Lyapunov-based Consistent Discretization of Quasi-Continuous High Order Sliding Modes

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Abstract In this chapter we propose an explicit discretization scheme for class of disturbed systems controlled by homogeneous quasi-continuous High Order Sliding Mode controllers which are equipped with a homogeneous Lyapunov function. Such a Lyapunov function is used to construct the discretization scheme that preserves important features from the original continuous-time system: asymptotic stability, finite-time convergence, and the Lyapunov function itself.

1 Introduction

Discretization of continuous-time models has become a fundamental step in most of the processes to design a control systems. It is required, e.g., for numerical simulation, for implementation by means of digital electronics, or for designing of sampled-data controllers [27, 11]. For the case of linear systems we can obtain exact dsicretized models, however, exact discretization of nonlinear systems is in general impossible due to the lack of explicit solutions. Hence, approximating discretization techniques must be used in the nonlinear setting. Nonetheless, for many nonlinear systems it is not a trivial task: first of all, standard discretization techniques usually impose some smoothness requirements on the system; and secondly, they do not preserve some relevant characteristics from the continuous-time system.

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For the particular case of High Order Sliding Mode (HOSM) systems, which are non-smooth by nature, it is well known that standard discretization methods produce undesirable behaviors in the discrete-time approximation [8, 36, 21, 18, 1, 16, 9, 24, 32].

That is why several new strategies to discretize sliding-mode systems have been designed, e.g.: the implicit discretization of standard sliding modes [8, 36] and HOSM controllers [5]; the discrete-time redesign of the robust exact differentiator of arbitrary order proposed in [19]; the consistent implicit or semi-implicit discretization algorithms for finite-time and fixed-time stable systems developed in [29], which is based on an adequate transformation of the system; the digital implementation of sliding-mode controllers based on the discretization of differential inclusions by means of the implicit Euler method, see e.g. [1, 16].

In this chapter we propose a technique to discretize a class of systems controlled by quasi-continuous HOSM controllers whose origin is asymptotically stable. It is well known that one of the advantages of quasi-continuous HOSM is that the only discontinuity is at the origin [23]. The proposed technique is based on the discretization procedure provided in [34, 33][1] for homogeneous systems without disturbances. On one hand we particularize the method for quasi-continuous HOSM, nonetheless, on the other hand we extend the method by allowing time-varying disturbances in the model. A relevant feature of the proposed method is that it takes advantage of the information provided by the Lyapunov function for the closed-loop system. As in [34] the discretization scheme has the following properties:

1. Lyapunov function preservation: the Lyapunov function of the continuous-time system is also a Lyapunov function for discrete-time system, guaranteeing this way that the origin of the obtained discrete-time system is Lyapunov stable;
2. Consistency: the origin of the discrete-time approximating system is finite-time stable (this means that the discretization is consistent in the sense described in [29]);
3. Independence of the discretization step: the properties of stability and consistency of the obtained discrete-time systems are not affected by the size of the discretization step.

Chapter organization: In Section 2 we state the problem to be solved and provide some definitions and preliminary results. In Section 3 we analyze the dynamics of the studied system by projecting it on a level set of its Lyapunov function. The discretization scheme, proposed in this chapter, is introduced and explained in Section 4. In Section 5, we present some examples of the proposed discretization method. In Section 6 some conclusions are stated.

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[1] Some parts of Lemma 2, Theorem 3, Theorem 5, and their proofs have been reproduced with permission from [Sanchez T, Polyakov A, Efimov D. Lyapunov-based consistent discretization of stable homogeneous systems. Int J Robust Nonlinear Control. 2021;31:3587–3605. https://doi.org/10.1002/rnc.5308] ©2020 John Wiley & Sons Ltd., and with permission from the IFAC License Agreement IFAC 2020#1150 of [T. Sanchez, A. Polyakov, D. Efimov, A Consistent Discretisation method for Stable Homogeneous Systems based on Lyapunov Function, IFAC-PapersOnLine 53(2), 5099-5104 (2020). DOI https://doi.org/10.1016/j.ifacol.2020.12.1141.].

Notation: The set of integer numbers is denoted by $\mathbb{Z}$. $\mathbb{R}_+^*$ denotes the set $\mathbb{R}_+ \setminus \{0\}$, analogously for the set $\mathbb{Z}$. For a function $V : \mathbb{R}^n \to \mathbb{R}_+$, which is continuous and positive definite, we denote the set $\mathscr{S}_V = \{x \in \mathbb{R}^n : V(x) = 1\}$. The class of functions $\eta : \mathbb{R}_+ \to \mathbb{R}_+$ with $\eta(0) = 0$, which are strictly increasing and continuous, is denoted by $\mathscr{K}$.

2 Problem statement and preliminaries

In this section we describe the class of systems to be studied in this chapter, we also give the statement of the problem to be solved, and we recall some important properties of homogeneous systems.

In this chapter we consider the following continuous-time system

$$\begin{aligned} \dot{x}_1(t) &= x_2(t) \\ &\vdots \\ \dot{x}_{n-1}(t) &= x_n(t) \\ \dot{x}_n(t) &= d_1(t) + d_2(t)u(x(t)) \end{aligned} \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the state, and $u(x(t)) \in \mathbb{R}$ is the control signal. The disturbances $d_1(t), d_2(t) \in \mathbb{R}$ are piece-wise continuous functions such that

$$|d_1(t)| \leq \bar{d}_1 , \quad \underline{d}_2 \leq |d_2(t)| \leq \bar{d}_2 , \tag{2}$$

for all $t \in \mathbb{R}$ for some known constants $\bar{d}_1, \underline{d}_2, \bar{d}_2 \in \mathbb{R}_+^*$.

For (1), we consider the following sub-class of quasi-continuous controllers $u(x)$ proposed in [7],

$$u(x) = -k_n \frac{\sigma_n(x)}{\bar{\sigma}_n(x)} , \tag{3}$$

where $\sigma_1(x) = x_1$, $\bar{\sigma}_1(x) = |x_1|$, and for $i \in \{2, \ldots, n\}$ we have that

$$r_i = n + 1 - i , \quad \sigma_i(x) = \lceil x_i \rfloor^{\frac{n}{r_i}} + k_{i-1}^{\frac{n}{r_i}} \sigma_{i-1} , \quad \bar{\sigma}_i(x) = |x_i|^{\frac{n}{r_i}} + k_{i-1}^{\frac{n}{r_i}} \bar{\sigma}_{i-1} , \tag{4}$$

where we used the notation $\lceil x \rfloor^p := |x|^p \operatorname{sign}(x)$.

For the closed-loop system (1), (3) it is also provided in [7] the Lyapunov function $V_n : \mathbb{R}^n \to \mathbb{R}_+$ given by the following construction for $i \in \{2, \ldots, n\}$,

$$V_i(x) = V_{i-1}(x) + W_i(x) , \tag{5}$$

where $W_i(x) = \frac{r_i}{2n}|x_i|^{\frac{2n}{r_i}} - \lceil \nu_{i-1} \rfloor^{\frac{2n-r_i}{r_i}} x_i + (1 - \frac{r_i}{2n})|\nu_{i-1}|^{\frac{2n}{r_i}}$, $\nu_i = -k_i \lceil \sigma_i \rfloor^{\frac{n-i}{n}}$, $\nu_1 = -k_1 \lceil x_1 \rfloor^{\frac{n-1}{n}}$, and $V_1(x) = \frac{1}{2}|x_1|^2$.

Theorem 1 ([7]). *Consider the closed-loop of* (1) *with* (3)*. There exist large enough* $k_i > 0$ *such that the origin of the system is globally finite-time stable and* V_n *given by* (5) *is a Lyapunov function for the system.*

A procedure to compute the gains k_i is given in [7]. Let us show three explicit examples of controllers (3) and their respective Lyapunov functions (5).

