the second-order tensor $\tilde{\mathbb{C}} := \underline{\mathfrak{M}}(\mathbb{C})$ defined by

$$\langle \tilde{\mathbb{C}}a, b \rangle_{\mathbb{R}^6} = \langle \mathbb{C}(\mathfrak{M}^{-1}a), \mathfrak{M}^{-1}b \rangle_{\mathbb{R}^{3 \times 3}} \quad \forall a, b \in \mathbb{R}^6.$$

Component-wise, the tensor $\tilde{\mathbb{C}}$ is given by

$$\tilde{\mathbb{C}}_{\alpha\beta} = \mathbb{C}_{ijkl} \mathfrak{M}_{ij\alpha}^{-1} \mathfrak{M}_{kl\beta}^{-1}.$$

B.2 axl isomorphism

A direct isomorphism can be established between $\mathfrak{so}(3)$ (which is a vector space of dimension 3) and \mathbb{R}^3 in the following way:

$$\text{axl} : \mathfrak{so}(3) \rightarrow \mathbb{R}^3, \quad \text{axl} \begin{pmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{pmatrix} := (a_1, a_2, a_3),$$

¹⁴N.B. The Voigt isomorphism is not an isometry between the two involved spaces. If we want to consider an isometry, we should change the transformation in the following way:

$$X \mapsto (X_{11}, X_{22}, X_{33}, \sqrt{2}X_{23}, \sqrt{2}X_{13}, \sqrt{2}X_{12})^T.$$

This transformation is known in the literature as the **Mandel isomorphism**.

¹⁵I.e. the map verifying $\mathfrak{M} \circ \mathfrak{M}^{-1} = \mathbb{1}_{\mathbb{R}^{3 \times 3}}$ or component-wise $\mathfrak{M}_{\alpha ij} \circ \mathfrak{M}_{kl\beta}^{-1} = \delta_{ij} \delta_{lk}$ and $\mathfrak{M}^{-1} \circ \mathfrak{M} = \mathbb{1}_{\mathbb{R}^6}$ or component-wise $\mathfrak{M}_{ij\alpha}^{-1} \mathfrak{M}_{\beta ij} = \tilde{\delta}_{\alpha\beta}$.

which component-wise gives

$$(\mathbf{axl} A)_k = -\frac{1}{2}\epsilon_{kij} A_{ij},$$

where ϵ_{kij} is the Levi-Civita tensor. For any $v \in \mathbb{R}^3$, the identity $Av = \mathbf{axl} A \times v$ holds. Its inverse operator is

$$\mathbf{anti} : \mathbb{R}^3 \rightarrow \mathfrak{so}(3), \quad (\mathbf{anti} a)_{ij} = -\epsilon_{ijk} a_k.$$

Following the same line as done for the isomorphism \mathfrak{M} , we introduce the isomorphism

$$\underline{\mathbf{Axl}} : \text{Sym}(\mathfrak{so}(3), \mathfrak{so}(3)) \rightarrow \text{Sym}(3), \quad \mathbb{C}_c \mapsto \tilde{\mathbb{C}}_c$$

defined by

$$\langle \tilde{\mathbb{C}}_c a, b \rangle_{\mathbb{R}^3} = \langle \mathbb{C}_c \mathbf{anti} a, \mathbf{anti} b \rangle$$

Component-wise, the tensor $\tilde{\mathbb{C}}_c$ is given by

$$(\tilde{\mathbb{C}}_c)_{mn} = (\mathbb{C}_c)_{ijhk} \epsilon_{ijm} \epsilon_{hkn}.$$

B.3 Computation of characters

In order to obtain the most convenient expression of $\chi(Q)$ for the numerical computation (that which involves the less number of powers) we express, thanks to the Cayley-Hamilton Theorem the trace of the powers of a given matrix Q in term of the powers of the trace of Q . For a matrix $Q \in \mathbb{R}^{3 \times 3}$ the Hamilton-Cayley theorem states that the identity

$$Q^3 - \text{tr}(Q) Q^2 + \frac{1}{2} \left((\text{tr} Q)^2 - \text{tr} Q^2 \right) Q - \det(Q) \mathbb{1} = 0 \quad (111)$$

holds. From (111) we easily derive that

$$Q^3 = \text{tr}(Q) Q^2 - \frac{1}{2} \left((\text{tr} Q)^2 - \text{tr} Q^2 \right) Q + \mathbb{1} \iff Q^2 = \text{tr}(Q) Q - \frac{1}{2} \left((\text{tr} Q)^2 - \text{tr} Q^2 \right) \mathbb{1} + Q^T$$

for $Q \in \text{SO}(3)$, from which we obtain

$$\text{tr} Q^2 = (\text{tr} Q)^2 - \frac{3}{2} \left((\text{tr} Q)^2 - \text{tr} Q^2 \right) + \text{tr} Q \iff \text{tr} Q^2 = (\text{tr} Q)^2 - 2 \text{tr} Q. \quad (112)$$

