

Formal Power Series and Differential Elimination

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CIMPA School: *Algebraic and Tropical Methods for Solving Differential Equations*

June 15, 2023



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getdrawings.com

Formal Power Serious

Ritt's Reduction Algorithm

It is a generalization of pseudo-division

Its efficiency relies on the fact that the derivative of a differential polynomial has degree one in its leading derivative

$$(v^n)' = n v^{n-1} v'$$

This would be false in difference algebra

If $p, f \in \mathcal{R}[x]$ are two polynomials

$$f = f_e x^e + \cdots + f_1 x + f_0 \quad p = p_d x^d + \cdots + p_1 x + p_0$$

Then there exist $q, g \in \mathcal{R}[x]$ with such that

$$p_d^{e-d+1} f = q p + g \quad (\deg(g, x) < d)$$

The polynomial $g = \text{prem}(f, p, x)$ is the pseudoremainder of f by p

Assume f, p are differential polynomials and a ranking is fixed

$$p = \dot{y}^2 + 8xy - y$$

It is possible to compute the pseudoremainder of f by $p, \dot{p}, \ddot{p}, \dots$

This process is called Ritt's reduction

The leading derivative \dot{y} of p plays the role of x

The initial $i_p (= 1)$ of p plays the role of p_d when reducing by p

The separant $s_p (= 2\dot{y})$ of p , which is the initial of every derivative of p , plays the role of p_d when reducing by any derivative of p

Specifications of Ritt's Reduction (v1)

There exists $d, e \in \mathbb{N}$ and $g, q_0, q_1, \dots \in \mathcal{F}\{y\}$ such that

$$i_p^d s_p^e f = g + q_0 p + q_1 \dot{p} + \dots$$

Assume f, p are differential polynomials and a ranking is fixed

$$p = \dot{y}^2 + 8xy - y$$

It is possible to compute the pseudoremainder of f by $p, \dot{p}, \ddot{p}, \dots$

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Specifications of Ritt's Reduction (v2)

There exists a power product h of the initial and the separant of p and $g \in \mathcal{F}\{y\}$ such that

$$hf = g \mod [p]$$

Assume f, p are differential polynomials and a ranking is fixed

$$p = \dot{y}^2 + 8xy - y$$

It is possible to compute the pseudoremainder of f by $p, \dot{p}, \ddot{p}, \dots$
This process is called Ritt's reduction

There exists a power product h of i_p, s_p and $g \in \mathcal{F}\{y\}$ such that

$$hf = g \mod [p]$$

Assume f, p are differential polynomials and a ranking is fixed

$$p = \dot{y}^2 + 8xy - y$$

It is possible to compute the pseudoremainder of f by $p, \dot{p}, \ddot{p}, \dots$
This process is called Ritt's reduction

There exists a power product h of i_p, s_p and $g \in \mathcal{F}\{y\}$ such that

$$hf = g \mod [p]$$

The process can be a **full** or a **partial** reduction

partial: g is free of $\ddot{y}, y^{(3)}, \dots$ (proper derivatives of \dot{y})

full: in addition, $\deg(g, \dot{y}) < \deg(p, \dot{y})$

Remark The property of the partial remainder relies on the fact that proper derivatives of p have degree 1 in their leading derivatives

$$\dot{p} = 2\dot{y}\ddot{y} + 8x\dot{y} - \dot{y} + 8y$$

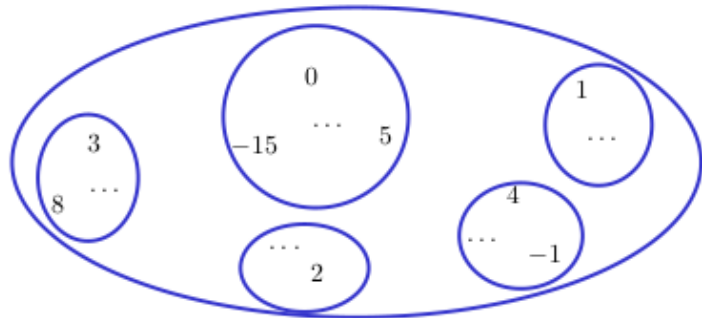
Solving in Abstract Differential Fields

It is the spirit of Ritt's *Differential Algebra* or Kolchin's *Differential Algebra and Algebraic Groups*