Example 1. For $n = 1$ we have the controller

$$u(x) = -k_1 \operatorname{sign}(x_1)\,,$$

and the Lyapunov function

$$V_1(x) = \tfrac{1}{2}x_1^2\,.$$

Example 2. For $n = 2$ we have the controller

$$u(x) = -k_2 \frac{\lceil x_2 \rfloor^2 + k_1^2 x_1}{x_2^2 + k_1^2 |x_1|}\,,$$

and the Lyapunov function

$$V_2(x) = \tfrac{1}{2}x_1^2 + \tfrac{1}{4}x_2^4 + k_1^3 \lceil x_1 \rfloor^{\frac{3}{2}} x_2 + \tfrac{3}{4}k_1^4 x_1^2\,.$$

Example 3. For $n = 3$ we have the controller

$$u(x) = -k_3 \frac{\lceil x_3 \rfloor^3 + k_2^3 \sigma_2(x)}{|x_3|^3 + k_2^3 \big(|x_2|^{\frac{3}{2}} + k_1^{\frac{3}{2}} |x_1| \big)}\,,$$

where $\sigma_2 = \lceil x_2 \rfloor^{\frac{3}{2}} + k_1^{\frac{3}{2}} x_1$. For this case, the Lyapunov function is

$$V_3(x) = \tfrac{1}{2}x_1^2 + W_2(x) + W_3(x)\,,$$

where

$$W_2(x) = \tfrac{1}{3}|x_2|^3 + k_1^2 \lceil x_1 \rfloor^{\frac{4}{3}} x_2 + \tfrac{2}{3}k_1^3 x_1^2\,,$$

$$W_3(x) = \tfrac{1}{6}x_3^6 + k_2^5 \lceil \sigma_2 \rfloor^{\frac{5}{3}} x_3 + \tfrac{5}{6}k_2^6 \sigma_2^2\,,$$

In Section 5 these examples are resumed to illustrate the discretization procedure developed in Section 4.

In order to make the exposition clearer, let us rewrite the closed-loop system (1), (3) as follows

$$\dot{x}(t) = f\big(x(t), d(t)\big)\,, \quad x(t) \in \mathbb{R}^n\,,\ d(t) \in \mathbb{R}^2\,, \tag{6}$$

where $f_i(x,d) = x_{i+1}$ for $i \in \{1,\ldots,n-1\}$, and $f_n(x,d) = d_1 - d_2 k_n \frac{\sigma_n(x)}{\bar{\sigma}_n(x)}$. Note that $f : \mathbb{R}^{n+2} \to \mathbb{R}^n$ is continuous except at $x = 0$. Let us also denote with $\mathscr{D}$ to the set of piece-wise continuous functions $d : \mathbb{R} \to \mathbb{R}^2$ satisfying (2).

Since the input d is unknown, a usual procedure in sliding mode control to analyze (6) is to replace it by the differential inclusion [22, 30]

$$
\begin{aligned}
\dot{x}_1(t) &= x_2(t)\\
&\vdots\\
\dot{x}_{n-1}(t) &= x_n(t)\\
\dot{x}_n(t) &\in [-\bar{d}_1, \bar{d}_1] - [\underline{d}_2, \bar{d}_2] k_n \tfrac{\sigma_n(x)}{\bar{\sigma}_n(x)}\,.
\end{aligned} \tag{7}
$$

Thus, the solutions of (7) are understood as the solutions of a differential inclusion

$$\dot{x} \in B(x)\,, \quad x \in \mathbb{R}^n\,, \tag{8}$$

associated with (7), where the set-valued map B satisfies the following *basic conditions* [10, p. 77]: for all $x \in \mathbb{R}^n$ the set $B(x)$ is nonempty, compact and convex, and the set-valued function B is upper-semicontinuous. In this context, a (generalized) solution of (7) is defined as a function $x : \Gamma \subset \mathbb{R}_+ \to \mathbb{R}^n$ which is absolutely continuous and satisfies (8) for almost all $t \in \Gamma$ [10, p. 50]. Moreover, the existence of solutions of the differential inclusion is guaranteed since B satisfies the basic conditions. Following [22], we refer to (8) as a *Filippov differential inclusion*, which is obtained by means of a kind of Filippov regularization of (7) [22]. In general, the solutions of (7) or (8) are non-unique, however, observe that the right-hand side of (6) satisfies the conditions[2] to guarantee uniqueness of solutions on $\mathbb{R}^n \setminus \{0\}$.

2.1 Homogeneity

Let us begin this section by recalling the definition of *Weighted Homogeneity*.

Definition 1 ([17, 26, 22]). Given a set of coordinates $(x_1, x_2, \ldots, x_n)$ for $\mathbb{R}^n$, $\Delta^{\mathbf{r}}(\varepsilon)x$ denotes the family of dilations characterized by the square diagonal matrix $\Delta^{\mathbf{r}}(\varepsilon) = \mathrm{diag}(\varepsilon^{\mathrm{r}_1}, \ldots, \varepsilon^{\mathrm{r}_n})$, where $\mathbf{r} = [r_1, \ldots, r_n]^\top$, $r_i \in \mathbb{R}_+^*$, and $\varepsilon \in \mathbb{R}_+^*$. The components of $\mathbf{r}$ are called the *weights* of the coordinates. Thus:

1. a function $V : \mathbb{R}^n \to \mathbb{R}$ is $\mathbf{r}$-homogeneous of degree $m \in \mathbb{R}$ if

$$V(\Delta^{\mathbf{r}}(\varepsilon)x) = \varepsilon^m V(x)\,, \quad \forall x \in \mathbb{R}^n,\ \forall \varepsilon \in \mathbb{R}_+^*\,;$$

2. a vector field $f : \mathbb{R}^n \to \mathbb{R}^n$, is $\mathbf{r}$-homogeneous of degree $\mu \in \mathbb{R}$ if

$$f(\Delta^{\mathbf{r}}(\varepsilon)x) = \varepsilon^\mu \Delta^{\mathbf{r}}(\varepsilon) f(x)\,, \quad \forall x \in \mathbb{R}^n,\ \forall \varepsilon \in \mathbb{R}_+^*\,;$$

3. a set-valued map $x \mapsto B(x) \subset \mathbb{R}^n$, is $\mathbf{r}$-homogeneous of degree $\mu \in \mathbb{R}$ if

$$B(\Delta^{\mathbf{r}}(\varepsilon)x) = \varepsilon^\mu \Delta^{\mathbf{r}}(\varepsilon) B(x)\,, \quad \forall x \in \mathbb{R}^n,\ \forall \varepsilon \in \mathbb{R}_+^*\,.$$

A differential inclusion (8) is said to be $\mathbf{r}$-homogeneous of degree $\mu \in \mathbb{R}$ if its vector-set field (or set-valued vector field) B is $\mathbf{r}$-homogeneous of degree μ.

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[2] Under the assumption of $d \in \mathscr{D}$, it can be seen that the right-hand side of (6) is locally Lipschitz in x on $\mathbb{R}^n \setminus \{0\}$.

Now, we recall some important features of $\mathbf{r}-$homogeneous differential inclusions[3].

Theorem 2 ([31, 26, 3]). *Let* (8) *be* $\mathbf{r}-$*homogeneous of degree* $\mu < 0$ *with* B *satisfying the basic conditions. If* $x = 0$ *is strongly globally asymptotically stable then*

1. $x = 0$ *is strongly globally finite-time stable;*
2. *for any positive integer* p *and any real* $m > p\max\{r_1, \ldots, r_n\}$ *there exists a positive definite function* $V : \mathbb{R}^n \to \mathbb{R}_+$ *such that*
   a. V *is of class* C^∞ *for all* $x \neq 0$ *and of class* C^p *for all* $x \in \mathbb{R}^n$*;*
   b. V *is* $\mathbf{r}-$*homogeneous of degree* m*;*
   c. *there exists a continuous positive definite function* $\bar{W} : \mathbb{R}^n \to \mathbb{R}_+$ *such that it is* $\mathbf{r}-$*homogeneous of degree* $m + \mu$*, and*

$$\frac{\partial V(x)}{\partial x} b \leq -\bar{W}(x)\,, \quad \forall x \in \mathbb{R}^n\,,\ \forall b \in B(x)\,. \tag{9}$$

Remark 1. It is important to mention that (7) is an $\mathbf{r}$-homogeneous differential inclusion of degree $\mu = -1$ with weights $\mathbf{r} = [n, n-1, \ldots, 1]^\top$. Also note that (since (6) describes the closed-loop system (1), (3)), for any function d the vector field f is such that

$$f(\Delta^{\mathbf{r}}(\varepsilon)x, d) = \varepsilon^{\mu}\Delta^{\mathbf{r}}(\varepsilon) f(x, d)\,, \quad \forall x \in \mathbb{R}^n,\ \forall \varepsilon \in \mathbb{R}_+^*\,. \tag{10}$$

Now, if V is as in Theorem 2, then the derivative of V along (6) is given by

$$\dot{V} = -W(x, d)\,, \quad W(x, d) := -\frac{\partial V(x)}{\partial x} f(x, d)\,, \tag{11}$$

where the function $W : \mathbb{R}^{n+m} \to \mathbb{R}$ satisfies the following

$$W(\Delta^{\mathbf{r}}(\varepsilon)x, d) = \varepsilon^{m+\mu} W(x, d)\,, \quad \forall x \in \mathbb{R}^n,\ \forall d \in \mathbb{R}^2,\ \forall \varepsilon \in \mathbb{R}_+^*\,,$$

which is a direct consequence of the fact that $\frac{\partial V(\Delta^{\mathbf{r}}(\varepsilon)x)}{\partial x} = \varepsilon^m \frac{\partial V(x)}{\partial x} \Delta^{-\mathbf{r}}(\varepsilon)$ (see, e.g. [35, Prop. 1]) and (10).