Having the expression of $\text{tr} Q^2$ as a polynomial of the powers of $\text{tr} Q$, we can calculate the same relation also for $\text{tr} Q^3$:

$$\begin{aligned} \text{tr} Q^3 &= \text{tr} \left(\text{tr}(Q) Q^2 - \frac{1}{2} \left((\text{tr} Q)^2 - \text{tr} Q^2 \right) Q + \mathbb{1} \right) = \text{tr} Q \text{tr} Q^2 - \frac{1}{2} (\text{tr} Q)^2 \text{tr} Q + \frac{1}{2} \text{tr} Q^2 \text{tr} Q + 3 \\ &= -\frac{1}{2} (\text{tr} Q)^3 + \frac{3}{2} \text{tr} Q^2 \text{tr} Q + 3 = -\frac{1}{2} (\text{tr} Q)^3 + \frac{3}{2} \left((\text{tr} Q)^2 - 2 \text{tr} Q \right) \text{tr} Q + 3 \\ &= -\frac{1}{2} (\text{tr} Q)^3 + \frac{3}{2} (\text{tr} Q)^3 - 3 (\text{tr} Q)^2 + 3 = (\text{tr} Q)^3 - 3 (\text{tr} Q)^2 + 3. \end{aligned} \quad (113)$$

Now, observing that

$$Q^4 = QQ^3 = \text{tr}(Q) Q^3 - \frac{1}{2} \left((\text{tr} Q)^2 - \text{tr} Q^2 \right) Q^2 + Q,$$

we find

$$\begin{aligned} \text{tr} Q^4 &= \text{tr} \left(\text{tr}(Q) Q^3 - \frac{1}{2} \left((\text{tr} Q)^2 - \text{tr} Q^2 \right) Q^2 + Q \right) \\ &= \text{tr} Q \text{tr} Q^3 - \frac{1}{2} \left((\text{tr} Q)^2 - \text{tr} Q^2 \right) \text{tr} Q^2 + \text{tr} Q \\ &= \text{tr} Q \left((\text{tr} Q)^3 - 3 (\text{tr} Q)^2 + 3 \right) - \frac{1}{2} \left((\text{tr} Q)^2 - \left((\text{tr} Q)^2 - 2 \text{tr} Q \right) \right) \left((\text{tr} Q)^2 - 2 \text{tr} Q \right) + \text{tr} Q \\ &= (\text{tr} Q)^4 - 3 (\text{tr} Q)^3 + 3 \text{tr} Q - \text{tr} Q \left((\text{tr} Q)^2 - 2 \text{tr} Q \right) + \text{tr} Q \\ &= (\text{tr} Q)^4 - 3 (\text{tr} Q)^3 + 3 \text{tr} Q - (\text{tr} Q)^3 + 2 (\text{tr} Q)^2 + \text{tr} Q \\ &= (\text{tr} Q)^4 - 4 (\text{tr} Q)^3 + 2 (\text{tr} Q)^2 + 4 \text{tr} Q. \end{aligned} \quad (114)$$

Thus, thanks to relations (112), (113) and (114), we can compute the expression of the characters for all the considered elasticity frameworks as function of the powers of $\text{tr} Q$.

B.3.1 Classical elasticity

In classical elasticity framework we found

$$\chi(Q) = \frac{1}{8} [(\text{tr } Q)^4 + 2(\text{tr } Q)^2 \text{tr } Q^2 + 2 \text{tr } Q^4 + 3(\text{tr } Q^2)^2],$$

thus

$$\begin{aligned} \chi(Q) &= \frac{1}{8} [(\text{tr } Q)^4 + 2(\text{tr } Q)^2 \text{tr } Q^2 + 2 \text{tr } Q^4 + 3(\text{tr } Q^2)^2] \\ &= \frac{1}{8} [(\text{tr } Q)^4 + 2(\text{tr } Q)^2 ((\text{tr } Q)^2 - 2 \text{tr } Q) + 2((\text{tr } Q)^4 - 4(\text{tr } Q)^3 + 2(\text{tr } Q)^2 + 4 \text{tr } Q) + 3((\text{tr } Q)^2 - 2 \text{tr } Q)^2] \\ &= \frac{1}{8} [(\text{tr } Q)^4 + 2(\text{tr } Q)^4 - 4(\text{tr } Q)^3 + 2(\text{tr } Q)^4 - 8(\text{tr } Q)^3 + 4(\text{tr } Q)^2 + 8 \text{tr } Q + 3(\text{tr } Q)^4 + 12(\text{tr } Q)^2 - 12(\text{tr } Q)^3] \\ &= \frac{1}{8} [8(\text{tr } Q)^4 - 24(\text{tr } Q)^3 + 16(\text{tr } Q)^2 + 8 \text{tr } Q] = (\text{tr } Q)^4 - 3(\text{tr } Q)^3 + 2(\text{tr } Q)^2 + \text{tr } Q. \end{aligned}$$

