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ARTICLE TYPE**Fixed-time Stabilization with a Prescribed Constant Settling Time by Static Feedback for Delay-Free and Input Delay Systems**Andrey Polyakov¹ | Miroslav Krstic²¹ Inria, University of Lille, FR-59000, France² University of California, San Diego, USA**Correspondence**

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Summary

A static non-linear homogeneous feedback for a fixed-time stabilization of a linear time-invariant (LTI) system is designed in such a way that the settling time is assigned exactly to a prescribed constant for all nonzero initial conditions. The constant convergence time is achieved due to a dependence of the feedback gain of the initial state of the system. The robustness of the closed-loop system with respect to measurement noise and exogenous perturbations is studied using the concept of Input-to-State Stability (ISS). Both delay-free and input delay systems are considered. Theoretical results are illustrated by numerical simulations.

KEYWORDS:

homogeneous system, fixed-time stabilization, robustness, input delay

1 | INTRODUCTION

The problem of regulation of a system to a desired set-point in a finite time can be solved using, for example, the methods of finite-time stabilization (see, e.g.,⁸ and references therein). Algorithms of finite-time regulation and stabilization for linear systems are well-known since 1950s (see, for example,^{10, 24, 13, 21, 15, 6, 3}). The settling time to a set-point may be uniformly bounded for all initial conditions (see, e.g.,^{26, 5, 3}). In³⁵, such a property of finite-time stable systems was called fixed-time stability.

Both time-independent (static) feedback laws (see, e.g.,^{35, 49}) and time-dependent regulators (see, e.g.,^{42, 47, 32}) are developed for fixed-time stabilization and regulation of LTI plants. For controllable systems, the settling time can be tuned arbitrary small. This immediately follows from the definition of controllability. For a control system topologically equivalent to the integrator chain, very simple schemes for tuning of the settling time are given, for example, in⁴² and³⁸. The time-dependent feedback⁴² is designed such that the closed-loop system converges to the origin *exactly* at a desired (prescribed) time $T > 0$ independently of the initial condition away from the origin. This property is, obviously, more strong than simply a fixed-time stabilization in a prescribed time $T > 0$. In the latter case, the system reaches the desired set-point *no later* than the time instant $t = T$. An assignment of the exact constant settling time may be useful for certain control problems^{42, 41}.

The time-dependent prescribed-time regulators are designed for various finite dimensional^{42, 16, 1} and infinite dimensional^{9, 44, 48} systems. Frequently (see, e.g.,^{42, 9}), the prescribed-time regulator has the form of a linear feedback with a *time-dependent gain tending to infinity* as the time tends to the prescribed time $T > 0$. This definitely impacts the robustness properties of the closed-loop system despite that the closed-loop system satisfies the ISS¹-like estimates⁴² on the prescribed interval of time $[0, T)$. In the delay-free case, the mentioned time-dependent controller rejects matched additive disturbances of unknown magnitude⁴², but it is very sensitive with respect to measurement noise^{32, 2}. The main reason of such sensitivity is the time dependence of the feedback gain which, independently of the stabilization error and the magnitude of the measurement

¹ISS=Input-to-State Stability⁴³

noise, infinitely amplifies the noise as time tends to the prescribed time T . To improve the robustness, a switching rule between time-dependent prescribed-time regulator and a static finite-time (sliding mode) stabilizer has been suggested in³².

The fixed-time stabilizer presented in³⁸ is a static (*time-independent*) nonlinear feedback, which can be interpreted as a linear control with a *state dependent feedback gain*. This gain tends to infinity as the norm of the stabilization error tends to zero. Such a control system admits a simple scheme for tuning of a required upper bound of the stabilization time, but it does not allow us to assign an exact (prescribed) constant settling time for all initial conditions (like for time-varying algorithm). Due to homogeneity, the static feedback controller is robust (in the ISS sense) with respect to a rather large class of perturbations^{3, 49, 37} including measurement noise. Moreover, comparing with the time-dependent stabilizer, it is expected to be less sensitive with respect to measurement noise since the feedback gain (amplifying the noise) does not tend to infinity in this case (due to non-zero stabilization error).

In this paper we design a global static feedback, which stabilizes the linear MIMO system such that the settling time of the closed-loop system to zero equals exactly to a prescribed time T for all non-zero initial conditions. To the best of authors' knowledge, the static (time-independent) feedback laws solving the mentioned problem have never been designed before, probably due to the following reason. The finite-time stability with a constant settling time is impossible for continuous autonomous ODE (Ordinary Differential Equation), since the settling-time function of any finite-time stable ODE is *strictly* decreasing along non-zero trajectories of the system^{7, Proposition 2.4}. So, it cannot be a constant for all non-zero initial conditions. Therefore, the considered prescribed-time stabilization problem is infeasible by a conventional static nonlinear feedback. Inspired by³⁸, to overcome this fundamental obstacle, we define the *gain of a static (time-invariant) nonlinear feedback as a function of the initial state*. Formally, in this case, the closed-loop system becomes a Functional Differential Equation (FDE), since its right-hand side depends on both current and previous (more precisely, initial) values of the state vector. However, this is a very particular class of FDE, since for any fixed initial condition, the FDE becomes an ODE and can be analyzed in the conventional way. Our design is essentially-based on an extension of homogeneity concept to such class of FDEs.

Homogeneity is a dilation symmetry widely utilized^{51, 39, 17, 6, 30, 25, 34, 3, 36} for finite-time stabilization and stability analysis. Any asymptotically stable homogeneous system of negative degree is finite-time stable. This paper extends the homogeneity-based analysis to a particular class of FDEs, which can be treated as autonomous ODEs with right-hand sides depended on the initial state. In this case, the vector field (the right-hand side of the FDE) may be homogeneous with respect to dilation of both actual and initial state vectors. We show that, under certain conditions, the asymptotically stable homogeneous FDE is fixed-time stable with a constant settling-time function. This novel result extends the existing knowledge about convergence rates of homogeneous systems.

In this paper, some novel static fixed-time controllers are designed for both delay-free and input delay LTI systems. In⁴⁶, the static fixed-time stabilizer (with non-constant settling time) for the integrator chain has been designed using transport PDE (Partial Differential Equation) as a model of the input delay and the back-stepping transformation^{23, 18}. However, a similar PDE-based analysis seems impossible for our fixed-time stabilizer due to its discontinuity. In the delay-free case, the analysis of the closed-loop dynamics is based on the Filippov's theory of discontinuous differential equations¹². Its analog for PDEs with discontinuous controllers is not yet well-developed, despite of some interesting recent contributions to this field³¹. We extend the results obtained in the delay-free case to the input delay LTI system by means of the Artstein's transformation⁴, which allows both the stability and the robustness analysis of the closed-loop system to be realized easily.

The paper is organized as follows. First, the problem statement is presented. Next, some preliminary remarks about the particular class of homogeneous FDEs are given. After that, a fixed-time stabilizer with a prescribed constant settling time is designed for LTI system. Finally, the numerical simulation examples and conclusions are presented.

Notation. \mathbb{R} is the field of reals; $\mathbb{R}_0^n = \mathbb{R}^n \setminus \{\mathbf{0}\}$, where $\mathbf{0}$ is the zero element of a vector space (e.g., $\mathbf{0} \in \mathbb{R}^n$ means that $\mathbf{0}$ is the zero vector); $\|\cdot\|$ is a norm in \mathbb{R}^n (to be specified later); a function $\sigma : [0, +\infty) \mapsto [0, +\infty)$ belongs to the class \mathcal{K} if σ is strictly increasing and $\sigma(0) = 0$; $\sigma \in \mathcal{K}^\infty$ if $\sigma \in \mathcal{K}$ and $\sigma(s) \rightarrow +\infty$ as $s \rightarrow +\infty$; the function $\beta : [0, +\infty) \times [0, +\infty) \mapsto [0, +\infty)$ belongs to the class \mathcal{KL} if the function $s \mapsto \beta(r, s)$ is decreasing to zero for any fixed $r \geq 0$, but the function $r \mapsto \beta(r, s)$ belongs to the class \mathcal{K} for any fixed $s \geq 0$; the matrix norm for $A \in \mathbb{R}^{n \times n}$ is defined as $\|A\| = \sup_{x \neq \mathbf{0}} \frac{\|Ax\|}{\|x\|}$; $\lambda_{\min}(P)$ denote a minimal eigenvalue of a symmetric matrix $P = P^\top \in \mathbb{R}^{n \times n}$; $P > 0$ means that the symmetric matrix P is positive definite; $C^1(\Omega_1, \Omega_2)$ denotes the set of continuously differentiable functions $\Omega_1 \subset \mathbb{R}^n \mapsto \Omega_2 \subset \mathbb{R}^m$; $L^\infty(\mathbb{R}, \mathbb{R}^k)$ is the Lebesgue space of measurable uniformly essentially bounded functions $\mathbb{R} \mapsto \mathbb{R}^k$ with the norm defined by the essential supremum, i.e., $\|q\|_{L^\infty(\mathbb{R}, \mathbb{R}^k)} = \text{ess sup}_{\tau \in (t_0, t)} \|q(\tau)\|$ for $q \in L^\infty(\mathbb{R}, \mathbb{R}^k)$; $W^{1,\infty}(\Gamma, \mathbb{R}^n) = \{\phi \in L^\infty(\Gamma, \mathbb{R}^n) : \dot{\phi} \in L^\infty(\Gamma, \mathbb{R}^n)\}$ is the Sobolev space of absolutely continuous functions $\Gamma \subset \mathbb{R} \mapsto \mathbb{R}^n$; we write $\stackrel{a.e.}{=}$ (resp. $\stackrel{a.e.}{\leq}$ or $\stackrel{a.e.}{\in}$) if an identity (resp., inequality or inclusion) holds almost everywhere.

2 | PROBLEM STATEMENT

Let us consider the system

$$\dot{x}(t) = Ax(t) + Bu(t - \tau), \quad t > 0, \quad x(0) = x_0 \in \mathbb{R}^n, \quad (1)$$

where $x(t) \in \mathbb{R}^n$ is the state variable, $u(t - \tau) \in \mathbb{R}^m$ is the control signal, $A \in \mathbb{R}^{n \times n}$ and $B \in \mathbb{R}^{n \times m}$ are known matrices, the time shift $\tau \geq 0$ models a delay of a transmission of the input signal to the plant. We restrict the class of admissible control signals $u \in L^\infty((-\tau, +\infty), \mathbb{R}^m)$, so the differential equation in (1) is assumed to be fulfilled almost everywhere. Notice that u has to be defined on the time interval $(-\tau, +\infty)$ to guarantee the well-posedness of the system (1). The whole state vector $x(t)$ is assumed to be available (measured or estimated) for the control purposes. The pair $\{A, B\}$ is assumed to be controllable.

First, we study the delay-free case ($\tau = 0$). For a given constant $T > 0$, we need to design a nonlinear feedback

$$u(t) = \tilde{K}(x(t), x_0)x(t), \quad \tilde{K} \in C^1(\mathbb{R}_0^n \times \mathbb{R}^n, \mathbb{R}^{m \times n}) \quad (2)$$

such that the closed-loop system (1), (2) is *fixed-time stable with the constant settling time* $T > 0$. This means that the system is Lyapunov stable and $x(t) = \mathbf{0}$, $\forall t \geq T$, $\forall x_0 \in \mathbb{R}^n$, but $x(t) \neq \mathbf{0}$ for $t \in [0, T)$ if $x_0 \neq \mathbf{0}$. Therefore, any trajectory of the system initiated away from the origin will reach the origin exactly at the time instant $t = T$.

The second goal of the paper is to study the robustness (in the sense of Input-to-State Stability⁴³) of the closed-loop system with respect to additive disturbances and measurement noise (for $\tau = 0$):

$$\dot{x}(t) \stackrel{a.e.}{=} Ax(t) + B\tilde{K}(x(t) + q_1(t), x_0 + q_0)(x(t) + q_1(t)) + q_2(t), \quad t > 0, \quad x(0) = x_0, \quad (3)$$

where $q_0 \in \mathbb{R}^n$ models the measurement error of the initial state $x(0) = x_0$, $q_1 \in L^\infty(\mathbb{R}, \mathbb{R}^k)$ is measurement noise of the state $x(t)$ for $t > 0$ and $q_2 \in L^\infty(\mathbb{R}, \mathbb{R}^k)$ is the exogenous disturbance.

Finally, the third goal is to solve the above problems for the input delay system (1) with $\tau > 0$. In this case, the control value generated at time t affects the system in the future instant of time $t + \tau$. Since the plant model (1) is valid only for $t > 0$, the control signal $u(t)$ can be generated based on the state measurements (similarly to (2)) only for $t > 0$, but for $t \in (-\tau, 0)$ it has to be initialized as follows:

$$u(\theta) = \phi(\theta) \quad \text{for } \theta \in (-\tau, 0), \quad \phi \in L^\infty((-\tau, 0), \mathbb{R}^m), \quad (4)$$

where, dependently of the control application, the initial control signal ϕ can be assumed to be uncertain or assigned as needed for reaching the control goal. Due to the input delay, we restrict the desired settling time to $T + \tau$. Below we show that the problem of the fixed-time stabilization with a constant settling time $T + \tau$ is feasible only if the control signal on $(-\tau, 0)$ is initialized by the zero ($\phi = \mathbf{0}$). Notice that, for an arbitrary selected or unknown initial function $\phi \neq \mathbf{0}$, the fixed-time stabilization with the prescribed time $T + \tau$ remains possible. However, the constant value of the settling time cannot be guaranteed anymore. The settling time can be just bounded by the prescribed constant $T + \tau$ in this case.

