

Hurwitz Derivation Formula

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This text illustrates quite an interesting derivation formula due to A. Hurwitz [3] and applied in [2, Lemma 2.3]. Computations are performed using the Python/sympy version of the DifferentialAlgebra package [1] and a small extra Python package Hurwitz which is based on it.

1 Louis Example

1.1 The Problem

We start with the following (autonomous) *differential polynomial*

$$p(y, \dot{y}, \ddot{y}) = y \ddot{y} + \dot{y}^2 - 6y. \tag{1}$$

```
[1]: from sympy import *
from DifferentialAlgebra import *
from Hurwitz import *
init_printing ()
```

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```
[2]: x = var ('x')
      y = indexedbase ('y')
```

```
[3]: R = DifferentialRing(derivations = [x], blocks = [y], notation = 'jet')
```

```
[4]: p = y*y[x,x] + y[x]**2 - 6*y
```

In order to solve the nonlinear ODE $p = 0$ or integrate it numerically, we would need two initial values (since p has order 2) such that $y_0 \neq 0$ to avoid the cancellation of the *separant* $\partial p / \partial \dot{y}$ which is equal to y . In this section however, we are concerned by the following problem: computing solutions of the ODE where vanilla methods do not apply i.e. such that $y_0 = 0$. As we shall see, such solutions exist but are not unique: they are the functions $y(x) = 0$ (the *singular solution*) and $y(x) = x^2$ (a particular case of the *general solution*).

```
[5]: R.evaluate (p, {y:0}).doit ()
```

```
[5]: 0
```

```
[6]: R.evaluate (p, {y:x**2})
```

```
[6]:  $x^2 \frac{d^2}{dx^2} x^2 - 6x^2 + \left( \frac{d}{dx} x^2 \right)^2$ 
```

```
[7]: expand (R.evaluate (p, {y:x**2}).doit ())
```

```
[7]: 0
```

1.2 Singular and General Solution

Our example features a single differential polynomial. In this context, the notions of general and singular solutions have a nice theoretical description and solution in differential algebra, by means of the *Low Power Theorem* initiated by Ritt in [5, chap. III] and quite extended by Kolchin in [4, chap. IV]. The simplest way to apply it consists in providing the argument `singsol='essential'` to the `RosenfeldGroebner` simplifier (this argument is meaningful only in the case of a system made of a single differential polynomial):

```
[8]: ideal = R.RosenfeldGroebner([p], singsol='essential')
      [ C.equations (solved = True) for C in ideal ]
```

```
[8]:  $\left[ \left[ y_{x,x} = \frac{-y_x^2 + 6y}{y} \right], [y = 0] \right]$ 
```