B.3.2 Second gradient elasticity

For the second gradient model we found

Cases $(V_1, \mathcal{G}, \varphi)$ and $(\underline{V}_1, \underline{\mathcal{G}}, \underline{\varphi})$:

$$\begin{aligned} \langle \Pi_{V_1}, \varphi_Q \rangle &= (\Pi_{V_1})_{iajbkcl dme} Q_{ia} Q_{jb} Q_{kc} Q_{ld} Q_{me} \\ &= \frac{1}{4} (\delta_{ia} \delta_{jb} \delta_{kc} \delta_{ld} \delta_{me} + \delta_{ia} \delta_{kb} \delta_{jc} \delta_{ld} \delta_{me} + \delta_{ia} \delta_{jb} \delta_{kc} \delta_{md} \delta_{le} + \delta_{ia} \delta_{kb} \delta_{jc} \delta_{md} \delta_{le}) Q_{ia} Q_{jb} Q_{kc} Q_{ld} Q_{me} \\ &= \frac{1}{4} (\delta_{ia} \delta_{jb} \delta_{kc} \delta_{ld} \delta_{me} Q_{ia} Q_{jb} Q_{kc} Q_{ld} Q_{me} + \delta_{ia} \delta_{kb} \delta_{jc} \delta_{ld} \delta_{me} Q_{ia} Q_{jb} Q_{kc} Q_{ld} Q_{me} \\ &\quad + \delta_{ia} \delta_{jb} \delta_{kc} \delta_{md} \delta_{le} Q_{ia} Q_{jb} Q_{kc} Q_{ld} Q_{me} + \delta_{ia} \delta_{kb} \delta_{jc} \delta_{md} \delta_{le} Q_{ia} Q_{jb} Q_{kc} Q_{ld} Q_{me}) \\ &= \frac{1}{4} (\underbrace{\delta_{ia} Q_{ia}}_{\text{tr } Q} \underbrace{\delta_{jb} Q_{jb}}_{\text{tr } Q} \underbrace{\delta_{kc} Q_{kc}}_{\text{tr } Q} \underbrace{\delta_{ld} Q_{ld}}_{\text{tr } Q} \underbrace{\delta_{me} Q_{me}}_{\text{tr } Q} + \underbrace{\delta_{ia} Q_{ia}}_{\text{tr } Q} \underbrace{Q_{jb} \delta_{bk} Q_{kc} \delta_{cj}}_{\text{tr } Q^2} \underbrace{\delta_{ld} Q_{ld}}_{\text{tr } Q} \underbrace{\delta_{me} Q_{me}}_{\text{tr } Q}) \\ &\quad + \underbrace{\delta_{ia} Q_{ia}}_{\text{tr } Q} \underbrace{\delta_{jb} Q_{jb}}_{\text{tr } Q} \underbrace{\delta_{kc} Q_{kc}}_{\text{tr } Q} \underbrace{\delta_{el} Q_{ld} \delta_{dm} Q_{me}}_{\text{tr } Q^2} + \underbrace{Q_{jb} \delta_{bk} Q_{kc} \delta_{cj}}_{\text{tr } Q^2} \underbrace{\delta_{ia} Q_{ia}}_{\text{tr } Q} \underbrace{\delta_{dm} Q_{me} \delta_{el} Q_{ld}}_{\text{tr } Q^2}) \\ &= \frac{1}{4} ((\text{tr } Q)^5 + (\text{tr } Q)^3 \text{tr } Q^2 + (\text{tr } Q)^3 \text{tr } Q^2 + \text{tr } Q (\text{tr } Q^2)^2) = \frac{1}{4} ((\text{tr } Q)^5 + 2(\text{tr } Q)^3 \text{tr } Q^2 + \text{tr } Q (\text{tr } Q^2)^2), \end{aligned}$$

