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(Conceptual) explanations in logic

For here it is for the empirical scientist to know the fact and for the mathematical to know the *reason why* [our emphasis] Aristotle (1993).

Abstract

To explain phenomena in the world is a central human activity and one of the main goals of rational inquiry. There are several types of explanation: one can explain by drawing an analogy, as one can explain by dwelling on the causes (see e.g. see Woodward (2004)). Amongst these different kinds of explanation, in the last decade philosophers have become receptive to those explanations which explain by providing *the reasons why* a statement is true; these explanations are often called *conceptual explanations* (e.g. see Betti (2010)). The main aim of the paper is to propose a logical account of conceptual explanations. We will do so by using the resources of proof theory, in particular sequent rules analogous to *deep inferences* (e.g. see Brünnler (2004)). The results we provide not only shed light on conceptual explanations themselves, but also on the role that logic and logical tools might play in the burgeoning field of inquiry concerning explanations. Indeed, we conclude the paper by underling interesting links between the present research and some other existing works on explanations and logic that have arise in recent years, e.g. see Arieli et al. (2022); Piazza et al. (2023).

1 Introduction

To explain phenomena in the world is a characteristically human enterprise and a central goal of rational inquiry; it is thus no surprise that the notion of explanation has been one of the most intensely discussed topics in philosophy of science over the past century, and computer science over the past decade. Though the term covers a wide range of diverse cases – from explaining how to build an Ikea bookcase, or what a symbol means, to explaining a new concept to a child – doubtless the main ones for human endeavor are explanations that have a deductive form and aim to shed light on why a certain phenomenon occurs or why a certain proposition is true. Archetypal examples are *causal explanations* (Woodward, 2004; Pearl, 2000), which explain their target phenomena by providing their causes. However, it has recently become increasingly clear that plenty of compelling examples of explanations-why cannot be captured by causal accounts. In physics as well as in mathematics, several types of explanations arose that do not seem to rely on any causal mechanism: very naturally, the idea that causation although certainly being a key ingredient of explanation, is probably not the full story, started to spread; non-causal explanations, namely explanations that in one way or another go beyond causation, have become a new thrilling and thriving subject of research.¹

¹E.g. see Lange (2017).

Figure 1: Examples of reasons-conclusion relation

	Toy examples		
Informal	Example 1.1.	Example 1.2.	Example 1.3.
Formal	$\neg(p \vee q)$ because $\neg p$ and $\neg q$	$\exists x(Vx)$ because $\exists x(F_1(x) \wedge F_2(x))$	$\forall x(SC(x) \rightarrow IC(x))$ be- cause $\forall x(SC(x) \rightarrow C(x))$ $\forall x(SC(x) \rightarrow T(x))$

Here we focus on a type of non-causal explanation that has been receiving increasing attention in the philosophical literature: *conceptual explanations* (e.g. see Betti (2010); Detlefsen (1988); Hunnean (2010); Mancosu et al. (2023)). Recognition of such explanations dates back millennia; as illustrated by the epigraph, a long tradition of scholars, including Aristotle, Proclus, Leibniz, Arnauld and Nicole, Bolzano, Frege,² has argued for their importance for the scientific inquiry. Conceptual explanations bear a striking analogy to causal explanations: where the latter seeks to explain by providing the causes, the former explains why a proposition is true by identifying the reasons for its truth. Instances of conceptual explanations range from stock, toy examples to more involved, real-life ones. The argument which explains why (1.1.) it is not the case that it is raining or it is windy because it is neither raining nor it is windy (together with the definition of the “or”) is an example of conceptual explanation. Indeed it displays the reasons, rather than the causes, for the conclusion, with the relation between reasons and conclusion holding in virtue of the concepts - *not* and *or* - that they contain. Similarly, the explanation of why (1.2.) a certain animal is a vixen that evokes that animal being a female as well as it being a fox (together with the definition of vixen), is an example of conceptual explanation. Indeed it displays the reasons, rather than the causes, of why that animal is a vixen, with the relation between reasons and conclusion holding in virtue of the concepts - *vixen*, *female* and *fox* - that they contain. Finally, the explanation why (1.3.) all Stanford Computer Science graduates are ideal candidates for a particular tech job in terms of their coding competences and teamwork skills (together with the stipulation of what an ideal candidate for that position is) is another example of a conceptual explanation in that it explains why a certain conclusion is true by bring out the reason(s) why it is true.

Indeed, beyond toy examples, there are many instances of conceptual explanation with more refined logical structure, in particular involving quantifiers. This is especially the case for *mathematical explanations* that take the form of proofs in mathematics that not only show a theorem to be true, but also seem to provide the reason(s) why it is true: as frequently noted, these could be argued to count among conceptual explanations (Betti, 2010; Mancosu et al., 2023). A simple example is the (elementary) proof which explains why (1.4) zero or the successor of any natural number is a natural number by emphasizing that zero is a natural number, and any successor of a natural number is a natural number. The literature on mathematical explanations is replete with less trivial examples (see e.g. Mancosu et al. (2023)). An old one dates back to Bolzano (2014), who considers the theorem which states that (1.5.) given any two circles A and B , one with center a and radius ab , and the other with center b and radius ab , then there always exists a point c where they intersect such that $l(ac) = l(cb) = l(ab)$. There exists a proof of this theorem that crucially relies on the property that, for any two points a and b , there always exists a third point c such that $l(ab) = l(bc) = l(ac)$. Bolzano argues that this proof is explanatory insofar as it

²E.g. see Detlefsen (1988).

relies on the relation between the property of the points - the reason - and the property of the circles - the conclusion. In its turn, this relation holds in virtue of the concepts involved, namely the concepts of point, radius, circle.

Or, another example, from one of the groundbreaking articles in the recent literature on mathematical explanation (Steiner, 1973) concerns Pythagoras' theorem. Out of the many proofs of the theorem, one that is often argued to be explanatory crucially relies on a property of similar triangles. Following Steiner (1973), but also a more recent and detailed analysis put forward by Poggiolesi (2024), this proof is explanatory in that it relies on the relation between the property of the similar triangles - the reason - and the property of the right triangle - the conclusion. In its turn, this relation holds in virtue of the concepts the sentences it connects contain, namely the concepts of similarity, and right angled triangle. These mathematical explanatory proofs are paradigmatic examples of conceptual explanation, in that they display the features of this type of explanation.

Despite their widespread relevance, to date conceptual explanations have received little attention in logic. This absence of systematic study is all the more striking given the recent interest in logic in non-causal explanations, be it in the field of XAI, e.g. see Darwiche and Hirth (2023); Shih et al. (2018), or as for abductive reasoning Arieli et al. (2022); Arieli and Strasser (2015); Piazza et al. (2023); Pulcini and Varzi (2021). As a result, this is an important lack of the logical literature and the main goal of the present paper is to fill this lack. In particular we aim at developing a logical theory of conceptual explanations, which is strong enough to encompass the several different cases of conceptual explanations.

There are (at least) two main questions that a logical theory of (conceptual) explanations need to address, namely (i) what is the structure of conceptual explanations? And (ii) what are the reasons for a certain conclusion? By relying on previous works developed in Poggiolesi (2024, 2016b, 2018), which is, as far as we know, the only one that have previously addressed analogous questions, we provide an answer to both (i) and (ii). As for (i) we put forward a modelization of conceptual explanations in terms of proofs. In particular, we introduce explanatory rules which are such that not only is the conclusion inferable from the premisses, but also the premisses represent the reasons why the conclusion is true. The concatenation of these rules represent the logical structure behind conceptual explanations. Note that, as the examples above show (see Figures 1 and 2), reasons are often linked to their conclusion by operating inside formulas. Take for instance the example 1.4., where $\forall x((Zx \vee SNx) \rightarrow Nx)$ is explained by $\forall x(Zx \rightarrow Nx)$ and $\forall x(SNx \rightarrow Nx)$. The link between these formula occurs deep inside the formulas themselves, i.e. the connective \vee inside the explanandum is broken into two and thus give rise to $\forall x(Zx \rightarrow Nx)$ and $\forall x(SNx \rightarrow Nx)$. As a consequence, explanatory rules will have the form of deep inferences, namely a recently introduced variation of the sequents calculus (e.g. see Brännler (2004); Guglielmi and Bruscoli (2009); Pimentel et al. (2019)) where rules operate deep inside formulas. Although the introduction of deep inferences have been motivated by cornerstone results of structural proof theory, in this context they reveal a profound philosophical significance.

As for (ii), we will set out features which establish when some formulas can be seen as the reasons why another is true. In other terms, by relying on deep insights which can be found in the philosophical litterature, we will set out those conditions which are necessary and sufficient to ensure that some formulas can be seen as the reasons why another is true. Moreover, we will show that the answer to this question goes hand in hand with the answer to the previous question, namely we will prove that our explanatory rules provides all and only those relations from reasons to conclusion that satisfies the features mentioned above.

As already said, the present work relies on the work developed in Poggiolesi (2024, 2016b, 2018); it also extends it in three related significant ways. It takes into account

Figure 2: Example of reasons-conclusion relation

	Mathematical examples		
Informal	Example 1.4	Example 1.5.	Example 1.6.
Formal	$\forall x(Zx \vee SNx \rightarrow Nx)$ because $\forall x(Zx \rightarrow Nx)$ and $\forall x(SNx \rightarrow Nx)$	$\forall x\forall y(\exists z(Circ(z, x, xy)) \wedge \exists w(Circ(w, y, xy)) \rightarrow \exists k(Point(k) \wedge k \in z \wedge k \in w \wedge l(kx) = l(ky) = l(xy)))$ because $\forall x\forall y((Point(x) \wedge Point(y)) \rightarrow \exists k(Point(k) \wedge l(kx) = l(ky) = l(xy)))$	$\Rightarrow \forall x\forall y\forall z\forall w(Rtr(xyz) \wedge xwz // xwy) \rightarrow xz^2 + xy^2 = zy^2)$ because $\forall A\forall B\forall C\forall H(Sim(ABC, AHC) \rightarrow AC^2 = HC.BC)$ and $\forall A\forall B\forall C\forall H(Sim(ABC, AHB) \rightarrow AB^2 = BH.BC)$

intricate examples of conceptual explanations; it has explanatory rules that go deep inside formulas; and finally it relies on the sequent calculus rather than on natural deduction. As a result, the present paper offers a very general theory of (conceptual) explanations which doesn't have any analogue in the literature. We will dedicate two sections to underline how this theory relates to other existing works in explanations, and how it paves the way for interesting paths of futures research.

The paper is organised in the following way. In *Section 2* we will clarify the formal framework where we will develop our account along with some characteristics of conceptual explanations as well as the related relation of reasons and conclusion. Whilst in *Section 3* we will set out the conditions under which some formulas are the formal reasons of another, in *Section 4* we will formalize the notion of conceptual explanation, via the notion of formal explanation. *Section 5* will serve to prove soundness and completeness between formal reasons-conclusion relation and formal explanation and in *Section 6* we will prove some results concerning our formal theory of explanation. Whilst in *Section 7* we will emphasize some interesting links with other related works, in *Section 8* we will draw some conclusions and sketch directions of future research.