Though terribly abstract, it is sometimes the simplest and the most general point of view to have

To be combined with Seidenberg's *Embedding Theorem* which states that every abstract differential field is isomorphic to a field of formal power series

A. Seidenberg. *Differential Algebra and the Analytic Case*. 1958



$\mathbb{Z}/5\mathbb{Z}$

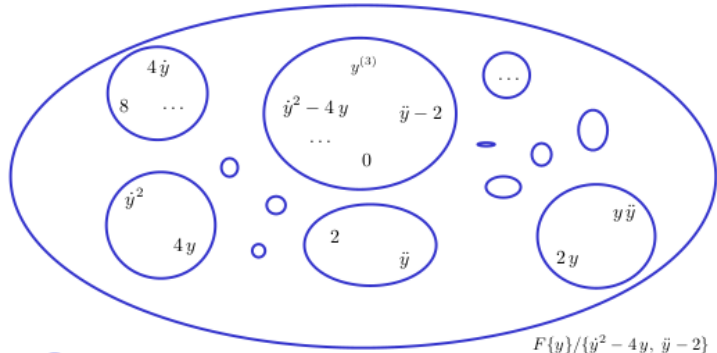
$$\begin{array}{c} \textcircled{a} \\ \textcircled{a} \end{array} + \begin{array}{c} \textcircled{b} \\ \textcircled{b} \end{array} = \textcircled{a+b}$$

$$\begin{array}{c} \textcircled{a} \\ \textcircled{a} \end{array} \times \begin{array}{c} \textcircled{b} \\ \textcircled{b} \end{array} = \textcircled{a \times b}$$

$\mathbb{Z}/5\mathbb{Z}$ is a set of five equivalence classes

It is endowed with a ring structure by defining the sum and the product of two classes

The definitions make sense because the class of $a + b$ (resp. $a \times b$) depends on the classes of a and b , not of a and b



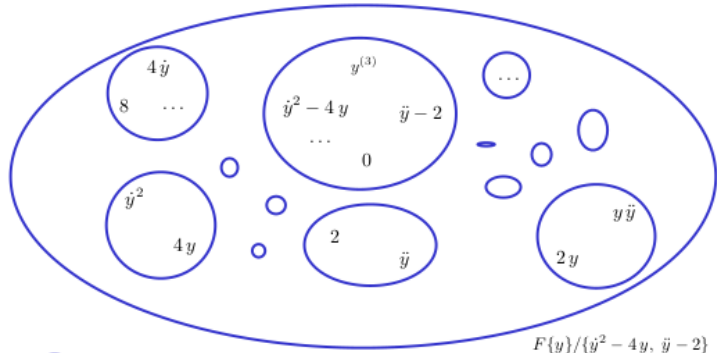
$$\begin{array}{lcl} \begin{array}{c} a \\ a \end{array} + \begin{array}{c} b \\ b \end{array} & = & \begin{array}{c} a+b \\ a+b \end{array} \\ \begin{array}{c} a \\ a \end{array} \times \begin{array}{c} b \\ b \end{array} & = & \begin{array}{c} ab \\ ab \end{array} \end{array} \quad \delta \begin{array}{c} a \\ a \end{array} = \begin{array}{c} \dot{a} \\ \dot{a} \end{array}$$

$$\mathfrak{P} = \{y^2 - 4y, y'' - 2\}$$

$\mathcal{F}\{y_1, \dots, y_n\}/\mathfrak{P}$ is a set of infinitely many equivalence classes