3 | PRELIMINARIES

3.1 | Stability notions

Let us consider the system

$$\dot{x}(t) = f(x(t), x(t_0)), \quad t > t_0, \quad x(t_0) = x_0 \in \mathbb{R}^n, \quad (5)$$

where $x(t)$ is the system state and $f : \mathbb{R}^n \times \mathbb{R}^n \mapsto \mathbb{R}^n$ is locally bounded. On the one hand, the system (5) is well-posed, since it can be rewritten in the form of the conventional ODE

$$\begin{cases} \dot{x} = f(x, r), \\ \dot{r} = \mathbf{0}, \end{cases} \quad t > t_0, \quad x(t_0) = r(t_0) = x_0. \quad (6)$$

For simplicity, we assume that $f \in C(\mathbb{R}_0^n \times \mathbb{R}^n, \mathbb{R}^n)$. However, the results presented in this section are also valid for discontinuous equations and inclusions studied in¹². If $f \in C(\mathbb{R}_0^n \times \mathbb{R}^n, \mathbb{R}^n)$ so this system (as well as the system (5)) has classical (possible non-unique) solutions $t \mapsto (x(t), r(t))$ on $\mathbb{R}_0^n \times \mathbb{R}^n$ and Filippov solutions on $\mathbb{R}_0^n \times \mathbb{R}^n$ provided that f is locally bounded.

On the other hand, the differential equation (5) is not a dynamical system in the sense that *its solutions do not satisfy the so-called semi-group property*²⁸. Namely, if $x(t, t_0, x_0)$ with $t \geq t_0$ denotes a solution of the system (5) then, in the general case, $x(t, s, x(s, t_0, x_0)) \neq x(t + s, t_0, x_0)$, where $s > t_0$. So, the classical results of the stability theory (such as the Lyapunov function method) cannot be directly applied to the system (5). However, the stability notions can be introduced in the conventional way.

Since, in this paper, we deal only with a global uniform stability, then, for shortness, we omit the words "global uniform" when we discuss stability issues.

Definition 1.⁸ The system (5) is said to be

- Lyapunov stable if there exists $\varepsilon \in \mathcal{K}$ such that

$$\|x(t)\| \leq \varepsilon(\|x_0\|), \quad \forall t \geq t_0, \quad \forall x_0 \in \mathbb{R}^n; \quad (7)$$

- asymptotically stable if there exists $\beta \in \mathcal{KL}$

$$\|x(t)\| \leq \beta(\|x_0\|, t - t_0), \quad \forall x_0 \in \mathbb{R}^n, \quad \forall t \geq t_0. \quad (8)$$

- finite-time stable if it is Lyapunov stable and there exists a locally bounded function $\tilde{T} : \mathbb{R}^n \mapsto \mathbb{R}_+$ such that for any $x_0 \in \mathbb{R}^n \setminus \{\mathbf{0}\}$ it holds

$$x(t) = \mathbf{0}, \quad \forall t \geq t_0 + \tilde{T}(x_0), \quad (9)$$

for any solution $x(t)$ of (5) with $x(t_0) = x_0$, but $x(t) \neq \mathbf{0}$ for all $t \in [0, \tilde{T}(x_0))$, at least, for one solution $x(t)$ of (5) with $x(t_0) = x_0$;

- fixed-time stable if it is finite-time stable and there exists $T_{\max} > 0$ such that

$$\exists T_{\max} > 0 \quad : \quad \tilde{T}(x_0) \leq T_{\max}, \quad \forall x_0 \in \mathbb{R}^n; \quad (10)$$

The function \tilde{T} from the above definition is known as the *settling time function*⁷ and its value $\tilde{T}(x_0)$ is referred to as the *settling time* for the given initial state x_0 . In this paper, we study *finite-time stable systems with constant settling-time functions*, i.e., $\tilde{T}(x_0) = \text{const}$ for all $x_0 \neq \mathbf{0}$.

Let us consider the system

$$\dot{x}(t) \stackrel{a.e.}{=} \tilde{f}(x(t), x_0, q(t)), \quad t > t_0, \quad x(t_0) = x_0 \in \mathbb{R}^n, \quad (11)$$

where $x(t)$ is the system state, $q \in L^\infty(\mathbb{R}, \mathbb{R}^k)$ and \tilde{f} is a locally bounded measurable function such that the system (11) has a Filippov solution¹² for any $x_0 \in \mathbb{R}^n$, any $t_0 \in \mathbb{R}$ and any $q \in L^\infty(\mathbb{R}, \mathbb{R}^k)$.

Definition 2.⁴³ A system is said to be Input-To-State Stable (ISS) if there exist $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}$ such that

$$\|x(t)\| \leq \beta(\|x_0\|, t - t_0) + \gamma(\|q\|_{L^\infty(t_0, t)}), \quad \forall x_0 \in \mathbb{R}^n, \quad \forall t \geq t_0, \quad \forall q \in L^\infty(\mathbb{R}, \mathbb{R}^k), \quad \forall t_0 \in \mathbb{R}, \quad (12)$$

where x is a state of the system at the time $t \geq t_0$, x_0 is the initial state and q is an exogenous input/perturbation.

In control theory, the ISS is frequently interpreted as a robustness of the system with respect to a perturbation, which is modelled by an exogenous input q in the right-hand side. Since the right-hand side of (11) depends on the initial condition then the behavior of the perturbed system on the infinite horizon could also depend on the initial state x_0 . Therefore, *in the case of the system (11), the function γ in the ISS definition may depend on x_0 as well.*

3.2 | Homogeneous Systems

3.2.1 | Linear dilations

Let us recall that a *family of operators* $\mathbf{d}(s) : \mathbb{R}^n \mapsto \mathbb{R}^n$ with $s \in \mathbb{R}$ is a *group* if $\mathbf{d}(0)x = x$, $\mathbf{d}(s)\mathbf{d}(t)x = \mathbf{d}(s+t)x$, $\forall x \in \mathbb{R}^n$, $\forall s, t \in \mathbb{R}$. A *group \mathbf{d}* is

- continuous* if the mapping $s \mapsto \mathbf{d}(s)x$ is continuous, $\forall x \in \mathbb{R}^n$;
- linear* if $\mathbf{d}(s)$ is a linear mapping (i.e., $\mathbf{d}(s) \in \mathbb{R}^{n \times n}$), $\forall s \in \mathbb{R}$;
- a *dilation* in \mathbb{R}^n if $\liminf_{s \rightarrow +\infty} \|\mathbf{d}(s)x\| = +\infty$ and $\limsup_{s \rightarrow -\infty} \|\mathbf{d}(s)x\| = 0$, $\forall x \neq \mathbf{0}$.

Any linear continuous group in \mathbb{R}^n admits the representation³³

$$\mathbf{d}(s) = e^{sG_d} = \sum_{j=1}^{\infty} \frac{s^j G_d^j}{j!}, \quad s \in \mathbb{R}, \quad (13)$$

where $G_d \in \mathbb{R}^{n \times n}$ is a generator of \mathbf{d} . A continuous linear group (13) is a dilation in \mathbb{R}^n if and only if G_d is an anti-Hurwitz matrix³⁶. In this paper we deal only with continuous linear dilations. The weighted dilation⁵¹ is the most popular dilation in control theory^{39, 17, 30, 25, 3}. It corresponds to the linear dilation with a diagonal generator (i.e., G_d is a diagonal anti-Hurwitz matrix). The standard (Euler) dilation is given by $\mathbf{d}(s) = e^s I_n$, $s \in \mathbb{R}$.

A dilation \mathbf{d} in \mathbb{R}^n is

- i) *monotone* if the function $s \mapsto \|\mathbf{d}(s)x\|$ is strictly increasing, $\forall x \neq \mathbf{0}$;
- ii) *strictly monotone* if $\exists \beta > 0$ such that $\|\mathbf{d}(s)x\| \leq e^{\beta s} \|x\|$, $\forall s \leq 0, \forall x \in \mathbb{R}^n$.

The monotonicity of the dilation guarantees the uniqueness of the homogeneous projection on the unit sphere³⁶, i.e., for any $x \in \mathbb{R}_0^n$ there exists a unique pair $(s_0, x_0) \in \mathbb{R} \times \mathcal{S}$ such that $x_0 = \mathbf{d}(s_0)x$, where $\mathcal{S} = \{x \in \mathbb{R}^n : \|x\| = 1\}$. Below we use this simple property in order to design the so-called canonical homogeneous norm.

Since for the linear continuous dilation we have $\frac{d}{ds}\mathbf{d}(s) = \mathbf{G}\mathbf{d}(s)$, $s \in \mathbb{R}$, then the following result is the straightforward consequence of the existence of the quadratic Lyapunov function for an asymptotically stable LTI system.

Corollary 1. A linear continuous dilation in \mathbb{R}^n is strictly monotone with respect to the weighted Euclidean norm $\|x\| = \sqrt{x^\top P x}$ with $0 < P \in \mathbb{R}^{n \times n}$ if and only if

$$P\mathbf{G}_d + \mathbf{G}_d^\top P > 0, \quad P > 0. \quad (14)$$

This corollary implies that any continuous linear dilation in \mathbb{R}^n is monotone with respect to a properly selected weighted Euclidean norm.

3.2.2 | Canonical homogeneous norm

Any linear continuous and monotone dilation in a normed vector space introduces also an alternative norm topology defined by the canonical homogeneous norm³⁶.

Definition 3 (*Canonical homogeneous norm*). Let a linear dilation \mathbf{d} in \mathbb{R}^n be continuous and monotone with respect to a norm $\|\cdot\|$. A function $\|\cdot\|_d : \mathbb{R}^n \mapsto [0, +\infty)$ defined as follows: $\|\mathbf{0}\|_d = 0$ and

$$\|x\|_d = e^{s_x}, \quad \text{where } s_x \in \mathbb{R} : \|\mathbf{d}(-s_x)x\| = 1, \quad x \neq \mathbf{0} \quad (15)$$

is said to be a canonical \mathbf{d} -homogeneous norm in \mathbb{R}^n

The function $\|\cdot\|_d$ introduced by the above definition is a norm in a vector space homeomorphic to \mathbb{R}^n (see,³⁶, Theorem 7.1). By construction, $\|x\|_d = 1 \Leftrightarrow \|x\| = 1$. Due to the monotonicity of the dilation, it holds $\|x\|_d < 1 \Leftrightarrow \|x\| < 1$ and $\|x\|_d > 1 \Leftrightarrow \|x\| > 1$.

For standard dilation $\mathbf{d}_1(s) = e^s I_n$ we, obviously, have $\|x\|_{d_1} = \|x\|$. In other cases, $\|x\|_d$ with $x \neq \mathbf{0}$ is implicitly defined by a nonlinear algebraic equation $\|\mathbf{d}(-\ln \|x\|_d)x\| = 1$, which always have a unique solution due to monotonicity of the dilation. In some particular cases³⁸, this implicit equation has explicit solution even for non-standard dilations. The basic properties (such as local Lipschitz continuity and differentiability) of the canonical homogeneous norm are characterized by the following lemma

Lemma 1.³⁶ If a linear continuous dilation \mathbf{d} in \mathbb{R}^n is monotone with respect to a norm $\|\cdot\|$ then

- 1) $\|\cdot\|_d : \mathbb{R}^n \mapsto \mathbb{R}_+$ is single-valued and continuous on \mathbb{R}^n ;
- 2) there exist $\sigma_1, \sigma_2 \in \mathcal{K}_\infty$ such that

$$\sigma_1(\|x\|_d) \leq \|x\| \leq \sigma_2(\|x\|_d), \quad \forall x \in \mathbb{R}^n; \quad (16)$$

- 3) $\|\cdot\|$ is locally Lipschitz continuous on $\mathbb{R}^n \setminus \{\mathbf{0}\}$ provided that the linear dilation \mathbf{d} is strictly monotone
- 4) $\|\cdot\|_d$ is continuously differentiable on $\mathbb{R}^n \setminus \{\mathbf{0}\}$ provided that $\|\cdot\|$ is continuously differentiable on $\mathbb{R}^n \setminus \{\mathbf{0}\}$ and \mathbf{d} is strictly monotone.

Below we use the canonical homogeneous norm as Lyapunov function for fixed-time stability analysis. In this case, it is important to know how to compute the derivative of $\|\cdot\|_d$. For the \mathbf{d} -homogeneous norm $\|x\|_d$ induced by the weighted Euclidean norm $\|x\| = \sqrt{x^\top P x}$ we have³⁶

$$\frac{\partial \|x\|_d}{\partial x} = \|x\|_d \frac{x^\top \mathbf{d}^\top (-\ln \|x\|_d) P \mathbf{d} (-\ln \|x\|_d)}{x^\top \mathbf{d}^\top (-\ln \|x\|_d) P \mathbf{G}_d \mathbf{d} (-\ln \|x\|_d) x}. \quad (17)$$

3.2.3 | Homogeneous vector field

Below we study various systems being symmetric with respect to linear dilations. The dilation symmetry introduced by the following definition is known as a generalized homogeneity^{51, 20, 39, 6, 36}.

