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# GENERALIZED STEIN'S LEMMA AND ASYMPTOTIC EQUIPARTITION PROPERTY FOR SUBALGEBRA ENTROPIES

LI GAO AND MIZANUR RAHAMAN

ABSTRACT. The quantum Stein's lemma is a fundamental result of quantum hypothesis testing in the context of distinguishing two quantum states. A recent conjecture, known as the "generalized Stein's lemma", asserts that this result is true in a general framework where one of the states is replaced by convex sets of quantum states. In this work, we show that the assertion of the generalized Stein's lemma is true for the setting where the second hypothesis is the state space of any subalgebra  $\mathcal{N}$ . This is obtained through a strong asymptotic equipartition property for smoothing subalgebra entropies that applies for any fixed smoothing parameter  $\varepsilon \in (0, 1)$ . As an application in resource theory, we show that the relative entropy of a subalgebra is the asymptotic dilution cost under suitable operations. This provides a scope to establish a connection between different quantum resources.

## 1. INTRODUCTION

Hypothesis testing is one of the most fundamental tasks in information theory. It involves making decisions based on experimental data from random variables, using tools and techniques from statistics and probability theory. In the setting of quantum information, and in particular in the context of quantum state discrimination, hypothesis testing is deeply connected with many important tasks of quantum information processing ([14, 21, 6]).

When identifying whether a given quantum state is one of the two possibilities, state  $\rho$  or state  $\sigma$ , one can make two types of errors: the observer finds that the state is  $\sigma$ , when in reality it was  $\rho$  (type I error), and the other type of error where the observer finds the state to be  $\rho$ , when it was actually  $\sigma$  (type II error). In the setting of **asymmetric hypothesis testing** one defines the optimized type II error  $\beta_\varepsilon(\rho\|\sigma)$  as the minimum probability of type II error, while requiring that the probability of type I error is bounded by a small parameter  $\varepsilon$ . The celebrated **quantum Stein's lemma** ([18, 24]) asserts that when the source emits i.i.d. copies of one of two quantum states  $\rho$  and  $\sigma$ , then for any  $0 < \varepsilon < 1$ ,

$$\beta_\varepsilon(\rho^{\otimes n}\|\sigma^{\otimes n}) \sim 2^{-nD(\rho\|\sigma)}, \text{ as } n \rightarrow \infty,$$

where  $D(\rho\|\sigma) = \text{tr}(\rho(\log \rho - \log \sigma))$  is the quantum relative entropy of  $\rho$  and  $\sigma$ .

In a more intricate situation known as composite hypothesis testing ([4, 5, 6]), instead of two quantum states  $\rho$  and  $\sigma$ , one wishes to distinguish between a state  $\rho$  and a whole family of quantum states. For example, to test whether a quantum device produces entanglement, one would like to distinguish a bipartite state against all the separable states. In the context of composite hypothesis testing, when the family of quantum states fulfils certain axioms motivated by the framework of resource theories, Brandão and Plenio's ([6]) claimed to prove a **generalized Stein's lemma** showing that the asymptotic error exponent equals the regularized relative entropy of  $\rho$  with respect to the given family of quantum states (for the exact description see [6] and

section 2). This seminal result was of fundamental importance as it established that any reasonable resource theory (which satisfies Axioms 1–5 listed in section 2) can be reversibly manipulated.

However, a gap has recently been found in the claimed proof of the generalized quantum Stein’s lemma ([2, 3]), which puts the main results of Brandão and Plenio’s work into question. This also puts into question many results in the literature that were directly or indirectly dependent on this fact, in particular the reversibility of quantum entanglement and reversibility of general quantum resources under asymptotically resource non-generating operations ([6, 7, 5]). In conclusion, the generalized quantum Stein’s lemma introduced in [6], is not known to be correct.