and

$$\begin{aligned} \langle \Pi_{\underline{V}_1}, \varphi_Q \rangle &= (\Pi_{\underline{V}_1})_{iajbkcl dme} Q_{ia} Q_{jb} Q_{kc} Q_{ld} Q_{me} \\ &= \frac{1}{4} (\delta_{ia} \delta_{jb} \delta_{kc} \delta_{ld} \delta_{me} + \delta_{ja} \delta_{ib} \delta_{kc} \delta_{ld} \delta_{me} + \delta_{ia} \delta_{jb} \delta_{kc} \delta_{md} \delta_{le} + \delta_{ja} \delta_{ib} \delta_{kc} \delta_{md} \delta_{le}) Q_{ia} Q_{jb} Q_{kc} Q_{ld} Q_{me} \\ &= \frac{1}{4} (\delta_{ia} \delta_{jb} \delta_{kc} \delta_{ld} \delta_{me} Q_{ia} Q_{jb} Q_{kc} Q_{ld} Q_{me} + \delta_{ja} \delta_{ib} \delta_{kc} \delta_{ld} \delta_{me} Q_{ia} Q_{jb} Q_{kc} Q_{ld} Q_{me} \\ &\quad + \delta_{ia} \delta_{jb} \delta_{kc} \delta_{md} \delta_{le} Q_{ia} Q_{jb} Q_{kc} Q_{ld} Q_{me} + \delta_{ja} \delta_{ib} \delta_{kc} \delta_{md} \delta_{le} Q_{ia} Q_{jb} Q_{kc} Q_{ld} Q_{me}) \\ &= \frac{1}{4} (\underbrace{\delta_{ia} Q_{ia}}_{\text{tr } Q} \underbrace{\delta_{jb} Q_{jb}}_{\text{tr } Q} \underbrace{\delta_{kc} Q_{kc}}_{\text{tr } Q} \underbrace{\delta_{ld} Q_{ld}}_{\text{tr } Q} \underbrace{\delta_{me} Q_{me}}_{\text{tr } Q} + \underbrace{\delta_{aj} Q_{jb} \delta_{bi} Q_{ia}}_{\text{tr } Q^2} \underbrace{\delta_{kc} Q_{kc}}_{\text{tr } Q} \underbrace{\delta_{ld} Q_{ld}}_{\text{tr } Q} \underbrace{\delta_{me} Q_{me}}_{\text{tr } Q}) \\ &\quad + \underbrace{\delta_{ia} Q_{ia}}_{\text{tr } Q} \underbrace{\delta_{jb} Q_{jb}}_{\text{tr } Q} \underbrace{\delta_{kc} Q_{kc}}_{\text{tr } Q} \underbrace{\delta_{el} Q_{ld} \delta_{dm} Q_{me}}_{\text{tr } Q^2} + \underbrace{\delta_{aj} Q_{jb} \delta_{bi} Q_{ia}}_{\text{tr } Q^2} \underbrace{\delta_{kc} Q_{kc}}_{\text{tr } Q} \underbrace{\delta_{dm} Q_{me} \delta_{el} Q_{ld}}_{\text{tr } Q^2}) \\ &= \frac{1}{4} ((\text{tr } Q)^5 + (\text{tr } Q)^3 \text{tr } Q^2 + (\text{tr } Q)^3 \text{tr } Q^2 + \text{tr } Q (\text{tr } Q^2)^2) = \frac{1}{4} ((\text{tr } Q)^5 + 2(\text{tr } Q)^3 \text{tr } Q^2 + \text{tr } Q (\text{tr } Q^2)^2). \end{aligned}$$

Cases $(V_2, \mathcal{G}, \varphi_2)$ and $(\underline{V}_2, \underline{\mathcal{G}}, \underline{\varphi}_2)$:

$$\langle \Pi_{V_2}, \varphi_Q \rangle = (\Pi_{V_2})_{iajbkcl dmenf} Q_{ia} Q_{jb} Q_{kc} Q_{ld} Q_{me} Q_{nf}$$

