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Fixed-time parameter estimation via the discrete-time DREM method

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Abstract: A simple fixed-time converging estimation algorithm is presented for a linear regression using the dynamic regressor extension and mixing (DREM) method within a discrete-time setting, with a persistently exciting regressor and bounded measurement noises. The solution is based on Kreisselmeier’s filters and is computationally simpler than the existing analogs.

Keywords: on-line parameter estimation, fixed-time estimation, DREM

1. INTRODUCTION

The parameter identification problem for dynamic or static systems is an essential issue in many scientific disciplines (Ljung, 1987). It appears everywhere with online measurements and where a mathematical model should be realized. Frequently, this leads to applying linear regression methods with respect to unknown parameters, mainly due to the existence of many well-matured approaches to solving this problem (Yan, 2009). The parameters can be identified in such a case if the regressor is sufficiently exciting (Sastry and Bodson, 1989; Ioannou and Kokotovic, 1983). Most tools for finding the parameters in linear regression (the least-squares or the gradient-descent algorithms are popular examples) also have some known shortages. Among them, it is necessary to mention the difficulty of accelerating the convergence process to the ideal values (even in a continuous-time setting, this property is mainly predefined by the level of excitation in the regressor) and the non-monotonicity of the convergence (Narendra and Annaswamy, 1987; Efimov and Fradkov, 2015; Efimov et al., 2019). The latter fact means that despite a decreasing norm of the parameter vector estimation error, the respective discrepancies for each parameter may demonstrate a complex oscillatory behavior before settling down in a vicinity of the true value, which may postpone a real-time utilization of the obtained estimates. That is why a solution to these drawbacks recently proposed by the dynamical regressor extension and mixing (DREM) method (Aranovskiy et al., 2017) has quickly gained popularity (Belov et al., 2018; Aranovskiy et al., 2019; Vediakova et al., 2021). Its additional advantage is that the excitation constraints can be slightly relaxed.

This note studies discrete-time systems since measurements are usually available in discrete time. Many DREM-based solutions are formulated in the continuous-time setting (Wang et al., 2020, 2019; Ortega et al., 2020),

and only some have an extension in the discrete-time environment (Ortega et al., 2021b,a). In all these works, the DREM method is used to reduce the initial linear regression problem for a vector of unknown parameters to a series of interrelated scalar linear regression problems, with a *posterior application of different iterative schemes* for asymptotic or finite/fixed-time evaluation of the values of parameters (Belov et al., 2018; Aranovskiy et al., 2019; Vediakova et al., 2021; Ortega et al., 2020; Wang et al., 2019).

In this paper, we show that a posterior application of iterative schemes is not necessary with the DREM procedure. We show that a proper choice of dynamics extension scheme at the first step of the DREM procedure ensures that the novel regressor is strictly positive, allowing for point-wise (algebraic) estimation. A point-wise estimator can be of limited practical interest due to its noise sensitivity, even if it provides the fixed-time (independent in initial conditions) convergence. To this end, we analyze the worst-case upper bound on the estimation error and study its dependence on tuning parameters. Moreover, we show that the point-wise estimator combined with a (non-)linear filter can provide faster transients than a standard iterative estimator, *e.g.*, the gradient-descent one, having a comparable steady-state performance.

To summarize, the contribution of this note is twofold:

- We show that (under proper tuning) the DREM procedure allows for a point-wise algebraic estimator providing fixed-time convergence independently of initial conditions;
- We show that for noised measurements, the point-wise algebraic estimator combined with a filter can outperform standard iterative tools.

The rest of the paper is organized as follows. The problem statement is given in Section 2, and Section 3 briefly de-

scribes the DREM procedure. In Section 4, we discuss how the DREM procedure can preserve the initial excitation. Section 5 presents this note's main result, and Section 6 contains simulations illustrating our results. Finally, Section 7 contains some concluding remarks.

Notation

- The sets of nonnegative real and integer numbers are denoted by \mathbb{R}_+ and \mathbb{N} , respectively. Also, $\mathbb{N}^* := \mathbb{N} \setminus \{0\}$. The set of real $n \times m$ -matrices is denoted by $\mathbb{R}^{n \times m}$. The n -identity matrix is denoted by I_n .
- For a vector $x \in \mathbb{R}^n$, $\|x\|$ denotes its Euclidean norm.
- The rounding function to the greatest integer smaller than $s \in \mathbb{R}$ is denoted by $\lfloor s \rfloor = \text{floor}(s)$.

2. ESTIMATION PROBLEM STATEMENT

Assume that a vector linear regression model is given in discrete-time:

$$\bar{y}_k = \bar{\phi}_k^\top \bar{\theta} + \bar{v}_k, \quad k \in \mathbb{N}, \quad (1)$$

where $\bar{y}_k, \bar{v}_k \in \mathbb{R}$ are the output and the bounded measurement noise, $\bar{\phi}_k, \bar{\theta} = (\theta_1 \dots \theta_n)^\top \in \mathbb{R}^n$ are the regressor and the unknown parameter vectors. The signals available for online measurements are \bar{y}_k and $\bar{\phi}_k$.