One of the few instances that the generalized Stein’s lemma is still known to hold true is the case of quantum coherence, where the second hypothesis concerns with incoherent states that are diagonal in a fixed orthonormal basis  $\{|i\rangle\}$  of  $\mathbb{C}^m$ ,

$$\mathcal{I}_n := \text{conv} \left\{ \bigotimes_{j=1}^n |j\rangle\langle j| \right\}.$$

In this scenario, by the series of works of Winter-Yang [29], Zhao et al. ([31, 32]), Chitambar ([9]), Regula et al. ([26]), the asymptotic error exponent of testing the i.i.d copies  $\rho^{\otimes n}$  of a fixed state  $\rho \in B(\mathbb{C}^m)$  against the sets  $\mathcal{I}_n$  is proven to be the relative entropy of coherence  $C(\rho) = \inf_{\sigma \in \mathcal{I}} D(\rho||\sigma)$ . Also, an earlier work of Hayashi-Tomamichel ([16]) proved this result for a particular subalgebra  $\mathcal{N} = 1_A \otimes B(\mathcal{H}_B) \subset B(\mathcal{H}_A \otimes \mathcal{H}_B)$ .

**1.1. Our Contributions:** In this work, we establish the generalized Stein’s lemma for the state spaces of an arbitrary von Neumann algebra acting on a finite-dimensional Hilbert space. In fact we prove a stronger result that works for any fixed error parameter  $\varepsilon \in (0, 1)$ , instead of letting  $\varepsilon \rightarrow 0$ .

**Theorem 1.** (*Generalized Stein’s lemma for subalgebras*) *Let  $\mathcal{N} \subseteq B(\mathcal{H})$  be a von Neumann subalgebra. Then for any  $\rho \in B(\mathcal{H})$  and  $\varepsilon \in (0, 1)$ , we have*

$$\lim_{n \rightarrow \infty} \frac{1}{n} D_H^\varepsilon(\rho^{\otimes n} || \mathcal{N}^{\otimes n}) = D(\rho || \mathcal{N}).$$

We refer to the preliminary section 2 for the precise definition of hypothesis testing relative entropy  $D_H^\varepsilon$ . This results extend the generalized Stein’s lemma beyond the setting of coherence. Note that the above statement implies the strong converse in the sense that it applies to any fixed error parameter  $\varepsilon \in (0, 1)$ , instead of vanishing error  $\varepsilon \rightarrow 0$ .

After finishing this article, we learned that Yamasaki and Kuroiwa in a very recent article [30] announced a proof for the generalized Stein’s lemma that solves this full generality. However, there seems to be some issues regarding their proof as explained in the latest version of their article. Our work provides a different, algebraic approach for the case of subalgebras by making a connection with the asymptotic equipartition property of conditional entropies, which can be of independent interests.

Our key ingredient is the following dilation theorem connecting smoothed subalgebra entropies with smoothed conditional entropies.

**Theorem 2.** *Let  $\mathcal{N} \subseteq B(\mathcal{H}_A)$  be a von Neumann subalgebra and  $\rho \in B(\mathcal{H})$  be a quantum state. Then*

i) For  $\epsilon \in [0, 1)$

$$D_{\max}^{\epsilon}(\rho||\mathcal{N}) = \inf_{\sigma \in B(\mathcal{H}_A)} D_{\max}^{\epsilon}(V\rho V^*||1_E \otimes \sigma) = -H_{\min}^{\epsilon}(E|A)_{V\rho V^*}.$$

$$D_{\min}^{\epsilon}(\rho||\mathcal{N}) = \inf_{\sigma \in B(\mathcal{H}_A)} D_{\min}^{\epsilon}(V\rho V^*||1_E \otimes \sigma) = -H_{\max}^{\epsilon}(E|A)_{V\rho V^*}.$$

ii) For  $\alpha \in [1/2, \infty]$ ,

$$D_{\alpha}(\rho||\mathcal{N}) = \inf_{\sigma \in B(\mathcal{H}_A)} D_{\alpha}(V\rho V^*||1_E \otimes \sigma) = -H_{\alpha}(E|A)_{V\rho V^*}.$$

where  $V : \mathcal{H}_A \rightarrow \mathcal{H}_A \otimes \mathcal{H}_E$  is the Stinespring dilation of the trace preserving conditional expectation  $\mathcal{E} : B(\mathcal{H}_A) \rightarrow \mathcal{N}$ .