```
[9]: Cgnr1 = ideal[0]
      Csing = ideal[1]
```

Two regular differential chains are produced. This means that none of them is redundant: the differential ideals they define are pairwise disjoint. The first one is the general solution. The second one corresponds to a truly singular solution. Therefore, $y(x) = 0$ is singular and $y(x) = x^2$ necessarily is a particular case of the general solution. Readers who want to

enter details may compute the *preparation equation* of p with respect to the second regular differential chain:

[10]: `Csing.preparation_equation(p)`

[10]: $y_{x,x}y + y_x^2 - 6y = z_{1x}^2 + z_{1x}z_1 - 6z_1$

The last term of this preparation equation is z_1^q with $q = 1$ (q is the *low power*). This is sufficient to conclude that the second regular differential chain defines an essential component [4, chap. IV, sect. 15, Theorem 6].

1.3 Reformulation of the Problem

But let us come back to our existence problem of power series solutions for an initial value $y_0 = 0$ which cancels the separant. In such situations, the number of needed initial values cannot always be clearly read from the order of the differential polynomial. In general, this number depends on the values of the initial values! A way to overcome this difficulty consists in providing arbitrarily many initial values $y_0, \dot{y}_0, \ddot{y}_0, \dots$ and encode them by means of a formal power series

$$\bar{y} = y_0 + y_1 x + y_2 \frac{x^2}{2} + \dots$$

Notice $y_0, y_1, y_2 = \bar{y}(0), \bar{y}'(0), \bar{y}''(0)$ are initial values. Of course, initial values must be compatible with the differential polynomial. For instance the three first initial values y_0, y_1, y_2 must satisfy¹ $p(y_0, y_1, y_2) = 0$. Since $p(y_0, y_1, y_2)$ is the first term of $p(\bar{y}, \bar{y}', \bar{y}'')$ (which is a series in x), we can express this compatibility constraint as $p(\bar{y}, \bar{y}', \bar{y}'') \bmod x = 0$.

Let us generalize these remarks. Hurwitz Lemma deals, more generally, with non autonomous differential polynomials of arbitrary order n i.e. with differential polynomials $p(x, y, \dot{y}, \dots, y^{(n)})$. Given a formal power series \bar{y} (which encodes initial values), it (more precisely [2, Lemma 2.3] which is based on it) provides two non negative integers β and δ such that, if

$$p(x, \bar{y}, \bar{y}', \dots, \bar{y}^{(n)}) = 0 \bmod x^\beta$$

then there exists a unique formal power series $\bar{\bar{y}}$ such that

$$\begin{aligned} \bar{\bar{y}} &= \bar{y} \bmod x^\delta, \\ p(x, \bar{\bar{y}}, \bar{\bar{y}}', \dots, \bar{\bar{y}}^{(n)}) &= 0. \end{aligned}$$

With words, if the series \bar{y} provides sufficiently many (i.e. δ) compatible initial values then these initial values can be prolonged into a unique formal power series $\bar{\bar{y}}$. The two bounds δ and β are strongly related but need not be the same.

¹In the classical/vanilla setting, a numerical integrator would compute $\ddot{y}(0) = y_2$ from p using y_0 and y_1 so that the three initial values would necessarily satisfy $p(y_0, y_1, y_2) = 0$.

1.4 Illustration over the Example

Let us come back to our example (1). In order to compute the two bounds, one first computes the valuation k of the separant of p at $y = \bar{y}$. The separant is denoted f_n in [2]

$$f_n(x, \bar{y}, \bar{y}', \bar{y}'') = c_0 x^k + c_1 x^{k+1} + \dots \quad (c_0 \neq 0)$$

The Lemma assumes k is finite so that we cannot handle here the singular solution $\bar{y} = 0$. Let us assume $y_1 \neq 0$. Then $k = 1$:

```
[11]: ybar = Add(*[y[i]*x**i/factorial(i) for i in range (10)])
      ybar = ybar.subs ({y[0]:0})
      ybar
```

```
[11]:  $\frac{x^9 y_9}{362880} + \frac{x^8 y_8}{40320} + \frac{x^7 y_7}{5040} + \frac{x^6 y_6}{720} + \frac{x^5 y_5}{120} + \frac{x^4 y_4}{24} + \frac{x^3 y_3}{6} + \frac{x^2 y_2}{2} + x y_1$ 
```

```
[12]: H = Hurwitz (p, R)
      H.k ({y:ybar})
```

```
[12]: 1
```

The valuation k being fixed, one can also determine an integer $0 \leq r \leq k$ which gives the first nonzero coefficient $A(q)$ of $p^{(2k+2+q)}(0, y_0, y_1, \dots)$ (here q is an indeterminate standing for a number of differentiations):

```
[13]: q = var ('q')
      k, r, A = H.kra (q, {y:ybar})
      (k, r, A)
```

```
[13]: (1, 1, qy1 + 6y1)
```

The question now is: are there non negative integer values for q which annihilate $A(q)$? Since we have assumed $y_1 \neq 0$, we see that $A(q)$ has no non negative integer root. From this we can deduce γ which is defined as an integer strictly bigger than any non negative integer root of $A(q)$. We may thus choose $\gamma = 0$. From all these numbers, we get

$$\begin{aligned} \beta &= 2k + 2 + \gamma + r = 5, \\ \delta &= n + 2k + 2 + \gamma = 6. \end{aligned}$$

```
[14]: H.beta (0, {y:ybar})
```

```
[14]: 5
```

```
[15]: H.delta (0, {y:ybar})
```

```
[15]: 6
```

Let us have a look at $p(x, \bar{y}, \bar{y}', \bar{y}'') \pmod{x^\beta}$