2 Formal explanations

In order to provide a formal structure which underly conceptual explanation, we start from an idea that is both ancient and central in the literature: explanations can be seen as deductive *arguments* which, starting from true premisses - be them the causes or the reasons - explain a certain conclusion.³ Of course not any deductive argument constitutes an explanation, but some of them do, namely those which have an explanatory power. The perspective that we will develop here consists in a formalization of this central idea along the following lines: explanations can be seen as *proofs* which, starting from true premisses, the reasons, not only prove that a certain conclusion is true, but also explain why it is such. This perspective naturally arises from the observation that proofs are deductive arguments; moreover, it is supported by the fact that mathematical explanations, a notable subset of conceptual explanations, actually are proofs of mathematical theorems that show why those theorems are true.

³E.g. see Aristotle (1993); Hempel (1965, 1942).

Let us pursue this perspective further. Since proofs are standardly formalized in logic by means of *derivations*, we will formalize conceptual explanations as a special type of derivations. More precisely, we will introduce a metalinguistic relation that we will call *formal explanation*,⁴ denoted by the symbol \Vdash , and which will represent the formal counterpart of conceptual explanations as well as a special case of the standard notion of derivation. As derivations are introduced via inferential rules, formal explanations will be introduced via explanatory rules, namely rules where not only is the conclusion inferable from their premise(s), but also such that the premisses are the reasons why the conclusion is true. We will consider explanatory rules, and related formal explanation relation, in first-order classical logic.

Definition 2.1. The language of first-order logic, \mathcal{L} , is composed by: variables (x_0, x_1, x_2, \dots), constants (c_0, c_1, c_2, \dots), predicates ($P_0^k, P_1^k, P_2^k, \dots$), logical connectives (\neg, \wedge, \vee), quantifiers (\forall, \exists), and parentheses: ($,)$. We take the symbols \top, \perp and \rightarrow to be defined as usual. For the sake of simplicity we do not use the identity symbol nor the functional symbols. Also we will use the symbols \circ and \odot in the following way: $\circ = \{\wedge, \vee\}$ and $\odot = \{\forall, \exists\}$. The set of well-defined formulas, \mathcal{WF} , is constructed in the standard way. A closed formula, or a sentence, is a formula where no free variable occurs. The set of closed formulas of \mathcal{L} will be denoted by \mathcal{CF} .

Definition 2.2. Given, the multiset $M \subseteq \mathcal{WF}$ and formula $A \in \mathcal{WF}$, we use the standard notation, $M \models A$, to mean that A logically follows from M in first-order classical logic. The notation $M \vdash A$ means that there exists a derivation from M to A in (the standard sequent calculus for) first-order classical logic.

In order to properly spell out the notion of (conceptual) explanation we are after, we introduce some notable distinctions that help identifying different types of deductive explanations. Here we start with the following two, namely the distinction between *total/partial* explanations, and the distinction between *immediate/mediate* explanations.⁵ A total explanation is one which provides all the reasons why something is true. In other terms, the multiset of all, and only, those formulas each of which contributes to explain C is a total explanation of C . On the other hand, each proper sub-multiset of the total explanation of C is a partial explanation of C .⁶

Let us now move to the distinction immediate/mediate. Whilst an immediate explanation is one that involves a single explanatory step, i.e. a step that does not seem to be further reducible, a mediate explanation includes several sequential immediate steps. In this paper we will first deal with the notions of total and immediate formal explanation and then generalize it to the mediate case.

There exists a third distinction that is linked to the notion of total explanation and that arises both in the causal and conceptual framework. To illustrate it, we start from the causal case, where it is most well-known and then move to the non-causal one. Consider the following notorious example.⁷ Billy and Suzy throw rocks at a bottle. The glass shatters. A causal explanation of why the glass shattered is that Suzy threw her rock at it. Indeed since Suzy threw her rock first, her rock arrived first too and shattered the glass; Billy's rock sailed through the air. Billy's throw is thus not a cause, but only a potential cause of why the bottle shattered. Potential causes are central for total explanations: if Billy's rock hit

⁴This name has already been used in Poggiolesi (2018). Here we employ it in a broader sense.

⁵E.g. see Lewis (1973); Schaffer (2016).

⁶E.g. see also Poggiolesi (2020).

⁷E.g. see Menzies and Beebe (2020).

Figure 3: The sequent calculus \mathbf{Gcl}^+ .

$$\begin{array}{c}
p, M \Rightarrow N, p \qquad \frac{M \Rightarrow N}{P \Rightarrow Q \mid M \Rightarrow N} \text{ }^{cw} \\
\\
\frac{M \Rightarrow N, F}{\neg F, M \Rightarrow N} \text{ }^{-L} \qquad \frac{F, M \Rightarrow N}{M \Rightarrow N, \neg F} \text{ }^{-R} \qquad \frac{F, G, M \Rightarrow N}{F \wedge G, M \Rightarrow N} \text{ }^{\wedge L} \qquad \frac{M \Rightarrow N, F \quad M \Rightarrow N, G}{M \Rightarrow N, F \wedge G} \text{ }^{\wedge R} \\
\\
\frac{\forall x F, F(x/t), M \Rightarrow N}{\forall x F, M \Rightarrow N} \text{ }^{\forall L} \qquad \frac{M \Rightarrow N, \forall x F(x/y)}{M \Rightarrow N, \forall x F} \text{ }^{\forall R}
\end{array}$$

where in $\forall R$ y does not occur in M nor in N .

the bottle at the same time as Suzy's rock, it would have been part of the total explanation of why the glass shattered.

A distinction analogous to that between causes and potential causes also arises in the conceptual framework. Consider indeed the following situation. Billy is Jane's brother and Suzy is Jane's sister. Jane has a niece. Thus the reason why Jane has a niece is that her sister has a girl. Indeed a niece is the girl of someone's brother or someone's sister and Suzy, Jane's sister, has a girl. Jane's brother could have had a girl, but he does not. Hence Jane's brother having a girl is merely a potential reason of why Jane has a niece. Potential reasons are also central for total explanations: if Jane's brother had a girl, his having a girl would have been part of the total explanation of why Jane has a niece. We rephrase this distinction between reasons and potential reasons as the one between reasons and *conditions*.⁸ So, for example, we will say that under the condition that Jane's brother does not have a girl, the total reason why Jane has a niece is that her sister has a child.

We will use the notation $N \mid M \models A$ to denote the fact that under the condition that none of the formulas in the multiset N holds, the formulas in M are the total reasons why A is true. In order to define the notion of formal explanation, we work with the classical sequent calculus for first-order logic, implemented with the metalinguistic symbol “|”, for conveying conditions, and the related rule *cw* which allows to introduce conditions beside standard sequents. Conditions only play a role in explanatory rules - no inferential rule operates on conditions - hence, the sequent calculus \mathbf{Gcl}^+ (see Figure 3) is equivalent to the classical sequent calculus for first-order logic \mathbf{Gcl} . The notion of sequent, its interpretation, and the interpretation of inferential rules are standard (e.g. see Troelstra and Schwichtenberg (1996)). We call *c-sequent* a sequent that only contains closed formulas.

We will add to \mathbf{Gcl}^+ explanatory rules. As already underlined, explanatory rules provide the reasons why their conclusion is true; but the link between reasons and conclusion involve to look deep inside formulas (see examples in Figures 1 and 2). So in order to be able to introduce explanatory rules, we will first need to introduce the notations necessary to work inside formulas. We will do so with the notions of *context* and related *formula in a context*. Roughly speaking, a context is the part of a formula where one does not focus on, and is denoted with the notation $C[\cdot]$. For example consider the first-order formula $F = \exists x(Sx \wedge Tx) \vee \forall x \forall y(Px \rightarrow Qx \wedge Ry)$ and suppose we want to focus on a particular part of F , say $Qx \wedge Ry$. We denote this fact by rewriting F as $C[Qx \wedge Ry]$, where $C[\cdot]$ is the context and $Qx \wedge Ry$ is the formula in the context $C[\cdot]$. Note that when working in

⁸Here we borrow vocabulary from Genco (2021).

an explanatory framework, negation needs to be taken into account with special attention. This is true also for the notion of context and to see this clearly, one can use the following example. Consider the two formulas $\neg(p \vee q)$ and $\neg(\neg p \vee \neg q)$. As it emerges from Poggiolesi (2016a, 2022), whilst the (total) reasons of the former amount to the formulas $\neg p, \neg q$, the (total) reasons of the latter amount to the formulas p, q . If we took a negation (or any odd number of consecutive negations) in front of a disjunction to be a context, and the reasons of a disjunction to be its disjuncts, we would get that the reasons for $\neg(p \vee q)$ are indeed $\neg p, \neg q$, whilst the reasons for $\neg(\neg p \vee \neg q)$ are $\neg\neg p, \neg\neg q$, which is an undesirable result. To avoid it, we will define contexts only on an even consecutive number of negations, and we will treat the negation of a disjunction with special rules that involve the use of the symbol of *converse of a formula*, which we introduce here below.

Definition 2.3. The converse of a formula A , written A^* , is defined as follows:

$$A^* = \begin{cases} \neg^{n-1}E, & \text{if } A = \neg^n E \text{ and } n \text{ is odd} \\ \neg^{n+1}E, & \text{if } A = \neg^n E \text{ and } n \text{ is even} \end{cases}$$

where the main connective in E is not a negation, $n \geq 0$ and 0 is taken to be an even number. For any multiset M , $(M)^* := \{B^* \mid B \in M\}$.

Definition 2.4. The set Co of contexts is inductively defined in the following way:

- $[\cdot] \in Co$,
- if $C[\cdot] \in Co$, then $\neg\neg C[\cdot], D \circ C[\cdot], C[\cdot] \circ D, \odot x C[\cdot] \in Co$,
- if $C[\cdot] \in Co$ and $C[\cdot] \neq \overbrace{\neg \dots \neg}^{2n}[\cdot]$, where $n \geq 0$, then $\neg C[\cdot] \in Co$.

Definition 2.5. For all contexts $C[\cdot]$, and formulas F , we define $C[F]$, a formula in a context, as follows:

- if $C[\cdot] = [\cdot]$, then $C[F] = F$,
- if $C[\cdot] = \neg\neg D[\cdot]$, then $C[F] = \neg\neg D[F]$,
- if $C[\cdot] = D' \circ D[\cdot], D[\cdot] \circ D', \odot x D[\cdot], \neg D[\cdot]$, then $C[F] = D' \circ D[F], D[F] \circ D', \odot x D[F], \neg D[F]$, respectively.

Once formulas are considered in contexts, they will naturally have a polarity which is either positive or negative and that is defined as standard, e.g. see Troelstra and Schwichtenberg (1996).

Definition 2.6. We define the set of contexts with positive \mathcal{P} and negative polarities \mathcal{N} simultaneously by an inductive definition given by the three clauses (i)-(iii) below.