Problem 1. Our goal is to estimate $\bar{\theta}$ from the measurements \bar{y}_k and $\bar{\phi}_k$ in fixed time, taking into account the presence of the noise \bar{v}_k .

3. DYNAMICAL REGRESSOR EXTENSION AND MIXING (DREM)

The DREM procedure consists in finding n stable causal filters $H_i(z)$, where $i \in \{1, \dots, n\}$, and z is the time shift operator (*i.e.*, $z^h \bar{y}_k = \bar{y}_{k-h}$ for any $k, h \in \mathbb{N}$, $k \geq h$), whose auxiliary role is also to filter the noise. For $i \in \{1, \dots, n\}$, denote

$$\begin{aligned} Y_k^i &= H_i(z) \bar{y}_k, \quad \Phi_k^i = H_i(z) \bar{\phi}_k, \quad V_k^i = H_i(z) \bar{v}_k, \quad (2) \\ Y_k &= (Y_k^1 \dots Y_k^n)^\top, \quad V_k = (V_k^1 \dots V_k^n)^\top \in \mathbb{R}^n, \\ \Phi_k &= (\Phi_k^1 \dots \Phi_k^n)^\top \in \mathbb{R}^{n \times n}. \end{aligned}$$

That leads to a new extended regression

$$Y_k = \Phi_k \bar{\theta} + \bar{V}_k, \quad \bar{V}_k = V_k + \epsilon_k, \quad k \in \mathbb{N},$$

where $\epsilon_k \in \mathbb{R}^n$ is an exponentially decaying term coming from the initialization of the filters (the filters can be applied directly to (1), or to the auxiliary regression problem $\bar{\phi}_k \bar{y}_k = \bar{\phi}_k \bar{\phi}_k^\top \bar{\theta} + \bar{\phi}_k \bar{v}_k$). Finally, for

$$\phi_k = \det(\Phi_k), \quad y_k = \text{adj}(\Phi_k) Y_k, \quad v_k = \text{adj}(\Phi_k) \bar{V}_k \quad (3)$$

this yields the element-wise scalar linear regression:

$$y_{k,i} = \phi_k \theta_i + v_{k,i}, \quad k \in \mathbb{N}, \quad i \in \{1, \dots, n\}, \quad (4)$$

where $y_{k,i} \in \mathbb{R}$ and $\phi_k \in \mathbb{R}$ are known signals, $\theta_i \in \mathbb{R}$ is the unknown constant parameter to be estimated, and $v_{k,i} \in \mathbb{R}$ is an unknown bounded measurement distortion. Note that the estimation of each θ_i is now explicitly independent of other components of $\bar{\theta}$, and the interconnection is hidden in $\phi_k \in \mathbb{R}$.

For $k \in \mathbb{N}$, let $\hat{\theta}_k \in \mathbb{R}^n$ denote an estimate of $\bar{\theta}$ at time k . Define $e_k := y_k - \phi_k \hat{\theta}_k = \phi_k \tilde{\theta}_k + v_k$ as the measured regression error, where $\tilde{\theta}_k := \bar{\theta} - \hat{\theta}_k$ is the parameter

estimation error. It remains now to design an algorithm to compute $\hat{\theta}_k$, which is a simpler problem than (1), since in (4) the estimation of each θ_i can be treated separately. Moreover, in scalar linear regression, the convergence of $\tilde{\theta}_k$ is elementwise monotone and can be accelerated by tuning the estimation procedures (Aranovskiy et al., 2017).

One of the main issues of the DREM methodology is the excitation of the common regressor ϕ_k , and its relation with the excitation of $\bar{\phi}_k$ in the original problem statement, which is obviously predefined by the choice of the filters H_i in (2). Let us present a solution to this problem.

4. KREISSELMEIER'S REGRESSOR EXTENSION

Let us introduce a possible selection of the filters H_i , $i \in \{1, \dots, n\}$ that keep the excitation level of the original regression for (4). To this end, let us characterize the admissible excitation levels (Narendra and Annaswamy, 1987):

Definition 1. (PE). A bounded signal $\bar{\phi} : \mathbb{N} \rightarrow \mathbb{R}^n$ is called (ℓ, μ) -persistently exciting (PE) if there exist $\ell \in \mathbb{N}^*$ and $\mu > 0$ such that

$$\sum_{i=0}^{\ell-1} \bar{\phi}_{k+i} \bar{\phi}_{k+i}^\top \geq \mu I_n, \quad \forall k \in \mathbb{N}.$$

It is said to be *intervally exciting* (IE) if the above inequality is satisfied for $k = 0$ only.