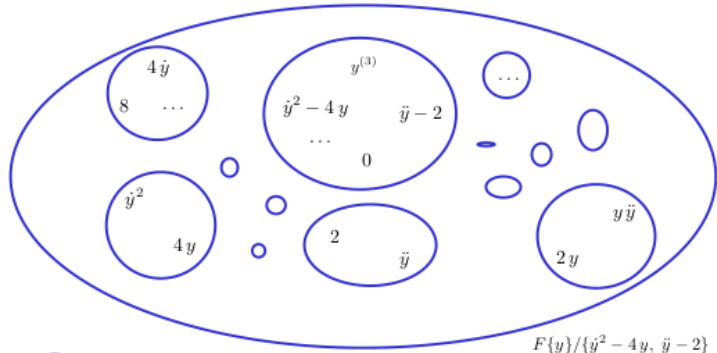
It can be endowed with a differential ring structure

By considering pairs of equivalence classes, one can define the differential field of fractions of $\mathcal{F}\{y\}/\mathfrak{P}$ provided that \mathfrak{P} is prime



$$\begin{array}{l} \textcircled{a} + \textcircled{b} = \textcircled{a+b} \\ \textcircled{a} \times \textcircled{b} = \textcircled{ab} \end{array} \quad \delta \textcircled{a} = \textcircled{\dot{a}}$$

The solution in $\mathcal{F}\{y\}/\mathfrak{P}$ of $\mathfrak{P} = \{\dot{y}^2 - 4y, \ddot{y} - 2\}$ is ...



$$\begin{array}{l} \textcircled{a} + \textcircled{b} = \textcircled{a+b} \\ \textcircled{a} \times \textcircled{b} = \textcircled{ab} \end{array} \quad \delta \textcircled{a} = \textcircled{\dot{a}}$$

The solution in $\mathcal{F}\{y\}/\mathfrak{P}$ of $\mathfrak{P} = \{\dot{y}^2 - 4y, \ddot{y} - 2\}$ is ...

$$y = \textcircled{y}$$

By the *Embedding Theorem* [Seidenberg, 1958], every such abstract differential field is isomorphic to a field of formal power series

Reduction to the Autonomous Case Almost Always Possible

A differential polynomial p in **non autonomous** if p explicitly depends on x

$$x \dot{y} + 8(x - 2)y^2 - 1 = 0 \quad (1)$$

In Ritt and Kolchin differential algebra, there is no such thing as a non autonomous differential polynomial

At the price of one extra differential indeterminate and one extra differential equation (per derivation), the reduction process is always possible provided that coefficients belong to $\mathcal{F}[x]$ or $\mathcal{F}(x)$

$$\begin{aligned} x \dot{y} + 8(x - 2)y^2 - 1 &= 0 \\ \dot{x} - 1 &= 0 \end{aligned} \quad (2)$$

Eq (1) has no formal power series solution centered at $x = 0$

Syst (2) has no formal power series solution for i.v. $x(0) = x_0 = 0$

Summary The reduction process transforms issues on expansion points into issues on initial values

Solving in Formal Power Series: Principle and Easy Case

Many tropical differential geometry problems consider non autonomous equations with coefficients in $\mathcal{F}[[x]]$

Such settings address the existence problem of formal power series solutions centered at the origin

In the easy case, we address a simpler but related problem: does there exist initial values for which formal power series solutions exist?

Recall the Brachistochrone equation

$$y \dot{y}^2 + y = D \quad (D \text{ nonzero constant})$$

It has formal power series solution but not for $y_0 = 0$

The Renaming Step

A differential equation states an equality between functions for every value of the independent variable

$$\dot{y}(x)^2 = 4y(x) \quad (\forall x)$$

The “renaming step” (next slide) expresses the fact that, the equality holds in particular for at $x = 0$ i.e. for “initial values” $y_0, y_1, \dots = y(0), \dot{y}(0), \dots$

$$y_1^2 = 4y_0$$

It is sometimes convenient to say we look for a “non differential solution” of

$$\dot{y}^2 = 4y$$

Looking for a formal power series centered at $x = \alpha$

$$\bar{y} = y_0 + y_1(x - \alpha) + y_2 \frac{(x - \alpha)^2}{2} + y_3 \frac{(x - \alpha)^3}{6} + \dots$$

solution of

$$p(x, y) = \dot{y}^2 + 8xy - y$$

Step 1: differentiate p

$$\dot{y}^2 + 8xy - y = 0$$

$$2\dot{y}\ddot{y} + 8x\dot{y} - \dot{y} + 8y = 0$$

$$2\dot{y}y^{(3)} + 2\ddot{y}^2 + 8x\ddot{y} - \ddot{y} + 16\dot{y} = 0$$

Looking for a formal power series centered at $x = \alpha$

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$$2\dot{y}y^{(3)} + 2\ddot{y}^2 + 8x\ddot{y} - \ddot{y} + 16\dot{y} = 0$$