- $[\cdot] \in \mathcal{P}$;

if $G^+ \in \mathcal{P}, G^- \in \mathcal{N}$, and F is any formula, then:

- (ii) $\neg G^-, F \wedge G^+, G^+ \wedge F, F \vee G^+, G^+ \vee F, \forall x G^+, \exists x G^+ \in \mathcal{P}$.
- (iii) $\neg G^+, F \wedge G^-, G^- \wedge F, F \vee G^-, G^- \vee F, \forall x G^-, \exists x G^- \in \mathcal{N}$

Figure 4: Explanatory propositional rules.

$$\begin{array}{c}
\frac{M \Rightarrow N, C[F]}{M \Rightarrow N, C[\neg\neg F]} \neg\neg \\
\\
\frac{M \Rightarrow N, C[F] \quad M \Rightarrow N, C[G]}{M \Rightarrow N, C[F \circ G]} \circ_1 \qquad \frac{M \Rightarrow N, C[F_j] \mid M \Rightarrow N, C[F_i]}{M \Rightarrow N, C[F_1 \circ F_2]} \circ_2 \\
\\
\frac{M \Rightarrow N, C[F^*] \quad M \Rightarrow N, C[G^*]}{M \Rightarrow N, C[\neg(F \circ G)]} \neg\circ_1 \qquad \frac{M \Rightarrow N, C[F_j^*] \mid M \Rightarrow N, C[F_i^*]}{M \Rightarrow N, C[\neg(F_1 \circ F_2)]} \neg\circ_2
\end{array}$$

where $i, j = \{1, 2\}$ and $j \neq i$.

We assume explanatory propositional rules not to distinguish between formulas which are FOL-equiv,⁹ and to only apply to c-sequents. Their application is conditioned by Definition 2.9.

whenever these objects are in Co . We say that a formula F is positive (resp. negative) in a context $C[F]$ if $C[\cdot] \in \mathcal{P}$ (resp. $C[\cdot] \in \mathcal{N}$).

The last ingredient needed to introduce explanatory rules is obtained by defining the scope of quantifiers formulas in contexts lie within. We can also classify these scopes of quantifiers according to the kind of quantifiers they are, as well the polarity of the formula which occurs in their scope.

Definition 2.7. If $C[\cdot]$ is a context, the *scope of a context*, $SC(C)$ and the inverse scope $SC^{inv}(C)$ are defined inductively in the following way:

- if $C[\cdot] = [\cdot]$ or $C[\cdot] \neq \overbrace{\neg \dots \neg}^{2n}[\cdot]$ for $n \geq 0$ then $SC(C) = SC^{inv}(C) = \emptyset$,
- if $C[\cdot] = D' \circ D[\cdot]$ or $D[\cdot] \circ D'$, then $SC(C) = SC(D)$ and $SC^{inv}(C) = SC^{inv}(D)$,
- if $C[\cdot] = \forall x D[\cdot]$, then $SC(C) = \forall x.(SC(D))$ and $SC^{inv}(C) = \exists x.(SC^{inv}(D))$
- if $C[\cdot] = \exists x D[\cdot]$, then $SC(C) = \exists x.(SC(D))$ and $SC^{inv}(C) = \forall x.(SC^{inv}(D))$
- if $C[\cdot] = \neg D[\cdot]$, then $SC(C) = SC^{inv}(D)$ and $SC^{inv}(C) = SC(D)$.

Definition 2.8. We say that a context $C[\cdot]$ has

- a *positive universal scope* (PUS) if $C \in \mathcal{P}$ and $SC(C) = \emptyset$ or $SC(C) = \forall x_1, \dots, \forall x_n$,
- a *negative universal scope* (NUS) if $C \in \mathcal{N}$ and $SC(C) = \emptyset$ or $SC(C) = \forall x_1, \dots, \forall x_n$,
- a *positive existential scope* (PES) if $C \in \mathcal{P}$ and $SC(C) = \emptyset$ or $SC(C) = \exists x_1, \dots, \exists x_n$,
- a *negative existential scope* (NES) if $C \in \mathcal{N}$ and $SC(C) = \emptyset$ or $SC(C) = \exists x_1, \dots, \exists x_n$.

We now have all the elements to introduce explanatory rules. In Figure 4 we introduce explanatory rules for propositional connectives which satisfy the following restrictions.

Definition 2.9. We assume the application of explanatory propositional rules¹⁰ to obey the following restrictions:

¹⁰Reading the rules bottom-up.

- rule \circ_1 can be applied on a formula of the form $C[F \circ G]$ if: $\left\{ \begin{array}{l} C \text{ has PUS and } \circ = \wedge, \text{ or} \\ C \text{ has NUS and } \circ = \vee, \text{ or} \\ C \text{ has PES and } \circ = \vee, \text{ or} \\ C \text{ has NES and } \circ = \wedge. \end{array} \right.$
- rule \circ_2 can be applied on a formula of the form $C[F \circ G]$ if: $\left\{ \begin{array}{l} C \text{ has PES and } \circ = \vee, \text{ or} \\ C \text{ has NES and } \circ = \wedge. \end{array} \right.$
- rule $\neg\circ_1$ can be applied on a formula of the form $C[\neg(F \circ G)]$ if: $\left\{ \begin{array}{l} C \text{ has PUS and } \circ = \vee, \text{ or} \\ C \text{ has NUS and } \circ = \wedge, \text{ or} \\ C \text{ has PES and } \circ = \wedge, \text{ or} \\ C \text{ has NES and } \circ = \vee. \end{array} \right.$
- rule $\neg\circ_2$ can be applied on a formula of the form $C[\neg(F \circ G)]$ if: $\left\{ \begin{array}{l} C \text{ has PES and } \circ = \wedge, \text{ or} \\ C \text{ has NES and } \circ = \vee. \end{array} \right.$

Let us spend some words to clarify these rules which extend the rules presented in Poggiolesi (2018); Genco (2021), inside contexts of the first-order level. First of all, it is easy to remark that there is no single rule for negation. This is so because explanations notoriously go from (potentially) true formulas to (potentially) true formulas, and hence, there can be no rule which acts as the rules for negation in the standard sequent calculus where formulas are shifted from one side of the sequent to another, e.g. where we infer $\neg F$, from the fact that F is false. Instead negation is spread over the other connectives: either it is analyzed when it is double, or when it is in front of conjunction and disjunction. Note that, for reasons mentioned some paragraphs above (when introducing contexts) and which are thoroughly discussed in Poggiolesi (2016b), the connective of negation in explanatory context should be carefully treated; this is why in the rules $\neg\circ_1$ as well as $\neg\circ_2$, the symbol of converse occurs.

Let us now move to comment on those rules that does not involve conditions: these are $\neg\neg, \circ_1$ and $\neg\circ_1$. Each of them stands as a straightforward generalization of standard rules concerning classical connectives: standard rules can now be applied deep inside formulas. This is so because now rules are more than simply inferential rules, namely they are explanatory rules, rules which provide the (total) reason(s) why their conclusion is true, and the relation between reason(s) and conclusion might hold in virtue of elements that lie inside formulas. Note however that application of rules deep inside formulas involves some limitations to preserve an adequate notion of explanation. To see this clearly, let us consider some paradigmatic examples. The following is an instance of rule $\neg\circ_1$:

$$\frac{\frac{\Rightarrow \neg p \quad \Rightarrow \neg q}{\Rightarrow \neg(p \vee q)}}{\Rightarrow \neg(p \vee q)}_{\neg\circ_1}$$

The rule can be applied since \circ is a disjunction, the context is empty and thus the polarity of the formula the rule acts on is positive. Thanks to the rule $\neg\circ_1$, we can explain, totally and immediately, the formula $\neg(p \vee q)$ by means of the formulas $\neg p$ and $\neg q$, which are its reasons. The rule matches the example 1.1. in Figure 1, and thus stands as an adequate instance of the rule. Let us now move to the following instance of the rule \circ_1 :

$$\frac{\frac{\Rightarrow \forall x(Zx \rightarrow Nx) \quad \Rightarrow \forall x(SNx \rightarrow Nx)}{\Rightarrow \forall x((Zx \vee SNx) \rightarrow Nx)}}{\Rightarrow \forall x((Zx \vee SNx) \rightarrow Nx)}_{\circ_1}$$

The rule can be applied since \circ is a disjunction with a negative polarity, in the scope of universal quantifiers (see Definition 2.9). Thanks to the rule \circ_1 , we can explain, totally and immediately, the formula $\forall x((Zx \vee SNx) \rightarrow Nx)$ by the formulas $\forall x(Zx \rightarrow Nx)$ and $\forall x(SNx \rightarrow Nx)$, which represent the reasons why it is true. The rule matches example 1.4. of Figure 2 and thus stands as a an adequate instance of the rule.

Finally, consider the following formula $\forall x(Nx \rightarrow Ex \vee Ox)$, which can be seen as formalizing the sentence “any natural number is either odd or even.” Suppose that one focuses on the disjunction and would like to apply a rule on it. Since disjunction occurs with a positive polarity inside an universal quantifier, none of the rules of the calculus can actually be applied to it. But this again matches our intuitions, as it would be incorrect to claim that the reasons why any natural number is either even or odd are the facts that any natural number is even, as well as that any natural number is odd.

Let us now move to the rules which involves conditions, namely the rules \circ_2 , $\neg\circ_2$. These rules naturally emerge since we deal with total explanations, i.e. explanations where all the reasons why a conclusion is true need to be evoked. In this setting, if conditions are not mentioned, equivocation between total and partial explanations could arise (e.g. see Poggiolesi (2016b)). Consider the example where John got into the University and he is rich or he passed the entrance exam. Suppose also that John got into the University, he is rich, but he did not pass the entrance exam. In this example, the explanation why it is true that John got into the University and he is rich or he passed the entrance exam is that John got into University and he is rich. However, if nothing is said about the passing exam, the explanation remains ambiguous: it is indeed unclear whether the explanandum is true also because John passed the entrance exam. In order to disambiguate the explanation, conditions are introduced. Thus we say that, under the condition that it is not the case that John got into University and passed the entrance exam, it is true that John got into the University and he is rich or he passed the entrance exam, because John got into University and he is rich. On formal terms, let us denote the sentence “John gets into the University and he is either rich or it has passed the entrance exam,” with the formula $p \wedge (q \vee r)$. Let us apply on this formula, focussing on the disjunction, the following instance of the rule \circ_2 , we get:

$$\frac{\Rightarrow p \wedge q \mid \Rightarrow p \wedge r}{\Rightarrow p \wedge (q \vee r)}$$

The rule can be applied since \circ is a disjunction with a positive polarity, in the scope of no quantifier (see Definition 2.9). Thanks to the rule \circ_2 , we can explain the formula $p \wedge (q \vee r)$ by the formula $p \wedge r$, which represents the total reason why it is true under the condition that the formula $p \wedge q$ does not hold. The rule matches what we have just been discussing and thus stands as a an adequate instance of the rule.

In Figure 5, we propose explanatory rules for quantifiers. Since we are providing a logical theory of conceptual explanations in first-order logic, we believe that we get a more elegant and harmonious overall theory if explanatory rules for quantifiers are displayed. However, we strongly underline that in all the examples of conceptual explanations that can be found in the literature (e.g. see examples 1.2.-1.6.) quantifiers are typically left untouched, the explanation occurring inside them (which thus become contexts). The explanatory rules of quantifiers are finitary rules, which we take to be a proof-theoretical desirable feature. Explanatory rules for quantifiers, like the explanatory rules for propositional connectives, extend inferential intuitions concerning the universal and the existential quantifiers at the explanatory level. Roughly speaking, the rule for the universal quantifier explains this

Figure 5: Explanatory first-order rules.

$$\begin{array}{c}
\frac{M \Rightarrow N, C[Fy]}{M \Rightarrow N, C[\odot x.Fx]} \odot_1 \\
\frac{M \Rightarrow N, C[F^*y]}{M \Rightarrow N, C[\neg(\odot x.Fx)]} \neg\odot_1 \\
\frac{M \Rightarrow N, C[\odot x.Fx], C[Ft]}{M \Rightarrow N, C[\odot x.Fx]} \odot_2 \\
\frac{M \Rightarrow N, C[\neg(\odot x.Fx)], C[F^*t]}{M \Rightarrow N, C[\neg(\odot x.Fx)]} \neg\odot_2
\end{array}$$

where in \odot_1 and $\neg\odot_1$ y does not occur free in M nor in N .