Moreover, the solutions of (6) are in the set of solutions of the Filippov differential inclusion (8). Therefore, by Theorem 2, the derivative of V along the solutions of (6) satisfies $\dot{V} = -W(x, d) \leq -\bar{W}(x)$, with $\bar{W}$ as given in (9). In such a case, there exists $\alpha \in \mathbb{R}_+^*$ such that [15, 26]

$$\dot{V} \leq -\alpha V^{\frac{m+\mu}{m}}(x)\,. \tag{12}$$

As stated in Theorem 2, $\bar{W}$ is $\mathbf{r}-$homogeneous, hence, the constant parameter α in (12) can be computed as follows

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[3] Following [3], we use the term *strong stability* (which involves all the solutions) in Theorem 2 to contrast with the term of *weak stability*, which claims properties of some solutions [10].

$$\alpha = \inf_{x \in \mathscr{S}_V} \bar{W}(x)\,. \tag{13}$$

We know from Theorem 2 that the degree of homogeneity of $\bar{W}$ is $m+\mu$, which is strictly positive if the homogeneity degree of V is restricted to $m > -\mu$. Observe that this is always the case for (5) since $m = 2n$ and $\mu = -1$.

The properties explained so far prove the following result (analogous to those in [13, 15, 26] for unperturbed systems).

Lemma 1. *Let* (8) *be a Filippov differential inclusion associated with* (7)*. Also let* (8) *and* V *be as in Theorem 2. Then, in* (6)*, for all* $x(0) \in \mathbb{R}^n$*, for all* $t \in \mathbb{R}_+$*, and for any* $d \in \mathscr{D}$*, the following holds (with* α *as given in* (13)*):* $V(x(t)) \leq \bar{V}(x(0),t)$ *where*

$$\bar{V}(x(0),t) = \begin{cases} \left(V^{\frac{-\mu}{m}}(x(0)) - \frac{-\mu}{m}\alpha t\right)^{\frac{m}{-\mu}}, & t < \frac{m}{-\mu\alpha}V^{\frac{-\mu}{m}}(x(0)) \\ 0, & t \geq \frac{m}{-\mu\alpha}V^{\frac{-\mu}{m}}(x(0)) \end{cases}. \tag{14}$$

From Lemma 1, we can see that the trajectories of (1) converge to the origin in finite-time. Moreover, the convergence time $T(x(0))$ to the origin, for the initial condition $x(0)$, is bounded as follows $T(x(0)) \leq \frac{m}{-\alpha\mu}V^{\frac{-\mu}{m}}(x(0))$.

2.2 Problem statement

As already mentioned in the introduction, an exact discretization for a nonlinear system is (in general) not possible, this due to the lack of knowledge of the exact solution of the system. However, any suitable discretization scheme should preserve relevant properties of the solutions, e.g., the type of convergence of the trajectories to the origin in case of asymptotic stability. If we are able to extract some relevant information from a Lyapunov function, e.g., stability properties and convergence rates, just as it is done in Lemma 1, then such a Lyapunov function should be used to develop a discretization scheme. Hence, the problem to be solved in this chapter is:

Develop a discretization scheme, for (6) *(which describes the closed-loop system* (1)*,* (3)*) such that: if the origin of* (6) *is finite-time stable, then the generated discrete-time approximating system preserves the finite-time stability property of the continuous-time system.*

In this chapter we solve this problem by taking advantage of the information provided by the homogeneous Lyapunov function of the system. This is done by making a homogeneous projection of the dynamics of the system on a level set of the Lyapunov function. Thus, the evolution of the system's trajectories can be determined from the trajectories of the projected dynamics and an expansion computed by using the information of the decaying rate of the Lyapunov function along the solutions of the system (see Section 3 for more details).

We have to mention that, in the literature, there exist some discretization methods that also utilize the idea of projecting the trajectories of the system onto level sets of the Lyapunov functions. Unfortunately, although some of those methods are able to keep the same Lyapunov function for the discrete-time system, they cannot guarantee that the convergence rates are preserved as well, see [12] and the references therein. In general, another disadvantage of those methods is that the projection is not explicit (i.e., an algebraic equation must be solved to find the projection). In contrast, in the discretization method of this chapter, the projection onto the level set of the Lyapunov function is explicit, which represents a procedural advantage.

3 Projected dynamics

In this section we compute and analyze the projection of the dynamics of (6) onto a unitary level set of its Lyapunov function. The developments of this section constitute the fundamentals for the construction of the discretization method proposed in Section 4.

Let V be as in Theorem 2, and define the following change of variable

$$y = \Delta^{\mathbf{r}}(V^{\frac{-1}{m}}(x))x\,, \quad \forall x \in \mathbb{R}^n \setminus \{0\}\,. \tag{15}$$

Observe that, according to [28, p. 159], (15) constitutes the *homogeneous projection* of the point x over the level set $\{x \in \mathbb{R}^n : V(x) = 1\}$, thus, $y \in \mathscr{S}_V$ for all $x \in \mathbb{R}^n \setminus \{0\}$. By taking the derivative of (15) along (6) we obtain

$$\dot{y} = \Delta^{\mathbf{r}}(V^{\frac{-1}{m}}(x))\left(I - \tfrac{1}{m}V^{-1}(x)Gx\tfrac{\partial V(x)}{\partial x}\right)f(x,d)\,, \tag{16}$$

where I is the $n \times n$ identity matrix, and $G := \mathrm{diag}(r_1, \ldots, r_n)$. Recall from Remark 1 that f in (6) satisfies (10). From (15) we obtain $x = \Delta^{\mathbf{r}}(V^{\frac{1}{m}}(x))y$, which can be substituted in (16) to obtain

$$\begin{aligned}
\dot{y} &= \Delta^{\mathbf{r}}(V^{\frac{-1}{m}}(x))f(x,d) - \tfrac{1}{m}V^{-1}(x)\Delta^{\mathbf{r}}(V^{\frac{-1}{m}}(x))G\Delta^{\mathbf{r}}(V^{\frac{1}{m}}(x))y\tfrac{\partial V(x)}{\partial x}f(x,d)\,,\\
&= V^{\frac{\mu}{m}}(x)f(y,d) + \tfrac{1}{m}\tfrac{W(x,d)}{V(x)}Gy\,,\\
&= V^{\frac{\mu}{m}}(x)f(y,d) + \tfrac{1}{m}\tfrac{V^{\frac{m+\mu}{m}}(x)W(y,d)}{V(x)}Gy\,,
\end{aligned}$$

(where W is given by (11)), therefore,

$$\dot{y} = V^{\frac{\mu}{m}}(x)\left[f(y,d) + \tfrac{1}{m}W(y,d)Gy\right]. \tag{17}$$

Equation (17) describes the dynamics (6) projected onto $\mathscr{S}_V$. However, note that we cannot recover the trajectories of (6) directly from the trajectories of (17) since (15) is not bijective. Here is where we can exploit the information provided by the Lyapunov function. Thus, we proceed to study the dynamics of V, i.e., its derivative along the trajectories of (6). Thus, considering (15), we obtain from (11) that

$$\dot{V} = -W\big(\Delta^{\mathbf{r}}(V^{\frac{1}{m}}(x))y, d\big) = -V^{\frac{m+\mu}{m}}(x)W(y,d)\,. \tag{18}$$

Note that (17) and (18) still depend on x, thus, we introduce two auxiliary equations that will be useful for our purposes. Thus, we define a function $v:\mathbb{R}_+ \to \mathbb{R}_+$ such that it is solution to the differential equation (cf. (18))

$$\dot{v}(t) = -v^{\frac{m+\mu}{m}}(t)W(z(t),d(t))\,, \tag{19}$$

where the function $z:\mathbb{R}_+ \to \mathbb{R}^n$ is the solution to the following system (cf. (17))

$$\dot{z}(t) = v^{\frac{\mu}{m}}(t)\big[f(z(t),d(t)) + \tfrac{1}{m}W(z(t),d(t))Gz(t)\big]\,. \tag{20}$$

From the developments made up to this point, we are now ready to state the main results of this section. The first one of these results consists in verifying that the set $\mathscr{S}_V$ is positively invariant with respect to the trajectories of (20).