$$\begin{aligned}
&= \frac{1}{8} \left((\operatorname{tr} Q)^6 + 2(\operatorname{tr} Q)^4 \left((\operatorname{tr} Q)^2 - 2 \operatorname{tr} Q \right) + (\operatorname{tr} Q)^2 \left((\operatorname{tr} Q)^2 - 2 \operatorname{tr} Q \right)^2 \right. \\
&\quad \left. + 2 \left((\operatorname{tr} Q)^4 - 4(\operatorname{tr} Q)^3 + 2(\operatorname{tr} Q)^2 + 4 \operatorname{tr} Q \right) \left((\operatorname{tr} Q)^2 - 2 \operatorname{tr} Q \right) + 2 \left((\operatorname{tr} Q)^2 - 2 \operatorname{tr} Q \right)^3 \right) \\
&= \frac{1}{8} \left((\operatorname{tr} Q)^6 + 2(\operatorname{tr} Q)^6 - 4(\operatorname{tr} Q)^5 + (\operatorname{tr} Q)^6 + 4(\operatorname{tr} Q)^4 - 4(\operatorname{tr} Q)^5 \right. \\
&\quad \left. + 2(\operatorname{tr} Q)^6 - 8(\operatorname{tr} Q)^5 + 4(\operatorname{tr} Q)^4 + 8(\operatorname{tr} Q)^3 - 4(\operatorname{tr} Q)^5 + 16(\operatorname{tr} Q)^4 - 8(\operatorname{tr} Q)^3 - 16(\operatorname{tr} Q)^2 \right. \\
&\quad \left. + 2(\operatorname{tr} Q)^6 - 16(\operatorname{tr} Q)^3 - 12(\operatorname{tr} Q)^5 + 24(\operatorname{tr} Q)^4 \right) \\
&= \frac{1}{8} \left(8(\operatorname{tr} Q)^6 - 32(\operatorname{tr} Q)^5 + 48(\operatorname{tr} Q)^4 - 16(\operatorname{tr} Q)^3 - 16(\operatorname{tr} Q)^2 \right) \\
&= (\operatorname{tr} Q)^6 - 4(\operatorname{tr} Q)^5 + 6(\operatorname{tr} Q)^4 - 2(\operatorname{tr} Q)^3 - 2(\operatorname{tr} Q)^2.
\end{aligned}$$

B.3.3 Non symmetric theories

In this case we have

$$\begin{aligned}
\bar{\Pi} &= \frac{1}{2} (\delta_{ia} \delta_{jb} \delta_{kc} \delta_{ld} + \delta_{ka} \delta_{lb} \delta_{ic} \delta_{jd}) Q_{ia} Q_{jb} Q_{kc} Q_{ld} = \frac{1}{2} (\delta_{ia} \delta_{jb} \delta_{kc} \delta_{ld} Q_{ia} Q_{jb} Q_{kc} Q_{ld} + \delta_{ka} \delta_{lb} \delta_{ic} \delta_{jd} Q_{ia} Q_{jb} Q_{kc} Q_{ld}) \\
&= \frac{1}{2} \left(\underbrace{\delta_{ia} Q_{ia}}_{\operatorname{tr} Q} \underbrace{\delta_{jb} Q_{jb}}_{\operatorname{tr} Q} \underbrace{\delta_{kc} Q_{kc}}_{\operatorname{tr} Q} \underbrace{\delta_{ld} Q_{ld}}_{\operatorname{tr} Q} + \underbrace{Q_{ia} \delta_{ak} Q_{kc} \delta_{ci}}_{\operatorname{tr} Q^2} \underbrace{Q_{jb} \delta_{bl} Q_{ld} \delta_{dj}}_{\operatorname{tr} Q^2} \right) = \frac{1}{2} \left((\operatorname{tr} Q)^4 + (\operatorname{tr} Q^2)^2 \right),
\end{aligned}$$

and so the Cayley-Hamilton Theorem yields

$$\begin{aligned}
\chi(Q) &= \frac{1}{2} \left((\operatorname{tr} Q)^4 + (\operatorname{tr} Q^2)^2 \right) = \frac{1}{2} \left((\operatorname{tr} Q)^4 + \left((\operatorname{tr} Q)^2 - 2 \operatorname{tr} Q \right)^2 \right) \\
&= \frac{1}{2} \left((\operatorname{tr} Q)^4 + (\operatorname{tr} Q)^4 - 4(\operatorname{tr} Q)^3 + 4(\operatorname{tr} Q)^2 \right) = (\operatorname{tr} Q)^4 - 2(\operatorname{tr} Q)^3 + 2(\operatorname{tr} Q)^2.
\end{aligned}$$