We assume explanatory first-order rules not to distinguish between formulas which are FOL-equiv. Their application is conditioned by Definition 2.10.

quantifier by using the *eigenvariable*,¹¹ i.e. it explains why any object x has a property A via the fact that if one picks a random object y , y has the property A . This seems what happens in mathematical contexts, where if a mathematician aims to explain why all triangles have a certain property, she will not work with all triangles, rather she will pick a triangle with no particular assumption on it - this is what the eigenvariable stands for - and prove that that triangle enjoys the property at issue. Since no particular assumption was engaged, she can generalize the explanation to all triangles. The rule for the existential quantifier explains this quantifier via one of its instances; however, in order for the premisses of this rule to be the reasons of its conclusion, the existential itself needs to be repeated in the premisses. This move is analogous to that adopted in the rules $\forall L$ and $\exists R$ of the classical sequent calculus for first-order logic, e.g. see Troelstra and Schwichtenberg (1996).

Definition 2.10. We assume the application of explanatory first-order rules¹² to obey the following restrictions:

- rule \odot_1 can be applied on a formula of the form $C[\odot x.Fx]$ if: $\left\{ \begin{array}{l} C \text{ has PUS and } \circ = \forall, \text{ or} \\ C \text{ has NUS and } \circ = \exists, \text{ or} \\ C \text{ has PES and } \circ = \exists, \text{ or} \\ C \text{ has NES and } \circ = \forall. \end{array} \right.$
- rule \odot_2 can be applied on a formula of the form $C[\odot x.Fx]$ if $C \in \mathcal{P}$ and $\odot = \exists$, or $C \in \mathcal{N}$ and $\odot = \forall$.
- rule $\neg\odot_1$ can be applied on a formula of the form $C[\neg(\odot x.Fx)]$ if: $\left\{ \begin{array}{l} C \text{ has PUS and } \circ = \exists, \text{ or} \\ C \text{ has NUS and } \circ = \forall, \text{ or} \\ C \text{ has PES and } \circ = \forall, \text{ or} \\ C \text{ has NES and } \circ = \exists. \end{array} \right.$
- rule $\neg\odot_2$ can be applied on a formula of the form $C[\odot x.Fx]$ if $C \in \mathcal{P}$ and $\odot = \forall$, or $C \in \mathcal{P}$ and $\odot = \exists$.

We will call \mathbf{Gcl}^E the sequent calculus composed by the rules of Figures 3 and 4, whilst we will call \mathbf{Gcl}^{EQ} the sequent calculus composed by the rules of Figures 3, 4 and 5. In what follows, for us to keep on working with closed formulas, we will mainly deal with the calculus \mathbf{Gcl}^E , leaving results concerning \mathbf{Gcl}^{EQ} for future research.

¹¹See Troelstra and Schwichtenberg (1996).

¹²Reading the rules bottom-up.

Definition 2.11. A *mixed derivation* in \mathbf{Gcl}^E is a finite (upwardgrowing) tree with a single root. The nodes of the tree are labelled by sequents or sequents with a bar and the top nodes are labelled by initial sequents. For each non-terminal node, its label is connected with the labels of the immediate predecessor nodes according with one of the logical rules or one of the explanatory rules, or the rule *cw*. The root of the tree is the conclusion of the whole derivation and its label is a theorem of the sequent calculus, in symbol $\vdash_{\mathbf{Gcl}^E}^* M \Rightarrow N$. A *derivation* in \mathbf{Gcl}^E is a mixed derivation where only logical rules have been applied; we denote it as usual with the symbol $\vdash_{\mathbf{Gcl}^E}$.

In the calculus \mathbf{Gcl}^E it is thus possible to construct standard derivations, but also derivations which contain explanatory steps that we call *mixed derivations*.

Let S, S', \dots be multisets of c-sequents. Then, $(S)^* = \{(M \Rightarrow N)^* \mid M \Rightarrow N \in S\}$, where the converse of a c-sequent, $(M \Rightarrow N)^*$, corresponds to the formulas $\bigwedge M, \bigvee N^*$.

Definition 2.12. For any multisets of c-sequents S and S' (which might be empty), and for any c-sequent $M \Rightarrow N$, we say that under the condition $(S')^*$, there exists a *total and immediate formal explanation* from S to $M \Rightarrow N$, in symbols $S' \mid S \Vdash M \Rightarrow N$ if, and only if, one of the explanatory rules of Figure 4 links S', S and $M \Rightarrow N$.

Definition 2.13. For any multisets of c-sequents S and S' (which might be empty), and for any c-sequent $M \Rightarrow N$, we say that under the condition $(S')^*$, there exists a *total and mediate formal explanation* from S to $M \Rightarrow N$, in symbols $S' \mid S \Vdash_m M \Rightarrow N$ if, and only if:

- $S' \mid S \Vdash M \Rightarrow N$,
- $S'' \mid S''' \Vdash M' \Rightarrow N'$ and $S'''' \mid S''''', M' \Rightarrow N' \Vdash_m M \Rightarrow N$, and $S'' \cup S'''' = S'$ and $S''' \cup S''''' = S$.

3 Eliminability of the explanatory rules in the calculus \mathbf{Gcl}^E

In the previous section, we have introduced the calculus \mathbf{Gcl}^E which is a calculus composed by the sequent calculus \mathbf{Gcl}^+ plus explanatory rules for the classical propositional connectives. In \mathbf{Gcl}^E not only one can construct standard derivations (denoted by the symbol \vdash), but also derivations with explanatory steps (denoted by the symbol \vdash^*), as well as formal explanations (denoted by the symbol \Vdash). As for standard derivations, \mathbf{Gcl}^E is equivalent to \mathbf{Gcl} and it keeps the same properties as \mathbf{Gcl} .

Lemma 3.1. For any sequent $M \Rightarrow N$, $\vdash_{\mathbf{Gcl}} M \Rightarrow N$ if, and only if, $\vdash_{\mathbf{Gcl}^E} M \Rightarrow N$.

Proof. Straightforward. □

Lemma 3.2. The structural rules of weakening and contraction are height-preserving admissible in \mathbf{Gcl}^E . The logical rules of \mathbf{Gcl}^E are height-preserving invertible (and given a logical rule \mathcal{R} , we will call $\bar{\mathcal{R}}$ its inverse).

Proof. The proof is the same as that developed in \mathbf{Gcl} , see (Troelstra and Schwichtenberg, 1996, Ch. 3.5). □

As for explanatory rules, we need to show that they do not allow us to derive any new formula, i.e. explanatory rules serve to build *derivations with an explanatory power*, not to prove new theorems. To get this result, we show that any explanatory rule can also be performed by several applications of the standard inferential rules.

Lemma 3.3. *For any sequent $M \Rightarrow N$, and for any mixed derivation d of $M \Rightarrow N$, namely $d \vdash^* M \Rightarrow N$ which contains only one application of an explanatory rule, one can construct a derivation d' with the same end-sequent, namely $d' \vdash M \Rightarrow N$.*

Proof. We reason by induction on the height of the derivation. We divide the explanatory rules into two groups: explanatory rules without conditions, namely $\neg\neg, \circ_1, \neg\circ_1$ and explanatory rules with conditions, namely $\circ_2, \neg\circ_2$. We start analyzing the rules of the first group. Suppose that the main formula of the premise of the explanatory rule is of the form $F[B]$. We apply on the context F as many rules \bar{R} as necessary to unfold the context itself and reach the formula B .¹³ Once arrived to B , given that explanatory rules do not distinguish between formulas which are FOL-equiv, we might need to further apply \bar{R} -rules to further decompose the formulas composing F . We then apply the standard logic rules to get from B , or any formula FOL-equivalent to B , to the desired conclusion, and then we also use the logical rules to reconstruct the context F . Here is a simple example of the procedure. Consider the following instance of the explanatory rule $\neg\neg$:

$$\frac{M \Rightarrow N, \forall x(Qx \wedge Px \rightarrow Rx)}{M \Rightarrow N, \forall x(Px \wedge Qx \rightarrow \neg\neg Rx)} \neg\neg$$

We obtain the desired result in the following way:

$$\frac{\frac{\frac{\frac{\frac{\frac{\frac{M \Rightarrow N, \forall x(Qx \wedge Px \rightarrow Rx)}{M \Rightarrow N, Qc \wedge Pc \rightarrow Rc} \forall R}{Qc \wedge Pc, M \Rightarrow N, Rc} \rightarrow R}{Pc, Qc, M \Rightarrow N, Rc} \wedge L}{\neg Rc, Pc, Qc, M \Rightarrow N} \neg L}{Pc, Qc, M \Rightarrow N, \neg\neg Rc} \neg R}{Pc \wedge Qc, M \Rightarrow N, \neg\neg Rc} \wedge L}{M \Rightarrow N, Pc \wedge Qc \rightarrow \neg\neg Rc} \rightarrow R}{M \Rightarrow N, \forall x(Px \wedge Qx \rightarrow \neg\neg Rx)} \forall R$$

As for the rules of the second group, namely those explanatory rules with conditions, one needs to consider the mixed derivation d , which will necessarily contain an application of the rule cw . We substitute the derivation d with a derivation d' with no application of the rule cw . Then we continue the procedure as above. \square

Proposition 3.4. *For any sequent $M \Rightarrow N$, and for any mixed derivation d of $M \Rightarrow N$, namely $d \vdash^* M \Rightarrow N$, one can construct a derivation d' with the same end-sequent, namely $d' \vdash M \Rightarrow N$.*

Proof. Considering the derivation d from top, by several applications of the previous Lemma 3.3. \square

¹³If A is empty, this first step of the procedure can be skipped.

4 From reasons to conclusions

The main aim of this paper is to develop a logical theory of conceptual explanations. To do so, there are two central questions that need to be addressed. The first question - namely what kind of structure underlies conceptual explanations? - has been answered in Section 2, with the introduction of a novel metalinguistic relation called *formal explanation*, which has been defined via explanatory rules in the sequent calculus. We use this section to answer the second question that concerns the relation between reasons and their conclusion; what kind of features need to be satisfied for some formulas to count as the (total and immediate) reasons of another? As we have done for the first question, in order to answer this second question, we will rely and extend Poggiolesi's work; in particular, whilst Poggiolesi (2016b) settles sufficient and necessary conditions to identify the reasons why a truth is such by only considering toy examples, here we will generalize her results to also encompass more intricate examples (see Figures 1 and 2). In particular, we will introduce the metalinguistic relation $N \mid M \models A$ to be read as: under certain conditions N , M are the total and immediate reasons of why A is true. Then, by working in first-order logic, we will put out those features that are sufficient and necessary to establish that, under certain conditions N , M are the total and immediate reasons of why A is true. The discussion will proceed into two main stages. First, we will introduce the features in a more informal way, for then moving to the formal definitions.