Persistent excitation is a widely used property and in the noise-free case, interval excitation can be related with the conventional identifiability/observability condition of $\bar{\theta}$ in (1).

Within the context of the DREM procedure, Kreisselmeier's dynamic regressor extension (the filter in (2) applied to the auxiliary regression problem) is given by

$$\begin{aligned} \Phi_{k+1} &= \lambda \Phi_k + (1 - \lambda) \bar{\phi}_k \bar{\phi}_k^\top, \quad (5) \\ Y_{k+1} &= \lambda Y_k + (1 - \lambda) \bar{\phi}_k \bar{y}_k, \end{aligned}$$

where $\lambda \in (0, 1)$, $Y_0 \in \mathbb{R}^n$ and $\Phi_0 \in \mathbb{R}^{n \times n}$, $\Phi_0 = \Phi_0^\top \geq 0$. After the mixing step, it yields the decoupled linear regression equation (4). Specifically, the choice $\Phi_0 = 0$ and $Y_0 = 0$ guarantees that $\epsilon_k \equiv 0$ for all $k \in \mathbb{N}$.

Lemma 1. (Excitation Preservation). Consider a (ℓ, μ) -PE signal $\bar{\phi}$ and the Kreisselmeier's dynamic extension (5). Then the signal ϕ defined in (3) is (ℓ, α) -PE, with

$$\alpha = (\mu(1 - \lambda) \lambda^{\ell-1})^n.$$

Moreover,

$$\phi_k \geq \alpha, \quad \forall k \geq \ell; \quad \liminf_{k \rightarrow \infty} \phi_k \geq \alpha \left(\frac{1}{1 - \lambda^\ell} \right)^n.$$

The proof of Lemma 1 is omitted for brevity. In the continuous-time setting, a similar result on the choice of filters is given in (Aranovskiy et al., 2023).

Remark 1. As it has been shown in (Vedyakov et al., 2019), even if ϕ does not satisfy the needed excitation constraints after transformation, the excitation can always be ensured by simple additional manipulations. For example, let us recall one of the solutions provided in (Vedyakov et al., 2019). Both sides of the equation (4)

can be multiplied by ϕ_k and summed, leading to a new linear regression equation:

$$\tilde{y}_k = \tilde{\phi}_k \bar{\theta} + \tilde{v}_k, \quad k \in \mathbb{N},$$

where

$$\tilde{y}_k = \sum_{i=0}^k \phi_i y_i, \quad \tilde{\phi}_k = \sum_{i=0}^k \phi_i^2, \quad \tilde{v}_k = \sum_{i=0}^k \phi_i v_i.$$

The new scalar regressor $\tilde{\phi}$ is a non-decreasing signal of time, and under very mild requirements (e.g., $\phi_k \neq 0$ for some $k \in \mathbb{N}$), it has to show a minimal excitation. It is worth highlighting that such a procedure may increase the noise intensity.

Remark 2. If $\bar{\phi}$ in (1) is IE, then $\tilde{\phi}$ resulting from (3), (5) cannot be identically zero, and if $\bar{\phi}$ is PE, then $\tilde{\phi}$ is strictly positive for all $k \geq \ell$. Thus, we consider only steps $k \in \mathbb{N}$ when $\phi_k \neq 0$. Otherwise, the linear regression (4) reads $y_k = v_k$, and there is no reason to update the estimate for such samples, i.e., $\hat{\theta}_k = \hat{\theta}_{k-1}$ for $\phi_k = 0$ (recall that the signal ϕ is known).

5. PARAMETER ESTIMATION FOR NOISED MEASUREMENTS

In the noise-free case, after applying the DREM method and obtaining the decoupled linear regression (4), the estimation problem has a trivial solution

$$\hat{\theta}_k = \frac{y_k}{\phi_k}, \quad (6)$$

which can be unconditionally applied as soon as $\phi_k \neq 0$ for some $k \in \mathbb{N}$. Such a scenario is always realizable with an IE-function ϕ in (4) (since (4) is identifiable).

In the presence of noise, using the established properties of Kreisselmeier's filters (5), the following simple result can be obtained, which presents the main outcome of this note. Its proof is omitted for brevity.

Theorem 3. Let the regressor signal $\bar{\phi}$ in (1) be (ℓ, μ) -PE and the Kreisselmeier's dynamic extension filters (5) be used in (2). Then the estimate $\hat{\theta}_k$ can be computed by (3), (6), for all $k \geq \ell$, and with α defined in Lemma 1.

$$\sup_{k \geq \ell} \|\tilde{\theta}_k\| \leq \frac{\sup_{k \geq \ell} \|v_k\|}{\alpha},$$

$$\limsup_{k \rightarrow +\infty} \|\tilde{\theta}_k\| \leq (1 - \lambda^\ell)^n \frac{\limsup_{k \rightarrow +\infty} \|v_k\|}{\alpha}$$

Thus, the DREM-based algorithm (5), (3), (6) guarantees a fixed-time convergence of the estimate $\hat{\theta}_k$ to the true value $\bar{\theta}$ (the convergence time $\ell \geq n$ is independent of initial conditions), while the estimation error is proportional to the noise v amplitude.