```
[16]: serie = collect (expand (R.evaluate (p, {y:ybar}).doit ()), x)
      rem (serie, x**5, x)
```

$$[16]: x^4 \left(\frac{y_1 y_5}{4} + \frac{5y_2 y_4}{8} + \frac{5y_3^2}{12} - \frac{y_4}{4} \right) + x^3 \left(\frac{5y_1 y_4}{6} + \frac{5y_2 y_3}{3} - y_3 \right) + x^2 \left(2y_1 y_3 + \frac{3y_2^2}{2} - 3y_2 \right) + x(3y_1 y_2 - 6y_1) + y_1^2$$

The initial values are compatible with the differential equation if and only if this series is identically zero. Looking at the last term, this would imply $y_1 = 0$ which would contradict our hypothesis. Thus the differential equation has no solution \bar{y} with $y_0 = 0$ and $y_1 \neq 0$.

Let us thus study the case $y_0 = y_1 = 0$ and assume $y_2 \neq 0$:

```
[17]: ybar = ybar.subs ({y[1]:0})
ybar
```

$$[17]: \frac{x^9 y_9}{362880} + \frac{x^8 y_8}{40320} + \frac{x^7 y_7}{5040} + \frac{x^6 y_6}{720} + \frac{x^5 y_5}{120} + \frac{x^4 y_4}{24} + \frac{x^3 y_3}{6} + \frac{x^2 y_2}{2}$$

```
[18]: k, r, A = H.kra (q, {y:ybar})
(k, r, A)
```

$$[18]: \left(2, 2, 8qy_2 + \binom{q}{2}y_2 + 28y_2 - 6 \right)$$

```
[19]: A.combsimp ()
```

$$[19]: \frac{q^2 y_2 + 15qy_2 + 56y_2 - 12}{2}$$

We have found $k = r = 2$ and a degree two polynomial $A(q)$ which may have arbitrary high positive integer roots, depending on y_2 . Without any further information on y_2 , the analysis is not so easy. In order to get some insight, let us have a look at the lowest terms of $p(x, \bar{y}, \bar{y}', \bar{y}'')$:

```
[20]: serie = collect (expand (R.evaluate (p, {y:ybar}).doit ()), x)
rem (serie, x**4, x)
```

$$[20]: x^3 \left(\frac{5y_2 y_3}{3} - y_3 \right) + x^2 \left(\frac{3y_2^2}{2} - 3y_2 \right)$$

Indeed, we see that y_2 must be equal to 2. Let us restart computations:

```
[21]: ybar = ybar.subs ({y[2]:2})
ybar
```

$$[21]: \frac{x^9 y_9}{362880} + \frac{x^8 y_8}{40320} + \frac{x^7 y_7}{5040} + \frac{x^6 y_6}{720} + \frac{x^5 y_5}{120} + \frac{x^4 y_4}{24} + \frac{x^3 y_3}{6} + x^2$$

```
[22]: k, r, A = H.kra (q, {y:ybar})
(k, r, A)
```

$$[22]: \left(2, 2, 16q + 2 \binom{q}{2} + 50 \right)$$

```
[23]: solve (A.combsimp())
```

```
[23]: [-10, -5]
```

Here we see that $A(q)$ has no non negative integer root. We can thus choose $\gamma = 0$ which gives us $\beta = 8$:

```
[24]: H.beta (0, {y:ybar})
```

```
[24]: 8
```

and the following truncated series:

```
[25]: serie = collect (expand (R.evaluate (p, {y:ybar}).doit ()), x)
      rem (serie, x**8, x)
```

```
[25]: x7  $\left(\frac{y_3 y_6}{60} + \frac{y_4 y_5}{40} + \frac{11 y_7}{840}\right) + x^6 \left(\frac{7 y_3 y_5}{90} + \frac{7 y_4^2}{144} + \frac{5 y_6}{72}\right) + x^5 \left(\frac{7 y_3 y_4}{24} + \frac{3 y_5}{10}\right) + x^4 \left(\frac{5 y_3^2}{12} + y_4\right) + \frac{7 x^3 y_3}{3}$ 
```

Obviously, this series is identically zero if and only if $y_3 = y_4 = \dots = y_7 = 0$. Thus for these initial values, the formal power series \bar{y} exists and is unique. Observe that [2, Lemma 2.3] does not tell us that all further coefficients are zero (so that $\bar{y} = x^2$).

1.5 Numerical Integration

The knowledge of a truncated power series solution $(y_0, y_1, y_2) = (0, 0, 2)$ permits us to integrate numerically the ODE. The order two differential polynomial is converted into a vector of two order one ODE. Denote $Y = (y, \dot{y})$. The coordinates of Y are denoted $Y = (Y_0, Y_1)$. The system to be integrated is

$$\dot{Y} = f(x, Y) = \begin{pmatrix} Y_1 \\ 6 - \frac{Y_1}{2Y_0} \end{pmatrix}. \quad (2)$$

The initial values are $Y = (0, 0)$. In order to avoid a division by zero, the truncated power series can be used as long as $Y_0 = 0$ (which happens at the two first steps of the integration process). See Figure 1.