The first feature that we need to consider in order to model the relation which links (total) reasons to their conclusion amounts to the observation, widespread in philosophy (e.g. see Jansson (2017); Kim (1994), Woodward (2004)), that this is a *dependence* relation. This dependence can, in its turn, be conveyed in the following terms. In a relation that goes from the (total) reasons to their conclusion not only does the conclusion follow from its reasons, but it is also the case that if the reasons were modified somehow (under certain conditions), then this change would affect the conclusion. Conveyed in logical terms this becomes: not only it is the case that the conclusion logically follows from the (total) reasons, but also the negation of the conclusion needs to logically follow from the negation of some (even all) the (total) reasons (under certain conditions).

Let us consider this idea of dependency expressed in logical terms on the background of one of the examples introduced above. Take for instance example 1.3. (Figure 1). It logically follows from their coding competences and teamwork skills that Stanford Computer Science graduates are ideal candidates for a particular tech job. However there seems to be more than just a logical consequence relation between these relata: indeed, if one of the premisses (or even both) were modified, this change would affect the conclusion. Suppose for example that Stanford Computer Science graduates do not have teamwork skills, then it follows that they no longer are ideal candidate for the tech job.

The conclusion is thus dependent on its reasons; however, this is not all. Indeed, any explanatory relation, such as the one we are examining, is asymmetric: there is a direction from what explains to what is explained. The dependency does not provide such a directionality. To see this clearly, one can consider any case with an unique reason. Example 1.5. above (see Figure 2) perfectly fits this type of scenario: we indeed have that a property of circles is explained by an unique reason, namely a property of points. Note that these two properties can be proved to be equivalent, as in line with our formulation of dependency. Yet, despite their equivalence and following Bolzano's intuitions, it is the property of points that explains the property of circles and not viceversa. We need to find an ingredient that determines this directionality or asymmetry. Poggiolesi¹⁴ relies on a strong and illustrious

¹⁴On this point, see further work in Poggiolesi (2018, 2024).

philosophical tradition¹⁵ in identifying the missing ingredient with *complexity*: the simplest reasons explain the more complex conclusion, and it would be absurd to go the other way. Moreover, increase in complexity from the reasons to their conclusion should be of a particular type:¹⁶ the formulas by means of which a sentence is explained should correspond to a decomposition of the sentence itself. Although this insight is clear, deep as well as supported by a brilliant philosophical tradition, problems arise when we try to formalize it. The first notions that would seem to naturally serve the purpose are logical complexity and associated relation of subformula; however, they turn out to be inadequate for an explanatory framework. There are two main kinds of counterexample which can be evoked to show that standard complexity and subformula are not adequate for an explanatory framework.¹⁷ The first type of counterexample concerns the use of negation. Consider the example 1.1. above, formalized in Figure 1. In this case, the (total and immediate) reasons of the formula $\neg(p \vee q)$ are $\neg p$ and $\neg q$. Although $\neg p$ and $\neg q$ would need to be less complex¹⁸ and subformulas of $\neg(p \vee q)$, according to the standard notion of logical complexity and subformula, they are not. The second type of counterexample to the notions of logical complexity and subformula in an explanatory framework arise when considering cases like 1.2-1.6, namely cases where, as already underlined, one needs to look deep inside a formula. Consider in particular the case 1.4. (Figure 2.), which logically is the most compelling.¹⁹ In this case the (total and immediate) reasons of the formula $\forall x((Zx \vee SNx) \rightarrow Nx)$ are the formulas $\forall x(Zx \rightarrow Nx)$ and $\forall x(SNx \rightarrow Nx)$. Although these two latter formulas would need to be logically less complex and subformulas of their conclusion, according to the standard notion of logical complexity and subformula, they are not. In particular, whilst the standard notion of subformula only allows to break a formula along its main connective, the present example vividly shows that the possibility of breaking a formula from the inside, i.e. by breaking a connective that is not the main one and leaving the rest of the formula untouched, should also be taken into account.

Given this analysis, in what follows we will enrich the notions of complexity and subformula so for them to fit in an explanatory framework. In particular, we will first introduce the notion of *e-complexity*, that extends the standard notion of logical complexity by providing a more explanatory compelling way of counting connectives. Consequently, and in accordance with the new notion of e-complexity, we will define another relation of subformula, called *e-subformula*, that extends the standard notion of subformula in a way which is adequate for an explanatory framework.

Definition 4.1. Let $A \in \mathcal{WF}$, the e-complexity of A , $ecm(A)$, is defined in the following way:

- $ecm(Pt) = ecm(\neg Pt) = 0$
- $ecm(\neg\neg A) = ecm(A) + 1$
- $ecm(A \circ B) = ecm(\neg(A \circ B)) = ecm(A) + ecm(B) + 1$

¹⁵E.g. see Betti (2010); Detlefsen (1988).

¹⁶See Rumberg (2013).

¹⁷Other features of logical complexity and subformulas that are not adequate for an explanatory framework will naturally emerge during the discussion.

¹⁸Here we mean that the sum of the logical complexity of $\neg p$ and $\neg q$ is lower than the logical complexity of $\neg(p \vee q)$.

¹⁹Although examples 1.2., 1.3. as well as 1.5. and 1.6., all display an analogous logical form, their explanatory power rely on the use of concepts the analysis of which goes beyond the purpose of the present paper.

$$- ecm(\odot xAx) = ecm(\neg \odot xAx) = ecm(Ax) + 1$$

Definition 4.1 relies on a previous definition of complexity for an explanatory framework provided in Poggiolesi (2016b) and extends it at the first-order level. Let us briefly illustrate the main insight behind it. It is a notion that basically aims at depicting a hierarchy of first-order formulas that lies at the background of an explanatory framework. Since in an explanatory framework, one goes from truth to truths, e-complexity tracks relationships among the truths expressed by the formulas if they were true. Let us now see how it works. Consider first conjunction, disjunction and quantifiers: in these cases, e-complexity coincides with the standard notion of logical complexity. If, for example, A and B express truths, then the truth expressed by $A \wedge B$ is obtained from the previous truths using a single operation. Thus conjunction increases of one the e-complexity of the sum of the e-complexity of A and the e-complexity of B . Analogously, if A expresses a truth, then the truth expressed by $\forall xAx$ is obtained from the previous truth using a single operation. Thus universal quantifier increases of one the e-complexity of the formula it is applied to. However, this is not so for the case of negation. Let us see this first with the case of literals. Since (at most) one of Pc and $\neg Pc$ will express a truth, then only one of these formulas will ever be an object of an explanatory hierarchy. Thus, there seems to be no reason to count $\neg Pc$ as more complex than Pc : in other terms, $\neg Pc$ can no longer be seen as constructed from Pc , since if one is true, the other is false. We should rather look at them as two formulas on the same level and this is precisely what the e-complexity does. Analogous reasoning can be applied to the e-complexity of more complex formulas like $A \wedge B$ and $\neg(A \wedge B)$, or $\forall xAx$ and $\neg(\forall xAx)$. We can no longer count the complexity of $\neg(A \wedge B)$ as the complexity of $A \wedge B$ plus one, as standard logical complexity does, since if $\neg(A \wedge B)$ is true, then $A \wedge B$ is false and thus it cannot be constructed from it. We should rather think of $A \wedge B$ and $\neg(A \wedge B)$ as two formulas that lie at the same level of an explanatory hierarchy and thus have the same e-complexity. Let us finally move to the case of double negation. In this case, the negation counts since $ecm(\neg\neg A) = ecm(A) + 1$. But this is in harmony with what has been said up to now as $\neg\neg A$ and A may both express truths, and thus the former can be seen as constructed from the latter by means of a single operation.

Note that as we have seen in the previous Sections and as we will see again shortly, in an explanatory framework one might work with contexts (see Definitions 2.4) and formulas in contexts (see Definitions 2.5). The e-complexity of contexts, and formulas in contexts can be defined as follows.

Definition 4.2. We define the e-complexity of a context $ecm(C[.]) = ecm(C[Pc])$ for any predicate P and constant c in \mathcal{L} .

Definition 4.3. We define the e-complexity of a formula in context, $ecm(C[F])$ as a pair of numbers (m, n) such that $m = ecm(C[.])$ and $n = ecm(F)$. Accordingly, given the formulas in a context $C_1[F_1], \dots, C_k[F_k]$ and $D[G]$, such that $ecm(C_1[F_1]) = (m, n_1), \dots, ecm(C_k[F_k]) = (m, n_k)$ and $ecm(D[G]) = (m, n)$, where $n = n_1 + \dots + n_k + 1$, $C_1[F_1], \dots, C_k[F_k]$ will be said to be *immediately less g-complex* than $D[G]$.

Note also that now we have the notion of e-complexity, we can look at the relation between a formula A , and its converse A^* (see Definition 2.3), under a novel light. Indeed each formula and its converse are such that their conjunction corresponds to a contradiction and they have the same e-complexity.²⁰

²⁰Whilst a formula and its negation are such that their conjunction corresponds to a contradiction but they do not necessarily have the same e-complexity.

We now move to the new notion of subformula we aim at proposing, that will be called *e-subformula*, and that will be working in parallel with the notion of e-complexity (just as logical complexity and subformula do). There are three main ideas that motivate the new notion of e-subformula. The first idea is related to the fact, already underlined in the previous sections as well as here above, that in an explanatory framework, where the new notion of e-subformula belongs, relations amongst formulas, as the relation between reasons and their conclusion, might involve connections that go deep inside formulas. The standard subformula allows to connect formulas by only working with the main connective; we will enrich it by also allowing to work with connectives inside formulas and for this reason we will use again the notions of context, and formula in a context. The second and third ideas behind the new notion of e-subformula are linked to the novel way of counting the complexity of a formula, namely by the notion of e-complexity. Consider indeed a formula F and dwell an any formula E which is equivalent to F by associativity and commutativity of conjunction and disjunction, change of orders of identical quantifiers, and substitution of variables. Not only are F and E logically equivalent, they also are equivalent from an explanatory point of view. Indeed, E and F convey the same “state of affairs,” and occupy the same place in the explanatory hierarchy, i.e. they have the same e-complexity. Hence if F is a subformula of F' , then (any) E should be as well. We will render this feature by closing the relation of e-subformula under associativity and commutativity of conjunction and disjunction, change of orders of identical quantifiers, and substitution of variables.²¹

Note that a reasoning analogous to the one just developed also applies any formula F , and its converse F^* . Although F and F^* are of course not equivalent, yet they share a deep relation: they convey the same “state of affairs” and they occupy the same place in the explanatory hierarchy, i.e. they have the same e-complexity. Either F is true or F^* is, yet they represent the two sides of the same coin. As a result, whenever a formula F' is a e-subformula of a formula F , also its converse will be.

Now that we have clarified the main insights behind the new notion of *e-subformula*, we introduce it in a formal way via the following definitions.

Definition 4.4. Given the formula F and G of \mathcal{L} , we say that F is *FOL-equiv* to G if, and only if, F can be obtained from G by associativity and commutativity of conjunction and disjunction, substitution of variables, and change of orders of identical quantifiers.