Step 2: rename $y^{(i)}$ as y_i

$$y_1^2 + 8xy_0 - y_0 = 0$$

$$2y_1y_2 + 8xy_1 - y_1 + 8y_0 = 0$$

$$2y_1y_3 + 2y_2^2 + 8xy_2 - y_2 + 16y_1 = 0$$

Looking for a formal power series centered at $x = \alpha$

$$\bar{y} = y_0 + y_1(x - \alpha) + y_2 \frac{(x - \alpha)^2}{2} + y_3 \frac{(x - \alpha)^3}{6} + \dots$$

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solution of

$$p(x, y) = \dot{y}^2 + 8xy - y$$

Step 1: differentiate p

Step 2: rename $y^{(i)}$ as y_i

$$y_1^2 + 8xy_0 - y_0 = 0$$

$$2y_1y_2 + 8xy_1 - y_1 + 8y_0 = 0$$

$$2y_1y_3 + 2y_2^2 + 8xy_2 - y_2 + 16y_1 = 0$$

Step 3: evaluate at $x = \alpha$ and denote $p_i = p^{(i)}(\alpha, y_0, y_1, y_2, \dots)$

$$p_0 \quad y_1^2 + 8\alpha y_0 - y_0 = 0$$

$$p_1 \quad 2y_1y_2 + 8\alpha y_1 - y_1 + 8y_0 = 0$$

$$p_2 \quad 2y_1y_3 + 2y_2^2 + 8\alpha y_2 - y_2 + 16y_1 = 0$$

Looking for a formal power series centered at $x = \alpha$

$$\bar{y} = y_0 + y_1(x - \alpha) + y_2 \frac{(x - \alpha)^2}{2} + y_3 \frac{(x - \alpha)^3}{6} + \dots$$

solution of

$$p(x, y) = \dot{y}^2 + 8xy - y$$

Step 1: differentiate p

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Looking for a formal power series centered at $x = \alpha$

$$\bar{y} = y_0 + y_1(x - \alpha) + y_2 \frac{(x - \alpha)^2}{2} + y_3 \frac{(x - \alpha)^3}{6} + \dots$$

solution of

$$p(x, y) = \dot{y}^2 + 8xy - y$$

Step 1: differentiate p

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Step 3: evaluate at $x = \alpha$ and denote $p_i = p^{(i)}(\alpha, y_0, y_1, y_2, \dots)$

$$p_0 \quad y_1^2 + 8\alpha y_0 - y_0 = 0$$

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$$p_2 \quad 2y_1 y_3 + 2y_2^2 + 8\alpha y_2 - y_2 + 16y_1 = 0$$

Fact

$$p(x, \bar{y}) = p_0 + p_1(x - \alpha) + p_2 \frac{(x - \alpha)^2}{2} + p_3 \frac{(x - \alpha)^3}{6} + \dots$$

Looking for a formal power series centered at $x = \alpha$

$$\bar{y} = y_0 + y_1(x - \alpha) + y_2 \frac{(x - \alpha)^2}{2} + y_3 \frac{(x - \alpha)^3}{6} + \dots$$

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Step 3: evaluate at $x = \alpha$ and denote $p_i = p^{(i)}(\alpha, y_0, y_1, y_2, \dots)$

$$\begin{aligned} p_0 \quad y_1^2 + 8\alpha y_0 - y_0 &= 0 \\ p_1 \quad 2y_1 y_2 + 8\alpha y_1 - y_1 + 8y_0 &= 0 \\ p_2 \quad 2y_1 y_3 + 2y_2^2 + 8\alpha y_2 - y_2 + 16y_1 &= 0 \end{aligned}$$