Definition 4. ²⁰ A vector field $g : \mathbb{R}^n \mapsto \mathbb{R}^n$ is \mathbf{d} -homogeneous of degree $\mu \in \mathbb{R}$ if

$$g(\mathbf{d}(s)x) = e^{\mu s} \mathbf{d}(s)g(x), \quad \forall s \in \mathbb{R}, \quad \forall x \in \mathbb{R}^n. \quad (18)$$

The homogeneity of a mapping is inherited by other mathematical objects induced by this mapping. In particular, solutions of \mathbf{d} -homogeneous system²

$$\dot{x} = g(x), \quad t > 0, \quad x(0) = x_0 \in \mathbb{R}^n \quad (19)$$

are symmetric with respect to the dilation \mathbf{d} in the following sense^{51, 20, 6}

$$x(t, \mathbf{d}(s)x_0) = \mathbf{d}(s)x(e^{\mu s}t, x_0), \quad (20)$$

where $x(\cdot, z)$ denotes a solution of (19) with $x(0) = z \in \mathbb{R}^n$ and $\mu \in \mathbb{R}$ is the homogeneity degree of g . The mentioned symmetry of solutions implies many useful properties of homogeneous system such as equivalence of local and global results. For example, local asymptotic (Lyapunov or finite-time stability) is equivalent to global asymptotic (resp., Lyapunov or finite-time) stability.

3.2.4 | Homogeneous FDE

It is well known^{50, 29, 6} that an asymptotically stable system (19) is finite-time stable provided that g is \mathbf{d} -homogeneous of negative degree. The following theorem shows that a homogeneous FDE (5) may be fixed-time stable with a constant settling-time function.

Theorem 1. Let $f \in C(\mathbb{R}_0^n \times \mathbb{R}^n, \mathbb{R}^n)$ be locally bounded on $\mathbb{R}^n \times \mathbb{R}^n$ and the system

$$\dot{x} = f(x, r), \quad t > 0, \quad x(0) = x_0, \quad r \neq \mathbf{0} \quad (21)$$

be asymptotically stable for any $r \in \mathbb{R}_0^n$. Let $\mathbf{d}_1, \mathbf{d}_2$ be linear dilations in \mathbb{R}^n such that

- for any $r \in \mathbb{R}^n$ the vector field

$$x \mapsto f(x, r) \quad (22)$$

is \mathbf{d}_1 -homogeneous of negative degree $\mu < 0$;

- the vector field

$$\begin{pmatrix} x \\ r \end{pmatrix} \mapsto \begin{pmatrix} f(x, r) \\ \mathbf{0} \end{pmatrix}, \quad x, r \in \mathbb{R}^n \quad (23)$$

is $\tilde{\mathbf{d}}$ -homogeneous of degree 0, where

$$\tilde{\mathbf{d}}(s) = \begin{pmatrix} \mathbf{d}_2(s) & \mathbf{0} \\ \mathbf{0} & \mathbf{d}_2(s) \end{pmatrix}, \quad s \in \mathbb{R}. \quad (24)$$

If the system (5) is Lyapunov stable then it is

- finite-time stable with a discontinuous (at least at $\mathbf{0}$) settling-time function $\tilde{T} : \mathbb{R}^n \mapsto [0, +\infty)$;
- fixed-time stable provided that \tilde{T} is bounded on some compact set $S \subset \mathbb{R}_0^n$ such that $\bigcup_{s \in \mathbb{R}} \mathbf{d}_2(s)S = \mathbb{R}_0^n$, moreover, the settling time is a constant for all $x_0 \neq \mathbf{0}$ if and only if \tilde{T} is constant on S .

Proof. On the one hand, since for any fixed $r \neq \mathbf{0}$ the system (21) is asymptotically stable then \mathbf{d}_1 -homogeneity with negative degree $\mu < 0$ implies its finite-time stability (see^{6, 30}), i.e., there exists a settling-time function $T_r : \mathbb{R}^n \mapsto \mathbb{R}_+$. This means that $T_{x_0}(x_0) < +\infty$ for any $x_0 \neq \mathbf{0}$. Hence, the Lyapunov stability of the system (5) implies its finite-time stability with the settling-time function $\tilde{T} : \mathbb{R}^n \mapsto [0, +\infty)$ defined as follows

$$\tilde{T}(x_0) = T_{x_0}(x_0) \quad \text{for } x_0 \neq \mathbf{0} \quad (25)$$

and $\tilde{T}(\mathbf{0}) = \mathbf{0}$.

On the other hand, $\tilde{\mathbf{d}}$ -homogeneity of the system (21) implies the following dilation symmetry of solutions

$$x_{\mathbf{d}_2(s)r}(t, \mathbf{d}_2(s)x_0) = \mathbf{d}_2(s)x_r(t, x_0), \quad t \geq 0 \quad (26)$$

²A system $\dot{x} = g(x)$ is \mathbf{d} -homogeneous if it is governed by a \mathbf{d} -homogeneous vector field g .

where $x_{\tilde{r}}(t, z)$ denotes the solution of the system (21) with the initial condition $x(0) = z$ and the vector of parameters (in the right-hand side) $\tilde{r} \in \mathbb{R}^n$.

The discovered symmetry implies that the simultaneous scaling of x_0 and r by $\mathbf{d}_2(s)$ in (21) does not change the settling time of the corresponding solution:

$$\tilde{T}(\mathbf{d}_2(s)x_0) = T_{\mathbf{d}_2(s)x_0}(\mathbf{d}_2(s)x_0) = T_{x_0}(x_0) = \tilde{T}(x_0) \quad (27)$$

Therefore, the settling time function of the system (5) has a constant value along any homogeneous curve $\Gamma_{\mathbf{d}_2}(x_0) = \{x \in \mathbb{R}^n : x = \mathbf{d}_2(s)x_0, s \in \mathbb{R}\}$. In this case, \tilde{T} is always discontinuous at zero since $\tilde{T}(\mathbf{0}) = 0$ and $\tilde{T}(x_0) \neq 0$ for $x_0 \neq \mathbf{0}$. However, taking into account $\bigcup_{s \in \mathbb{R}} \mathbf{d}_2(s)S = \mathbb{R}_0^n$, a boundedness of \tilde{T} on the compact S implies the uniform boundedness of \tilde{T} on \mathbb{R}^n , i.e., the system is fixed-time stable. Moreover, if $\tilde{T}(x_0) = \text{const}$ for all $x_0 \in S$ then, due to homogeneity, $\tilde{T}(x_0) = \text{const}$ for all $x_0 \in \mathbb{R}_0^n$. ■

The proven theorem discovers a novel property of a homogeneous system. Namely, a homogeneous system with a right-hand side dependent of the initial condition may be globally fixed-time stable and the settling-time function may be a constant (outside of the origin). In the case the classical ODE, a fixed-time stability is possible for locally homogeneous system^{3, 36}.

Below we design a feedback law u for the system (1) with $\tau = 0$ such that the closed-loop system satisfies the above theorem for $S = \{x \in \mathbb{R}^n : \|x\| = 1\}$ and $\mathbf{d}_2(s) = e^s I_n, s \in \mathbb{R}$.

4 | PRESCRIBED-TIME STABILIZATION BY STATIC HOMOGENEOUS FEEDBACK

4.1 | Initial State Dependent Homogeneous Feedback

Inspired by³⁸, let us consider the system (1) and define the homogeneous feedback as follows

$$u = K_0 x + K \mathbf{d}(-\ln T) \mathbf{d} \left(-\ln \left\| \frac{x}{\|x_0\|_{\mathbf{d}}} \right\|_{\mathbf{d}} \right) x \quad \text{for } x_0 \neq \mathbf{0}, \quad (28)$$

where $K_0, K \in \mathbb{R}^{m \times n}$ are feedback gains to be defined, \mathbf{d} is a dilation in \mathbb{R}^n , $T > 0$ is a prescribed settling time of the system, $\|\cdot\|_{\mathbf{d}}$ is a canonical homogeneous norm induced by a norm $\|\cdot\|$ in \mathbb{R}^n to be defined below. For $x_0 = \mathbf{0}$ we assign $u = K_0 x$.

The key difference between the feedback (28) and the homogeneous controller studied in⁴⁹ is the dependence of the feedback gain of the initial state x_0 and the prescribed settling time $T > 0$. As in⁴⁹, the linear term $K_0 x$ is selected such that the linear vector field $x \mapsto (A + BK_0)x$ is \mathbf{d} -homogeneous. Moreover, \mathbf{d} is constructed such that, for any fixed x_0 , the right-hand side of the closed-loop system (1), (28) is \mathbf{d} -homogeneous of degree $\mu = -1$. The closed-loop system is standard homogeneous of degree 0 with respect to simultaneous scaling of x and x_0 by $\mathbf{d}_2(s) = e^s I_n, s \in \mathbb{R}$. Therefore, in the view of Theorem 1, the asymptotic stability of the closed-loop system implies its fixed-time stability. Below we prove, by the direct Lyapunov method, that the settling time of the system is equal to the constant $T > 0$ for any initial state $x(0) = x_0 \neq \mathbf{0}$. Since the feedback (28) is discontinuous at $x = \mathbf{0}$ (for $x_0 \neq \mathbf{0}$) then solutions of the closed-loop system (1), (28) are defined in the sense of Filippov¹².

The dependence of the control gain on the initial state of the system simplifies the fixed-time control design. This can be easily illustrated in the scalar case. Indeed, let us consider the controllable system (1) with $\tau = 0$ and $n = m = 1$. In this case, $A \in \mathbb{R}, B \in \mathbb{R}_0$, the linear dilation \mathbf{d} in \mathbb{R} is given by the standard dilation $\mathbf{d}(s) = e^s$ with $s \in \mathbb{R}$, so $\|x\|_{\mathbf{d}} = \|x\| = |x|$ with $x \in \mathbb{R}$. According to⁴⁹, to homogenize the system, the gain $K_0 \in \mathbb{R}$ has to be selected as $K_0 = -\frac{A}{B}$. Therefore, for $K = 1$, the closed-loop system (1), (28) in the considered case has the form

$$\dot{x} = -\frac{|x_0|}{T} \frac{x}{|x|}, \quad t > 0, \quad x(0) = x_0.$$

Hence, we derive $|x(t)| = |x_0| - \frac{|x_0|}{T} t$ as long as $x(t) \neq 0$ and $x(t) \rightarrow 0$ as $t \rightarrow T$ independently of $x_0 \neq \mathbf{0}$. Therefore, the controller (28) solves the problem of the fixed-time stabilization with the constant settling time $T > 0$ at least in the scalar case. The stability analysis in the vector case is essentially based on the homogeneity theory.

Lemma 2 (Well-posedness of delay-free control system). The feedback law (28) is locally bounded and for any $x \neq \mathbf{0}$ it holds

$$u \rightarrow K_0 x \quad \text{as } x_0 \rightarrow \mathbf{0}. \quad (29)$$

For any $x_0 \neq \mathbf{0}$, the closed-loop system (1), (28) has a global-in-time Filippov solution $x : \mathbb{R}_+ \mapsto \mathbb{R}^n$ being a unique classical solution as long as $x(t) \neq \mathbf{0}$. For $x_0 = \mathbf{0}$, the closed-loop system has the unique zero solution.

Proof. According to Filippov's method¹², to define a solution of the closed-loop system, the discontinuous feedback is regularized at $x = \mathbf{0}$ as follows

$$u \in \|x_0\| \mathbf{Kd}(-\ln T)\mathcal{B} \quad \text{for } x = \mathbf{0}, \quad (30)$$

where $\mathcal{B} = \{x \in \mathbb{R}^n : \|x\| \leq 1\}$ is the unit ball. The right-hand side of the closed-loop system (1), (28) becomes a differential inclusion with an upper semi-continuous right-hand side, which is single-valued at $x \neq \mathbf{0}$ and set-valued (compact- and convex-valued) at $x = \mathbf{0}$. In this case, the system has a Filippov solution (defined at least locally in time) for any $x_0 \neq \mathbf{0}$ (see,¹² for more details). Since, by definition of the canonical homogeneous norm, we have $\|\mathbf{d}(-\ln \|z\|_{\mathbf{d}})z\| = 1$ then

$$\|u - K_0 x\|_{\mathbb{R}^m} \leq \|\mathbf{Kd}(-\ln T)\|_m \cdot \|x_0\| \quad (31)$$

where $\|\cdot\|_{\mathbb{R}^m}$ is a norm in \mathbb{R}^m , $\|\cdot\|$ is a norm utilized for the definition of $\|\cdot\|_{\mathbf{d}}$ and $\|\mathbf{Kd}(-\ln T)\|_m = \sup_{\|x\|=1} \|\mathbf{Kd}(-\ln T)x\|_{\mathbb{R}^m}$. Moreover, since the canonical homogeneous norm $\|\cdot\|_{\mathbf{d}}$ is locally Lipschitz continuous on \mathbb{R}_0^n then the right-hand side of the closed-loop system (1), (28) is locally Lipschitz continuous away from the origin ($x \neq \mathbf{0}$). So, for any $x_0 \neq \mathbf{0}$, the closed-loop system has a classical solution $x(t), t \geq 0$ defined uniquely as long as $0 < \|x(t)\| < +\infty$. Since the right-hand side of the closed-loop system satisfies the estimate

$$\|Ax + Bu\| \leq \|A + BK_0\| \cdot \|x\| + \|BK\mathbf{d}(-\ln T)\| \cdot \|x_0\| \quad (32)$$

with respect to $\|x\|$ then, in the view of Winter's theorem (see, e.g.,¹⁴), the solution $x(t)$ is defined globally in time (i.e., for all $t \geq 0$). For $x_0 = \mathbf{0}$, the (regularized) closed-loop system becomes linear $\dot{x} = (A + BK_0)x, t > 0, x(0) = \mathbf{0}$, so it has the unique zero solution. ■

Notice that, in the view of the proven lemma, Lyapunov stability of the zero solution should imply the uniqueness of all solutions of the closed-loop system.