We refer to, again, the preliminary section 2 for the definitions of smooth relative entropies  $D_{\max}^{\epsilon}$ ,  $D_{\min}^{\epsilon}$  and conditional entropies  $H_{\min}^{\epsilon}$ ,  $H_{\max}^{\epsilon}$  appearing above. Such a dilation theorem is inspired from the work [20] for the case  $\mathcal{N}$  being the diagonal subalgebra.

In addition to establishing the generalized Stein's lemma in the setting of subalgebras, we also explore the resource theory in this scenario. In the context of quantum coherence, free or incoherent states are those diagonal in a priori fixed computational basis. We introduce a notion of generalized coherence with respect to any given subalgebra of a system, where the incoherent states are defined to be from the given subalgebra of the system. Two fundamental problems in resource theory are dilution and distillation of resourceful state. In this work, we show that the cost of diluting a state  $\rho$  from maximal incoherent state under suitable incoherent operations is given by relative entropy of subalgebra  $D(\rho||\mathcal{N})$ . More precisely, we prove the following

**Theorem 3 (Informal).** *Let  $\mathcal{N} \subseteq B(\mathcal{H})$  be a subalgebra. The asymptotic  $\mathcal{N}$ -coherent cost for a state  $\rho$  under maximal incoherent operations is given by  $D(\rho||\mathcal{N})$*

We refer to section 4 for the precise meaning of these quantities.

The rest of the paper is organized as follows. Section 2 reviews the definitions of various entropic quantity, the statement of generalized Stein's lemma, subalgebra entropies as well as the quantum asymptotic equipartition property. In Section 3, we establish the dilation Theorem 2, and based on that, prove Theorem 1. Section 4 is devoted to the resource theory of subalgebra coherence and obtain the asymptotic dilution cost. We conclude the paper with a discussion on several further questions and directions.

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## 2. DEFINITIONS, NOTATIONS AND PRELIMINARIES

**2.1. Relative entropies.** Throughout the paper, we consider  $\mathcal{H}, \mathcal{K}$  to be finite dimensional Hilbert spaces and denote  $B(\mathcal{H})$  as the set of linear operators on  $\mathcal{H}$ . The standard matrix trace on  $B(\mathcal{H})$  is denoted as  $\text{tr}$ . We say  $\rho \in B(\mathcal{H})$  is a state (resp. substate) if  $\rho$  is positive semi-definite and  $\text{tr}(\rho) = 1$  (resp.  $\text{tr}(\rho) \leq 1$ ). The set of all states (resp. substate) is denoted as  $S(\mathcal{H})$  (resp.  $S_{\leq}(\mathcal{H})$ ).

Let  $\rho$  be a substate and  $\sigma$  be a semi-definite operators in  $B(\mathcal{H})$  with support projection  $\text{supp}(\rho) \leq \text{supp}(\sigma)$ . We recall the definitions of the following relative entropies

i) **Relative entropy:**

$$D(\rho||\sigma) = \text{tr}(\rho \log \rho - \rho \log \sigma)$$

ii) **Sandwiched Rényi relative entropy:** for  $\alpha \in [1/2, 1) \cup (1, \infty]$ ,

$$D_\alpha(\rho||\sigma) = \frac{1}{\alpha - 1} \log \text{tr}[(\sigma^{\frac{1-\alpha}{2\alpha}} \rho \sigma^{\frac{1-\alpha}{2\alpha}})^\alpha].$$

iii) **Max relative entropy:**

$$D_{\max}(\rho||\sigma) := D_\infty(\rho||\sigma) = \log \inf\{ \lambda > 0 \mid \rho \leq \lambda \sigma \}.$$

vi) **Min relative entropy:**

$$D_{\min}(\rho||\sigma) := D_{1/2}(\rho||\sigma) = -2 \log \text{tr} |\rho^{1/2} \sigma^{1/2}| = -\log F(\rho, \sigma)$$

v) **Hypothesis testing relative entropy:**

$$D_H^\varepsilon(\rho||\sigma) = -\log \inf\{ \text{tr}(Q\sigma) \mid 0 \leq Q \leq 1, \text{tr}(Q\rho) \geq 1 - \varepsilon \}.$$