Notice the separant $s_p = 2\dot{y}$, the initial of every derivative of p , which provides the leading coefficient of p_1, p_2, \dots

$$\bar{y} = y_0 + y_1(x - \alpha) + y_2 \frac{(x - \alpha)^2}{2} + y_3 \frac{(x - \alpha)^3}{6} + \dots \quad (3)$$

$$p(x, y) = \dot{y}^2 + 8xy - y \quad (4)$$

Steps 1, 2, 3: differentiate, rename, evaluate at $x = \alpha$

$$\begin{aligned} p_0 \quad y_1^2 + 8\alpha y_0 - y_0 &= 0 \\ p_1 \quad 2y_1 y_2 + 8\alpha y_1 - y_1 + 8y_0 &= 0 \end{aligned}$$

Step 4: solve and substitute the solution in \bar{y}

$$\bar{y} = y_0 + y_1(x - \alpha) + y_2 \frac{(x - \alpha)^2}{2} + y_3 \frac{(x - \alpha)^3}{6} + \dots \quad (3)$$

$$p(x, y) = \dot{y}^2 + 8xy - y \quad (4)$$

Steps 1, 2, 3: differentiate, rename, evaluate at $x = \alpha$

$$\begin{aligned} p_0 \quad y_1^2 + 8\alpha y_0 - y_0 &= 0 \\ p_1 \quad 2y_1 y_2 + 8\alpha y_1 - y_1 + 8y_0 &= 0 \end{aligned}$$

Step 4: solve and substitute the solution in \bar{y}

Example Looking for y_2 ? Compute the partial remainder g of \ddot{y} by p

$$\dot{p} = 2\dot{y}\ddot{y} + 8x\dot{y} - \dot{y} + 8y$$

Thus

$$\underbrace{(2\dot{y})}_{h} \ddot{y} = \underbrace{-8x\dot{y} + \dot{y} - 8y}_g \pmod{[p]}$$

and

$$y_2 = \frac{g(\alpha, y_0, y_1)}{h(\alpha, y_0, y_1)}$$

$$\bar{y} = y_0 + y_1(x - \alpha) + y_2 \frac{(x - \alpha)^2}{2} + y_3 \frac{(x - \alpha)^3}{6} + \dots \quad (3)$$

$$p(x, y) = \dot{y}^2 + 8xy - y \quad (4)$$

Steps 1, 2, 3: differentiate, rename, evaluate at $x = \alpha$

$$\begin{aligned} p_0 \quad y_1^2 + 8\alpha y_0 - y_0 &= 0 \\ p_1 \quad 2y_1 y_2 + 8\alpha y_1 - y_1 + 8y_0 &= 0 \end{aligned}$$

Step 4: solve and substitute the solution in \bar{y}

Summary Every solution of the following system on initial values can be prolonged to a differential solution \bar{y}

$$y_1^2 + 8\alpha y_0 - y_0 = 0, \quad 2y_1 \neq 0.$$

Reformulation Every “non differential” zero of p which does not annihilate the initial and separant of p can be prolonged into a differential zero

A Famous Ritt Example

The solution set of

$$\dot{y}^2 - 4y = 0$$

can be decomposed in two cases

$$\underbrace{\begin{cases} \dot{y}^2 = 4y, \\ 2\dot{y} \neq 0 \end{cases}}_{y(x)=(x+c)^2} \quad \text{and} \quad \underbrace{y = 0}_{y(x)=0}$$

The general solution $y(x) = (x + c)^2$ corresponds to initial values

$$y_0, y_1, y_2 = c^2, 2c, 2$$

One of these formal power series solutions ($y(x) = x^2$) satisfies

$$2\dot{y} \neq 0 \quad 2y_1 = 0.$$

The study of these formal power series solutions belong to the difficult case

Generalization of the Easy Case to Systems

$$\mathcal{F}\{y_1, \dots, y_n\}$$

$A = \{p_1, \dots, p_r\}$ a triangular set of differential polynomials

ODE case: the elements of A must be pairwise partially reduced

PDE case: in addition, A must be coherent

Note The coherence condition is due to A. Rosenfeld (1959) who generalizes a result of A. Seidenberg. Kolchin's definition hides its algorithmic nature