Remark 4. According to the estimates given in Theorem 3, the choice $\lambda^* = \frac{\ell-1}{\ell}$ minimizes the parameter estimation error gain α^{-1} with respect to the noise. Augmenting the value of λ can improve the asymptotic precision of the proposed algorithm.

Several final comments are in order:

- (1) Roughly speaking, the DREM method provides a direct measurement of the unknown parameter vector $\bar{\theta}$ for $k \geq \ell$:

$$\mathcal{Y}_k = \bar{\theta} + \mathcal{V}_k, \quad \text{with } \mathcal{Y}_k = \phi_k^{-1} y_k, \quad \mathcal{V}_k = \phi_k^{-1} v_k.$$

Note that for a (ℓ, μ) -PE signal $\bar{\phi}$, the linear regression problem (1) has the sliding-window solution for $k \geq \ell - 1$:

$$\hat{\theta}_k = \left(\sum_{i=0}^{\ell-1} \bar{\phi}_{k-i} \bar{\phi}_{k-i}^\top \right)^{-1} \sum_{i=0}^{\ell-1} \bar{\phi}_{k-i} \bar{y}_{k-i}, \quad (7)$$

then (5), (3), (6) realize such a scheme with a forgetting factor λ (with exponentially weighted moving average). The iterative application of (6) or (7) for $k > \ell$ is originated from the noise filtering need.

- (2) Any other additional noise filtering can be applied to \mathcal{Y}_k to attenuate the influence of \mathcal{V}_k . In the existing literature, gradient algorithms are frequently used for (4) (Belov et al., 2018; Aranovskiy et al., 2019; VEDIKOVA et al., 2021; Ortega et al., 2020; Wang et al., 2019), which serve merely for implicit noise-canceling, being probably not the best choice for such a purpose. Indeed, the conventional selection is the least-squares algorithm:

$$\hat{\theta}_k = \hat{\theta}_{k-1} + \gamma_k \phi_k (y_k - \phi_k \hat{\theta}_{k-1}), \quad \gamma_k \in (0, 2\phi_k^{-2}).$$

For $\gamma_k = \phi_k^{-2}$, it is reduced to (6), while for $\gamma_k = \rho \phi_k^{-2}$ with $\rho \in (0, 1)$, to a low-pass filtering of (6). In general, many more options are available, as the moving-average filter or nonlinear designs that cannot be directly applied to the linear regression (4), e.g., the median filter. See Section 6 for an illustrative comparison of selected methods.

- (3) Note that Kreisselmeier's dynamic extension (5) filters the noise \bar{v}_k already, or other noise filters can be applied to get (1).

6. EXAMPLE

We consider the parameter estimation problem for an ARX model. The model is given by

$$y_k = -a_1 y_{k-1} - a_0 y_{k-2} + b_0 u_k + \bar{v}_k, \quad (8)$$

where y and u are measured scalar output and input signals, respectively, \bar{v} is the unmeasured noise, and b_0 , a_0 , and a_1 are unknown constants to be estimated. This model can be written in the form (1) setting for all $k \in \mathbb{N}$

$$\bar{\phi} = (-y_{k-2} \quad -y_{k-1} \quad u_k)^\top,$$

$$\bar{\theta} = (a_0 \quad a_1 \quad b_0)^\top,$$

and $\bar{y}_k = y_k$.

The input signal is chosen as

$$u_k = A_1 \sin(\omega_1 k + \psi_1) + A_2 \sin(\omega_2 k + \psi_2), \quad (9)$$

where A_1 , A_2 , ψ_1 , ψ_2 , ω_1 , and ω_2 are known parameters of the input signal.

We apply the DREM procedure with the Kreisselmeier's dynamic regressor extension (5) with zero initial conditions. The procedure yields the element-wise scalar linear regression equations (4) for $i = 1, 2, 3$.

Standard results in system identification, see, e.g., Sastry and Bodson (1989), show that for $\omega_1 \neq \omega_2$, the input signal (9) provides excitation to the model (8) and $\bar{\phi}$ is persistently exciting. Thus, the Excitation Preservation Lemma applies and the signal ϕ is also PE and strictly

separated from zero for $k \geq T_0$ for some $T_0 \in \mathbb{N}$; a conservative estimate of T_0 is the period of the input signal u . Then the direct estimation (6) is applied,

$$\hat{\theta}_{k,i}^{\text{dir}} = \frac{y_{k,i}}{\max\{\phi_k, \varepsilon_\phi\}}, \quad i = 1, 2, 3, \quad (10)$$

where $\varepsilon_\phi > 0$ is a small constant introduced to ensure the feasibility of implementing the direct estimation for the initial transients of (5).