```
[26]: from scipy.integrate import solve_ivp
      import numpy as np
      def f (x,y) :
          if y[0] == 0 :
              return np.array ([y[1], 2], dtype=np.float64)
          else :
              return np.array ([y[1], 6 - y[1]**2/y[0]], dtype=np.float64)
```

```
[27]: ode_sol = solve_ivp (f, [0,1.5], np.array ([0,0], dtype=np.float64), rtol=1e-8)
```

Here are the commands which produced Figure 1:

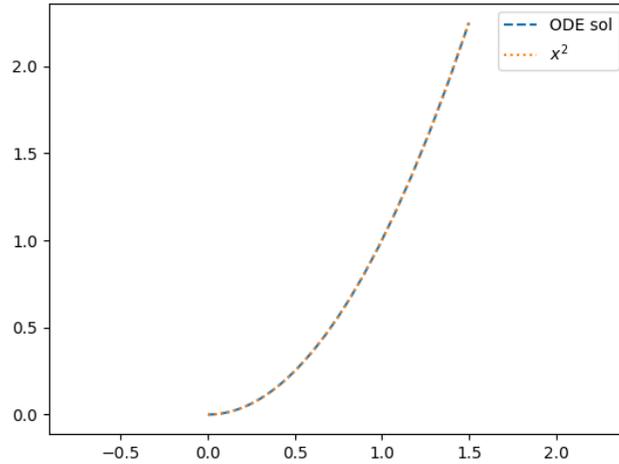


Figure 1: The curve obtained by numerical integration of ODE (2) and the parabola fit.

```
[28]: import matplotlib.pyplot as plt
plt.axis('equal')
plt.plot(ode_sol.t, ode_sol.y[0], linestyle='dashed', label='ODE sol')
plt.plot(ode_sol.t, [x**2 for x in ode_sol.t], linestyle='dotted',
        ↪label='$x^2$')
plt.legend()
plt.show()
```

2 Michael Example

Consider the following (linear) differential polynomial where a is a constant

$$p = x \dot{y} - a y.$$

One solution is the zero function $y(x) = 0$. If a is a non negative integer then there also exist power series solutions $y(x) = c x^a$ where c is a constant, This example was used by Michael Singer in [6] to show that the problem “*given a polynomial ODE system, determine if there exists a nonzero formal power series solution*” is undecidable.

This example is also particular because the separant of p , which is $f_n = x$ does not depend on y so that the valuation $k = 1$ of $f_n(x, \bar{y})$ does not depend on \bar{y} .