Theorem 2 (*Homogeneous stabilization with constant settling time*). For any controllable pair $\{A, B\}$ one holds

- 1) the linear algebraic equation

$$AG_0 - G_0A + BY_0 = A, \quad G_0B = \mathbf{0} \quad (33)$$

has a solution $Y_0 \in \mathbb{R}^{m \times n}, G_0 \in \mathbb{R}^{n \times n}$ such that

- the matrix

$$G_{\mathbf{d}} = I_n + \mu G_0 \quad (34)$$

is anti-Hurwitz for $\mu \leq 1/\tilde{n}$, where $\tilde{n} \in \mathbb{N}$ is a minimal number such that $\text{rank}[B, AB, \dots, A^{\tilde{n}-1}B] = n$;

- the matrix $G_0 - I_n$ is invertible and the matrix

$$A_0 = A + BY_0(G_0 - I_n)^{-1} \quad (35)$$

satisfies the identity

$$A_0G_{\mathbf{d}} = (G_{\mathbf{d}} + \mu I_n)A_0 \quad \text{and} \quad G_{\mathbf{d}}B = B; \quad (36)$$

- 2) the linear algebraic system

$$A_0X + XA_0^{\top} + BY + Y^{\top}B^{\top} + G_{\mathbf{d}}X + XG_{\mathbf{d}}^{\top} = \mathbf{0}, \quad G_{\mathbf{d}}X + XG_{\mathbf{d}}^{\top} > \mathbf{0}, \quad X = X^{\top} > \mathbf{0} \quad (37)$$

always has a solution $X \in \mathbb{R}^{n \times n}, Y \in \mathbb{R}^{m \times n}$;

- 3) the closed-loop system (1), (28) with $\tau = 0$, with

$$K_0 = Y_0(G_0 - I_n)^{-1}, \quad K = YX^{-1}, \quad (38)$$

with the dilation $\mathbf{d}(s) = e^{sG_{\mathbf{d}}}$ and with the canonical homogeneous norm $\|\cdot\|_{\mathbf{d}}$ induced by the formula (15) using the weighted Euclidean norm

$$\|x\| = \sqrt{x^{\top} \mathbf{d}^{\top}(-\ln T) X^{-1} \mathbf{d}(-\ln T) x}, \quad (39)$$

is fixed-time stable with the constant settling time $T > 0$;

- 4) all solutions of the closed-loop system (1), (28) are unique.

Proof. The claims 1) and 2) are proven in⁴⁹. For any constant $r > 0$, using the formula (17) we derive

$$\frac{d}{dt} \|x/r\|_{\mathbf{d}} = \|x/r\|_{\mathbf{d}} \frac{(x/r)^{\top} \mathbf{d}^{\top} (-\ln \|x/r\|_{\mathbf{d}}) \mathbf{d}^{\top} (-\ln T) X^{-1} \mathbf{d} (-\ln T) \mathbf{d} (-\ln \|x/r\|_{\mathbf{d}}) (\dot{x}/r)}{(x/r)^{\top} \mathbf{d}^{\top} (-\ln \|x/r\|_{\mathbf{d}}) \mathbf{d}^{\top} (-\ln T) X^{-1} G_{\mathbf{d}} \mathbf{d} (-\ln T) \mathbf{d} (-\ln \|x/r\|_{\mathbf{d}}) (x/r)}. \quad (40)$$

The identity (36) implies that

$$A_0 \mathbf{d}(s) = e^s \mathbf{d}(s) A_0 \quad \text{and} \quad \mathbf{d}(s) B = e^s B, \quad \forall s \in \mathbb{R} \quad (41)$$

Hence, for the closed-loop system (1), (28) with $x_0 \neq \mathbf{0}$ we have

$$\frac{d}{dt} \left\| \frac{x}{\|x_0\|_{\mathbf{d}}} \right\|_{\mathbf{d}} = e^{-\ln T} \frac{x^{\top} \mathbf{d}^{\top} (-\ln \|x/\|x_0\|_{\mathbf{d}}\|_{\mathbf{d}}) \mathbf{d}^{\top} (-\ln T) X^{-1} (A_0 + BK) \mathbf{d} (-\ln T) \mathbf{d} (-\ln \|x/\|x_0\|_{\mathbf{d}}\|_{\mathbf{d}}) x}{x^{\top} \mathbf{d}^{\top} (-\ln \|x/\|x_0\|_{\mathbf{d}}\|_{\mathbf{d}}) \mathbf{d}^{\top} (-\ln T) X^{-1} G_{\mathbf{d}} \mathbf{d} (-\ln T) \mathbf{d} (-\ln \|x/\|x_0\|_{\mathbf{d}}\|_{\mathbf{d}}) x}. \quad (42)$$

Using (37) we derive

$$\frac{d}{dt} \left\| \frac{x}{\|x_0\|_{\mathbf{d}}} \right\|_{\mathbf{d}} = -\frac{1}{T} \quad (43)$$

as long as $x(t) \neq \mathbf{0}$. For $t = 0$ and $x(0) = x_0 \neq \mathbf{0}$ we have $\|x(0)/\|x_0\|_{\mathbf{d}}\|_{\mathbf{d}} = 1$. Since the derivative of the function $t \mapsto \|x(t)/\|x_0\|_{\mathbf{d}}\|_{\mathbf{d}}$ is negative (for $x(t) \neq \mathbf{0}$) then $\|x(t)/\|x_0\|_{\mathbf{d}}\|_{\mathbf{d}} \leq 1$ for all $t \geq 0$. This is equivalent to $\|x(t)\| \leq \|x_0\|$, so the closed-loop system is Lyapunov stable. Moreover, the system is fixed-time stable such that $x(t) = \mathbf{0}$ for $t \geq T$ and $x(t) \neq \mathbf{0}$ for all $t \in [0, T)$ if $x_0 \neq \mathbf{0}$. Finally, by Lemma 2 any solution of the system unique as long as $x(t) \neq \mathbf{0}$, but the proven Lyapunov stability guarantee the uniqueness of the solution after the reaching of the origin. The proof is complete. ■

By the proven theorem, for any controllable pair $\{A, B\}$ the algebraic equations (34), (37) are feasible, and any controllable linear plant can be stabilized to zero exactly in a prescribed time by means of the static homogeneous feedback (28).

4.2 | Robust stabilization of the delay-free LTI system

If x_0 in (28) is replaced with a non-zero constant vector, then the corresponding closed-loop system (1) is homogeneous and ISS (in the conventional sense⁴³) with respect to measurement noise (in the view of results^{40,3}). So, the obtained static feedback has some ISS properties with respect to measurement noise, but the robustness (namely, the asymptotic gain γ in Definition 2) depends essentially on the initial state x_0 . Indeed, considering the term

$$\tilde{K}(x, x_0) = K_0 + K \mathbf{d} (-\ln T) \mathbf{d} \left(-\ln \left\| \frac{x}{\|x_0\|_{\mathbf{d}}} \right\|_{\mathbf{d}} \right) \quad (44)$$

as a state-dependent gain of the feedback $u = \tilde{K}(x, x_0)x$ we conclude that $\tilde{K} \rightarrow K_0$ as $x_0 \rightarrow \mathbf{0}$. Since the matrix $A + BK_0$ is nilpotent then the stability margin³ of the matrix $A + B\tilde{K}(x, x_0)$ tends to zero as $x_0 \rightarrow \mathbf{0}$. This badly impacts the robustness of the system. For example, the delay-free closed-loop system (1), (28) with matched additive exogenous perturbation and the zero initial condition has the form

$$\dot{x} = (A + BK_0)x + Bq(t), \quad t > 0, \quad x(0) = x_0 = \mathbf{0}, \quad q \in L^{\infty}(\mathbb{R}, \mathbb{R}^m). \quad (45)$$

Since the matrix $A + BK_0$ is nilpotent and $\{A, B\}$ is controllable then, for $m = 1$, this system is equivalent to the perturbed integrator chain $y^{(n)}(t) = q(t)$, which has unbounded solution for any arbitrary small positive perturbation q .

To eliminate this drawback, we modify the feedback law (28) as follows

$$u_{\text{ct}}(x, x_0) = K_0 x + K \mathbf{d} (-\ln T) \mathbf{d} \left(-\ln \min \left\{ 1, \left\| \frac{x}{\|x_0\|_{\mathbf{d}}} \right\|_{\mathbf{d}} \right\} \right) x \quad \text{for} \quad x_0 \neq \mathbf{0} \quad (46)$$

with

$$u_{\text{ct}}(x, \mathbf{0}) = u_{\text{in}}(x) := (K_0 + K \mathbf{d} (-\ln T))x \quad \text{for} \quad x_0 = \mathbf{0}. \quad (47)$$

Obviously, for $x \neq \mathbf{0}$, it holds

$$u_{\text{ct}}(x, x_0) \rightarrow u_{\text{in}}(x) \quad \text{as} \quad x_0 \rightarrow \mathbf{0}. \quad (48)$$

Since $u_{\text{ct}}(x, x_0) = u_{\text{in}}(x)$ for $\|x\| \geq \|x_0\|$, the above limit is uniform on any compact from \mathbb{R}_0^n and $u_{\text{ct}} \in C(\mathbb{R}_0^n \times \mathbb{R}^n, \mathbb{R}^m)$.

Such a modification of the controller allows us to improve the robustness of the feedback law (28) preserving all stability properties of the closed-loop system.

³The stability margin of a Hurwitz matrix $M \in \mathbb{R}^{n \times n}$ is the maximum of real parts of eigenvalues of M .

Theorem 3. Let the parameters of the control (28) be defined as in Theorem 2. Then the closed-loop system (1), (46) is fixed-time stable with the constant settling time $T > 0$ and it has a unique solutions for any $x_0 \in \mathbb{R}^n$. All solutions of the perturbed closed-loop system:

$$\dot{x} \stackrel{a.e.}{=} Ax + Bu_{ct}(x + q_1, x_0 + q_0) + q_2, \quad t > t_0, \quad x(t_0) = x_0, \quad q = (q_0, q_1, q_2) \in \mathbb{R}^n \times L^\infty(\mathbb{R}, \mathbb{R}^n) \times L^\infty(\mathbb{R}, \mathbb{R}^n), \quad (49)$$

admit the ISS-like estimate

$$\|x(t)\| \leq \|x_0 + q_0\| \left(\beta \left(\frac{\|x_0\|}{\|x_0 + q_0\|}, t - t_0 \right) + \gamma_1 \left(\frac{\|q_1\|_{L^\infty(t_0, t)}}{\|x_0 + q_0\|} \right) + \gamma_2 \left(\frac{\|q_2\|_{L^\infty(t_0, t)}}{\|x_0 + q_0\|} \right) \right) \quad \text{for } x_0 + q_0 \neq \mathbf{0} \quad (50)$$

where $\beta \in \mathcal{KL}$ and $\gamma_1, \gamma_2 \in \mathcal{K}$ are independent of x_0 and q , but dependent on $T > 0$. Moreover,

- there exist $C_i > 0$ dependent of $T > 0$ such that

$$\|x_0 + q_0\| \beta \left(\frac{\|x_0\|}{\|x_0 + q_0\|}, t - t_0 \right) \rightarrow \|x_0\| e^{-\alpha(t-t_0)} \quad \text{and} \quad \|x_0 + q_0\| \gamma_i \left(\frac{\|q_i\|_{L^\infty(t_0, t)}}{\|x_0 + q_0\|} \right) \rightarrow C_i \|q_i\|_{L^\infty(t_0, t)} \quad \text{as } \|x_0 + q_0\| \rightarrow 0, \quad (51)$$

where $i = 1, 2$ and $\alpha = \frac{1}{2T} \lambda_{\min}(X^{-1}G_d + G_d^T X^{-1})$;

- the system (49) is fixed-time stable with the settling time estimate $T_{\max} = \frac{\rho T}{\rho - 1}$ for $\rho > 1$ provided that $q_0 = \mathbf{0}, q_1 = \mathbf{0}, q_2 = B\gamma$ and

$$\|B\gamma(t)\| \leq \|x_0\| \frac{\lambda_{\min}(X^{-1/2}G_d X^{1/2} + X^{1/2}G_d^T X^{-1/2})}{2\rho T}, \quad \forall t \in \mathbb{R}. \quad (52)$$

Proof. The identity $\frac{d}{dt} \|x(t)/\|x_0\|_{\mathbf{d}} = -\frac{1}{T}$ proven in Theorem 2 holds also for the unperturbed closed-loop system (1), (28) since $\|x(t)/\|x_0\|_{\mathbf{d}} \leq 1$ for all $t \geq 0$. So, all conclusions of Theorem 2 remain valid.