Definition 4.5. Given a context $C[.]$ of \mathcal{L} , we say that $C[.]$ is *FOL-equiv* to $D[.]$ if, and only if, for any predicate P and any constant $c \in \mathcal{L}$, $C[Pc]$ is FOL-equiv to $D[Pc]$.

Definition 4.6. For any pair of formulas of \mathcal{L} F and G , we say that $F \cong G$ if, and only if, F is FOL-equiv to G or F is FOL-equiv to G^* .

Definition 4.7. For any pair of contexts of \mathcal{L} $C[.]$ and $D[.]$, we say that $C[.] \cong D[.]$ if, and only if, for any predicate P and any constant c in \mathcal{L} , $C[Pc]$ is FOL-equiv to $D[Pc]$ or $C[Pc]$ is FOL-equiv to $(D[Pc])^*$.

Definition 4.8. For any pair of multisets of formulas of \mathcal{L} M and N , such that $M = \{C_1[F_1], \dots, C_n[F_n]\}$ and $N = \{D_1[G_1], \dots, D_n[G_n]\}$, we say that $M \cong N$, if, and only if, $F_1 \cong G_1, \dots, F_n \cong G_n$ and $C_1 \cong D_1, \dots, C_n \cong D_n$.

Definition 4.9. For any pair of formulas in contexts of \mathcal{L} $C[F]$ and $D[G]$, we say that $D[G]$ is a e-subformula of $C[F]$ if, and only if, $C[.] \cong D[.]$, and:

²¹This is the feature that make our logical theory of conceptual explanations hyperintensional, e.g. see Berto and Nolan (2023); Leitgeb (2019).

- $F \cong G$,
- $F \cong \neg\neg F'$ and G is a e-subformula of F' ,
- $F \cong F' \circ F''$ and G is a e-subformula of F' or G is a e-subformula of F'' ,
- $F \cong \odot x F'$ and G is a e-subformula of $F'(t/x)$ for all t free for x in F' .

The notion of *immediate e-subformula* is analogous to that of immediate subformula.

Definition 4.10. M is a multiset of *distinguished immediate e-subformulas* of $C[F]$, if, and only if:

- $M \cong \{C[F']\}$ and $F \cong \neg\neg F'$,
- $M \cong \{C[F'], C[F'']\}$ and $F \cong (F' \circ F'')$,
- $M \cong \{C[\odot x F']\}$ and $F \cong F'(t/x)$, for all t free for x in F' .

Note that the distinguished immediate e-subformulas of $C[F]$ are always immediately less g-complex than $C[F]$ according to Definition 4.3, so that the notion of e-complexity and e-subformula go hand in hand.

We finally have all the ingredients to formally define the necessary and sufficient conditions which establish when, under certain conditions N , some formulas M are to be count as the total and immediate reasons of a formula A .

Definition 4.11. For any finite multisets of \mathcal{CF} $M = \{D_1[G_1], \dots, D_m[G_m]\}$ and $N = \{C_1[F_1], \dots, C_n[F_n]\}$ (which could be empty), and for any \mathcal{CF} $C[F]$, under the condition that N^* , M is a *total and immediate formal reason* of $C[F]$, in symbols $N \mid M \models C[F]$, if, and only if, for any E such that $SC(E) = SC(C)$ and $E \in \mathcal{P}$ if, and only if, $C \in \mathcal{P}$, we have:

1. $E[G_1], \dots, E[G_m] \models E[F]$,
2. for some non empty (possibly non proper) submultiset M' of M , such that $M' = \{D_{k_1}[G_{k_1}], \dots, D_{k_r}[G_{k_r}]\}$, we have that $(E[F_1])^*, \dots, (E[F_n])^*, (E[G_{k_1}])^*, \dots, (E[G_{k_r}])^*, M^-/E \models (E[F])^*$.
3. $N \cup M$ is a multiset of distinguished immediate e-subformulas of $F[B]$.

where $M^- = M - M'$ and $M^-/E = \{E[G_z] \mid D_z[G_z] \in M^-\}$.

Definition 4.11 represents the formal counterpart of the features discussed in this section. Conditions 1. and 2. are meant to capture the dependency of the relation. Obviously this dependence holds amongst the formulas (in contexts) at issue, independently from the contexts these formulas belong to. For this reason, universal quantification over any context $E[\cdot]$, whose scope and polarity are the same as that of the formula to be explained, needs to be added.²² Condition 3. amounts to the directionality or asymmetry of the explanatory relation at issue: this is conveyed via the new notion of e-subformula.

Let us evaluate some relations of reasons to related conclusions which emerge from this definition. Consider the formula $\neg(p \vee q)$ that we have discussed in the example 1.1. above (see Figure 1). $\neg p, \neg q$ can be taken to be the total and immediate reasons why $\neg(p \vee q)$ is

²²Although it arose in a different framework, a similar dependence relation has been investigated in Humberstone (2013).

true. Definition 4.11 confirms this intuition: indeed $\neg(p \vee q)$ is a classical logical consequence of $\neg p$ and $\neg q$. However, it is also the case that if we modify a subset of the reasons and we consider, say, p and $\neg q$, instead of $\neg p$, $\neg q$, it logically follows that $p \vee q$. Finally, $\{\neg p, \neg q\}$ is the multiset of distinguished immediate e-subformulas of $\neg(p \vee q)$.

Let us now move to the formula $\forall x((Zx \vee SNx) \rightarrow Nx)$ that is central in the example 1.4 (see Figure 2). In particular we have said that its total and immediate reasons are the formulas $\forall x(Zx \rightarrow Nx)$ and $\forall x(SNx \rightarrow Nx)$. Definition 4.11 confirms this intuition: indeed for any context $E[.]$, such that $SC(E) = \forall x$ and $E \in \mathcal{N}$, we have that $E[Zx \vee SNx]$ logically follows from $E[Zx]$ and $E[SNx]$. However, it is also the case that if we modify the reasons, so we consider, say $E[Zx]$ and $(E[SNx])^*$, then it logically follows that $(E[Zx \vee SNx])^*$. Finally, $\{\forall x(Zx \rightarrow Nx), \forall x(SNx \rightarrow Nx)\}$ is a multiset of distinguished immediate e-subformulas of $\forall x((Zx \vee SNx) \rightarrow Nx)$.

Finally, consider the formula $\forall x(Nx \rightarrow Ex \vee Ox)$, taken into account also in Section 2, that could be seen as formalizing the sentence “for any x , if x is a natural number, then it is an odd or an even number.” Although the formulas $\forall x(Nx \rightarrow Ex)$ - for any x if x is a natural number, then it is even - and $\forall x(Nx \rightarrow Ox)$ - for any x if x is a natural number, then it is odd - are both e-subformulas of $\forall x(Nx \rightarrow Ex \vee Ox)$, it would be rather weird to think of them as its reasons, if only because they are false. Definition 4.11 confirms this fact in that it can be easily checked that condition 2. of Definition 4.11 does not hold between the well-formed closed formula $\forall x(Nx \rightarrow Ex \vee Ox)$ and the formulas $\forall x(Nx \rightarrow Ex)$ and $\forall x(Nx \rightarrow Ox)$. We can extend the definition of total and immediate formal reasons-conclusion to total and mediate formal reasons-conclusion in the following way.

Definition 4.12. For any multisets of \mathcal{CF} M and N (which could be empty), and for any \mathcal{CF} F , under the condition that N^* , M is a *total and mediate formal reason* of F , $N \mid M \Vdash_m F$, if, and only if:

- $N \mid M \Vdash F$, or
- $N' \mid M' \Vdash G$ and $N'' \mid G, M'' \Vdash_m F$, where $M' \cup M'' = M$, and $N' \cup N'' = N$.

5 Explanatory rules provide all, and only, relations from total and immediate reasons to conclusion (in FOL)

We will use this section to prove an equivalence between the new two metalinguistic relations that have been introduced in this paper, namely the metalinguistic relations \Vdash and \Vdash_m . In particular, Theorem 5.7 establishes that if a rule is explanatory then its premisses represent the total and immediate reasons of its conclusion, according to Definition 4.11. Theorem 5.13 will prove that if some formulas count as the total and immediate reasons of a certain conclusion (according to Definition 4.11), then there exists an explanatory rule which convey this relation. Other lemmas and definitions serve to prove these main theorems.

Lemma 5.1. *The following rules are admissible in the calculus \mathbf{Gcl} :*

when $C \in \mathcal{P}$:

$$\frac{C[F_i], M \Rightarrow N}{C[F_1 \wedge F_2], M \Rightarrow N} \wedge 1$$

$$\frac{M \Rightarrow N, C[F_i]}{M \Rightarrow N, C[F_1 \vee F_2]} \vee 1$$

$$\frac{M \Rightarrow N, C[F_i]}{M \Rightarrow N, C[F_1 \wedge F_2]} \wedge 2$$

$$\frac{C[F_i], M \Rightarrow N}{C[F_1 \vee F_2], M \Rightarrow N} \vee 2$$

$$\frac{C[F_i^*], M \Rightarrow N}{C[\neg(F_1 \vee F_2)], M \Rightarrow N} \neg \vee 1$$

$$\frac{M \Rightarrow N, C[F_i^*]}{M \Rightarrow N, C[\neg(F_1 \wedge F_2)]} \neg \wedge 1$$

$$\frac{M \Rightarrow N, C[F_i^*]}{M \Rightarrow N, C[\neg(F_1 \vee F_2)]} \neg \wedge 2$$

$$\frac{C[F_i^*], M \Rightarrow N}{C[\neg(F_1 \wedge F_2)], M \Rightarrow N} \neg \vee 2$$

where $i = \{1, 2\}$.

Proof. We prove in detail the admissibility of the rules $\wedge 1$ and $\wedge 2$ by induction on the construction of the context $C[\cdot]$, and subinduction on the height of the derivation of the premise of the rule. The admissibility of any other rule can be proved analogously.

We distinguish cases according to the form of $C[\cdot]$. If $C[\cdot] = [\cdot]$, then from the premise $F_i, M \Rightarrow N$ we obtain the desired result thanks to the rule $\wedge L$. As for the rule $\wedge 2$, since F_i has a negative polarity in $C[\cdot]$, it can be thought of as $\neg(F_i)$. Thus we first apply the inverse of the rule $\neg R$ ²³ obtaining $F_i, M \Rightarrow N$. We then apply the rules $\wedge L$ and $\neg R$ to get the desired result.