```
[29]: x = var('x')
a,q = var('a,q')
y = indexedbase('y')
```

```
[30]: R = DifferentialRing(derivations = [x], blocks = [y,a], parameters = [a],
        ↪notation = 'jet')
```

[31]: `p = x*y[x] - a*y`

[32]: `H = Hurwitz (p, R)`
`k, r, A = H.kra(q, {})`
`k, r, A`

[32]: (1, 1, $-a + q + 4$)

The integer r and the polynomial $A(q)$ do not either but $A(q)$ depends on a :

[33]: `solve (A, q)`

[33]: `[a - 4]`

The bound γ is defined as an integer bigger than any non negative integer root of $A(q)$. Thus if $a < 4$ then we can choose $\gamma = 0$ else we can choose $\gamma = a - 3$. Here are the corresponding bounds δ and β :

[34]: `H.delta (0, {}), H.delta (a-3, {})`

[34]: (5, $a + 2$)

Indeed, in both cases, [2, Lemma 2.3] tells us to look at sufficiently many initial values to decide the existence and uniqueness of a formal power series solution \bar{y} .

3 François Example

Consider the following order two non autonomous differential polynomial

$$p(x, y, \dot{y}, \ddot{y}) = x y \ddot{y} + \dot{y} + y^2 + 1.$$

Proposition 1 *The differential polynomial p admits a formal power series solution (which is then uniquely defined) if and only if its initial value $y_0 \notin E$ where $E = \{-1/a \mid a \in \mathbb{N}^*\}$. Moreover, if $y_0 = -1/a$ for some $a \in \mathbb{N}^*$ then $p(x, y, \dot{y}, \ddot{y}) = 0 \pmod{x^a}$ admits a solution.*

Proof Denote $p_a = p^{(a)}(0, y)$. Rename $y^{(i)}$ as y_i for any i . It is easy to prove by induction that we have $p_a = (a y_0 + 1) y_{a+1} + q_a$ where q_a is a nonempty sum of monomials of the form $c_{i,j} y_i y_j$ such that $c_{i,j} > 0$ and $i + j \leq a + 1$. Here are a few values:

i	p_i
0	$y_1 + y_0^2 + 1$
1	$(y_0 + 1) y_2 + 2 y_0 y_1$
2	$(2 y_0 + 1) y_3 + (2 y_1 + 2 y_0) y_2 + 2 y_1^2$
3	$(3 y_0 + 1) y_4 + (6 y_1 + 2 y_0) y_3 + 6 y_1 y_2 + 3 y_2^2$
\vdots	\vdots

For any $y_0 \notin E$, it is clear that the above system admits a unique solution. Let us assume $y_0 = -1/a$ for some $a \in \mathbb{N}^*$. Then the leading coefficient of p_a vanishes. Since it is the first

one to vanish, we see that $p(x, y, \dot{y}, \ddot{y}) = 0 \pmod{x^a}$ admits a solution. We now claim that $p(x, y, \dot{y}, \ddot{y}) = 0 \pmod{x^{a+1}}$ has no solution. The leading coefficients $i y_0 + 1$ of p_i ($0 \leq i < a$) are positive. It is easy to prove by induction, using the form of the polynomials q_i , that the equations $p_0 = \dots = p_{a-1} = 0$ force the variables y_1, \dots, y_a to have strictly negative values. But the equation $p_a = 0$ then forces y_a to have a strictly positive value. Thus the system $p_0 = \dots = p_a = 0$ is inconsistent and the claim is proved. \square

Let us consider a formal power series \bar{y} with $y_0 \neq 0$. Then we have:

```
[35]: p = x*y*y[x,x] + y[x] + y**2 + 1
```

```
[36]: ybar = Add (*[y[i]*x**i/factorial(i) for i in range (3)])
ybar
```

```
[36]:  $\frac{x^2 y_2}{2} + x y_1 + y_0$ 
```

```
[37]: H = Hurwitz (p,R)
k, r, A = H.kra(q, {y:ybar})
k, r, A
```

```
[37]: (1, 1, qy_0 + 4y_0 + 1)
```

Let us assume $y_0 = -1/a$ for some positive integer a . Then we can choose $\gamma = a - 3$ and the bounds provided by [2, Lemma 2.3] are sufficient to prove the non existence of any power series solution \bar{y} :

```
[38]: solve (A.subs({y[0]:-1/a}), q)
```

```
[38]: [a - 4]
```

```
[39]: H.beta (a-3, {y:ybar}), H.delta (a-3, {y:ybar})
```

```
[39]: (a + 2, a + 3)
```

References

- [1] François Boulier and al. DifferentialAlgebra. <https://codeberg.org/francois.boulier/DifferentialAlgebra>.
- [2] Jan Denef and Leonard Lipshitz. Power Series Solutions of Algebraic Differential Equations. *Mathematische Annalen*, 267:213–238, 1984.
- [3] A. Hurwitz. Sur le développement des fonctions satisfaisant à une équation différentielle algébrique. *Annales de l'École Normale Supérieure*, 6:327–332, 1889.
- [4] Ellis Robert Kolchin. *Differential Algebra and Algebraic Groups*. Academic Press, New York, 1973.

- [5] Joseph Fels Ritt. *Differential Algebra*, volume 33 of *American Mathematical Society Colloquium Publications*. American Mathematical Society, New York, 1950.
- [6] Michael Singer. The Model Theory of Ordered Differential Fields. *Journal of Symbolic Logic*, 43(1):82–91, 1978.