Making the change of variables $z = x/\|\tilde{x}_0\|$ with $\tilde{x}_0 = x_0 + q_0 \neq \mathbf{0}$ we derive

$$\begin{cases} \dot{z} \stackrel{a.e.}{=} Az + B \left(K_0 (z + q_1^{x_0}) + K_T \mathbf{d} \left(-\ln \min \left\{ \|z + q_1^{x_0}\|_{\mathbf{d}}, 1 \right\} \right) (z + q_1^{x_0}) \right) + q_2^{x_0}, \\ z(0) = z_0, \end{cases} \quad (53)$$

where $q_1^{x_0} = \frac{q_1}{\|\tilde{x}_0\|}, q_2^{x_0} = \frac{q_2}{\|\tilde{x}_0\|}, K_T = K\mathbf{d}(-\ln T)$ and $z_0 = \frac{x_0}{\|\tilde{x}_0\|}$. The system (53) is homogeneous in the bi-limit³(with the zero degree in ∞ -limit and negative degree in 0-limit. The unperturbed system (53) as well as its homogeneous approximations are globally asymptotically stable. This implies ISS with respect to $q_1^{x_0}$ and $q_2^{x_0}$ in the view of the results³. Hence, we derive the estimate (50). By construction, the stabilizer $u_{ct}(x + q_1, x_0 + q_0)$ tends (uniformly on compacts from \mathbb{R}^n) to the linear feedback $u_{lin}(x + q_1) = (K_0 + K_T)(x + q_1)$ as $x_0 + q_0 \rightarrow \mathbf{0}$. Moreover, $u_{ct}(x + q_1, x_0 + q_0) = u_{lin}(x + q_1)$ for $\|x + q_1\| \geq \|x_0 + q_0\|$ and

$$\frac{d\|x\|}{dt} \stackrel{a.e.}{=} \frac{x^T \mathbf{d}^T (-\ln T) X^{-1} \mathbf{d} (-\ln T) (Ax + q_2 + B(K_0 + K_T)(x + q_1))}{\|x\|} \stackrel{a.e.}{\leq} -\alpha \|x\| + \|B(K_0 + K_T)q_1\| + \|q_2\| \quad \text{if } \|x + q_1\| \geq \|x_0 + q_0\|,$$

where the identity (37) and the homogeneous identities $\mathbf{d}(-\ln T)(A + BK_0) = e^{-\ln T}(A + BK_0)\mathbf{d}(-\ln T)$, $\mathbf{d}(-\ln T)B = e^{-\ln T}B$ have been utilized to obtain the above estimate, and the norm $\|\cdot\|$ is defined as in Theorem 2. Hence, the property (51) holds.

Finally, assuming $q_0 = \mathbf{0}, q_1 = \mathbf{0}$ and $q_2 = B\gamma$ we derive $q_0 = \mathbf{0}, q_1^{x_0} = \mathbf{0}$ and $q_2^{x_0} = B\frac{\gamma}{\|x_0\|}$. In this case, repeating the proof of Theorem 2 we derive

$$\frac{d}{dt} \|z\|_{\mathbf{d}} \stackrel{a.e.}{=} -\frac{1}{T} + \frac{z^T \mathbf{d}^T (-\ln \|z\|_{\mathbf{d}}) \mathbf{d}^T (-\ln T) X^{-1} \mathbf{d} (-\ln T) B \frac{\gamma}{x_0}}{z^T \mathbf{d}^T (-\ln \|z\|_{\mathbf{d}}) \mathbf{d}^T (-\ln T) X^{-1} G_d \mathbf{d} (-\ln T) \mathbf{d} (-\ln \|z\|_{\mathbf{d}}) z} \quad (54)$$

Since $\|\mathbf{d}(-\ln \|z\|_{\mathbf{d}})z\| = 1$ or, equivalently,

$$z^T \mathbf{d}^T (-\ln \|z\|_{\mathbf{d}}) \mathbf{d}^T (-\ln T) X^{-1} \mathbf{d} (-\ln T) \mathbf{d} (-\ln \|z\|_{\mathbf{d}}) z = 1 \quad (55)$$

then

$$\frac{d}{dt} \|z\|_{\mathbf{d}} \stackrel{a.e.}{\leq} -\frac{1}{T} + \frac{2\|B\gamma\|}{\|x_0\| \lambda_{\min}(X^{-1/2}G_d X^{1/2} + X^{1/2}G_d^T X^{-1/2})} \leq -\frac{1}{T} + \frac{1}{\rho T} = -\frac{\rho - 1}{\rho T}. \quad (56)$$

Taking into account $\|z(0)\|_{\mathbf{d}} = \left\| \frac{x(0)}{x_0} \right\|_{\mathbf{d}} = 1$ we derive $z(t) = \mathbf{0}$ for $t \geq T_{\max}$. ■

The ISS-like estimate (50) and the property (51) highlight the specific continuous dependence of solutions of the closed-loop system on exogenous perturbations and measurement noise: *the closer the noisy measurement of the initial state $x_0 + q_0$ to the origin, the closer is the ISS property of the nonlinear system (1), (46) to the ISS properties of the linear system (1), (47).*

For small initial conditions the robustness properties of the closed-system (1) with the non-linear feedback (46) are close to the system with the linear feedback u_{lin} . However, the time-varying controller⁴² and the fixed-time controller³⁶ are known to be efficient in rejection of the matched additive disturbances. The estimate (52) shows that the static prescribed-time controller (46) rejects the matched perturbation γ of a magnitude proportional $\|x_0\|$.

The further modification of the obtained feedback law

$$u_{\text{fix}}(x, x_0) = K_0 x + K \mathbf{d}(-\ln T) \mathbf{d} \left(-\ln \min \left\{ 1, \left\| \frac{x}{\max\{\|x_0\|, 1\}} \right\|_{\mathbf{d}} \right\} \right) x \quad \text{for } x_0 \neq \mathbf{0}, \quad (57)$$

allows us to enlarge a class of matched perturbations to be rejected, but, in the disturbance-free case, it guarantees just the fixed-time stabilization (see Definition 1) with the prescribed upper bound $T_{\text{max}} = T > 0$ of the settling time. It is worth stressing that the settling-time estimate is exact for $\|x_0\| \geq 1$.

Theorem 4. Let the parameters of the control (28) be defined as in Theorem 2. The closed-loop system (1), (57) is fixed-time stable such that the settling-time function admits the representation

$$\tilde{T}(x_0) = \begin{cases} T & \text{if } \|x_0\| > 1, \\ T\|x_0\|_{\mathbf{d}} & \text{if } \|x_0\| \leq 1. \end{cases} \quad (58)$$

Moreover, the perturbed closed-loop system (49), (57) is

- ISS with respect to the additive measurement noise $q_1 \in L^\infty(\mathbb{R}, \mathbb{R}^n)$ and additive exogenous perturbations $q_2 \in L^\infty(\mathbb{R}, \mathbb{R}^n)$ such that

$$\|x(t)\| \leq \max\{1, \|x_0 + q_0\|\} \left(\beta \left(\frac{\|x_0\|}{\max\{1, \|x_0 + q_0\|\}}, t - t_0 \right) + \gamma_1 \left(\frac{\|q_1\|_{L^\infty(t_0, t)}}{\max\{1, \|x_0 + q_0\|\}} \right) + \gamma_2 \left(\frac{\|q_2\|_{L^\infty(t_0, t)}}{\max\{1, \|x_0 + q_0\|\}} \right) \right) \quad (59)$$

for some $\beta \in \mathcal{KL}$ and $\gamma_1, \gamma_2 \in \mathcal{K}_\infty$;

- fixed-time stable with $T_{\text{max}} = \frac{\rho T}{\rho - 1}$ provided that $\rho > 1$, $q_1 = \mathbf{0}$, $q_2 = B\gamma$,

$$\|B\gamma(t)\| \leq \max\{1, \|x_0\|\} \frac{\lambda_{\min}(X^{-1/2} G_{\mathbf{d}} X^{1/2} + X^{1/2} G_{\mathbf{d}}^T X^{-1/2})}{2\rho T}, \quad \forall t \in \mathbb{R}. \quad (60)$$

The proof of this theorem repeats the proof of Theorems 2 and Theorem 3 using the change of variables $z = \frac{x}{\max\{1, \|x_0 + q_0\|\}}$ instead of $z = \frac{x}{\|x_0 + q_0\|}$, respectively.

4.3 | Predictor-based stabilization of input delay LTI plant

The so-called predictor-based approach^{27, 4, 22, 23, 19} allows the delay-free control design ideas to be extended to input delay systems. The fixed-time stabilizer for the linear generalized homogeneous plant (the integrator chain) with input-delay has been proposed in⁴⁶ based on the technique developed in^{23, 19}, which consist in the modeling of the input delay using a transport PDE (Partial Differential Equation). The control design based on PDE models has one technical limitation: the theory of partial differential equations is not supported with a well-established common methodology for analysis and design of control systems with state-dependent discontinuities such as to Filippov's method¹² for discontinuous ODEs and sliding mode control system⁴⁵. Some ideas for possible expansion of the sliding mode (discontinuous) control methodology to infinite dimensional system can be found in³¹. However, this technique is far to be universal, well-recognized and easy-to-use. Since our controller (46) has the discontinuity at the origin, then the PDE-based design of fixed-time input delay controller is expected to be complicated. However, for linear plants, the predictor-based control design can be easily done using the well-known Artstein's transformation⁴:

$$y(t) = e^{A\tau} x(t) + \int_{-\tau}^0 e^{-A\theta} B u(t + \theta) d\theta, \quad t \geq 0, \quad (61)$$

where $\tau > 0$ is the input delay. The variable y is the so-called predictor variable, since it estimates the future state $x(t+h) = y(t)$ of the system (1). Notice that if $u \in L^\infty((-\tau, +\infty), \mathbb{R}^m)$ then

$$\dot{y}(t) \stackrel{a.e.}{=} e^{A\tau} \dot{x}(t) + \frac{d}{dt} \left(e^{At} \int_{t-\tau}^t e^{-A\sigma} Bu(\sigma) d\sigma \right) \quad (62)$$

$$\stackrel{a.e.}{=} e^{A\tau} (Ax(t) + Bu(t-\tau)) + Ae^{At} \int_{t-\tau}^t e^{-A\sigma} Bu(\sigma) d\sigma + e^{At} \frac{d}{dt} \int_{t-\tau}^t e^{-A\sigma} Bu(\sigma) d\sigma \quad (63)$$

$$\stackrel{a.e.}{=} A \left(e^{A\tau} x(t) + e^{At} \int_{t-\tau}^t e^{-A\sigma} Bu(\sigma) d\sigma \right) + e^{A\tau} Bu(t-\tau) + e^{At} e^{-At} Bu(t) - e^{At} e^{-A(t-\tau)} Bu(t-\tau). \quad (64)$$

Therefore, the dynamics of the predictor variable is governed by the ODE

$$\dot{y}(t) \stackrel{a.e.}{=} Ay(t) + Bu(t), \quad t \geq 0, \quad y(0) = y_0, \quad (65)$$

where $y_0 = e^{A\tau} x_0 + \int_{-\tau}^0 e^{-A\sigma} \phi(\sigma) d\sigma$ and $\phi \in L^\infty((-\tau, 0), \mathbb{R}^m)$ defines the control signal u (see, (4)) on the time interval $[-\tau, 0]$.

The Artstein's transformation reduces the problem of a control design for the input delay system to the same problem in the delay free case. By Theorem 3, the fixed-time stabilizer for the delay-free system (65) can be designed in the form of the discontinuous feedback (46). Following the Filippov regularization technique (see Lemma 2) we define

$$u(t) \stackrel{a.e.}{\in} \bar{u}_{ct}(y(t), y_0) := \begin{cases} u_{ct}(y(t), y_0) & \text{if } y(t) \neq \mathbf{0}, \\ \|y_0\| K \mathbf{d}(-\ln T) \mathcal{B} & \text{if } y(t) = \mathbf{0}, \end{cases} \quad (66)$$

where u_{ct} is given by (46) and $\mathcal{B} = \{x \in \mathbb{R}^n : \|x\| \leq 1\}$ is the unit ball.

Lemma 3 (Well-posedness of the input delay control system). For any $x_0 \neq \mathbf{0}$ and any $\phi \in L^\infty((-\tau, 0), \mathbb{R}^m)$, there exists a tuple $(x, y, u) \in W_{loc}^{1,\infty}(\mathbb{R}_+, \mathbb{R}^n) \times W_{loc}^{1,\infty}(\mathbb{R}_+, \mathbb{R}^n) \times L_{loc}^\infty((\tau, +\infty), \mathbb{R}^m)$ satisfying (1), (4), (66), (61).

Proof. By Lemma 2 the system (65),(66) is well-posed and has a Filippov solution $y : \mathbb{R}_+ \mapsto \mathbb{R}^n$. The Filippov's lemma¹¹ about measurable selector guarantees that there exists a measurable function $u \in L_{loc}^\infty(\mathbb{R}_+, \mathbb{R}^m)$ such that

$$\begin{cases} \dot{y}(t) \stackrel{a.e.}{=} Ay(t) + Bu(t), \\ u(t) \in \bar{u}_{ct}(y(t), y_0), \end{cases} \quad t \geq 0. \quad (67)$$

Let us extend the signal u to the time interval $(-\tau, +\infty)$ using the initial condition (4). Applying the inverse Artstein's transformation $x(t) = e^{-A\tau} \left(y(t) - \int_{-\tau}^0 e^{-A\sigma} u(t+\sigma) d\sigma \right)$ we derive $x(0) = x_0$ and

$$\dot{x}(t) = e^{-A\tau} \left(\dot{y} - \frac{d}{dt} \left(e^{At} \int_{t-\tau}^t e^{-A\sigma} Bu(\sigma) d\sigma \right) \right) = e^{-A\tau} \left(Ay(t) + Bu(t) - \frac{d}{dt} \left(e^{At} \int_{t-\tau}^t e^{-A\sigma} Bu(\sigma) d\sigma \right) \right) = Ax(t) + Bu(t-\tau).$$

Therefore, the constructed tuple (x, y, u) satisfies (1), (4), (66), (61). ■

The following theorem proves that the closed-loop input-delay system is fixed-time stable with a constant settling time.