References

- [1] Sophie Abramian, Boris Desmorat, Rodrigue Desmorat, Boris Kolev, and Marc Olive. Recovering the normal form and symmetry class of an elasticity tensor. *Journal of Elasticity*, 142:1–33, 2020.
- [2] Nicolas Auffray. On the algebraic structure of isotropic generalized elasticity theories. *Mathematics and Mechanics of Solids*, 20(5):565–581, 2015.
- [3] Nicolas Auffray, Qi-Chang He, and Hung Le Quang. Complete symmetry classification and compact matrix representations for 3d strain gradient elasticity. *International Journal of Solids and Structures*, 159:197–210, 2019.
- [4] Nicolas Auffray, Boris Kolev, and Marc Olive. Handbook of bi-dimensional tensors: Part I: Harmonic decomposition and symmetry classes. *Mathematics and Mechanics of Solids*, 22(9):1847–1865, 2017.
- [5] Nicolas Auffray, Hung Le Quang, and Qi-Chang He. Matrix representations for 3d strain-gradient elasticity. *Journal of the Mechanics and Physics of Solids*, 61(5):1202–1223, 2013.
- [6] Gabriele Barbagallo, Angela Madeo, Marco V. d’Agostino, Rafael Abreu, Ionel-Dumitrel Ghiba, and Patrizio Neff. Transparent anisotropy for the relaxed micromorphic model: macroscopic consistency conditions and long wave length asymptotics. *International Journal of Solids and Structures*, 2017.
- [7] Romesh C. Batra. On heat conduction and wave propagation in non-simple rigid solids. *Letters in Applied and Engineering Science*, 3:97–107, 1975.
- [8] Romesh C. Batra. Private communications. 2018.
- [9] Romesh C. Batra and Jiashi S. Yang. Second-order constitutive relations for transversely isotropic piezoelectric porous materials. *The Journal of the Acoustical Society of America*, 97(4):2595–2598, 1995.
- [10] Allan F. Bower. *Applied Mechanics of Solids*. CRC press, 2009.
- [11] Augustin L. Cauchy. De la pression ou tension dans un corps solide. *Exercices de Mathématiques*, 2:42–56, 1827.
- [12] Augustin L. Cauchy. Sur la condensation et la dilatation des corps solides. *Exercices de Mathématiques*, 2(1827):60–69, 1827.
- [13] Augustin L. Cauchy. Sur les équations qui expriment les conditions d’équilibre ou les lois du mouvement des fluides. *Exercices de Mathématiques*, pages 158–179, 1828.
- [14] Philippe G. Ciarlet. *Mathematical Elasticity: Three Dimensional Elasticity, vol. 1*. 1988.
- [15] Marco V. d’Agostino, Gabriele Barbagallo, Ionel-Dumitrel Ghiba, Bernhard Eidel, Patrizio Neff, and Angela Madeo. Effective description of anisotropic wave dispersion in mechanical band-gap metamaterials via the relaxed micromorphic model. *Journal of Elasticity*, 139:299–329, 2020.