Lemma 2. *Consider* (20) *with* $z(0) \in \mathscr{S}_V$*, and any continuous function* $v:\mathbb{R}_+ \to \mathbb{R}$*. If* $\mu < 0$ *and* $v(t) \neq 0$ *for all* $t \in [0,T)$ *for some* $T \in \mathbb{R}_+^*$*, then* $z(t) \in \mathscr{S}_V$ *for all* $t \in [0,T)$*, and any* $d \in \mathscr{D}$*.*

Proof. To verify that $\mathscr{S}_V$ is positively invariant, let us compute the derivative of $V(z)$ along the trajectories of (20), thus

$$\dot{V} = v^{\frac{\mu}{m}}\big[\tfrac{\partial V(z)}{\partial z}f(z,d) + \tfrac{1}{m}W(z,d)\tfrac{\partial V(z)}{\partial z}Gz\big]\,.$$

Since $\frac{\partial V(z)}{\partial z}f(z,d) = -W(z,d)$, we can see that if $W(z,d) = 0$ then $\dot{V}(z(t)) = 0$. Thus, without loss of generality, we assume that $W(z,d) \neq 0$. According to the equality[4]

$$\tfrac{\partial V(z)}{\partial z}Gz = mV(z)\,, \tag{21}$$

we obtain

$$\dot{V} = v^{\frac{\mu}{m}}W(z,d)\big[-1 + V(z)\big]\,.$$

Hence $\dot{V}(z(t)) = 0$ if and only if $V(z(t)) = 1$ for all $t \in [0,T)$. □

The second main result of this section is with respect to a useful representation of the solutions of (19).

Lemma 3. *Let* (6) *and* V *be as in Lemma 1. For any initial condition* $v(0) \in \mathbb{R}_+^*$*, and any* $d \in \mathscr{D}$*, there exists* $\Theta_d(v(0)) \in \mathbb{R}_+^*$ *such that the function* $v:\mathbb{R}_+ \to \mathbb{R}_+$ *given by*

$$v(t) = \begin{cases} \Big(v(0)^{\frac{-\mu}{m}} - \frac{-\mu}{m}\hat{W}_0(t)\Big)^{\frac{m}{-\mu}}, & \frac{-\mu}{m}\hat{W}_0(t) < v(0)^{\frac{-\mu}{m}} \\ 0, & \frac{-\mu}{m}\hat{W}_0(t) \geq v(0)^{\frac{-\mu}{m}} \end{cases}, \tag{22}$$

[4] This equation is known as the Euler's theorem for weighted homogeneous functions, see, e.g. [2, Proposition 5.4].

where $\hat{W}_0(t) := \int_0^t W(z(\tau), d(\tau))\, \mathrm{d}\tau$*, satisfies* (19) *with* $v(t) > 0$ *for all* t *in the interval* $[0, \Theta_d(v(0)))$*, and* $v(t) \to 0$ *as* $t \to \Theta_d(v(0))$*.*

Proof. This lemma is proven by direct integration of (19) to obtain (22). Nonetheless, let us provide some clarifying details. Since $W(z,d) > 0$ for all $z \in \mathscr{S}_V$ and all $d \in \mathscr{D}$, then v in (22) is strictly decreasing to zero, hence, for each $v(0) \in \mathbb{R}_+^*$ there exists a maximal $\Theta_d(v(0)) \in \mathbb{R}_+^*$ such that $\hat{W}_0(t) < v(0)^{\frac{-\mu}{m}}$ for all $t \in \left[0, \Theta_d(v(0))\right)$. Note that, $\left[0, \Theta_d(v(0))\right)$ is the interval of time such that the right-hand side of (20) is well-defined on it. $\square$

Note that $v : \mathbb{R}_+ \to \mathbb{R}_+$, given by (22), is a continuous function, also note that $v(t) \neq 0$ for all $t \in [0, \Theta_d(v(0))$ and for initial conditions $v(0) \in \mathbb{R}_+^*$. Therefore, (22) satisfies the hypothesis required in Lemma 2 for the function v.

The last results of this section (the following theorem and its corollary) constitute the fundamentals of the discretization technique that is developed in Section 4.

Theorem 3. *Let* (6) *and* V *be as in Lemma 1. Define* $\zeta = [v, z^\top]^\top \in Z$*, where* $Z = \mathbb{R}_+^* \times \mathscr{S}_V$*. Consider* (6) *with* $x \in \mathbb{R}^n \setminus \{0\}$*,* $d \in \mathscr{D}$*, and* (19)-(20) *with* $\zeta \in Z$*. The solutions of* (6) *and the solutions of* (19)-(20) *are equivalent with the homeomorphism* $\Phi : \mathbb{R}^n \setminus \{0\} \to Z$ *given by*

$$\Phi(x) = \begin{bmatrix} V(x) \\ \Delta^{\mathbf{r}}\left(V^{\frac{-1}{m}}(x)\right)x \end{bmatrix}. \tag{23}$$

Proof. Since V is a continuous function of x, we can ensure that Φ is also a continuous function of x, moreover, it has a continuous inverse $\Phi^{-1} : Z \to \mathbb{R}^n \setminus \{0\}$ given by

$$\Phi^{-1}(\zeta) = \Delta^{\mathbf{r}}\left(v^{\frac{1}{m}}\right)z. \tag{24}$$

The remaining steps of the proof are straightforward, it is only needed to note that $\zeta(t) = \Phi(x(t))$ satisfies (19)-(20) and $x(t) = \Phi^{-1}(\zeta(t))$ satisfies (6). $\square$

Corollary 1. *If* v *is a solution of* (19) *with initial condition* $v(0) = V(x(0))$*, and if* z *is a solution of* (20) *with initial condition* $z(0) = \Delta^{\mathbf{r}}\left(v^{\frac{-1}{m}}(0)\right)x(0)$*, for any* $x(0) \in \mathbb{R}^n \setminus \{0\}$*, then the function* $x : \mathbb{R}_+ \to \mathbb{R}^n$ *given by*

$$x(t) = \begin{cases} \Delta^{\mathbf{r}}\left(v^{\frac{1}{m}}(t)\right)z(t), & t < \Theta_d(v(0)), \\ 0, & t \geq \Theta_d(v(0)), \end{cases} \tag{25}$$

(with Θ_d *as given in Lemma 3) is solution of* (6) *for all* $t \in \mathbb{R}_+$*.*

4 Discretization scheme

In this section we describe the proposed discretization scheme. The main idea of the method is a consequence of the developments presented in Section 3. Indeed, observe that (20) represents the dynamics of (6) but projected on a *unit sphere* $\mathscr{S}_V$, and that v (as given in Lemma 3) characterize the decay of the Lyapunov function V evaluated along the trajectories of (6). So, the main idea is to compute a numerical solution[5] of (19)-(20), and next, to define the numerical solution of the original system by using (25).

Remark 2. Although, several different schemes can be used to obtain a numerical solution of (19)-(20), we restrict ourselves in this chapter to the explicit (also known as forward) Euler method taking into account that $v(t)$ is a nonegative variable and $z(t)$ belongs to a manifold for all $t \geq 0$.

To construct the discrete-time approximation of v, we see from Lemma 3 that for any $h \in \mathbb{R}_+$,

$$v(t+h) = \begin{cases} \left(v^{\frac{-\mu}{m}}(t) - \frac{-\mu}{m}\hat{W}(t)\right)^{\frac{m}{-\mu}}, & \frac{-\mu}{m}\hat{W}(t) < v^{\frac{-\mu}{m}}(t) \\ 0, & \frac{-\mu}{m}\hat{W}(t) \geq v^{\frac{-\mu}{m}}(t) \end{cases}, \tag{26}$$

where $\hat{W}(t) := \int_t^{t+h} W\big(z(\tau), d(\tau)\big)\,\mathrm{d}\tau$. Note that (26) give us the exact value of $v(t+h)$, but it requires the value of $\hat{W}(t)$. Hence, by defining a discrete-time approximation of $\hat{W}(t)$ we immediately obtain a discrete-time approximation of v. For example, if we use the forward Euler method with an integration step h, then we obtain the discrete-time approximation $v_k \in \mathbb{R}$ to $v(kh)$ given by

$$\boxed{v_{k+1} = \begin{cases} \left(v_k^{\frac{-\mu}{m}} - \frac{-\mu h}{m}W(z_k, d_k)\right)^{\frac{m}{-\mu}}, & \frac{-\mu h}{m}W(z_k, d_k) < v_k^{\frac{-\mu}{m}} \\ 0, & \frac{-\mu h}{m}W(z_k, d_k) \geq v_k^{\frac{-\mu}{m}} \end{cases},} \tag{27}$$

for all $k \in \mathbb{Z}_+$, where $d_k := d(kh)$, and $z_k \in \mathbb{R}^n$ is the discrete-time approximation to $z(kh)$ given by

$$\boxed{\begin{aligned} z_{k+1} &= \Delta^{\mathbf{r}}\big(V^{\frac{-1}{m}}(\tilde{z}_{k+1})\big)\tilde{z}_{k+1}, \\ \tilde{z}_{k+1} &= \begin{cases} z_k + h v_k^{\frac{\mu}{m}}\big(f(z_k, d_k) + \frac{1}{m}W(z_k, d_k)Gz_k\big), & v_{k+1} > 0 \\ z_k, & v_{k+1} = 0 \end{cases}. \end{aligned}} \tag{28}$$

Let us explain the main idea of (28). First, the term $\tilde{z}_{k+1}$ can be regarded as an explicit Euler discretization of (20); second, such a discretization is scaled by the factor $\Delta^{\mathbf{r}}\big(V^{\frac{-1}{m}}(\tilde{z}_{k+1})\big)$. Note that such scaling is necessary since we have to ensure that $z_k \in \mathscr{S}_V$ for all $k \in \mathbb{Z}_+$. Also note that, the condition $\tilde{z}_{k+1} \neq 0$ is necessary to have (28) well defined, this is why we require the following assumption.