Theorem 5. If all parameters of the control (66) are defined as in Theorem 2 then any solution of the system (1) with the control signal generated by the formulas (4), (66), (61) satisfies

$$x(t) = \mathbf{0}, \quad \forall t \geq T + \tau \quad (68)$$

independently of $x_0 \in \mathbb{R}^n$. Moreover, if $\phi = \mathbf{0}$ then the system (1), (4), (66), (61) is fixed-time stable (in the sense of Definition 1) with the constant settling time $T + \tau$ and $x(t) \neq \mathbf{0}$ for all $t \in [0, T + \tau)$ provided that $x_0 \neq \mathbf{0}$.

Proof. Let $(x, y, u) \in W_{loc}^{1,\infty}(\mathbb{R}_+, \mathbb{R}^n) \times W_{loc}^{1,\infty}(\mathbb{R}_+, \mathbb{R}^n) \times L_{loc}^\infty([-\tau, +\infty), \mathbb{R}^m)$ satisfy (1),(4), (61), (66). Then, due to Artstein's transformation, y satisfies the following differential inclusion

$$\dot{y}(t) \in \stackrel{a.e.}{=} Ay(t) + B\bar{u}_{ct}(y(t), y_0), \quad t \geq 0, \quad y(0) = y_0. \quad (69)$$

Since, by Theorem 3, the system is fixed-time stable with a constant settling time $T > 0$ and has the unique solution:

$$y(t) = \mathbf{0} \quad \text{for all } t \geq T$$

and $y(t) \neq \mathbf{0}$ for all $t \in [0, T)$ and all $y_0 \in \mathbb{R}_0^n$. Since $y(t) = \dot{y}(t) = \mathbf{0}, \forall t > T$ then, in the view of the equation (65), we have $Bu(t) \stackrel{a.e.}{=} \mathbf{0}, t > T$. Therefore,

$$\begin{cases} \mathbf{0} \stackrel{a.e.}{=} Bu(t), \\ \mathbf{0} = e^{A\tau}x(t) + \int_{-\tau}^0 e^{-A\theta}Bu(t+\theta)d\theta, \end{cases} \quad \forall t \geq T \Rightarrow x(t) = \mathbf{0}, \quad \forall t \geq T + \tau. \quad (70)$$

The identity $x(t) = y(t - \tau)$ holds for all $t \geq \tau$, i.e., $x(t) = \mathbf{0}, \forall t \geq T + \tau$.

Moreover, if $\phi = \mathbf{0}$ then $y_0 = e^{A\tau}x_0$ and $x_0 \neq \mathbf{0} \Rightarrow y_0 \neq \mathbf{0}$, so $y(t) \neq \mathbf{0}$ for all $t \in [0, T)$ independently of $x_0 \in \mathbb{R}_0^n$. To prove the constant convergence time, we just need to show that $x_0 \neq \mathbf{0} \Rightarrow x(t) \neq \mathbf{0}$ for all $t \in [0, \tau]$. Suppose the contrary, i.e., for some $x_0 \neq \mathbf{0}$ there exists $t^* \in (0, \tau]$ such that

$$x(t^*) = \mathbf{0}. \quad (71)$$

On the one hand, since $x(t^*) = e^{-A\tau} \left(y(t^*) - \int_{-\tau}^0 e^{-A\theta}Bu(t^* + \theta)d\theta \right)$ then

$$y(t^*) = \int_0^{t^*} e^{A(t^*-\sigma)}Bu(\sigma)d\sigma. \quad (72)$$

On the other hand, since $\phi = \mathbf{0}$ and y is satisfies (69), then, by Cauchy formula, we have

$$y(t^*) = e^{At^*}y_0 + \int_0^{t^*} e^{A(t^*-\sigma)}Bu(\sigma)d\sigma. \quad (73)$$

Hence, we derive $e^{At^*}y_0 = \mathbf{0}$, but this is possible if and only if $y_0 = \mathbf{0}$ (or, equivalently, $x_0 = \mathbf{0}$). We derive the contradiction.

Finally, since the system (69) is Lyapunov stable then there exists $\bar{\alpha} \in \mathcal{K}$ such that $\|y(t)\| \leq \bar{\alpha}(\|y_0\|), \forall t \geq 0$. Taking into account

$$\|u_{ct}(y, y_0)\|_{\mathbb{R}^m} \leq (\|K_0\| + \|K\mathbf{d}(-\ln T)\|) \cdot (\|y_0\| + \|y\|). \quad (74)$$

and using (61), we derive

$$\|x(t)\| \leq \|e^{-A\tau}\| \bar{\alpha}(\|y_0\|) + \int_{-h}^0 \|e^{-As}\| ds \cdot (\|K_0\| + \|K\mathbf{d}(-\ln T)\|) \cdot (\|y_0\| + \bar{\alpha}(\|y_0\|)). \quad (75)$$

Since $y_0 = e^{A\tau}x_0$ then there exists $\bar{\alpha} \in \mathcal{K}$ such that $\|x(t)\| \leq \bar{\alpha}(\|x_0\|), \forall t \geq 0$. The proof is complete. ■

Notice that it is impossible to assign the constant settling time for all $x_0 \neq \mathbf{0}$ if $\phi \neq \mathbf{0}$. Indeed, since the pair $\{A, B\}$ is controllable then $\int_{-\tau}^0 e^{-A\theta}B\phi(\theta)d\theta \neq \mathbf{0}$ for $\phi \neq \mathbf{0}$. In this case, taking $x_0 = -e^{-A\tau} \int_{-\tau}^0 e^{-A\theta}B\phi(\theta)d\theta \neq \mathbf{0}$ we derive that the unique solution of the closed-loop system satisfies $x(\tau) = y_0 = \mathbf{0}$. Hence, $x(t) = \mathbf{0}$ for all $t \geq \tau$. However, if $x_0 = -2e^{-A\tau} \int_{-\tau}^0 e^{-A\theta}B\phi(\theta)d\theta \neq \mathbf{0}$ and $x(\tau) \neq \mathbf{0}$ for any fixed-time stabilizing controller. Therefore, at least, the two different non-zero initial vectors $x_0 \neq \mathbf{0}$ corresponds two different settling times of the system to zero.

To analyze a robustness with respect to perturbations (such as measurement noise, additive disturbances, computational errors for the predictor variable, etc) we consider the system

$$\begin{cases} \dot{x}(t) \stackrel{a.e.}{=} Ax(t) + Bu(t - \tau) + q_2(t), \\ y(t) = e^{A\tau}x(t) + \int_{-\tau}^0 e^{-A\theta}Bu(t+\theta)d\theta, \quad t \geq 0, \quad x(0) = x_0, \\ u(t) \stackrel{a.e.}{\in} \bar{u}_{ct}(y(t) + q_1(t), y(0) + q_0), \end{cases} \quad (76)$$

where \bar{u}_{ct} is given above, $q_0 \in \mathbb{R}$ is the estimation error of the predictor's initial state, $q_1 \in L^\infty(\mathbb{R}, \mathbb{R}^n)$ models measurement noise and computational errors of the predictor variable y , but $q_2 \in L^\infty(\mathbb{R}, \mathbb{R}^n)$ is additive perturbations of the plant.

Theorem 6. Under the conditions of Theorem 5 and $\phi = \mathbf{0}$, any solution of the system (76), (4) satisfies the ISS-like estimate

$$\|x(t)\| \leq \|e^{A\tau}x_0 + q_0\| \left(\beta \left(\frac{\|e^{A\tau}x_0\|}{\|e^{A\tau}x_0 + q_0\|}, t - t_0 \right) \right) + \gamma_1 \left(\frac{\|q_1\|_{L^\infty(t_0, t)}}{\|e^{A\tau}x_0 + q_0\|} \right) + \gamma_2 \left(\frac{\|q_2\|_{L^\infty(t_0, t)}}{\|e^{A\tau}x_0 + q_0\|} \right), \quad (77)$$

where $\beta \in \mathcal{KL}$ and $\gamma_1, \gamma_2 \in \mathcal{K}$ are independent of x_0 , but depend on $T > 0$ and $\tau > 0$. Moreover, $\beta, \gamma_1, \gamma_2$ satisfy the property (51).

Proof. For $\phi = \mathbf{0}$, the predictor dynamics is described by

$$\begin{cases} \dot{y}(t) = Ay(t) + Bu(t) + e^{A\tau}q_2(t), & t > 0, & y(0) = e^{A\tau}x_0, \\ u(t) \in \bar{u}_{\text{ct}}(y(t) + q_1(t), e^{A\tau}x_0 + q_0). \end{cases} \quad (78)$$

The ISS of this system is studied in Theorem 3, where it is shown that there exist $\beta^y \in \mathcal{KL}$, $\gamma_1^y, \gamma_2^y \in \mathcal{K}$ such that

$$\|y(t)\| \leq \|e^{A\tau}x_0 + q_0\| \left(\beta^y \left(\frac{\|e^{A\tau}x_0\|}{\|e^{A\tau}x_0 + q_0\|}, t - t_0 \right) + \gamma_1^y \left(\frac{\|q_1\|_{L^\infty_{(t_0,t)}}}{\|e^{A\tau}x_0 + q_0\|} \right) + \gamma_2^y \left(\frac{\|e^{A\tau}q_2\|_{L^\infty_{(t_0,t)}}}{\|e^{A\tau}x_0 + q_0\|} \right) \right). \quad (79)$$

and $\beta^y, \gamma_1^y, \gamma_2^y$ satisfy the property (51). On the other hand, by Cauchy formula, we have

$$y(t) = e^{A\tau}y(t - \tau) + \int_{-\tau}^0 e^{-A\theta} Bu(t + \theta) d\theta + \int_{-\tau}^0 e^{-A\theta} q_2(t + \theta) d\theta, \quad t \geq \tau, \quad (80)$$

so

$$\left\| \int_{-\tau}^0 e^{-A\theta} Bu(t + \theta) d\theta \right\| \leq \|y(t)\| + \|e^{A\tau}\| \left(\|y(t - \tau)\| + \|q_2\|_{L^\infty_{(0,t)}} \tau \sup_{\theta \in [0,\tau]} \|e^{A\theta}\| \right). \quad (81)$$

On the one hand, using the formula (61) we derive

$$\|x(t)\| \leq \|e^{-A\tau}\| \left(\|y(t)\| + \left\| \int_{-\tau}^0 e^{-A\theta} Bu(t + \theta) d\theta \right\| \right) \leq \quad (82)$$

$$\|e^{-A\tau}\| \left(2\|y(t)\| + \|e^{A\tau}\| \left(\|y(t - \tau)\| + \|q_2\|_{L^\infty_{(0,t)}} \tau \sup_{\theta \in [0,\tau]} \|e^{A\theta}\| \right) \right), \quad \forall t \geq \tau. \quad (83)$$

On the other hand, since $\|u(t)\| \leq (\|K_0\| + \|K_d(-\ln T)\|)(\|y(t)\| + \|e^{A\tau}x_0 + q_0\|)$, $\forall t \geq 0$ then, taking into account $\phi = \mathbf{0}$, by the Artstein's transformation, we derive

$$\|x(t)\| \leq \|e^{-A\tau}\| \left(\|y(t)\| + C_1 \|e^{A\tau}x_0 + q_0\| + C_2 \sup_{\sigma \in [0,t]} \|y(\sigma)\| \right), \quad \forall t \in [0, \tau]. \quad (84)$$

for some $C_1, C_2 > 0$. Therefore, using (79) we derive that the ISS-like estimate (77) holds for $t \geq 0$ under a properly defined functions $\beta \in \mathcal{KL}$ and $\gamma_1, \gamma_2 \in \mathcal{K}$. ■

The matched perturbation $q_2 = B\gamma$ becomes mismatched for the predictor system (78), so it cannot be completely rejected as it was done in the delay-free case. Therefore, in the input delay case, the robustness of the prescribed-time stabilizers with respect to additive perturbations is proven only in the ISS-like sense.