In the sequel, we consider various filtering techniques applied to the direct estimate (10) and compare them with estimators applied to the element-wise linear regression models (4). The results illustrate that fixed-time direct estimation followed by filtering performs slightly better than estimators with asymptotic convergence.

Used approaches

For the element-wise linear regression (4), we consider the following two estimators for $i = 1, 2, 3$:

- the gradient estimator given by

$$\hat{\theta}_{k,i}^{\text{gr}} = \hat{\theta}_{k-1,i}^{\text{gr}} + \frac{\gamma_0}{1 + \gamma_0 \phi_k^2} \left(y_{k,i} - \phi_k \hat{\theta}_{k-1,i}^{\text{gr}} \right), \quad (11)$$

where $\gamma_0 > 0$ is the tuning coefficient;

- the least-squares estimator with forgetting given by

$$p_k = \frac{p_{k-1}}{\lambda_{ls} + \phi_k^2 p_{k-1}}, \quad (12)$$

$$\hat{\theta}_{k,i}^{\text{ls}} = \hat{\theta}_{k-1,i}^{\text{ls}} + p_k \phi_k \left(y_{k,i} - \phi_k \hat{\theta}_{k-1,i}^{\text{ls}} \right),$$

with $p_0 > 0$ and $\lambda_{ls} \in (0, 1)$ is the forgetting factor.

For the direct estimate, we consider the following filtering methods applied to $\hat{\theta}_{k,i}^{\text{dir}}$ given in (10), $i = 1, 2, 3$:

- the low-pass filtering given by

$$\hat{\theta}_{k,i}^{\text{low}} = \varrho \hat{\theta}_{k-1,i}^{\text{low}} + (1 - \varrho) \hat{\theta}_{k,i}^{\text{dir}}, \quad (13)$$

where $\varrho \in (0, 1)$ is the tuning coefficient;

- the moving-average filtering given by

$$\hat{\theta}_{k,i}^{\text{ma}} = \frac{1}{N} \sum_{j=0}^{N-1} \hat{\theta}_{k-j,i}^{\text{dir}}, \quad (14)$$

where $N = \min\{k, N_{\text{ma}}\}$, and N_{ma} is the length of the averaging window;

- the median filter given by

$$\hat{\theta}_{k,i}^{\text{med}} = \text{median}\left\{\hat{\theta}_{k-N+1,i}^{\text{dir}}, \dots, \hat{\theta}_{k,i}^{\text{dir}}\right\}, \quad (15)$$

where $N = \min\{k, N_{\text{med}}\}$, and N_{med} is the number of samples used in the median computation,

- the Kalman filter given by

$$\hat{\theta}_{k,i}^{\text{kal}} = \hat{\theta}_{k-1,i}^{\text{kal}} + \frac{P_{k-1} + Q}{P_{k-1} + Q + R_k} \left(\frac{y_k}{\phi_k} - \hat{\theta}_{k-1,i}^{\text{kal}} \right),$$

$$P_k = \frac{R_k}{P_{k-1} + Q + R_k} (P_{k-1} + Q),$$

$$R_k = \frac{R_0}{\max\{\phi_k^2, \varepsilon_\phi^2\}}, \quad (16)$$

where $Q \geq 0$ and $R_0 > 0$ are the process and observation noise covariances. When $Q \neq 0$, the case of time-varying parameters can be treated.

Simulation results

Consider the model (8), where $a_0 = 0.73$, $a_1 = -1.71$, and $b_0 = 0.12$. The initial conditions are zero, $y_{-1} = y_{-2} = 0$. The measurement noise \bar{v} is chosen as a uniform random variable, $\bar{v} \sim U(-0.05, 0.05)$.

For the inputs signal (9), we choose $A_1 = 2$, $A_2 = 1$, $\psi_1 = 0$, $\psi_2 = \frac{\pi}{3}$, $\omega_1 = \frac{2\pi}{132}$, and $\omega_2 = \frac{2\pi}{66}$. Following Remark 4, the value λ in (5) is chosen as $\lambda = \frac{T_1 - 1}{T_1} = \frac{131}{132}$, where $T_1 := \frac{2\pi}{\omega_1}$. The initial conditions in (5) are zeros, $\Phi_0 = 0$, $Y_0 = 0$.

For the direct estimation, the value ε_ϕ in (10) is chosen as $\varepsilon_\phi = 10^{-6}$. The initial values in the estimators (11), (12) and filters (13), (16) are zero; the initial values of p in (12) and P in (16) equal to one. The parameters of the estimators (11), (12) and filters (13), (14), (16) are chosen to provide equal transient time in the ideal noise-free scenario, where the transient time is measured as the step number when the estimation errors decays to 5% of the initial value. The values are $\gamma_0 = 585$, $\lambda_{ls} = e^{-\frac{3}{28}}$, $\varrho = e^{-\frac{3}{80}}$, $N_{\text{ma}} = 84$, and $Q = 1 \cdot 10^{-4}$, $R_0 = 2 \cdot 10^{-3} Q$. For the median filter (15), the number of used samples is equal to the length of the averaging window in (14), $N_{\text{med}} = N_{\text{ma}}$.