Assumption 1. Consider (6) and V as in Theorem 3. For all $z \in \mathscr{S}_V$, all $d \in \mathscr{D}$, and all $\tau \in \mathbb{R}_+^*$,

[5] In this chapter, we mean by *numerical solution* a sequence $\{z_k\}_{k\in\mathbb{Z}_+}$ such that $z_0 = z(0)$, and for some $h \in \mathbb{R}_+^*$, z_k approximates $z(kh)$.

$$z+\tau\big(f(z,d)+\tfrac{1}{m}W(z,d)Gz\big)\neq 0\,.$$

In the following lemma we state some sufficient conditions that can be helpful in verifying Assumption 1.

Lemma 4. *Assumption 1 holds in any of the following cases:*

1. *for all $z\in\mathscr{S}_V$ such that $\frac{\partial V(z)}{\partial z}z=0$ we have that $z^\top F(z,d)\geq 0$, where $F(z,d):=f(z,d)+\frac{1}{m}W(z,d)Gz$;*
2. *$\frac{\partial V(z)}{\partial z}z\neq 0$ for all $z\in\mathscr{S}_V$;*
3. *the set $\{z\in\mathbb{R}^n : V(z)\leq 1\}$ is convex.*

Proof. From Lemma 2 we know that, for all $z\in\mathscr{S}_V$ and all $d\in\mathscr{D}$, $F(z,d)$ is tangent to $\mathscr{S}_V$. Hence, $\frac{\partial V(z)}{\partial z}F(z,d)=0$ for all $z\in\mathscr{S}_V$. On the other hand, if there exist $\tau\in\mathbb{R}_+^*$, $z\in\mathscr{S}_V$ and $d\in\mathscr{D}$ such that $z+\tau F(z,d)=0$, then the vector $F(z,d)$ is necessarily collinear to z but it has the opposite direction. Therefore, the following are necessary conditions to have $z+\tau F(z,d)=0$: $\frac{\partial V(z)}{\partial z}z=0$ and $z^\top F(z,d)<0$.

The analysis in the previous paragraph let us clearly see that a sufficient condition to guarantee that Assumption 1 holds is either $\frac{\partial V(z)}{\partial z}z\neq 0$ for all $z\in\mathscr{S}_V$ (which is satisfied, for example, if the function $z\mapsto\frac{\partial V(z)}{\partial z}z$ is positive definite), or $z^\top F(z,d)\geq 0$ for all z such that $\frac{\partial V(z)}{\partial z}z=0$. This proves the first two items of the lemma.

The third item of the lemma is proven as follows. On one hand, if the set $\{z\in\mathbb{R}^n : V(z)\leq 1\}$ is convex, the fact that V is homogeneous guarantees that the sets $\{z\in\mathbb{R}^n : V(z)\leq a\}$ are also convex for all $a\in\mathbb{R}_+^*$, hence, we have that the function V is *quasi-convex* (see, e.g. [4, Section 3.4.1]). On the other hand, $z=0$ is a global minimum of V since it is positive definite. From these reasoning we conclude that V is a *pseudo-convex* function (see, e.g. [6, Lemma 2.1]), therefore (by definition of pseudo-convexity), $\frac{\partial V(z)}{\partial z}z>0$ for all $z\in\mathscr{S}_V$, see also [25, p. 40]. $\square$

It is important to mention that in [34] it is wrongly stated that (in absence of disturbances) the first item in Lemma 4 is an equivalent condition to Assumption 1, however, this is only true for $n=2$. Now, we can state the following theorem, which is the main result of this section.

Theorem 4. *Let* (6) *and V be as in Lemma 1. Suppose that Assumption 1 holds. Consider the discrete-time approximation of* (6) *given by*

$$\boxed{x_{k+1}=\psi(x_k)=\begin{cases}\Delta^{\mathbf{r}}\big(v_{k+1}^{\frac{1}{m}}\big)z_{k+1}, & x_k\neq 0,\\ 0, & x_k=0,\end{cases}\quad k\in\mathbb{Z}_+\,,}\tag{29}$$

where v_{k+1} and z_{k+1} are given by (27) *and* (28)*, respectively, with $v_k=V(x_k)$, $z_k=\Delta^{\mathbf{r}}\big(V^{\frac{-1}{m}}(x_k)\big)x_k$, and $x_0=x(0)$. Then V is a Lyapunov function for* (29)*, and for all $h\in\mathbb{R}_+^*$ and all $x(0)\in\mathbb{R}^n\setminus\{0\}$, $x_k\to 0$ as $k\to\infty$. Moreover, $V(x_k)\leq\bar{V}(x_0,kh)$ for all $k\in\mathbb{Z}_+$, with $\bar{V}$ given by* (14)*.*

The proof of the theorem is completely analogous to the proof of Theorem 2 in [34]. It is clear from Theorem 4 that the solutions of (29) reach the origin in a finite number of steps.

Remark 3. Let us underline the main features of the proposed discretization scheme described in Theorem 4:

1. the discretization method is consistent, this means that the stability properties and the convergence rate from the solutions of the continuous-time system are preserved;
2. the Lyapunov function is preserved, i.e., the Lyapunov function from the continuous-time system is a Lyapunov function for the discrete-time approximating system as well;
3. the discretization method is explicit since the right-hand side of (29) does not depend on x_{k+1} but only on x_k.

Numerical convergence to the solutions

In this section we verify that the solutions of the discrete-time approximating system converge to the solutions of the continuous-time system. The methodology is the same as that given in [34].

First, consider the solution $x : [0,a] \to \mathbb{R}^n$ of (6) for some fixed time $a \in \mathbb{R}^*_+$. Second, suppose that the discretization step h is given by $h = a/N$, for some $N \in \mathbb{Z}^*_+$, thus it is clear that $h \to 0$ as $N \to \infty$.

Let $\{x_k\}_{k=0}^N$ be the sequence generated by means of some discretization of (6). Consider the step-function (associated to such a discretization method) defined as $t \mapsto \tilde{x}(t) := x_k$, for all $t \in [kh,(k+1)h)$. An essential requirement for any discretization technique is that its associated step-function converges uniformly on $[0,a]$ to the solution x as the step size h tends to zero (or equivalently, as N tends to infinity). The standard procedure to verify such a convergence property consists in confirming that the *global truncation error*[6] tends to zero as the step h tends to zero. Such a confirmation can be achieved by verifying the existence of a function $\eta \in \mathscr{K}$ such that the *local truncation error*[7] $E(t+h) := x(t+h) - x_{k+1}$ satisfies the following (see, e.g. [37] and [14, pp. 37 and 159])

$$|E(t+h)| \leq h\eta(h)\,. \tag{30}$$

Thus, in this section we demonstrate the convergence property of the discretization scheme proposed in Section 4, by means of the verification of a local truncation error estimate given by 30. It is important to see that taking into account Theorem 3, we only need to demonstrate that v_k and z_k converge to v and z, respectively. Thus,

---

[6] The global truncation error can be understood as the accumulation of the errors generated at each step in a given compact interval $[0,a]$, see, e.g. [20] or [14, p. 159].

[7] The local truncation error is the one-step error computed by assuming that $E(t) = 0$, i.e. $x(t) = x_k$.

we only have to analyze the local truncation error estimates of v_k and z_k as stated in the following theorem.

Theorem 5. *Assume that the hypotheses of Theorem 4 hold with $v(t) = v_k$ and $z(t) = z_k$ for some $t \in \mathbb{R}_+$. Assume also that $\min\{V(x(t+h)), v_{k+1}\} \geq b$ for some $b \in \mathbb{R}_+^*$. Then, there exist functions $\eta_v, \eta_z \in \mathscr{K}$ such that in* (27), (28)*:*

$$|E^v(t+h)| := |v(t+h) - v_{k+1}| \leq h\,\eta_v(h)\,, \tag{31}$$

$$|E^z(t+h)| := |z(t+h) - z_{k+1}| \leq h\,\eta_z(h)\,. \tag{32}$$

The proof of Theorem 5 is given below, but first we state the following lemma, which is used for the proof of the theorem. We use the following notation: the i-th element of the vector $\tilde{z}_k$ is denoted as $\tilde{z}_k^i$.