5 | NUMERICAL EXAMPLE

5.1 | Prescribed-time stabilization of the harmonic oscillator in the delay-free case

As an example, let us design the prescribed-time stabilizer for the harmonic oscillator in the delay-free ($\tau = 0$) case

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad (85)$$

The parameters of the prescribed-time stabilizer (46) are designed according to Theorem 2:

$$K_0 = (1 \ 0), \quad G_d = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}, \quad K = (-5.5 \ -3), \quad X = \begin{pmatrix} 1 & -2 \\ -2 & 5.5 \end{pmatrix}, \quad T = 1. \quad (86)$$

The simulation has been done in MATLAB using the zero-order-hold method and the consistent discretization of the homogeneous controller (46) realized in Homogeneous Control Systems Toolbox⁴ for MATLAB. The consistent discretization (see³⁷) allows the convergence rate (e.g., finite-time or fixed-time convergence) of the continuous-time control system to be preserved in the case of the sampled-time implementation of the controller. The sampling period for the simulation is $h = 0.01$. The simulation results show the prescribed-time convergence of the closed-loop system with $T = 1$. Indeed, independently of the selected

⁴<https://gitlab.inria.fr/polyakov/hcs-toolbox-for-matlab>

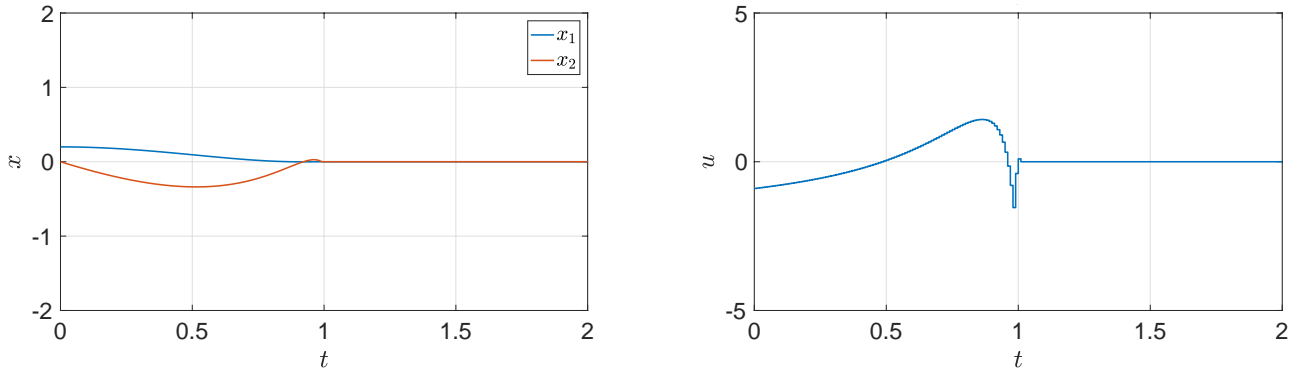


Figure 1 Stabilization at the prescribed-time $T = 1$ for $x_0 = (0.2 \ 0)^T$ and $\tau = 0$

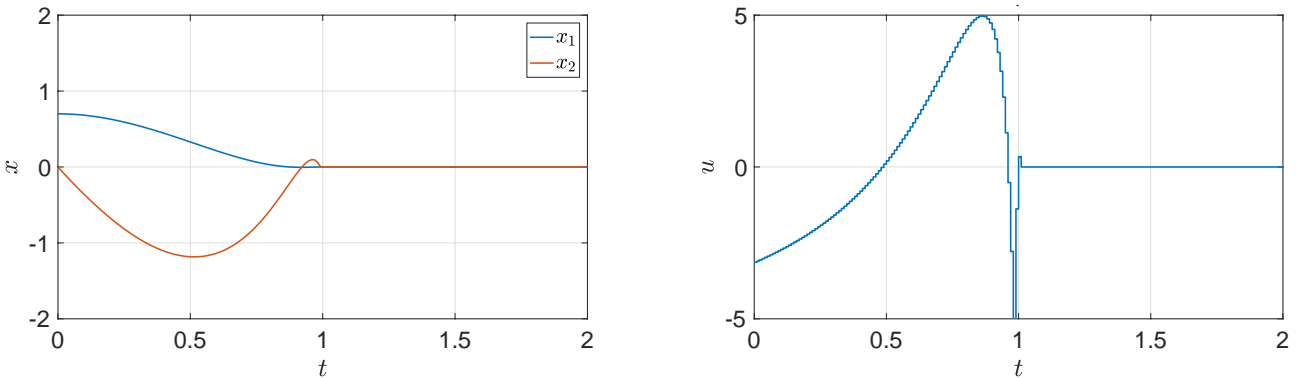


Figure 2 Stabilization at the prescribed-time $T = 1$ for $x_0 = (0.7 \ 0)^T$ and $\tau = 0$

initial condition (see Figures 1 and 2) the state of the closed-loop system converges to zero with the precision of the machine epsilon ($\approx 10^{-16}$) at the time instant $\tilde{T} = 1.01$, which perfectly corresponds to the prescribed settling time $T = 1$ (up to the sampling period). The simulations have been done for various initial conditions up to $\|x_0\| = 10^5$. The settling time remains equal to 1 (up to the sampling period h) in all simulation and various $h < T$. Notice that the settling-time error $\tilde{T} - T$ is proportional to the sampling period h for any real $h > 0$ (see³⁷). So, to have a reasonable precision of the settling time in practice, the sampling period of the controller must be essentially less than the required settling time.

To study robustness properties of the closed-loop system, the simulations have been done, first, for the system with matched additive perturbation $B \sin(5t)$. As claimed in Theorem 3, such a perturbation cannot be rejected by the prescribed-time controller (46) if the initial state is too small (see Figure 3). The larger initial condition, the larger matched perturbation can be rejected (see Figure 4). However, according to Theorem (3), the settling time is increased $T_{\max} = T \frac{\rho}{\rho-1}$, $\rho > 1$ in this case. The fixed-time stabilizer (57) rejects the considered matched perturbation for all initial conditions.

The ISS with respect to noisy measurements is quite opposite to the case of ISS with respect to additive perturbations in the sense that the smaller initial state x_0 , the less sensitive closed-loop system with respect to measurement noise (see Figures 5 and 6). The numerical simulations for this case have been done by adding a noise η of the magnitude 0.01 to the state measurements $\hat{x} = x + \eta$. The noise is simulated by MATLAB as a uniformly distributed (pseudo-)random variable $\eta \in [-0.01, 0.01]$. All state measurements (including the initial state x_0) are corrupted by the noise.

5.2 | Prescribed-time stabilization of the input delay system

Let the model of the controller harmonic studied above have the input delay $\tau = 0.5$. In this case, the prescribed-time feedback has to be calculated using the predictor variable y given by (61). To implement the method of consistent discretization³⁷ to the

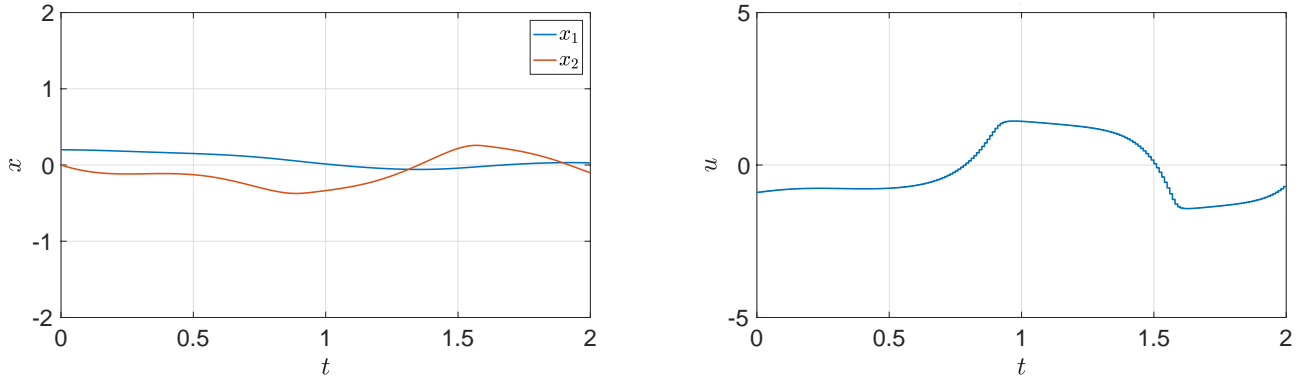


Figure 3 The case of matched additive disturbance $B \sin(5t)$ for $x_0 = (0.2 \ 0)^T$ and $\tau = 0$

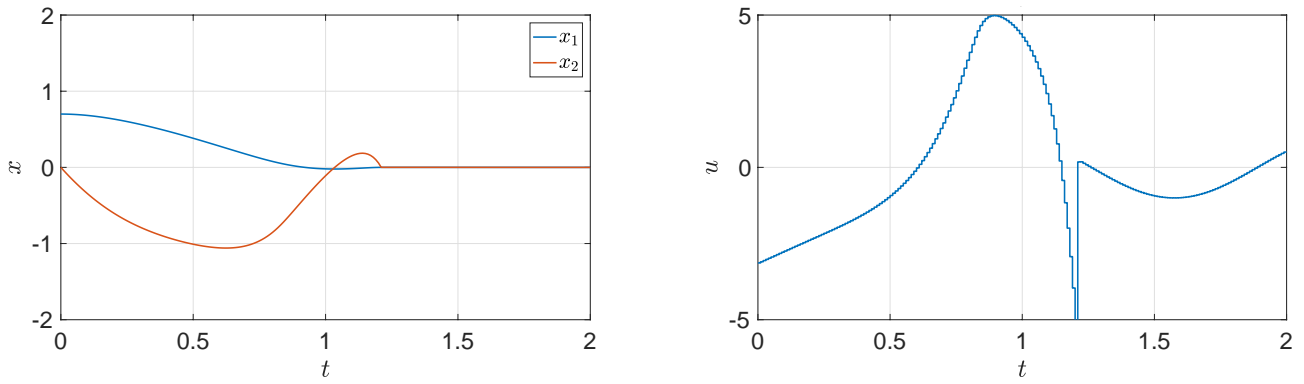


Figure 4 The case of matched additive disturbance $B \sin(5t)$ for $x_0 = (0.7 \ 0)^T$ and $\tau = 0$

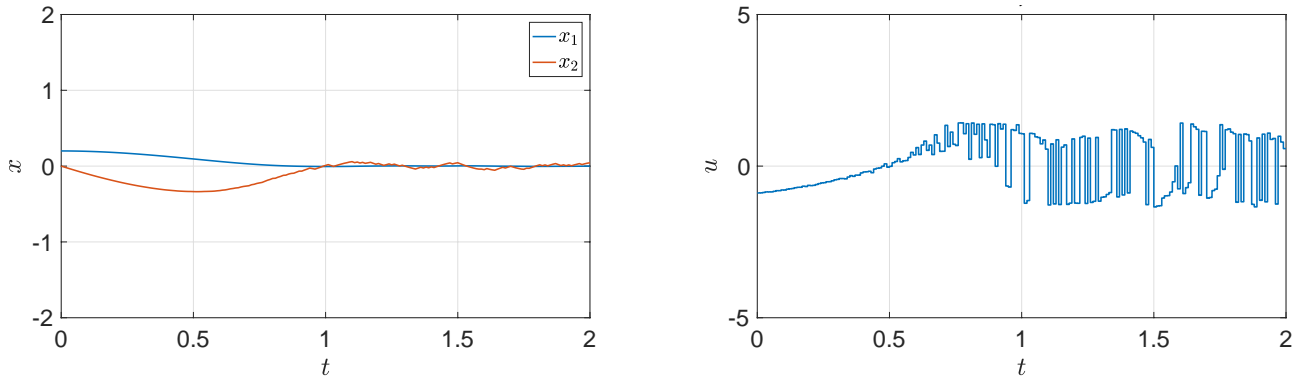


Figure 5 The case of noisy measurements for $x_0 = (0.2 \ 0)^T$ and $\tau = 0$

system (65), (66), the predictor variable has to be calculated exactly at the time instances

$$t_i = ih, \quad i = 0, 1, \dots, \quad (87)$$

where $h = 0.01$ is the sampling period. Since the control signal is a piece-wise constant function with the sampling period $h = 0.01$, the integral term in (61) admits the following exact representation

$$\int_{-\tau}^0 e^{-A\theta} u(t_i + \theta) d\theta = \sum_{j=1}^N \left(\int_{-jh}^{-(j-1)h} e^{-A\theta} d\theta \right) u(t_i - jh) = \sum_{j=1}^N A^{-1} (e^{jhA} - e^{(j-1)hA}) u(t_i - jh), \quad (88)$$

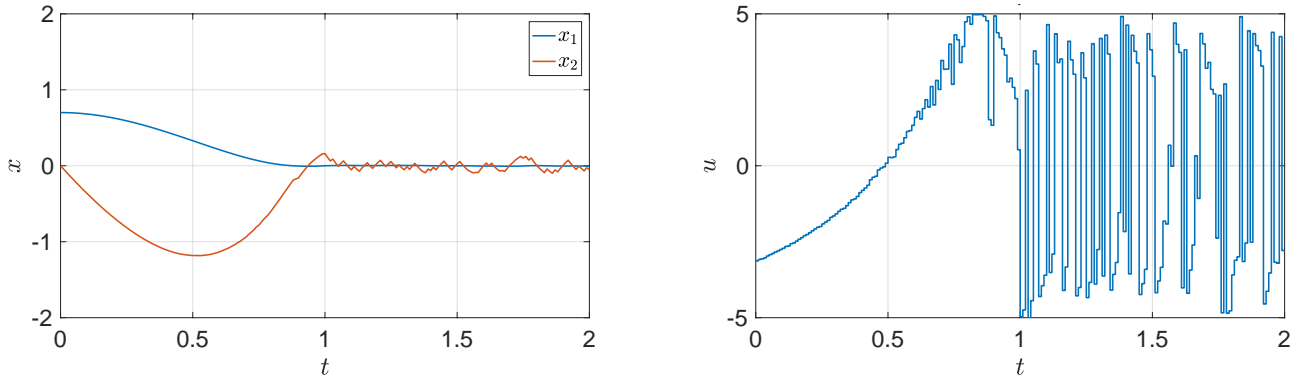


Figure 6 The case of noisy measurements for $x_0 = (0.7 \ 0)^T$ and $\tau = 0$

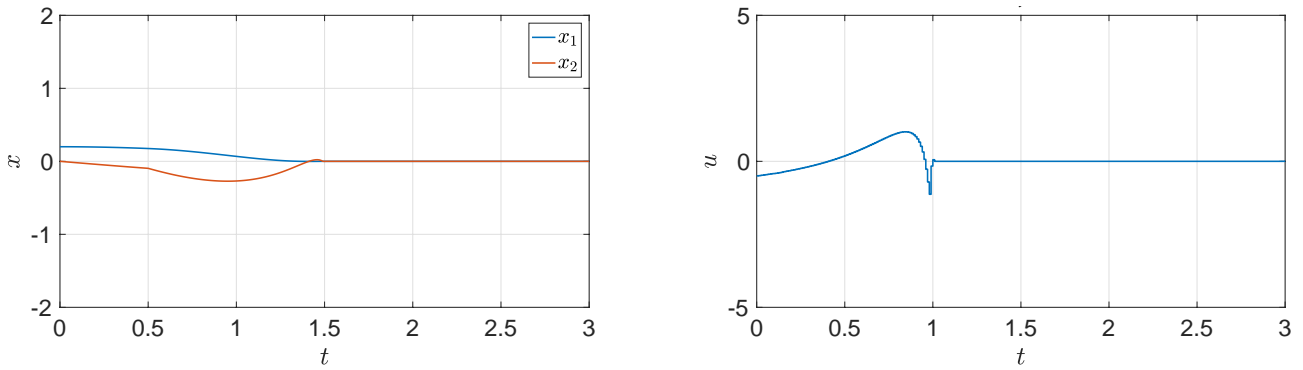


Figure 7 Stabilization at the prescribed-time $T = 1.5$ for $x_0 = (0.2 \ 0)^T$ and the input delay $\tau = 0.5$