If $C[\cdot] \neq [\cdot]$, then we distinguish cases according to the last applied rule \mathcal{R} on $C[F_i], M \Rightarrow N$ and on $M \Rightarrow N, C[F_i]$. (i) A rule \mathcal{R} has been applied on either M or N . In this case we apply the inductive hypothesis on the height of the derivation, and then by re-applying \mathcal{R} we get the desired result. (ii) A rule \mathcal{R} has been applied on $C[F_i]$ in the sequent $C[F_i], M \Rightarrow N$ (the case where \mathcal{R} has been applied on $C[F_i]$ in the sequent $M \Rightarrow N, C[F_i]$ is analogous). We distinguish the following subcases according to the form of C .

$$\frac{E, D[F_i], M \Rightarrow N}{E \wedge D[F_i], M \Rightarrow N} \rightsquigarrow^{24} \frac{E, D[F_1 \wedge F_2], M \Rightarrow N}{E \wedge D[F_1 \wedge F_2], M \Rightarrow N}$$

$$\frac{E, M \Rightarrow N \quad D[F_i], M \Rightarrow N}{E \vee D[F_i], M \Rightarrow N} \rightsquigarrow \frac{E, M \Rightarrow N \quad D[F_1 \wedge F_2], M \Rightarrow N}{E \vee D[F_1 \wedge F_2], M \Rightarrow N}$$

$$\frac{\forall x D[F_i], D[F_i], M \Rightarrow N}{\forall x D[F_i], M \Rightarrow N} \rightsquigarrow \frac{\forall x D[F_1 \wedge F_2], D[F_i], M \Rightarrow N}{\forall x D[F_1 \wedge F_2], D[F_1 \wedge F_2], M \Rightarrow N} \text{ i.h.}$$

Suppose finally that $C[F_i]$ is of the form $\neg D[F_i]$ ²⁵ and that the sequent $\neg D[F_i], M \Rightarrow N$ has been obtained from the sequent $M \Rightarrow N, D[F_i]$ by means of the rule $\neg L$. Then we consider the sequent $M \Rightarrow N, D[F_i]$ and we apply (since now $D \in \mathcal{N}$) the rule $\wedge 2$ obtaining the desired result. \square

Lemma 5.2. *For any pair of formulas $F, \neg\neg F \in \mathcal{CF}$, it holds that:*

²³All logical rules are invertible in **Gcl**, see Troelstra and Schwichtenberg (1996).

²⁴The symbol \rightsquigarrow means: the premise of the right side is obtained by induction hypothesis on the premise of the left side.

²⁵The case where $A[B_i]$ is of the form $\neg\neg F[B_i]$ is clearly analogous.

$$C[F] \models C[\neg\neg F]$$

Proof. By induction on the construction of $C[.]$. If $C[.] = [.]$, then it is trivial. If $C[.] \neq [.]$, then we need to distinguish cases. However, since F and $\neg\neg F$ are logically equivalent, it is straightforward to check that it holds for any case. \square

Definition 5.3. Given $G, G', F \in \mathcal{CF}$, by

$$G, G' \doteq F \text{ we denote } G, G' \models F \text{ and } G^*, G'^* \models F^*.$$

$$G \mid G' \doteq F \text{ we denote } G' \models F \text{ and } G^*, G'^* \models F^*.$$

$$\langle G \rangle G' \doteq F \text{ we denote } G'^* \models F^* \text{ and } G, G' \models F.$$

Lemma 5.4. For any $G, G', F \in \mathcal{CF}$:

$$G \mid G' \doteq F \text{ if, and only if, } \langle G^* \rangle G'^* \doteq F^*$$

Proof. Straightforward. \square

Lemma 5.5. For any context $C[.]$ that has PUS (see Definition 5.11) and for any formula $G, G', F \in \mathcal{CF}$, such that $F \in \{G \wedge G', \neg(G \vee G')\}$, then it holds that:

$$(a) \text{ if } G, G' \doteq F, \text{ then } C[G], C[G'] \doteq C[F],$$

$$(b) \text{ if } \langle G \rangle G' \doteq F, \text{ then } \langle C[G] \rangle C[G'] \doteq C[F].$$

For any context $C[.]$ that has NES (see Definition 5.11) and for any formula $G, G', F \in \mathcal{CF}$, such that $F \in \{G \wedge G', \neg(G \vee G')\}$, then it holds that:

$$(c) \text{ if } G, G' \doteq F, \text{ then } C[G], C[G'] \doteq C[F],$$

$$(d) \text{ if } \langle G \rangle G' \doteq F, \text{ then } C[G] \mid C[G'] \doteq C[F].$$

Proof. We prove (a)-(d) by (a common) induction on the the construction of $C[.]$. **We start from (a).** If $C[.] = [.]$, then it is trivial. Suppose $C[.] \neq [.]$, then we distinguish cases according to the form of C . We have (i) $C = \neg\neg D[.]$, (ii) $C = E \wedge D[.]$,²⁶ (iii) $C = E \vee D[.]$,²⁷ (iv) $C = \forall x D[.]$, (v) $C = \neg D[.]$.

(i). It is straightforward.

(ii). Suppose $G, G' \doteq F$ (the other option is to have $G^*, G'^* \doteq F$. This can be treated analogously). By i.h., one obtains $D[G], D[G'] \doteq D[F]$. In order to get the desired result, we exploit the sequent calculus **Gcl** in the following way:²⁸

²⁶The case $C = D[.] \wedge E$ is analogous.

²⁷The case $C = D[.] \vee E$ is analogous.

²⁸For the sake of simplicity, we use the multiplicative version of the rule $\wedge R$, as well as the rule of contraction on the left side of the sequent, which are both admissible rules in the calculus **Gcl**.

$$\begin{array}{c}
\frac{D[G], D[G'] \Rightarrow D[F] \quad E, E \Rightarrow E}{E, D[G], E, D[G'] \Rightarrow E \wedge D[F]} \wedge R' \\
\frac{E, D[G], E \wedge D[G'] \Rightarrow E \wedge D[F]}{E \wedge D[G], E \wedge D[G'] \Rightarrow E \wedge D[F]} \wedge L \\
\frac{E, D[G], E \wedge D[G'] \Rightarrow E \wedge D[F]}{E \wedge D[G], E \wedge D[G'] \Rightarrow E \wedge D[F]} \wedge L
\end{array}
\qquad
\begin{array}{c}
\frac{D[F] \Rightarrow D[G], D[G'] \quad E \Rightarrow E}{E, D[F] \Rightarrow D[G], E \wedge D[G']} \wedge R' \\
\frac{E, E, D[F] \Rightarrow E \wedge D[G], E \wedge D[G']}{E, D[F] \Rightarrow E \wedge D[G], E \wedge D[G']} \wedge R' \\
\frac{E, D[F] \Rightarrow E \wedge D[G], E \wedge D[G']}{E \wedge D[F] \Rightarrow E \wedge D[G], E \wedge D[G']} \wedge L
\end{array}$$

From $E \wedge D[G], E \wedge D[G'] \vdash E \wedge D[F]$ by completeness of **Gcl**, one gets $E \wedge D[G], E \wedge D[G'] \models E \wedge D[F]$. From $E \wedge D[F] \vdash E \wedge D[G] \vee E \wedge D[G']$ by completeness of **Gcl**, and the symbol of converse (see Definition 2.3), one gets $(E \wedge D[G])^*, (E \wedge D[G'])^* \models (E \wedge D[F])^*$. Thus we have $E \wedge D[G], E \wedge D[G'] \doteq E \wedge D[F]$.

(iii). Analogously to (ii).

(iv) In this case we further distinguish sub-cases according to the form of F . We thus have (iva) $F = G \wedge G'$, and (ivb) $F = \neg(G \vee G')$.

(iva). By i.h., one obtains $D[G], D[G'] \doteq D[G \wedge G']$. One gets the desired result, exploiting rule $\wedge 1$ of Lemma 5.1, as well as the sequent calculus **Gcl**, in the following way:²⁹

$$\begin{array}{c}
\frac{D[Gy], D[G'y] \Rightarrow D[Gy \wedge G'y]}{\forall x D[Gx], D[G'y] \Rightarrow D[Gy \wedge G'y]} \forall L' \\
\frac{\forall x D[Gx], \forall x D[G'y] \Rightarrow D[Gy \wedge G'y]}{\forall x D[Gx], \forall x D[G'y] \Rightarrow D[Gy \wedge G'y]} \forall L' \\
\frac{\forall x D[Gx], \forall x D[G'y] \Rightarrow D[Gy \wedge G'y]}{\forall x D[Gx], \forall x D[G'y] \Rightarrow \forall x D[Gx \wedge G'y]} \forall R
\end{array}
\qquad
\begin{array}{c}
\frac{\forall x D[Gx] \Rightarrow \forall x D[Gx]}{\forall x D[Gx \wedge G'y] \Rightarrow \forall x D[Gx]} \wedge 1 \\
\frac{\forall x D[Gx \wedge G'y] \Rightarrow \forall x D[Gx], \forall x D[G'y]}{\forall x D[Gx \wedge G'y] \Rightarrow \forall x D[Gx], \forall x D[G'y]} WR
\end{array}$$

From $\forall x D[Gx], \forall x D[G'y] \vdash \forall x D[Gx \wedge G'y]$ by completeness of **Gcl** one gets $\forall x D[Gx], \forall x D[G'y] \models \forall x D[Gx \wedge G'y]$. From $\forall x D[Gx \wedge G'y] \vdash \forall x D[Gx] \vee \forall x D[G'y]$ by completeness of **Gcl**, and the symbol of converse (see Definition 2.3), one gets $(\forall x D[Gx])^*, (\forall x D[G'y])^* \models (\forall x D[Gx \wedge G'y])^*$. Thus we have $\forall x D[Gx], \forall x D[G'y] \doteq \forall x D[Gx \wedge G'y]$.

(ivb). Analogously to (iia) by using the rule $\neg \vee 1$, whose admissibility has been shown in Lemma 5.1.

(v) Assuming $G, G' \doteq F$, we apply (c) getting $D[G], D[G'] \doteq D[F]$, where F has a negative polarity. However, by logic, this is equivalent to $\neg D[G], \neg D[G'] \doteq \neg D[F]$, which is the desired result and where D has a positive polarity.

The cases (b)-(d) can be treated analogously to case (a). □

Lemma 5.6. *For any context $C[\cdot]$ that has PES (see Definition 5.11) and for any formula $G, G', F \in \mathcal{CF}$, such that $F \in \{G \vee G', \neg(G \wedge G')\}$, then it holds that:*

- (a) if $G, G' \doteq F$, then $C[G], C[G'] \doteq C[F]$,
- (b) if $G \mid G' \doteq F$, then $C[G] \mid C[G'] \doteq C[F]$.

²⁹For the sake of simplicity, we use the version of the rule $\forall L$ without the repetition of the quantifier, as well as the weakening on the right. These rules are admissible in the calculus **Gcl**.

For any context $C[\cdot]$ that has NUS (see Definition 5.11) and for any formula $G, G', F \in \mathcal{CF}$, such that $F \in \{G \vee G', \neg(G \wedge G')\}$, then it holds that:

(c) if $G, G' \doteq F$, then $C[G], C[G'] \doteq C[F]$,

(d) if $G \mid G' \doteq F$, then $\langle C[G] \rangle C[G'] \doteq C[F]$,

Proof. The proof is analogous to the proof of Lemma 5.5. \square

Theorem 5.7. (Soundness) For any multisets of sequents S', S (where S' is possibly empty), and sequent $M \Rightarrow N$,

$$\text{if } S' \mid S \Vdash M \Rightarrow N, \text{ then } (S')^\tau \mid (S)^\tau \Vdash \bigwedge M \rightarrow \bigvee N$$

where $(S')^\tau, (S)^\tau$ are the standard translation of the multisets of sequents into multisets of formulas.

Proof. In order to prove the theorem, we should check the validity of each explanatory rule of Figure 4. The validity of the rule $\neg\neg$ follows from Lemma 5.2. We prove the validity of rule \circ_1 . The validity of the other rules can be proved analogously.