Lemma 5. *Consider* (28)*. Given $H \in \mathbb{R}_+^*$, under the assumptions of Theorem 5, there exist constants $\bar{\gamma}_i, \underline{\gamma}_i \in \mathbb{R}_+^*$ such that $\underline{\gamma}_i \leq |\tilde{z}_{k+1}^i| \leq \bar{\gamma}_i$ for all $h \leq H$ and all $i \in \{1, \ldots, n\}$.*

Proof. On one hand, we have that Assumption 1 guarantees the existence of the constants $\underline{\gamma}_i$. On the other hand, from (28) we can see that for every $i = 1, \ldots, n$,

$$|\tilde{z}_{k+1}^i| \leq \zeta_i + hv_k^{\frac{\mu}{m}}\left(\bar{f}_i + \tfrac{1}{m}\bar{\alpha} r_i \zeta_i\right),$$

where $\bar{D} = [-\bar{d}_1, \bar{d}_1] \times [\underline{d}_2, \bar{d}_2]$,

$$\bar{f}_i = \sup_{\substack{z \in \mathscr{S}_V \\ d \in \bar{D}}} |f_i(z,d)|\,, \quad \zeta_i = \sup_{z \in \mathscr{S}_V} |z_i|\,, \quad \bar{\alpha} = \sup_{\substack{z \in \mathscr{S}_V \\ d \in \bar{D}}} |W(z,d)|\,.$$

Now, since v_k is decreasing and $\mu < 0$, the hypotheses of the lemma ensure that

$$\bar{\gamma}_i = \zeta_i + Hb^{\frac{\mu}{m}}\left(\bar{f}_i + \tfrac{1}{m}\bar{\alpha} r_i \zeta_i\right).$$

$\square$

Proof of Theorem 5

First we analyze E^v given by (31). From (26) and (27) we have that (denoting $E^v = E^v(t+h)$)

$$|E^v| \leq v(t)\left|\left(1 - \tfrac{-\mu}{m} v^{\frac{\mu}{m}}(t)\hat{W}(t)\right)^{\frac{m}{-\mu}} - \left(1 - \tfrac{-\mu}{m} v^{\frac{\mu}{m}}(t) W(z_k, d_k) h\right)^{\frac{m}{-\mu}}\right|,$$

where $\hat{W}(t) := \int_t^{t+h} W(z(\tau), d(\tau))\,\mathrm{d}\tau$. Note that we have used the hypothesis $v(t) = v_k$. Now, we rewrite these inequalities as follows

$$|E^v| \leq hv(t)\frac{\left|\left(1 - \tfrac{-\mu}{m} v^{\frac{\mu}{m}}(t)\hat{W}(t)\right)^{\frac{m}{-\mu}} - \left(1 - \tfrac{-\mu}{m} v^{\frac{\mu}{m}}(t) W(z_k, d_k) h\right)^{\frac{m}{-\mu}}\right|}{h}\,.$$

Assume that d is continuous at t. Recall that W is continuous for all $z \in \mathbb{R}^n \setminus \{0\}$ and all $d \in \mathbb{R}^2$, moreover, $z(t) = z_k$ and $d(t) = d_k$. Hence, it is clear (e.g. by using the L'Hôpital's rule) that

$$\lim_{h \to 0} \frac{1}{h} \left| \left(1 - \tfrac{-\mu}{m} v^{\frac{\mu}{m}}(t) \hat{W}(t)\right)^{\frac{m}{-\mu}} - \left(1 - \tfrac{-\mu}{m} v^{\frac{\mu}{m}}(t) W(z_k, d_k) h\right)^{\frac{m}{-\mu}} \right| = 0\,.$$

Observe that $W(z,d)$ is positive and bounded for all $z \in \mathscr{S}_V$ and all $d \in \bar{D}$, therefore, there exists a function $\bar{\eta}_v \in \mathscr{K}$ (which does not depend on $z_k \in \mathscr{S}_V$) such that

$$\frac{1}{h} \left| \left(1 - \tfrac{-\mu}{m} v^{\frac{\mu}{m}}(t) \hat{W}(t)\right)^{\frac{m}{-\mu}} - \left(1 - \tfrac{-\mu}{m} v^{\frac{\mu}{m}}(t) W(z_k, d_k) h\right)^{\frac{m}{-\mu}} \right| \leq \bar{\eta}_v(h)\,.$$

Thus, the result of the theorem is obtained by taking $\eta_v(h) = v(t)\bar{\eta}_v(h)$.

Now, we analyze the error E^z, which is given by (32). Observe from (28) that

$$E^z(t+h) = z(t+h) - \Delta^{\mathbf{r}}\left(V^{\frac{-1}{m}}(\tilde{z}_{k+1})\right)\tilde{z}_{k+1},$$

which can be rewritten as

$$E^z(t+h) = z(t+h) - \tilde{z}_{k+1} + \left(I - \Delta^{\mathbf{r}}\left(V^{\frac{-1}{m}}(\tilde{z}_{k+1})\right)\right)\tilde{z}_{k+1}.$$

Hence, for $i = 1, \ldots, n$, we have that

$$E_i^z(t+h) = z_i(t+h) - \tilde{z}_{k+1}^i + \left(1 - V^{\frac{-r_i}{m}}(\tilde{z}_{k+1})\right)\tilde{z}_{k+1}^i\,. \tag{33}$$

Since $z(t+h) \in \mathscr{S}_V$, the term $1 - V^{\frac{-r_i}{m}}$ in (33) can be rewritten as follows

$$\begin{aligned} 1 - V^{\frac{-r_i}{m}}(\tilde{z}_{k+1}) &= \frac{1}{V^{\frac{r_i}{m}}(\tilde{z}_{k+1})}\left(V^{\frac{r_i}{m}}(\tilde{z}_{k+1}) - 1\right) \\ &= \frac{1}{V^{\frac{r_i}{m}}(\tilde{z}_{k+1})}\left(V^{\frac{r_i}{m}}(\tilde{z}_{k+1}) - V^{\frac{r_i}{m}}(z(t+h))\right). \end{aligned}$$

From Lemma 5, and for any $H \in \mathbb{R}_+^*$, we can ensure the existence of constants $a_1, a_2 \in \mathbb{R}_+^*$ such that $a_1 \leq V(\tilde{z}_{k+1}) \leq a_2$ for all $h \leq H$. Hence,[8]

$$\left|V^{\frac{r_i}{m}}(\tilde{z}_{k+1}) - V^{\frac{r_i}{m}}(z(t+h))\right| \leq L_i \left|V(\tilde{z}_{k+1}) - V(z(t+h))\right| \leq L_i L_v \left|\tilde{z}_{k+1} - z(t+h)\right|,$$

for some constants $L_i, L_v \in \mathbb{R}_+^*$. Thus, we obtain $\left|1 - V^{\frac{-r_i}{m}}(\tilde{z}_{k+1})\right| \leq c_i \left|\tilde{z}_{k+1} - z(t+h)\right|$ with $c_i := a_1^{\frac{-r_i}{m}} L_i L_v$. Hence, we find a bound for (33) as follows

[8] Since $[a_1, a_2] \subset \mathbb{R}$ is compact and $a_1 > 0$, the function $g : [a_1, a_2] \subset \mathbb{R}_+^* \to \mathbb{R}$ given by $g(V) = V^{\frac{r_i}{m}}$ is Lipschitz continuous. Also, z_k and z belong to a compact subset of $\mathbb{R}^n$ on which V is Lipschitz continuous.