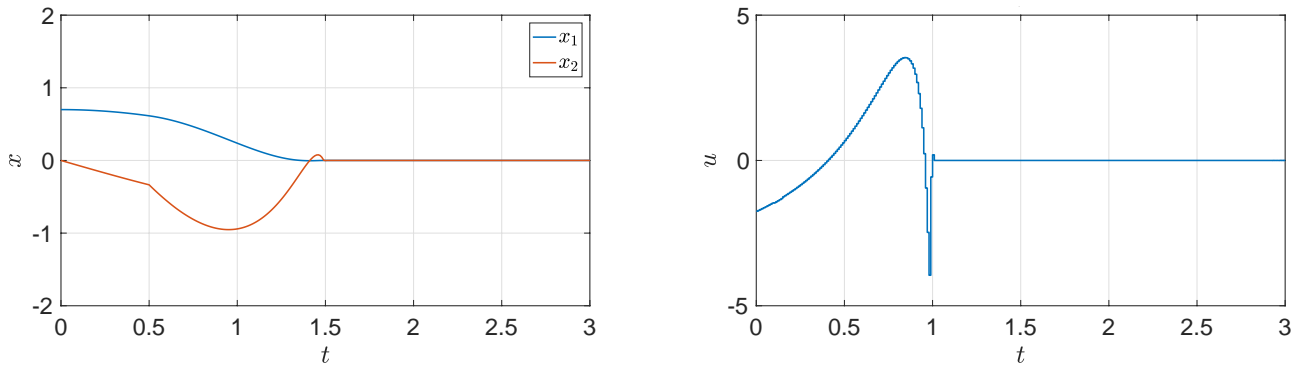


Figure 8 Stabilization at the prescribed-time $T = 1.5$ for $x_0 = (0.7 \ 0)^T$ and the input delay $\tau = 0.5$

where $N = \frac{\tau}{h} = 50$ and $A^{-1} = -A$ (for our model of the harmonic oscillator). Let the control for the predictor equation (65) be designed as for the delay-free system considered above. Due to the input delay the control signal generated at the time t impacts on the system at the time instant $t + \tau$. The control signal as well as the predictor variable converge to a steady state (e.g., to zero) at the prescribed-time $T = 1$, but the expected settling time of the system is $T + \tau = 1.5$. The numerical simulations show this prescribed converge time (see, Figures 7 and 8) for the closed-loop system.

Notice that the matched additive perturbation $B \sin(5t)$ of the original system becomes the mismatched additive perturbation $e^{A\tau} B \sin(5t)$ in the predictor equation (78). So, this perturbation cannot be rejected by the predictor-based stabilizer and just ISS with respect the additive perturbations can be guaranteed (see Figure 9 and 10). The conclusions about sensitivity with respect to measurement noise are the same as in the delay free case.

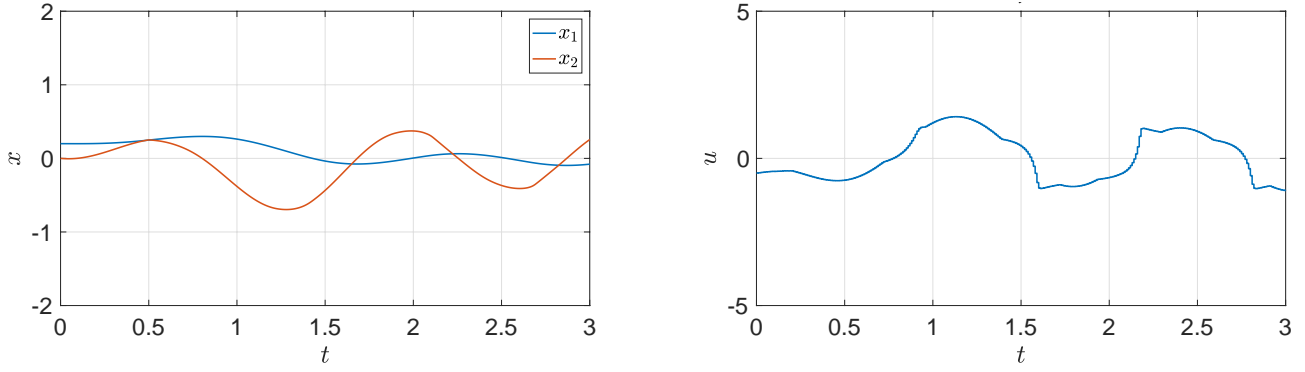


Figure 9 The case of matched additive disturbance $B \sin(5t)$ for $x_0 = (0.2 \ 0)^T$ and the input delay $\tau = 0.5$

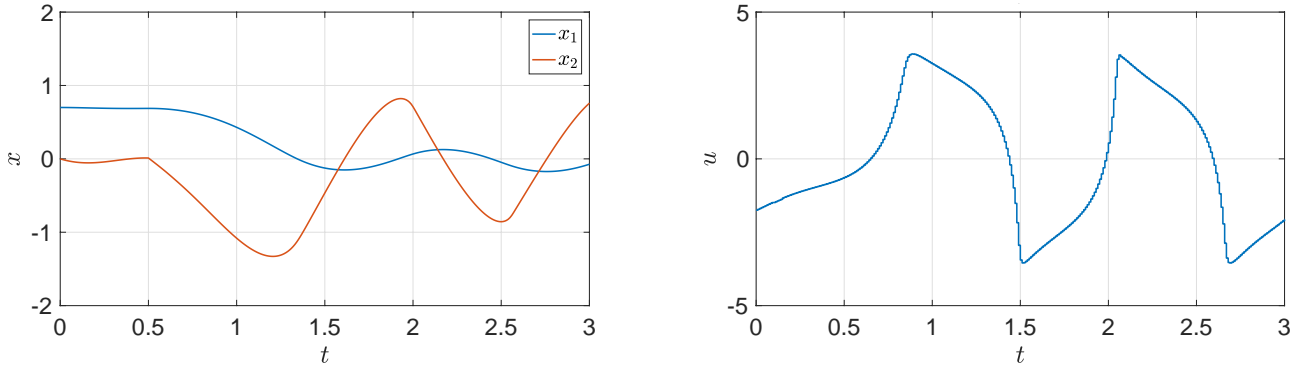


Figure 10 The case of matched additive disturbance $B \sin(5t)$ for $x_0 = (0.7 \ 0)^T$ and the input delay $\tau = 0.5$

5.3 | Comparison time-varying and static feedback laws in the delay-free case

Inspired by⁴², we design the time-varying feedback (prescribed-time) regulator in the form

$$u(t) = (K_0 + K \mathbf{d}(\ln \mu(t)))x(t), \quad \mu(t) = \frac{T}{T-t}, \quad t \in [0, T), \quad (89)$$

where the matrices K_0 , K and the linear continuous dilation \mathbf{d} are defined as before (see the formula (86)). Such a feedback steers the state of the system (1), (85) with $\tau = 0$ to zero in the prescribed time $T > 0$. In the disturbance-free case, both regulators provide a good enough precision (see Fig. 11). A small error of the time-varying algorithm in the delay-free case is caused by the sampled-time implementation of the feedback law. The sampling period is as before $h = 0.01$. In the case of the noisy measurements, the quality of the time-varying prescribed-time regulator degrades drastically (see Fig. 12). As before, the noise is simulated as a random variable uniformly distributed on $[-0.01, 0.01]$. The obtained simulation results strongly support our claim that the static prescribed-time regulator is less sensitive with respect to the measurement noise than the time-varying prescribed-time algorithm.

6 | CONCLUSIONS

In this paper, new fixed-time stabilizers are designed for LTI systems. The key feature of the new stabilizers is the dependence of the feedback gain on the initial condition. This allows the settling time of the closed-loop to have a prescribed constant settling time for all non-zero initial conditions. The obtained stabilizer does not have a time varying gain which tends to infinity as time tends to the settling time. This essentially improves the robustness properties of the closed-loop system with respect to measurement noise comparing to well-known time-varying prescribed-time stabilizers (like⁴²). The control laws are designed for both delay-free and input-delay cases. The theoretical results are illustrated by numerical simulations.

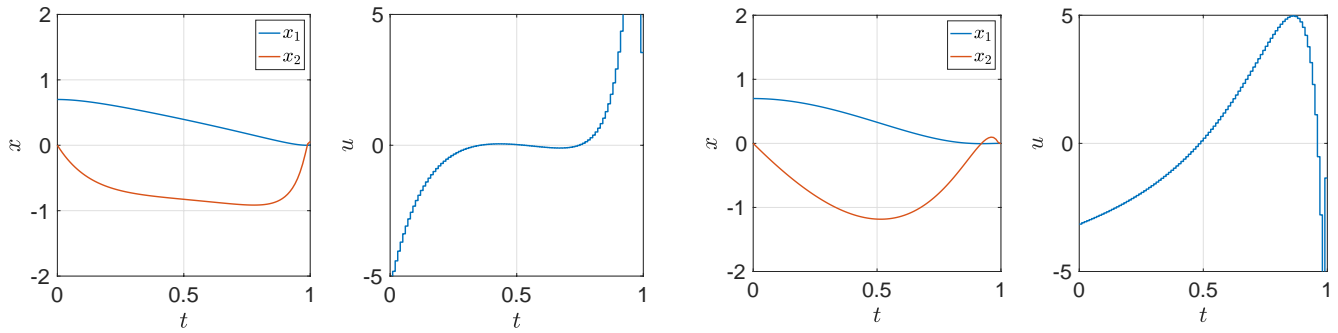


Figure 11 Comparison of time-varying (left) and static (right) controllers in the noise-free case

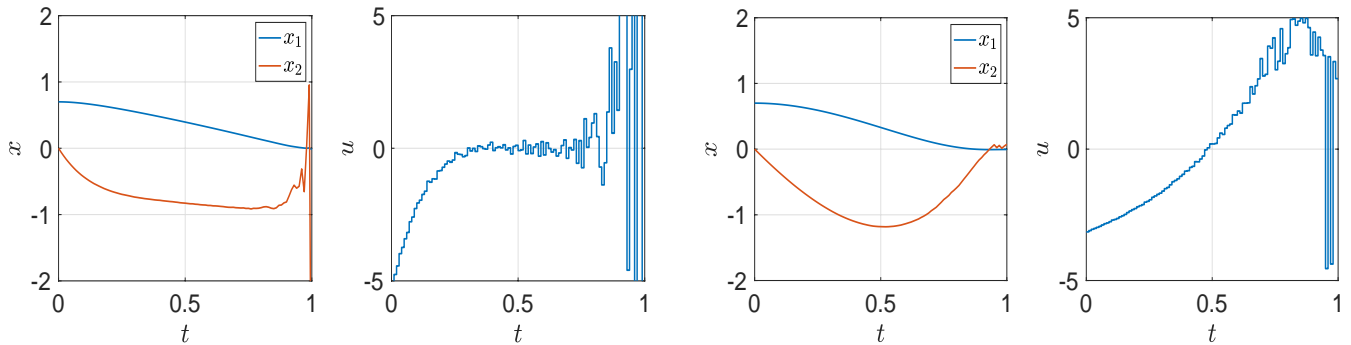


Figure 12 Comparison of time-varying (left) and static (right) controllers in the noisy case

The designed controller (46) steers the state of the system to zero *exactly* at the given instant of time $T > 0$ (independently of nonzero initial state). This means that the closed-loop system with nonzero initial condition cannot reach the origin before the desired instant of time $t = T$. Such a problem statement may be motivated by a rendezvous problem, where some object must meet another object exactly at the given place and the given instant of time, while the reaching of the meeting point before the given instant of time is prohibited. To the best of authors knowledge, the controller (46) is the only static feedback solving such a problem of fixed-time stabilization with the exact constant settling time. This controller is designed for linear MIMO plant and applicable to nonlinear system topologically equivalent to the linear one. A similar method of the control gain scaling dependently of the initial state seems to be possible for other homogeneous control systems. The rigorous study of this issue is the interesting problem for the future research.

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