Figures 1, 2, and 3 depict the estimation error transients for θ_1 , θ_2 , and θ_3 , respectively, of the considered methods in the ideal noise-free scenario. The figures depict the direct estimation $\hat{\theta}^{\text{dir}}$ (10), the gradient estimator $\hat{\theta}^{\text{gr}}$ (11), the least-squares estimator $\hat{\theta}^{\text{ls}}$ (12), the low-pass filtering $\hat{\theta}^{\text{low}}$ (13), the moving-average filtering $\hat{\theta}^{\text{ma}}$ (14), the median filtering $\hat{\theta}^{\text{med}}$ (15), and the Kalman filter $\hat{\theta}^{\text{kal}}$ (16). The dashed line corresponds to 5% of the initial error value; the estimators are tuned to have approximately equal transient time.

Denote by T_{fix} , the step number when ϕ_k overcomes the threshold ε_ϕ , where $T_{\text{fix}} = 10$ in the example. The direct estimate (10) converges in the fixed time T_{fix} . The median filter also has the fixed-time convergence property with convergence time $\min\{2T_{\text{fix}}, T_{\text{fix}} + \lfloor \frac{N_{\text{med}}}{2} \rfloor\}$. The moving-average filter converges in the fixed time $T_{\text{fix}} + N_{\text{ma}} - 1$, and other estimates converge asymptotically.

Figures 4, 5, and 6 depict the estimation error transients of the considered methods for noisy measurements for θ_1 , θ_2 , and θ_3 , respectively. As these figures focus on noise attenuation, we omit the initial transients and present the steady-state behavior of estimators. The results illustrate that all considered methods have comparable performance in alleviating the noise, where filtering of the direct estimate, such as moving average or median filtering, has a slightly better attenuation. These results are summarized in Table 1, where the mean squared error (MSE) and the mean absolute error (MAE) are given; these values are computed over 10^6 samples after the transients.

7. CONCLUSION

In this work, we showed that using Kreisselmeier's scheme for the dynamic extension step of the DREM procedure, the excitation of the original regressor is preserved. Then,