$$
\begin{aligned}
|E_i^z(t+h)| &\le |z_i(t+h) - \tilde{z}_{k+1}^i| + c_i\big|\tilde{z}_{k+1} - z(t+h)\big||\tilde{z}_{k+1}^i|\,, \\
&\le |\tilde{z}_{k+1} - z(t+h)| + c_i\big|\tilde{z}_{k+1} - z(t+h)\big||\tilde{z}_{k+1}^i|\,, \\
&\le (1 + c_i|\tilde{z}_{k+1}^i|)\big|\tilde{z}_{k+1} - z(t+h)\big|\,, \\
&\le \bar{c}_i\big|\tilde{z}_{k+1} - z(t+h)\big|\,, \quad \bar{c}_i := 1 + c_i\overline{\gamma}_i\,,
\end{aligned} \tag{34}
$$

with $\overline{\gamma}_i$ as given in Lemma 5. To analyze the term $\big|\tilde{z}_{k+1} - z(t+h)\big|$ define $F(x,d) := f(x,d) + \frac{1}{m}W(x,d)Gx$. Thus, from (20) and (28) we have that $z(t+h) = z(t) + \int_t^{t+h} \nu^{\frac{\mu}{m}}(\tau)F(z(\tau),d(\tau))\,\mathrm{d}\tau$ and $\tilde{z}_{k+1} = z_k + h\nu_k^{\frac{\mu}{m}}F(z_k,d_k)$, respectively. By the Taylor's theorem, there exists a function $h \mapsto R(t,h)$ such that $z(t+h) = z(t) + \nu^{\frac{\mu}{m}}(t)F(z(t),d(t))h + R(t,h)$, and $\frac{1}{h}R(t,h) \to 0$ as $h \to 0$. Since $z_k = z(t)$, $d(t) = d_k$, and $\nu_k = \nu(t)$,

$$
\begin{aligned}
\big|\tilde{z}_{k+1} - z(t+h)\big| &= \Big|z_k + h\nu_k^{\frac{\mu}{m}}F(z_k,d_k) - z(t) - \nu^{\frac{\mu}{m}}(t)F(z(t),d(t))h - R(t,h)\Big|\,, \\
&= |R(t,h)| = h\Big|\frac{1}{h}R(t,h)\Big|\,, \quad \lim_{h\to 0}\Big|\frac{1}{h}R(t,h)\Big| = 0\,.
\end{aligned} \tag{35}
$$

Therefore, from (34) and (35) we conclude that there exist $c \in \mathbb{R}_+^*$ and $\eta_z \in \mathscr{K}$ such that

$$
|E^z(t+h)| \le h\eta_z(h)\,, \quad \eta_z(h) \ge c\Big|\frac{1}{h}R(t,h)\Big|\,.
$$

□

5 Examples

In this section, firstly we resume the examples given in Section 2 to illustrate the discretization scheme proposed in Section 4. Secondly, in Example 7 we show a possible application of the proposed discretization scheme to construct a discrete-time implementation of the controller 3. The disturbances to be used in the examples are given by

$$
d_1(t) = A_1\sin(\omega_1 t)\,, \quad d_2(t) = 1 - A_2\cos(\omega_2 t)\,, \tag{36}
$$

where $\bar{d}_1 = A_1$, $\underline{d}_2 = 1 - A_2$ and $\bar{d}_2 = 1 + A_2$, with the parameters $A_1 = 1$, $\omega_1 = \pi$, $A_2 = 1/5$, and $\omega_2 = 10\pi$.

Example 4. For $n = 1$ consider (1) with the controller and the Lyapunov function given in Example 1. Observe that $\frac{\partial V_1(x)}{\partial x}x = x_1^2$. Hence, Lemma 4 guarantees that Assumption 1 holds. For the simulation, we use the gain $k_1 = 3$ and the initial condition $x(0) = 5$. Fig. 1 shows the behavior of the discretization scheme (29) with a step of $h = 0.1$. It is clear that the state of the system converges exactly to the origin in finite-time despite the disturbance.

Additional details of this example are given in [34], where it is even compared with the implicit discretization schemes from [8] and [1].

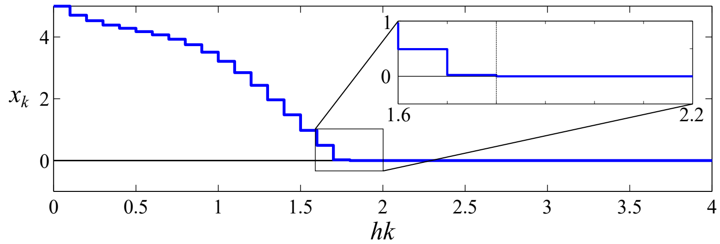

Fig. 1 Discrete-time approximation of (1), (3) for $n = 1$.

Example 5. Now consider (1) with $n = 2$, the controller and the Lyapunov function given in Example 2. Observe that

$$\frac{\partial V_2(x)}{\partial x}x = \left(1 + \tfrac{3}{2}k_1^4\right)x_1^2 + \tfrac{5}{2}k_1^3\lceil x_1\rfloor^{\frac{3}{2}}x_2 + x_2^4 .$$

By applying the Young's inequality to the term $\lceil x_1\rfloor^{\frac{3}{2}}x_2$ it can be verified that the function given by $x \mapsto \frac{\partial V_2(x)}{\partial x}x$ is positive definite if $k_1 < 8^{\frac{1}{4}}\left(3\left(5\left(\frac{5}{8}\right)^{\frac{1}{3}} - 4\right)\right)^{-\frac{1}{4}}$. Under this condition, Lemma 4 guarantees that Assumption 1 holds. For the simulation we use the initial conditions $x_1(0) = 2$, $x_2(0) = 2$ and the gains $k_1 = 1$, $k_2 = 4$. The discretization step is again set to $h = 0.1$. In Fig. 2 it can be seen the states of the discrete-time approximation (29) preserving the finite-time converge feature from the continuous-time model.

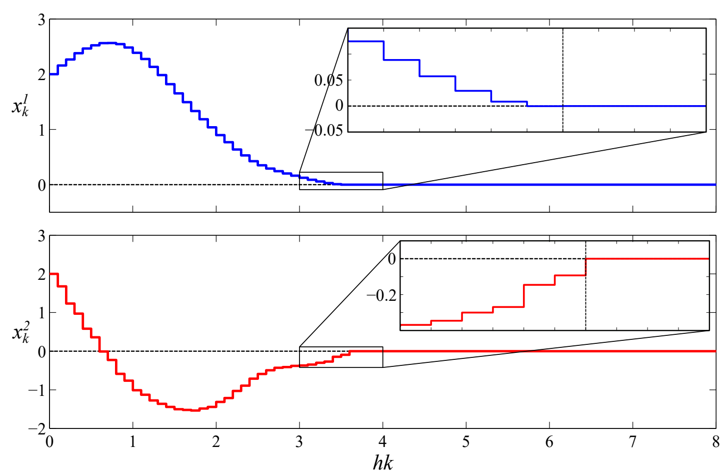

Fig. 2 Discrete-time approximation of (1), (3) for $n = 2$.

Example 6. Now, for the case $n = 3$, consider (1) with the controller and the Lyapunov function given in Example 3. To simulate the discretization scheme (29) we use the initial conditions $x_1(0) = 2$, $x_2(0) = 2$, $x_3(0) = 2$ and the gains $k_1 = 0.6$, $k_2 = 1.7$, $k_3 = 1200$. The discretization step is set to $h = 0.001$. Fig. 3 shows the states of the system converging exactly to the origin in finite-time.

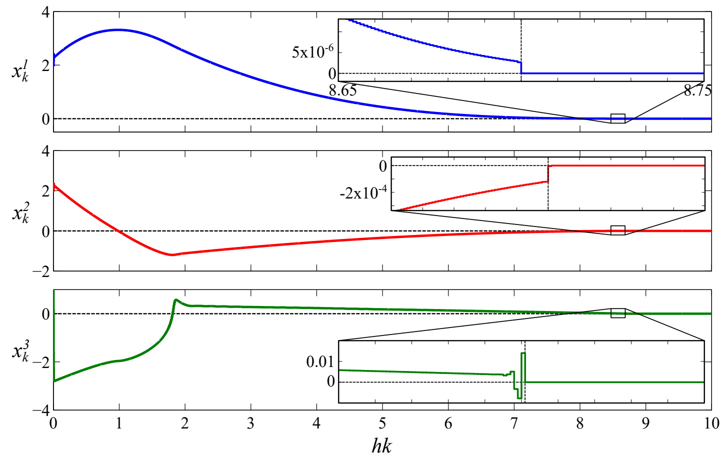

Fig. 3 Discrete-time approximation of (1), (3) for $n = 3$.

Since, for this example, Assumption 1 is not easily verifiable by means of Lemma 4, we confirm along the simulation that $\tilde{z}_k \neq 0$ for $k \geq 0$. This can be corroborated in Fig. 4, which shows the norm of $\tilde{z}_k$.

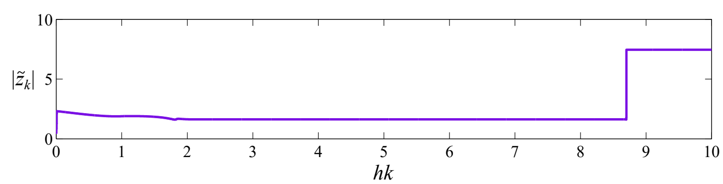

Fig. 4 Norm of $\tilde{z}_k$ in (28) for the discrete-time approximation of (1), (3) for $n = 3$.

