



**HAL**  
open science

# A new twist on the large size limit behaviour of networks of Hopfield-like neurons

Olivier C Faugeras, Etienne Tanré

► **To cite this version:**

Olivier C Faugeras, Etienne Tanré. A new twist on the large size limit behaviour of networks of Hopfield-like neurons. 2024. hal-04391216

**HAL Id: hal-04391216**

**<https://hal.science/hal-04391216>**

Preprint submitted on 12 Jan 2024

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

We revisit the problem of characterising the thermodynamic limit of a fully connected network of Hopfield-like neurons. Our contributions are that we provide a) a complete description of the mean-field equations as a set of stochastic differential equations depending on a mean ( $m(t)$ ) and covariance ( $K(t, s)$ ) functions, b) a provably convergent method for estimating these functions, and c) numerical results of this estimation as well as examples of the resulting dynamics. The mathematical tools are *the theory of Large Deviations, Itô stochastic calculus, and the theory of Volterra equations*. Starting from the equation of the membrane potential  $X_t$  of a single neuron with intrinsic dynamics  $g$ , submitted to Brownian noise  $W_t$

$$\begin{cases} dX_t &= g(X_t)dt + \lambda dW_t \\ \text{Law of } X_0 &= \mu_0, \end{cases} \quad (1)$$

we couple  $N$  such neurons with independent initial distribution (*Law of*  $(X_0^1, \dots, X_0^N) = \mu_0^{\otimes N}$ ) as

$$dX_t^i = (g(X_t^i) + \sum_j J_{ij} f(X_t^j)) dt + \lambda dW_t^i \quad 1 \leq i \leq N \quad (2)$$

$f$  is a regular bounded function (typically a sigmoid) that converts membrane potential to activities. The  $W^i$ s are independent Brownians. The synaptic weights  $J_{ij}$  are i.i.d. Gaussian random variables

$$J_{ij} \simeq \mathcal{N}\left(\frac{J}{N}, \frac{\sigma^2}{N}\right), \quad i, j = 1, \dots, N \quad (3)$$

The theory of large deviations allows us, inspired by [1], to write the mean-field equations:

$$\begin{cases} X_t &= X_0 + \int_0^t g(X_s) ds + \lambda B_t & B_t &= \int_0^t m_Q(s) ds + B_t^0 \\ B_t^0 &= W_t + \int_0^t \int_0^s \tilde{K}_Q^s(s, u) dB_u^0 ds & m_Q(t) &= \bar{J} \mathbb{E}_Q[f(X_t)], \quad \bar{J} = J/\lambda \\ K_Q(t, s) &= \bar{\sigma}^2 \mathbb{E}_Q[f(X_t)f(X_s)], \quad \bar{\sigma} = \sigma/\lambda & Q &\text{ is the law of } X_t, \quad W_t \text{ is a Brownian under } Q \end{cases} \quad (4)$$

$\tilde{K}_Q^t$  is a simple function of  $K_Q$ . The theory of Volterra equations allows us to compute  $B^0$  as a function of the Brownian  $W$  and the covariance  $K_Q$ :

$$B_t^0 = W_t + \int_0^t \tilde{W}_s ds + \int_0^t \left( \int_0^s H_Q^s(s, u) \tilde{W}_u du \right) ds \quad \text{where } \tilde{W}_t = \int_0^t \tilde{K}_Q^t(t, s) dW_s, \quad (5)$$

and  $H_Q^t(t, s)$  is the *resolvent kernel* of the corresponding Volterra equation in the variable  $B_t^0$  which is a simple function of  $\tilde{K}_Q$  and hence of  $K_Q$ . Equations (4) and (5) are a new description of the mean-field dynamics as a set of non-Markov stochastic equations.

The functions  $K_Q$  and  $m_Q$  are unknown functions of  $g, f, \bar{J}$  and  $\bar{\sigma}$ . We have proved that a certain map defined on the set of pairs of functions  $(m(t), K(t, s))$  has a unique fixed point, i.e.  $(m_Q(t), K_Q(t, s))$  which defines the dynamics (4) and characterized its rate of convergence.

We have implemented the corresponding algorithm in the Julia language using Monte Carlo methods and found experimentally that it converged in just a few iterations, allowing us to simulate the mean-field dynamics (4).

We show some of our numerical results in the Figure below for a network of Hopfield neurons, i.e.  $g(x) = -x, f$  a sigmoid. From left to right we use  $J = -6$  (inhibitory network),  $J = 0$  and  $J = 2$  (excitatory network). In all cases we have  $\sigma = \lambda = 1$ . The simulations run over 10 seconds. Due to lack of space we do not show any results for the spin-glass network.

To summarize we have generalized and set on a firm mathematical basis the work of Sompolinsky et al., e.g. [2], Helias et al., e.g. [4]. We have also extended the work of Ben Arous and Guionnet, e.g. [1], and clarified and complemented the work of Faugeras et al., e.g. [3].

## References

- [1] Gérard BenArous and Alice Guionnet. Large deviations for Langevin spin glass dynamics. *Probability Theory and Related Fields*, 102(4):455–509, 1995.
- [2] A. Crisanti and H. Sompolinsky. Path integral approach to random neural networks. *Physical Review E*, 98(6):062120, December 2018.
- [3] Olivier Faugeras, Jonathan Touboul, and Bruno Cessac. A constructive mean-field analysis of multi population neural networks with random synaptic weights and stochastic inputs. *Frontiers in Computational Neuroscience*, 3, 2009.
- [4] Alexander van Meegen, Tobias Kühn, and Moritz Helias. Large-Deviation Approach to Random Recurrent Neuronal Networks: Parameter Inference and Fluctuation-Induced Transitions. *Physical Review Letters*, 127(15):158302, October 2021.

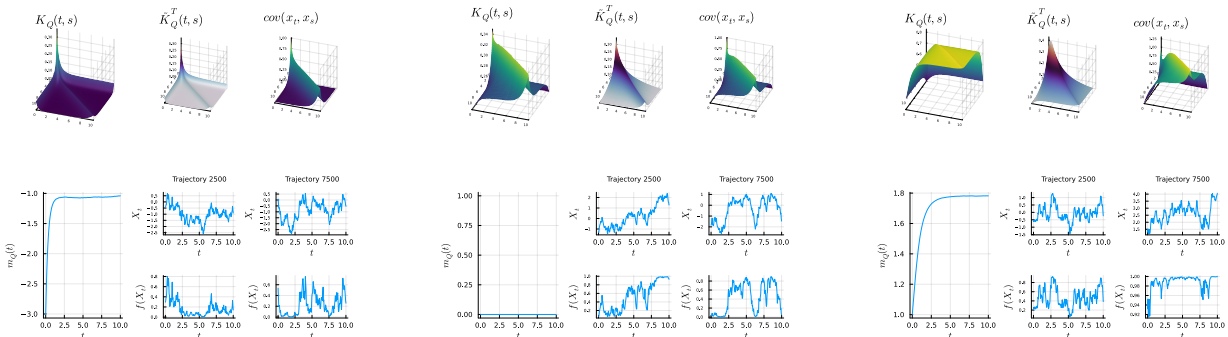


Figure 1: Results of the Hopfield network mean-field estimation