- [16] Marco V. d’Agostino, Gabriele Barbagallo, Ionel-Dumitrel Ghiba, Angela Madeo, and Patrizio Neff. A panorama of dispersion curves for the weighted isotropic relaxed micromorphic model. *Zeitschrift für Angewandte Mathematik und Mechanik*, 97(11):1436–1481, 2017.
- [17] Alexandre Danescu. The number of elastic moduli for anisotropic polynomial constitutive equations. *Mechanics Research Communications*, 24(3):289–301, 1997.
- [18] Alexandre Danescu and Angelo M. Tarantino. Electro-elastic interactions and second order anisotropic constitutive equations. *Meccanica*, 31(6):657–664, 1996.
- [19] Alexandre Danescu and Angelo M. Tarantino. Anisotropic polynomial constitutive equations for electroelastic crystals. *Zeitschrift für Angewandte Mathematik und Physik*, 48(6):857–873, 1997.
- [20] Anton Deitmar and Siegfried Echterhoff. *Principles of Harmonic Analysis*. Springer, 2014.
- [21] Félix Demore, Ginaluca Rizzi, Manuel Collet, Patrizio Neff, and Angela Madeo. Unfolding engineering metamaterials design: Relaxed micromorphic modeling of large-scale acoustic meta-structures. *Journal of the Mechanics and Physics of Solids*, 168:104995, 2022.
- [22] Boris Desmorat and Nicolas Auffray. Space of 2d elastic materials: a geometric journey. *Continuum Mechanics and Thermodynamics*, 31:1205–1229, 2019.
- [23] Boris Desmorat, Marc Olive, Nicolas Auffray, Rodrigue Desmorat, and Boris Kolev. Computation of minimal covariants bases for 2d coupled constitutive laws. *International Journal of Engineering Science*, 191:103880, 2023.
- [24] James Diebel. Representing attitude: Euler angles, unit quaternions, and rotation vectors. *Matrix*, 58(15-16):1–35, 2006.
- [25] Marek Elżanowski and Marcelo Epstein. The symmetry group of second-grade materials. *International Journal of Non-Linear Mechanics*, 27(4):635–638, 1992.
- [26] Manuel Ferretti, Angela Madeo, Francesco dell’Isola, and Philippe Boisse. Modeling the onset of shear boundary layers in fibrous composite reinforcements by second-gradient theory. *Zeitschrift für Angewandte Mathematik und Physik*, 65(3):587–612, 2014.
- [27] Andreas Fischle and Patrizio Neff. Grioli’s theorem with weights and the relaxed-polar mechanism of optimal Cosserat rotations. *Rendiconti Lincei*, 28(3):573–600, 2017.
- [28] Sandra Forte and Maurizio Vianello. Symmetry classes for elasticity tensors. *Journal of Elasticity*, 43(2):81–108, 1996.
- [29] Ionel-Dumitrel Ghiba, Patrizio Neff, Angela Madeo, and Ingo Münch. A variant of the linear isotropic indeterminate couple-stress model with symmetric local force-stress, symmetric nonlocal force-stress, symmetric couple-stresses and orthogonal boundary conditions. *Mathematics and Mechanics of Solids*, 22(6):1221–1266, 2017.
- [30] Martin Golubitsky, Ian Stewart, and David G. Schaeffer. *Singularities and Groups in Bifurcation Theory*, volume 2. Springer Science & Business Media, 2012.
- [31] Erik W. Grafarend and Wolfgang Kühnel. A minimal atlas for the rotation group $SO(3)$. *GEM-International Journal on Geomathematics*, 2(1):113–122, 2011.
- [32] Morton E. Gurtin. *The Linear Theory of Elasticity*. Springer Berlin Heidelberg, Berlin, Heidelberg, 1973.
- [33] Brian C. Hall. Lie groups, Lie algebras, and Representations. In *Quantum Theory for Mathematicians*, pages 333–366. Springer, 2013.
- [34] Yu-Ning Huang and Romesh C. Batra. Energy-momentum tensors in nonsimple elastic dielectrics. *Journal of Elasticity*, 42(3):275–281, 1996.
- [35] Fritz John. Rotation and strain. *Communications on Pure and Applied Mathematics*, 14(3):391–413, 1961.
- [36] Augustus Edward Hough Love. *A Treatise on the Mathematical Theory of Elasticity*. Dover Publications, Inc, New York, 1944.
- [37] Angela Madeo, Gabriele Barbagallo, Marco V. d’Agostino, and Philippe Boisse. Continuum and discrete models for unbalanced woven fabrics. *International Journal of Solids and Structures*, 94-95(2016):263–284, 2016.
- [38] Angela Madeo, Patrizio Neff, Ionel-Dumitrel Ghiba, Luca Placidi, and Giuseppe Rosi. Band gaps in the relaxed linear micromorphic continuum. *Zeitschrift für Angewandte Mathematik und Mechanik*, 95(9):880–887, 2014.
- [39] Jean Mandel. Plastic waves in an infinite three dimensional medium. *Journal de Mécanique*, 1:3–30, 1962.
- [40] Jerrold E. Marsden and Tudor S. Ratiu. Introduction to Mechanics and Symmetry. *Physics Today*, 48(12):65, 1995.
- [41] Ingo Münch and Patrizio Neff. Rotational invariance conditions in elasticity, gradient elasticity and its connection to isotropy. *Mathematics and Mechanics of Solids*, 23(1):3–42, 2018.
- [42] Ingo Münch, Patrizio Neff, Angela Madeo, and Ionel-Dumitrel Ghiba. The modified indeterminate couple stress model: Why Yang et al.’s arguments motivating a symmetric couple stress tensor contain a gap and why the couple stress tensor may be chosen symmetric nevertheless. *Zeitschrift für Angewandte Mathematik und Mechanik*, 97(12):1524–1554, 2017.
- [43] Ian A. Murdoch. Symmetry considerations for materials of second grade. *Journal of Elasticity*, 9(1):43–50, 1979.
- [44] Claude L.M.H. Navier. *Mémoire sur les lois de l’équilibre et du mouvement des corps solides élastiques*. 1827.
- [45] Patrizio Neff, Bernhard Eidel, Marco V. d’Agostino, and Angela Madeo. Identification of scale-independent material parameters in the relaxed micromorphic model through model-adapted first order homogenization. *Journal of Elasticity*, 139(2):269–298, 2020.
- [46] Patrizio Neff, Ionel-Dumitrel Ghiba, Angela Madeo, and Ingo Münch. Null-Lagrangians and the indeterminate couple stress model. *Proceedings in Applied Mathematics and Mechanics*, 16(1):379–380, 2016.
- [47] Patrizio Neff, Ionel-Dumitrel Ghiba, Angela Madeo, Luca Placidi, and Giuseppe Rosi. A unifying perspective: the relaxed linear micromorphic continuum. *Continuum Mechanics and Thermodynamics*, 26(5):639–681, 2014.