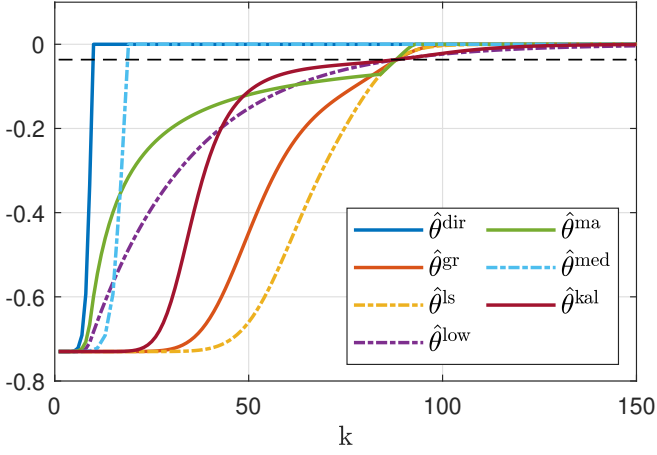


Fig. 1. Estimation error transients in the noise-free case for θ_1

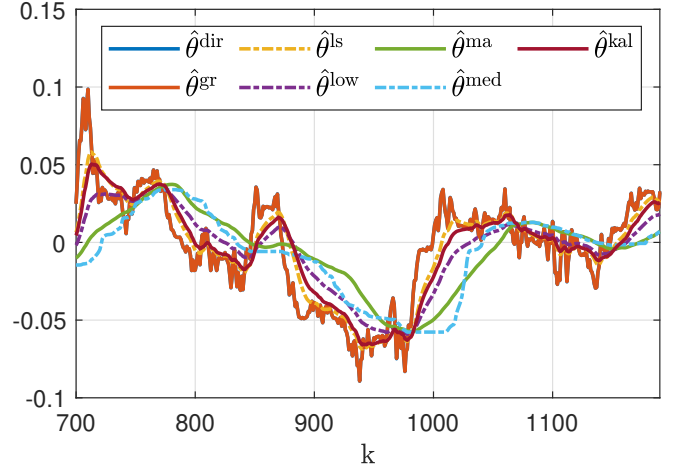


Fig. 4. Estimation errors in the noised case for θ_1

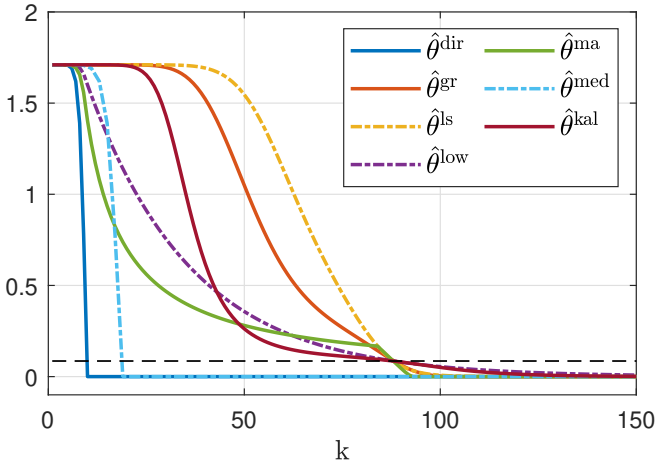


Fig. 2. Estimation error transients in the noise-free case for θ_2

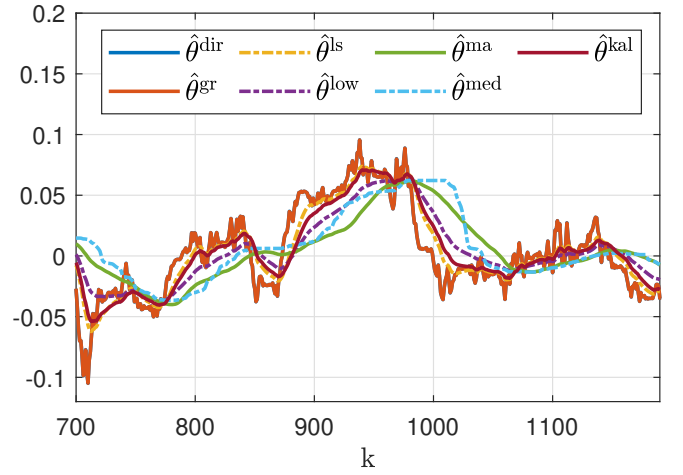


Fig. 5. Estimation errors in the noised case for θ_2

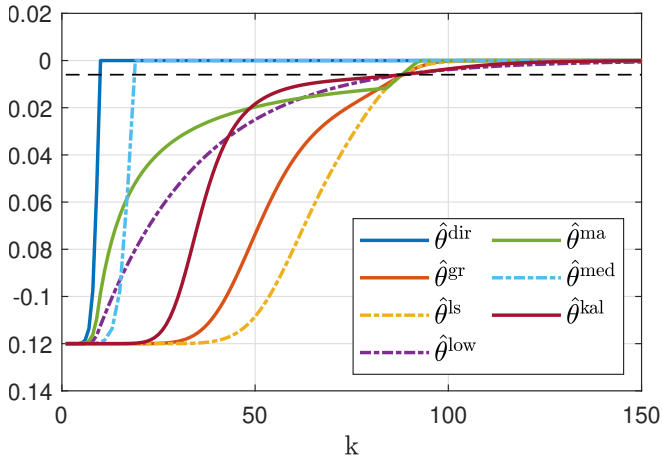


Fig. 3. Estimation error transients in the noise-free case for θ_3

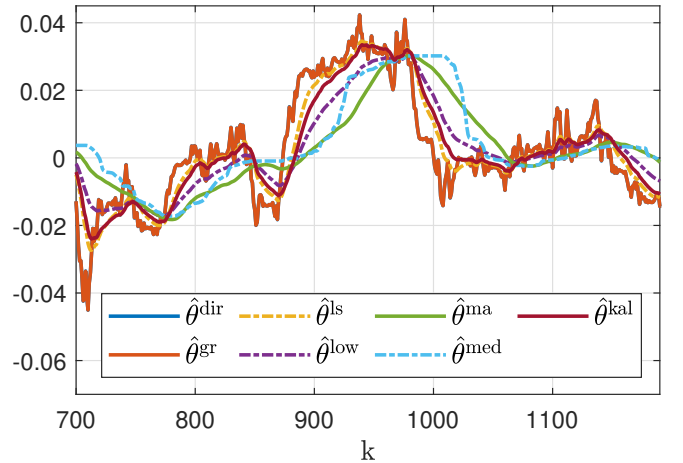


Fig. 6. Estimation errors in the noised case for θ_3

the DREM procedure generates a novel regressor, separated from zero, after the initial time interval. This property allows for a direct algebraic pointwise estimation of unknown parameters, also providing the worst-case noise propagation bound.

Finally, we demonstrated that such a direct estimate combined with a (nonlinear) low-pass filter represents a suitable alternative to the typical combination of the DREM procedure with an iterative estimator, such as the gradient of the least-squares one.