Consider the rule \circ_1 applied on a formula of the form $C[F \wedge G]$ such that C has PUS. Clearly, it holds that $F, G \doteq F \wedge G$. But, then by Lemma 5.5, we have $\bigwedge M \rightarrow \bigvee N \vee C[F], \bigwedge M \rightarrow \bigvee N \vee C[G] \doteq \bigwedge M \rightarrow \bigvee N \vee C[F \wedge G]$, where the context $\bigwedge M \rightarrow \bigvee N \vee C[\cdot]$ has PUS. Actually for Lemma 5.5 again, we have that, for any context $E[\cdot]$ that has PUS, it holds that $E[F], E[G] \doteq E[F \wedge G]$. Finally, $\{\bigwedge M \rightarrow \bigvee N \vee C[F], \bigwedge M \rightarrow \bigvee N \vee C[G]\}$ is a multiset of immediate distinguished e-subformulas of $\bigwedge M \rightarrow \bigvee N \vee C[F \wedge G]$ (also thinking of FOL-equivalent formulas). Hence we have the desired result.

Consider the rule \circ_1 applied on a formula of the form $C[F \wedge G]$ such that C has NES. Then the reasoning is the same as above and it thus crucially relies on Lemma 5.5.

Consider the rule \circ_1 applied on a formula of the form $C[F \vee G]$ such that C has PES. Then the reasoning is the same as above, except that one needs to use Lemma 5.6.

Consider the rule \circ_1 applied on a formula of the form $C[F \vee G]$ such that DC has NUS. Then the reasoning is the same as above, except that one needs to use Lemma 5.6. \square

Definition 5.8. For any context $C[\cdot]$, we define the related quantifiers-only-context $Qo(C)[\cdot]$, in the following way:

- if $C \in \mathcal{P}$, then $Qo(C)[\cdot] = SC(C)[\cdot]$
- if $C \in \mathcal{N}$, then $Qo(C)[\cdot] = SC(C)[\cdot]^*$

where $[\cdot]^*$ stands for $\neg(\cdot \wedge \top)$.

Lemma 5.9. Let $Qo(C)[\cdot]$ be the quantifiers-only-context related to $C[\cdot]$, then:

$$SC(Qo(C)) = SC(C)$$

Proof. Straightforward from Definition 5.8. \square

Lemma 5.10. For any multisets of \mathcal{CF} M and N (which could be empty), and for any \mathcal{CF} $C[F]$,

$$\text{if } N \mid M \models C[F], \text{ then } Qo(N) \mid Qo(M) \models Qo(C)[F]$$

where for any multiset of closed formulas P , $Qo(P) = \{Qo(E)[G] \mid E[G] \in P\}$.

Proof. By Definition 4.11. □

Definition 5.11. For any quantifier-only-context $Qo(C)[.]$, we say that $Qo(C)[.]$ is:

a *positive universal* if, and only if, $Qo(C)[.] = \forall x_1, \dots, \forall x_n[.]$, where $n \geq 0$.

a *negative universal* if, and only if, $Qo(C)[.] = \forall x_1, \dots, \forall x_n[.]^*$, where $n \geq 0$.

a *positive existential* if, and only if, $Qo(C)[.] = \exists x_1, \dots, \exists x_n[.]$, where $n \geq 0$.

a *negative existential* if, and only if, $Qo(C)[.] = \exists x_1, \dots, \exists x_n[.]^*$, where $n \geq 0$.

Lemma 5.12. For any multisets of \mathcal{CF} M and N (which could be empty), and for any \mathcal{CF} $C[F]$,

$$\text{if } Qo(N) \mid Qo(M) \models Qo(C)[F] \text{ then } (N)^\delta \mid (M)^\delta \models \Rightarrow C[F]$$

where for any multiset of \mathcal{CF} M , $M^\delta = \{\Rightarrow E[C] \mid E[C] \in M\}$.

Proof. We proceed by distinguishing cases based on the form of $Qo(C)[.]$ and F .

$[-]$ $Qo(C)[.]$ might be such that: (i) it is a positive universal; (ii) it is a positive existential; (iii) it is a negative universal; (iv) it is a negative existential; (v) $Qo(C)[.] = SC(C)[.]$, where $SC(C)$ corresponds to any finite sequence of universal and existential quantifiers that is not empty and is neither of the type $\forall x_1, \dots, \forall x_n$, nor of the type $\exists x_1, \dots, \exists x_n$; (vi) $Qo(C)[.] = SC(C)[.]^*$, where $SC(C)$ corresponds to any finite sequence of universal and existential quantifiers that is not empty and is neither of the type $\forall x_1, \dots, \forall x_n$, nor of the type $\exists x_1, \dots, \exists x_n$.

$[-]$ F can be of the following form: (a) $\neg\neg G$; (b) $G \wedge G'$; (c) $G \vee G'$; (d) $\neg(G \wedge G')$; (e) $\neg(G \vee G')$; (f) $\forall x Gx$; (g) $\neg\forall x Gx$; (h) $\exists x Gx$; (i) $\neg\exists x Gx$.

We check in detail the combination of (i)-(vi) with (a), (b) and (e). The other combinations can be treated analogously.

1. We combine (i)-(vi) with (a). In each case, we have that $Qo(C)[G] \models Qo(C)[\neg\neg G]$; at the syntactic level the explanatory rule $\neg\neg$ gives us what desired, namely $\Rightarrow C[G] \models \Rightarrow C[\neg\neg G]$.
2. We combine (i) with (b). We have that $Qo(C)[G], Qo(C)[G'] \models Qo(C)[G \wedge G']$. At the syntactic level, thanks to the explanatory rule \circ_1 , we get what desired, namely $\Rightarrow C[G], \Rightarrow C[G'] \models \Rightarrow C[G \wedge G']$.
3. We combine (iv) with (b). We have that $Qo(C)[G], Qo(C)[G'] \models Qo(C)[G \wedge G']$, $Qo(C)[G] \mid Qo(C)[G'] \models Qo(C)[G \wedge G']$ and $Qo(C)[G] \mid Qo(C)[G'] \models Qo(C)[G \wedge G']$. At the syntactic level, thanks to the explanatory rules \circ_1, \circ_2 , we get what desired, namely $\Rightarrow C[G], \Rightarrow C[G'] \models \Rightarrow C[G \wedge G']$, $\Rightarrow C[G] \mid \Rightarrow C[G'] \models \Rightarrow C[G \wedge G']$ and $\Rightarrow C[G] \mid \Rightarrow C[G'] \models \Rightarrow C[G \wedge G']$.
4. It is straightforward to check that the combination of (ii), (iii), (v), and (vi) with (b) does not give rise to any relation from reasons to conclusion.

5. We combine (i)-(vi) with (e), hence with a formula of the type $Qo(C)[\forall xAx]$ It is easy to check that there is no *closed* e-subformula of $Qo(C)[\forall xAx]$ such that it stands with $Qo(C)[\forall xAx]$ in a relation of total and immediate reasons-conclusion. Hence, this case does not need to be further analyzed.

□

Theorem 5.13. (*Completeness*) For any multisets of closed formulas N, N' (possibly empty), and formula $C[F]$,

$$\text{if } N' \mid N \models C[F], \text{ then } (N')^\delta \mid (N)^\delta \Vdash \Rightarrow C[F]$$

Proof. From Lemmas 5.10 and 5.12.

□

6 Related works

Recent years have witnessed an increasing interest in the notion of explanation from a logical point of view. Here we mention some trends (the list is not exhaustive) that could be seen as related to the present work. A first trend is that which is taking place with some common machine learning classifiers, where recent research has aimed at identifying the reasons behind the classification of instances, and thus has proposed explanations for this type of decision, e.g. see Darwiche and Hirth (2023); Shih et al. (2018). By considering the examples taken into account in this wide and still flourishing literature, and in particular by dwelling on the crucial notion of *sufficient reason*, we are led to believe that there are strong analogies with our approach. If this analogy is well-founded, then it could be proved fruitful in (at least) two ways. On the one hand, the semantics put forward by Darwiche and Hirth (2023) - which is mainly in terms of *prime implicants* - can be profitably employed to develop a semantic-approach for the present proposal, which is for now mainly syntactic. On the other hand, the power of the explanatory sequent calculus introduced here could be implemented to enrich the computational part of the work developed in, e.g. Shih et al. (2018).

Note that related to the work Darwiche and Hirth (2023); Shih et al. (2018) is the work of Liu and Lorini (2023, 2022). In a nutshell, Liu and Lorini introduce the reasons behind the classification of instances in the (modal) language, then developing an axiomatic system as well as a semantic for the new connective. Since the approach we propose only lies at the metalinguistic level, but it is lacking for its linguistic counterpart, the work of Liu and Lorini (2023, 2022) can be seen as a useful source of inspiration.

A third trend that one might be tempted to consider lies within metaphysics, where there is a growing interest towards the notion of (metaphysical) grounding (e.g. see Fine (2012)), namely a relation which shares some characteristics with our relation of reasons-conclusion. Despite this first similarity, there also are deep differences that separate these approaches. We mention the following two: on the one hand, the intrinsic metaphysical character of the study of the notion of grounding, which is absent from the present discussion; on the other hand, the fact that in this literature only toy examples are taken into account, while one of the main interest of our theory precisely relies in his capability of encompassing intricate examples as well. Hence, comparison and links with the metaphysical grounding literature

can be developed only once these two main differences have previously been investigated and clarified.

Last, but not least, another recent trend in the current literature concerns those explanations that are characterized by abductive reasoning, namely an inference to the best explanation. In this framework there are (at least) two recent lines of work, one developed by Arieli et al. (2022); Millson and Strasser (2019), whilst the other by Piazza et al. (2023); Pulcini and Varzi (2021). Despite their difference, these works have a strong common feature, namely they both develop new sequent calculi where several different rules are proposed to account for abductive reasoning. We thus have three proof-theoretical formal frameworks dealing with different notions of explanations. As a consequence, the study of their relations could open up for a novel and interesting connection between conceptual and abductive explanations, both at the conceptual and at the logical level.

7 Conclusions

The word *explanation* is an umbrella term which covers several different notions, such as causal, non-causal or abductive explanations. In this paper we have focussed on conceptual explanations, namely some deductive explanations-why, which come from a long and illustrious tradition in philosophy, bear several analogies with causal explanations, but still deserve a thorough formal study. The main aim of this paper has been to move the first steps towards filling this gap, by the introduction of a logical theory of the notion of (conceptual) explanation and related relation of reasons to conclusion. We have accomplished this task by using and enriching the standard tools of proof theory, namely the sequent calculus for classical first-order logic. In particular we have added to the standard inferential rules explanatory rules, i.e., rules whose premisses represent the (total and immediate) reasons why their conclusion is true. By means of these rules we can construct formal explanations, which represent the formalization of the notion of (conceptual) explanation. Not only do we believe that this research provides a valuable contribution *per se*, in that it fills an important gap in the logical literature, but it also naturally opens up several directions of future research, such as the formalization of the notion of explanation in logics other than classical logic, the applications of conceptual explanations to related fields such as explainable AI, or to related notions of explanations. Finally, it also opens up to the investigation of the value of explanatory rules in proof-theoretic semantics, e.g. see Francez (2015).

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