- [48] Patrizio Neff, Jena Jeong, and Hamidréza Ramézani. Subgrid interaction and micro-randomness - Novel invariance requirements in infinitesimal gradient elasticity. *International Journal of Solids and Structures*, 46(25-26):4261–4276, 2009.
- [49] Patrizio Neff, Johannes Lankeit, and Angela Madeo. On Grioli’s minimum property and its relation to Cauchy’s polar decomposition. *International Journal of Engineering Science*, 80:209–217, 2014.
- [50] Patrizio Neff and Ingo Münch. Curl bounds Grad on $SO(3)$. *ESAIM: Control, Optimisation and Calculus of Variations*, 14(1):148–159, 2008.
- [51] Patrizio Neff, Ingo Münch, Ionel-Dumitrel Ghiba, and Angela Madeo. On some fundamental misunderstandings in the indeterminate couple stress model. A comment on recent papers of A.R. Hadjesfandiari and G.F. Dargush. *International Journal of Solids and Structures*, 81:233–243, 2016.
- [52] Marc Olive and Nicolas Auffray. Symmetry classes for even-order tensors. *Mathematics and Mechanics of Complex Systems*, 1(2):177–210, 2013.
- [53] Marc Olive and Nicolas Auffray. Symmetry classes for odd-order tensors. *Zeitschrift für Angewandte Mathematik und Mechanik*, 94(5):421–447, 2014.
- [54] Marc Olive, Boris Kolev, and Nicolas Auffray. A minimal integrity basis for the elasticity tensor. *Archive for Rational Mechanics and Analysis*, 226(1):1–31, 2017.
- [55] Marc Olive, Boris Kolev, Rodrigue Desmorat, and Boris Desmorat. Characterization of the symmetry class of an elasticity tensor using polynomial covariants. *Mathematics and Mechanics of Solids*, 27(1):144–190, 2022.
- [56] Leonardo A. Perez Ramirez, Gianluca Rizzi, and Angela Madeo. Multi-element metamaterial’s design through the relaxed micromorphic model. In *Sixty Shades of Generalized Continua: Dedicated to the 60th Birthday of Prof. Victor A. Eremeyev*, pages 579–600. Springer, 2023.
- [57] Gianluca Rizzi, Manuel Collet, Félix Demore, Bernhard Eidel, Patrizio Neff, and Angela Madeo. Exploring metamaterials’ structures through the relaxed micromorphic model: switching an acoustic screen into an acoustic absorber. *Frontiers in Materials*, 7:589701, 2021.
- [58] Gianluca Rizzi, Marco V. d’Agostino, Patrizio Neff, and Angela Madeo. Boundary and interface conditions in the relaxed micromorphic model: Exploring finite-size metastructures for elastic wave control. *Mathematics and Mechanics of Solids*, 27(6):1053–1068, 2022.
- [59] Gianluca Rizzi, Patrizio Neff, and Angela Madeo. Metamaterial shields for inner protection and outer tuning through a relaxed micromorphic approach. *Philosophical Transactions of the Royal Society A*, 380(2231):20210400, 2022.
- [60] Gianluca Rizzi, Domenico Tallarico, Patrizio Neff, and Angela Madeo. Towards the conception of complex engineering metastructures: Relaxed-micromorphic modelling of low-frequency mechanical diodes/high-frequency screens. *Wave Motion*, 113:102920, 2022.
- [61] Mohammad Sarhil, Lisa Scheunemann, Jörg Schröder, and Patrizio Neff. Size-effects of metamaterial beams subjected to pure bending: on boundary conditions and parameter identification in the relaxed micromorphic model. *Computational Mechanics*, pages 1–23, 2023.
- [62] Yu I. Sirotin. Group Tensor Spaces (in russian). *Kristallografiya (translated as Soviet Phys. Cryst.)*, 5:171–179, 1960.
- [63] Geoffrey F. Smith and Ronald S. Rivlin. The strain-energy function for anisotropic elastic materials. *Transactions of the American Mathematical Society*, 88(1):175–193, 1958.
- [64] Thomas Chi-Tsai Ting. *Anisotropic Elasticity: Theory and Applications*. Number 45. Oxford University Press on Demand, 1996.
- [65] Isaac Todhunter. *A History of the Theory of Elasticity and of the Strength of Materials*. Cambridge University Press, 2014.
- [66] Woldemar Voigt. *Lehrbuch der Kristallphysik*. Vieweg+Teubner Verlag, Wiesbaden, 1966 edition, 1909.
- [67] Jendrik Voss, Gianluca Rizzi, Patrizio Neff, and Angela Madeo. Modeling a labyrinthine acoustic metamaterial through an inertia-augmented relaxed micromorphic approach. *Mathematics and Mechanics of Solids*, 0(0):1–25, 0.
- [68] Jiashi S. Yang and Romesh C. Batra. A second-order theory for piezoelectric materials. *The Journal of the Acoustical Society of America*, 97(1):280–288, 1995.