Table 1. MSE and MAE of selected methods

		$\hat{\theta}_{gr}$	$\hat{\theta}_{ls}$	$\hat{\theta}_{low}$	$\hat{\theta}_{ma}$	$\hat{\theta}_{med}$	$\hat{\theta}_{kal}$
θ_1	MSE · 10 ³	2.5	2.4	2.1	2.1	2.2	2.3
	MAE · 10 ²	4.0	3.9	3.7	3.7	3.7	3.8
θ_2	MSE · 10 ³	2.9	2.7	2.4	2.4	2.5	2.6
	MAE · 10 ²	4.3	4.1	3.9	3.9	4.0	4.1
θ_3	MSE · 10 ³	0.5	0.5	0.4	0.4	0.4	0.4
	MAE · 10 ²	1.8	1.7	1.6	1.6	1.6	1.7

REFERENCES

- Aranovskiy, S., Belov, A., Ortega, R., Barabanov, N., and Bobtsov, A. (2019). Parameter identification of linear time-invariant systems using dynamic regressor extension and mixing. *International Journal of Adaptive Control and Signal Processing*, 33(6), 1016–1030.
- Aranovskiy, S., Bobtsov, A., Ortega, R., and Pyrkin, A. (2017). Performance enhancement of parameter estimators via dynamic regressor extension and mixing. *IEEE Transactions on Automatic Control*, 62(7), 3546–3550. doi:10.1109/TAC.2016.2614889.
- Aranovskiy, S., Ushirobira, R., Korotina, M., and Vedyakov, A. (2023). On preserving-excitation properties of kreisselmeier’s regressor extension scheme. *IEEE Transactions on Automatic Control*, 68(2), 1296–1302.
- Belov, A., Aranovskiy, S., Ortega, R., Barabanov, N., and Bobtsov, A. (2018). Enhanced parameter convergence for linear systems identification: The DREM approach. In *Proc. European Control Conference (ECC)*, 2794–2799. Limassol.
- Efimov, D., Barabanov, N., and Ortega, R. (2019). Robust stability under relaxed persistent excitation conditions. *International Journal of Adaptive Control and Signal Processing*. doi:https://doi.org/10.1002/acs.2997.
- Efimov, D. and Fradkov, A. (2015). Design of impulsive adaptive observers for improvement of persistency of excitation. *Int. J. Adaptive Control and Signal Processing*, 29(6), 765–782.
- Ioannou, P. and Kokotovic, P. (1983). *Adaptive Systems with Reduced Models*. Springer Verlag, Secaucus, NJ.
- Ljung, L. (1987). *System Identification: Theory for the User*. Prentice-Hall, Upper Saddle River, NJ.
- Narendra, K.S. and Annaswamy, A.M. (1987). Persistent excitation in adaptive systems. *International Journal of Control*, 45(1), 127–160. doi:10.1080/00207178708933715.
- Ortega, R., Aranovskiy, S., Pyrkin, A.A., Astolfi, A., and Bobtsov, A.A. (2021a). New results on parameter estimation via dynamic regressor extension and mixing: Continuous and discrete-time cases. *IEEE Transactions on Automatic Control*, 66(5), 2265–2272. doi:10.1109/TAC.2020.3003651.
- Ortega, R., Nikiforov, V., and Gerasimov, D. (2020). On modified parameter estimators for identification and adaptive control. a unified framework and some new schemes. *Annual Reviews in Control*, 50, 278–293. doi:https://doi.org/10.1016/j.arcontrol.2020.06.002.
- Ortega, R., Nuño, E., and Bobtsov, A. (2021b). An algebraic, distributed state observer for continuous- and discrete-time linear time-invariant systems with time-varying communication graphs. *International Journal of Adaptive Control and Signal Processing*. doi:https://doi.org/10.1002/acs.3240.
- Sastry, S. and Bodson, M. (1989). *Adaptive Control: Stability, Convergence and Robustness*. Prentice-Hall, London.
- Vediakova, A., Vedyakov, A., Pyrkin, A., Bobtsov, A., and Gromov, V. (2021). Finite time frequency estimation for multi-sinusoidal signals. *European Journal of Control*, 59, 38–46.
- Vedyakov, A.A., Vediakova, A.O., Bobtsov, A.A., and Pyrkin, A.A. (2019). Relaxation for online frequency estimator of bias-affected damped sinusoidal signals based on dynamic regressor extension and mixing. *International Journal of Adaptive Control and Signal Processing*, 33(12), 1857–1867. doi:https://doi.org/10.1002/acs.3034.
- Wang, J., Efimov, D., Aranovskiy, S., and Bobtsov, A. (2019). Fixed-time estimation of parameters for non-persistent excitation. *European Journal of Control*.
- Wang, J., Efimov, D., and Bobtsov, A. (2020). On robust parameter estimation in finite-time without persistence of excitation. *IEEE Transactions on Automatic Control*, 65(4).
- Yan, X. (2009). *Linear Regression Analysis: Theory and Computing*. World Scientific, Singapore.