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A new twist on the large size limit behaviour of networks of Hopfield-like neurons

Olivier C Faugeras, Etienne Tanré

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We revisit the problem of characterising the thermodynamic limit of a fully connected network of Hopfield-like neurons. Our contributions are that we provide a) a complete description of the mean-field equations as a set of stochastic differential equations depending on a mean ($m(t)$) and covariance ($K(t, s)$) functions, b) a provably convergent method for estimating these functions, and c) numerical results of this estimation as well as examples of the resulting dynamics. The mathematical tools are *the theory of Large Deviations, Itô stochastic calculus, and the theory of Volterra equations*. Starting from the equation of the membrane potential X_t of a single neuron with intrinsic dynamics g , submitted to Brownian noise W_t

$$\begin{cases} dX_t &= g(X_t)dt + \lambda dW_t \\ \text{Law of } X_0 &= \mu_0, \end{cases} \quad (1)$$

we couple N such neurons with independent initial distribution (*Law of* $(X_0^1, \dots, X_0^N) = \mu_0^{\otimes N}$) as

$$dX_t^i = (g(X_t^i) + \sum_j J_{ij} f(X_t^j)) dt + \lambda dW_t^i \quad 1 \leq i \leq N \quad (2)$$

f is a regular bounded function (typically a sigmoid) that converts membrane potential to activities. The W^i s are independent Brownians. The synaptic weights J_{ij} are i.i.d. Gaussian random variables

$$J_{ij} \simeq \mathcal{N}\left(\frac{J}{N}, \frac{\sigma^2}{N}\right), \quad i, j = 1, \dots, N \quad (3)$$

The theory of large deviations allows us, inspired by [1], to write the mean-field equations:

$$\begin{cases} X_t &= X_0 + \int_0^t g(X_s) ds + \lambda B_t & B_t &= \int_0^t m_Q(s) ds + B_t^0 \\ B_t^0 &= W_t + \int_0^t \int_0^s \tilde{K}_Q^s(s, u) dB_u^0 ds & m_Q(t) &= \bar{J} \mathbb{E}_Q[f(X_t)], \quad \bar{J} = J/\lambda \\ K_Q(t, s) &= \bar{\sigma}^2 \mathbb{E}_Q[f(X_t)f(X_s)], \quad \bar{\sigma} = \sigma/\lambda & Q &\text{ is the law of } X_t, \quad W_t \text{ is a Brownian under } Q \end{cases} \quad (4)$$

\tilde{K}_Q^t is a simple function of K_Q . The theory of Volterra equations allows us to compute B^0 as a function of the Brownian W and the covariance K_Q :

$$B_t^0 = W_t + \int_0^t \tilde{W}_s ds + \int_0^t \left(\int_0^s H_Q^s(s, u) \tilde{W}_u du \right) ds \quad \text{where } \tilde{W}_t = \int_0^t \tilde{K}_Q^t(t, s) dW_s, \quad (5)$$

and $H_Q^t(t, s)$ is the *resolvent kernel* of the corresponding Volterra equation in the variable B_t^0 which is a simple function of \tilde{K}_Q and hence of K_Q . Equations (4) and (5) are a new description of the mean-field dynamics as a set of non-Markov stochastic equations.

The functions K_Q and m_Q are unknown functions of g, f, \bar{J} and $\bar{\sigma}$. We have proved that a certain map defined on the set of pairs of functions $(m(t), K(t, s))$ has a unique fixed point, i.e. $(m_Q(t), K_Q(t, s))$ which defines the dynamics (4) and characterized its rate of convergence.

We have implemented the corresponding algorithm in the Julia language using Monte Carlo methods and found experimentally that it converged in just a few iterations, allowing us to simulate the mean-field dynamics (4).

We show some of our numerical results in the Figure below for a network of Hopfield neurons, i.e. $g(x) = -x, f$ a sigmoid. From left to right we use $J = -6$ (inhibitory network), $J = 0$ and $J = 2$ (excitatory network). In all cases we have $\sigma = \lambda = 1$. The simulations run over 10 seconds. Due to lack of space we do not show any results for the spin-glass network.

To summarize we have generalized and set on a firm mathematical basis the work of Sompolinsky et al., e.g. [2], Helias et al., e.g. [4]. We have also extended the work of Ben Arous and Guionnet, e.g. [1], and clarified and complemented the work of Faugeras et al., e.g. [3].

References

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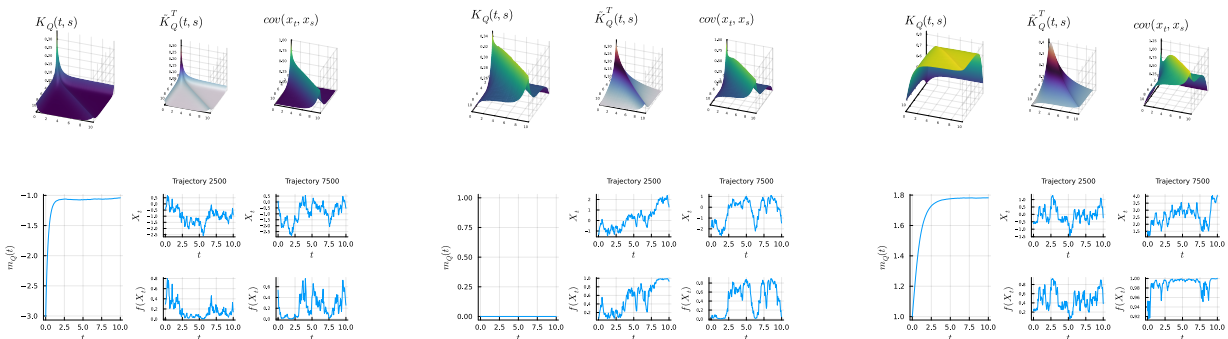


Figure 1: Results of the Hopfield network mean-field estimation