Example 7. In this example we show a possible application of the proposed discretization scheme to construct a discrete implementation of the controller (3).

Consider (1) for $n = 2$. Assume that the state x is measured at instants $t_k = kh$, $k \in \mathbb{Z}_+$, $h \in \mathbb{R}_+^*$, and the control signal must be constant for the interval $I_k = [kh, (k+1)h)$, i.e. $u(t) = u_k$ for all $t \in I_k$. It is well known that the standard discretization of $u(x(t))$ given by

$$u_k = u(x(t_k)), \tag{37}$$

generates numerical chattering as it can be seen in Fig. 5, which shows the behavior of the states of the system for $d_1 = 0$ and $d_2 = 1$, with the controller (3) discretized as in (37). For the simulation the initial conditions are $x_1(0) = 1, x_2(0) = 1$ and the gains $k_1 = 1$, $k_2 = 4$. The step for the control discretization is $h = 0.01$. The continuous-time dynamics is approximated by means of the explicit Euler discretization with a step of $h_s = 1\times 10^{-5}$.

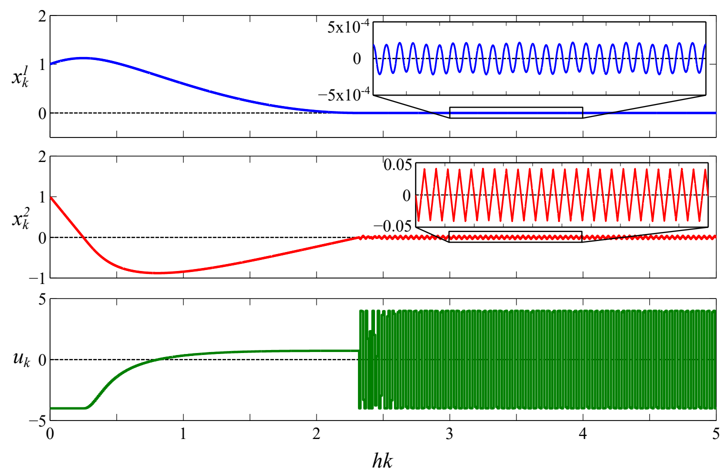

Fig. 5 States of (1) in closed loop with the discretization of (3) given by (37) (undisturbed case).

Now, as it is done with implicit discretization techniques for sliding mode controllers (see, e.g. [8, 1]) we consider the controller discretization

$$u_k = u(x_{k+1}). \tag{38}$$

We use the proposed discretization method to compute x_{k+1}. First, observe that the function $u : \mathbb{R}^n \to \mathbb{R}$ given by (3) is $\mathbf{r}-$homogeneous of zero degree. Hence, if $x = \Delta^{\mathbf{r}}(\varepsilon)z$, then $u(x) = u(z)$ for all $\varepsilon \in \mathbb{R}_+^*$. Thus, by considering (29), we can replace (38) with

$$u_k = u(z_{k+1}). \tag{39}$$

To compute z_{k+1} we use (28) and v_{k+1} given by (27) with the data

$$v_k = V(x(t_k)), \quad z_k = \Delta^{\mathbf{r}}\left(v_k^{-\frac{1}{m}}\right)x(t_k).$$

Now, note that both v_{k+1} and z_{k+1} given by (27) and (28), respectively, depend on the disturbance d, and it is generally unknown. Thus, we compute v_{k+1} and z_{k+1} by assuming the nominal case, i.e. with the disturbance d such that $d_1 = 0$ and $d_2 = 1$. Finally, observe that u is discontinuous at zero, hence we have to assign the value of $u(z_{k+1})$ for the case $z_{k+1} = 0$. Since $\lim_{\varepsilon \downarrow 0} u(\Delta^{\mathbf{r}}(\varepsilon) z) = u(z)$ for all $z \neq 0$, we set $u(z_{k+1}) = u(z_k)$ if $z_{k+1} = 0$.

In Fig. 6 we can see the states of the system with the controller (3) discretized as in (39). The disturbance signals are set as before, i.e. $d_1 \equiv 0$ and $d_2 \equiv 1$. The initial conditions are $x_1(0) = 1$, $x_2(0) = 1$ and the gains $k_1 = 1$, $k_2 = 4$. It can be seen that the numerical chattering has been considerably reduced with the proposed discretization scheme.

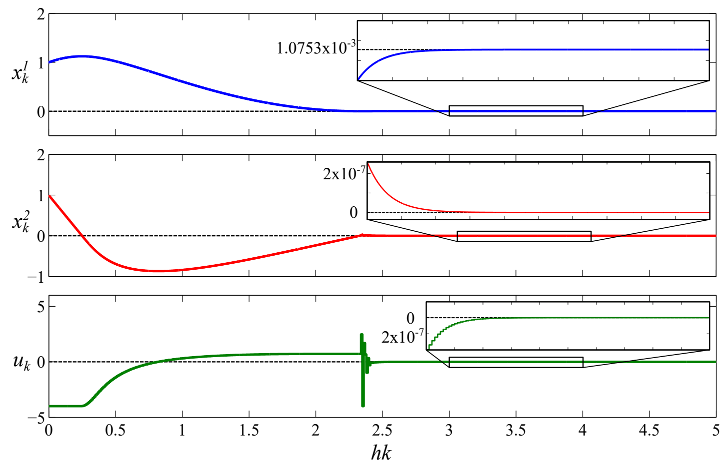

Fig. 6 States of (1) in closed loop with the discretization of (3) given by (39) (undisturbed case).

Now we repeat the simulations by considering the disturbances given in (36). In Fig. 7 and Fig. 8 we can appreciate the states of the system and the control signals with the two different methods. It is clear that the proposed Lyapunov-based discretization helps to reduce the numerical chattering effect in the state signals. It is also noticeable that the accuracy in the second state is improved with the Lyapunov-based method, but it is not for the first state.

6 Conclusion

We have provided in this chapter a discretization scheme for a class of systems controlled by a family of quasi-continuous HOSM controllers. Two of the most relevant properties of the method are that it preserves from the continuous-time system the finite-time convergence to the origin and the Lyapunov function. Another interesting feature of the technique is that both the discretization and the projection procedures are explicit. Finally, we have shown an example of the application of the proposed discretization-scheme to design a discrete-time implementation of a quasi-continuous controller that helps to reduce the numerical chattering effect.

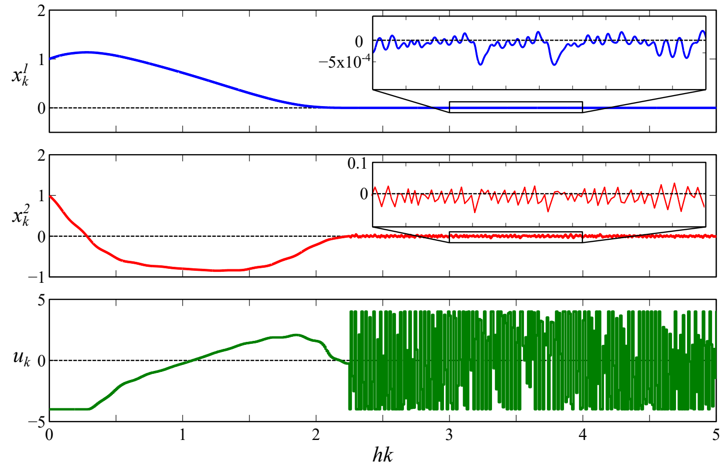

Fig. 7 States of (1) in closed loop with the discretization of (3) given by (37) (disturbed case).

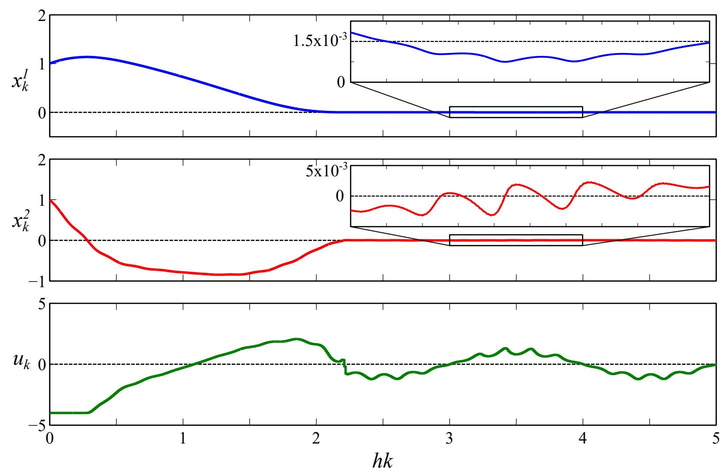

Fig. 8 States of (1) in closed loop with the discretization of (3) given by (39) (disturbed case).

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