A. Rosenfeld. *Specializations in Differential Algebra*. 1959

A. Seidenberg *An Elimination Theory in Differential Algebra*. 1956

Remark Up to some encoding, Gröbner bases (1965) are a particular case of coherent systems

The Pairwise Partially Reduced Condition

$\mathcal{F}\{y, z\}$ with a single derivation

$A = \{p_1, \dots, p_r\}$ a triangular set of differential polynomials

$H_A = i_1 \cdots i_r s_1 \cdots s_r$ the product of the initials and separants of A

The elements of A are not pairwise partially reduced

$$A \begin{cases} z &= \ddot{y}, \\ \dot{y}^2 &= 4y. \end{cases}$$

At steps 3 and 4, the constraint $y_2 = 2$ is missed (assuming $y_1 \neq 0$)

The Pairwise Partially Reduced Condition

$\mathcal{F}\{y, z\}$ with a single derivation

$A = \{p_1, \dots, p_r\}$ a triangular set of differential polynomials

$H_A = i_1 \cdots i_r s_1 \cdots s_r$ the product of the initials and separants of A

Thanks to Ritt's partial reduction, the elements of A can be made pairwise partially reduced under the assumption $\dot{y} \neq 0$

$$A \left\{ \begin{array}{l} z = 2, \\ \dot{y}^2 = 4y, \end{array} \quad \dot{y} \neq 0. \right. \quad (5)$$

Summary If the elements of A are pairwise partially reduced then every non differential zero of the following system can be prolonged into a differential zero

$$A = 0, \quad H_A \neq 0.$$

Remark The system may have no solution

The Coherence Condition

In $\mathcal{F}\{y, z\}$ endowed with $\partial/\partial x$ and $\partial/\partial t$

The elements of A are pairwise partially reduced and $H_A = 1$ but A is not coherent

$$A \begin{cases} y_x &= z, \\ y_t &= 0 \end{cases}$$

Not every non differential zero of $A = 0$ can be prolonged to a differential zero since $z_t = 0$ (A is not coherent)

The Coherence Condition

In $\mathcal{F}\{y, z\}$ endowed with $\partial/\partial x$ and $\partial/\partial t$

System A can be made coherent

$$A \begin{cases} y_x &= z, \\ y_t &= 0, \\ z_t &= 0 \end{cases}$$

Def z_t is a particular case of a Δ -polynomial

Test If all Δ -polynomials of A are reduced to zero by A then A is coherent

Summary If A is coherent and its elements are pairwise partially reduced then every non differential zero of following system can be prolonged into a differential zero

$$A = 0, \quad H_A \neq 0.$$

Specification of the Differential Elimination Algorithm

Given an input system Σ and a ranking, it is possible to compute an equivalent set of finitely many regular differential chains

$$A_1, \dots, A_\varrho$$

If this decomposition is empty ($\varrho = 0$) then $1 \in \{\Sigma\}$ (the perfect differential ideal) and Σ has no solution at all

If this decomposition is nonempty ($\varrho > 0$) then every non differential zero of

$$A_i = 0, \quad H_{A_i} \neq 0$$

can be prolonged into a differential zero of Σ

Hence there exist initial values for which Σ has formal power series solutions

Solving in Formal Power Series: the Difficult Case

$$p(x, y) = 0 \quad (\text{ord } p = n)$$

Difficult cases arise when the separant $s(x, y) = \partial p / \partial y^{(n)}$ does not vanish identically but vanishes at the prescribed initial values

At some point, the existence problem of a formal power series solution amounts to find nonnegative integer solutions to polynomials in m variables (the number of derivations)

The PDE case mostly leads to undecidability results by Matiassevich (1970) negative answer to Hilbert's 10th Problem

The ODE case ($m = 1$) seems to be algorithmic

J. Denef and L. Lipshitz. *Power Series Solutions of Algebraic Differential Equations*. 1984

A. Hurwitz. *Sur le développement des fonctions satisfaisant à une équation différentielle algébrique*. 1899

The Number of Initial Values Depends on the Initial Values

$$\begin{aligned}\bar{y} &= y_0 + y_1 x + y_2 \frac{x^2}{2} + y_3 \frac{x^3}{6} + \cdots \quad \text{solution of} \\ p(x, y) &= 0 \quad (\text{ord } p = n)\end{aligned}$$

The number of needed initial values depends on the initial values

The Number of Initial Values Depends on the Initial Values

$$\begin{aligned}\bar{\bar{y}} &= y_0 + y_1 x + y_2 \frac{x^2}{2} + y_3 \frac{x^3}{6} + \cdots \quad \text{solution of} \\ p(x, y) &= 0 \quad (\text{ord } p = n)\end{aligned}$$

Let us give ourselves infinitely many of them and encode them in a series

$$\bar{y} = y_0 + y_1 x + y_2 \frac{x^2}{2} + y_3 \frac{x^3}{6} + \cdots \quad (\text{i.v. encoding series})$$

The Number of Initial Values Depends on the Initial Values

$$\begin{aligned}\bar{\bar{y}} &= y_0 + y_1 x + y_2 \frac{x^2}{2} + y_3 \frac{x^3}{6} + \cdots \quad \text{solution of} \\ p(x, y) &= 0 \quad (\text{ord } p = n)\end{aligned}$$

Let us give ourselves infinitely many of them and encode them in a series

$$\bar{y} = y_0 + y_1 x + y_2 \frac{x^2}{2} + y_3 \frac{x^3}{6} + \cdots \quad (\text{i.v. encoding series})$$

From p and \bar{y} compute

δ the number of needed initial values

β the size of the polynomial system to solve

$$p_0 = p_1 = \cdots = p_{\beta-1} = 0$$

Thm Every solution of the polynomial system (if any) can be prolonged into a formal power series solution \bar{y}

Leading Coefficients are Now Given by a Polynomial $A(q)$

$$p(x, y) = y \ddot{y} + \dot{y}^2 - 6y$$

$$s(x, y) = y$$

$$\bar{y} = y_2 \frac{x^2}{2} + \cdots \quad (y_0 = y_1 = 0)$$

$$k = 2 \quad (\text{the valuation of } s(x, \bar{y}), \text{ assuming } y_2 \neq 0)$$

$$A(q) = y_2 q^2 + 15 y_2 q + 56 y_2 - 12$$

For $(y_0, y_1, y_2) = (0, 0, 2)$ we have $\beta = \delta = 8$

$$\begin{aligned} p_3 & \frac{7}{3} y_3 & = 0 & \frac{A(-3)}{3!} \\ p_4 & 1 y_4 + \frac{5}{12} y_3^2 & = 0 & \frac{A(-2)}{4!} \\ p_5 & \frac{3}{10} y_5 + \frac{7}{24} y_3 y_4 & = 0 & \frac{A(-1)}{5!} \\ p_6 & \frac{5}{72} y_6 + \frac{7}{144} y_4^2 + \frac{7}{90} y_3 y_5 & = 0 & \frac{A(0)}{6!} \\ p_7 & \frac{11}{840} y_7 + \frac{1}{40} y_4 y_5 + \frac{1}{60} y_3 y_6 & = 0 & \frac{A(1)}{7!} \end{aligned}$$

We have found $y(x) = x^2$

An Undecidability Result

The problem: *does a given ordinary differential polynomial system have any nonzero formal power series solution centered at the origin?* is algorithmically undecidable [Singer, 1978]

Singer's argument relies on the following system

$$\begin{aligned}x \dot{y} &= \alpha y \\ \dot{\alpha} &= 0\end{aligned}$$

which has a nonzero formal power series solution $y(x) = x^\alpha$ if and only if α is a nonnegative integer (Hilbert's 10th Problem)

Denef and Lipshitz formula permits to solve the existence and uniqueness problem of a formal power series solution under the assumption that we can determine k (over our example, we had to assume $y_2 \neq 0$)