If  $\text{supp}(\rho) \not\subseteq \text{supp}(\sigma)$ ,  $D$  and  $D_{\max}$  are defined as  $+\infty$ . All the logarithms above and in the following are in base 2. Recall the purified distance between two substates  $\rho$  and  $\sigma$  is

$$d(\rho, \sigma) = \sqrt{1 - F(\rho, \sigma)^2}, \quad F(\rho, \sigma) = \text{tr}(|\sqrt{\rho}\sqrt{\sigma}|) + \sqrt{(1 - \text{tr}(\rho))(1 - \text{tr}(\sigma))},$$

where  $F(\rho, \sigma)$  is the generalized fidelity. For two substates  $\rho$  and  $\rho'$ , we write  $\rho' \sim^\varepsilon \rho$  if  $d(\rho', \rho) \leq \varepsilon$ . For  $0 \leq \varepsilon \leq 1$ , we consider the smoothed version of max/min relative entropy,

vi) **Smooth max-relative entropy:**

$$D_{\max}^\varepsilon(\rho||\sigma) = \inf_{\rho' \sim^\varepsilon \rho} D_{\max}(\rho' || \sigma),$$

vii) **Smooth min-relative entropy:**

$$D_{\min}^\varepsilon(\rho||\sigma) = \sup_{\rho' \sim^\varepsilon \rho} D_{\min}(\rho' || \sigma),$$

where the infimum/supremum is over all substates  $\rho'$  that is  $\varepsilon$ -close to  $\rho$  in purified distance.

Imagine a source generating several i.i.d copies of one of two quantum states  $\rho, \sigma$ , and our task is to decide which one of them is being produced. In order to learn the identity of the state, the observer measures a two outcome POVM  $\{Q_n, 1 - Q_n\}$ , given  $n$  realizations of the unknown state. The state  $\rho$  is seen as the null hypothesis, while the state  $\sigma$  as the alternate hypothesis. There are two types of errors that one can make:

- Type I : Observer thinks it is  $\sigma$ , when it is actually  $\rho$ . This happens with probability  $\alpha_n(Q_n) = \text{tr}(\rho^{\otimes n}(1 - Q_n))$ .
- Type II : Observer thinks it is  $\rho$ , when it is actually  $\sigma$ . This happens with probability  $\beta_n(Q_n) = \text{tr}(\sigma^{\otimes n}(Q_n))$ .

In the setting of **asymmetric hypothesis testing**, the probability of type II error should be minimized, while only requiring that the probability of type I error is bounded by a small parameter  $\varepsilon$ . That is,

$$\beta_n(\varepsilon) = \min_{0 \leq Q_n \leq 1} \{ \beta_n(Q_n) : \alpha_n(Q_n) \leq \varepsilon \}.$$

Note that this quantity is precisely the **Hypothesis testing relative entropy** defined in the beginning of the section:

$$-\log \beta_n(\epsilon) = D_H^\epsilon(\rho^{\otimes n} \|\sigma^{\otimes n}).$$

The celebrated **Stein's lemma** ([18, 24]) asserts that for any  $\epsilon \in (0, 1)$ ,

$$\lim_{n \rightarrow \infty} \frac{1}{n} D_H^\epsilon(\rho^{\otimes n} \|\sigma^{\otimes n}) = D(\rho \|\sigma). \quad (2.1)$$

The limits sharing the same spirit for  $D_{\max}^\epsilon$  and  $D_{\min}^\epsilon$  are called **asymptotic equipartition property (AEP)** (see [27, 12])

$$\lim_{n \rightarrow \infty} \frac{1}{n} D_{\max}^\epsilon(\rho^{\otimes n} \|\sigma^{\otimes n}) = \lim_{n \rightarrow \infty} \frac{1}{n} D_{\min}^\epsilon(\rho^{\otimes n} \|\sigma^{\otimes n}) = D(\rho \|\sigma). \quad (2.2)$$

**2.2. Generalized Stein's lemma.** In the framework of composite hypothesis testing, the alternative hypothesis, instead of the usual i.i.d. form, is given by a more general convex set of quantum states. Indeed, when distinguishing  $\rho$  from a whole set of quantum states, one considers this generalized Stein scenario. The family of states satisfy some properties that are motivated from a general resource theory point of view and we outline them below.  $\mathcal{F}_n \subseteq \mathcal{S}(\mathcal{H}^{\otimes n}), n \in \mathbb{N}$ , satisfying the following properties:

- (1) Each  $\mathcal{F}_n$  is convex and closed.
- (2) Each  $\mathcal{F}_n$  contains  $\sigma^{\otimes n}$ , for a full rank state  $\sigma \in \mathcal{S}(\mathcal{H})$ .
- (3) If  $\rho \in \mathcal{F}_{n+1}$ , then  $\text{tr}_k(\rho) \in \mathcal{F}_n$ , for every  $k \in \{1, \dots, n+1\}$ .
- (4) If  $\rho \in \mathcal{F}_n$  and  $v \in \mathcal{F}_m$ , then  $\rho \otimes v \in \mathcal{F}_{n+m}$ .
- (5) If  $\rho \in \mathcal{F}_n$ , then  $P_\pi \rho P_\pi^* \in \mathcal{F}_n$  for every unitary  $P_\pi$  implementing on  $\mathcal{H}^{\otimes n}$  corresponding to any permutation  $\pi \in S_n$ .

The above axioms form the structure of **free states** in resource theory, motivated by the model of bipartite separable states. Indeed, the general framework of quantum resource theories ([10]) is designed to analyse various quantum resources under a unified format, the set of separable states being one of the resources. Here, one usually identifies a family of systems of interest, modelled by finite-dimensional Hilbert space  $\mathcal{H}$ , and over each  $\mathcal{H}$  a set  $\mathcal{F}_n \subseteq \mathcal{S}(\mathcal{H}^{\otimes n}), n \in \mathbb{N}$  of free states, i.e. quantum states that are easily available.

Apart from the separable states, another example that satisfies the above axioms is the sets of incoherent states

$$\mathcal{F}_n = \text{conv} \left\{ \bigotimes_{k=1}^n |j_k\rangle \langle j_k| \right\},$$

for a fixed orthonormal basis  $\{|j\rangle\}$  of  $\mathcal{H}$ .

Now given  $\mathcal{F}_n \subseteq \mathcal{S}(\mathcal{H}^{\otimes n}), n \in \mathbb{N}$ , the **relative entropy of resource** of any  $\rho \in \mathcal{S}(\mathcal{H})$  is defined as

$$D_{\mathcal{F}}(\rho) = \inf_{\sigma \in \mathcal{F}_1} D(\rho \|\sigma),$$

and the regularized **relative entropy of resource** is defined as

$$D_{\mathcal{F}}^\infty(\rho) = \lim_{n \rightarrow \infty} \frac{1}{n} D_{\mathcal{F}_n}(\rho^{\otimes n}).$$

In this context, the conjectured generalized Stein's lemma is stated as follows.

**Conjecture.** [Generalized quantum Stein's lemma:] For any family of sets of quantum sets  $\{\mathcal{F}_n\}_{n \in \mathbb{N}}$  satisfying the Axioms 1-5 in subsection 2, it holds that

$$\lim_{\epsilon \rightarrow 0} \liminf_{n \rightarrow \infty} \frac{1}{n} \min_{\sigma_n \in \mathcal{F}_n} D_H^\epsilon(\rho^{\otimes n} \|\sigma_n) = D_{\mathcal{F}}^\infty(\rho). \quad (2.3)$$

As stated earlier, an early paper by Brandão, and Plenio ([6]) claimed that the above equality is true, however, very recently it was recently shown [2] that their argument had a gap in the course of the proof and the equality in equation 2.3 can not be claimed and hence the above conjecture came to surface. Despite the gap, the generalized Stein's lemma is known to be true for a few scenarios: for quantum coherence, pseudo-entanglement, etc (see [2]). In the context of quantum coherence with  $\mathcal{N}$  being the diagonal subalgebra, this was established in [4] (see also [8] [32]). Another special case, where  $\mathcal{N} = \mathbb{C}1_A \otimes B(\mathcal{H}_B) \subseteq B(\mathcal{H}_A) \otimes B(\mathcal{H}_B)$ , for which the generalized Stein's lemma is known to be true was shown by [17]. Thus, our result captures all these known cases and provides a bigger subset where the gen. Stein's lemma can be realized. It is worth noting that the proof here does not rely on the arguments given in [6].

Let  $\mathcal{M} \subseteq B(\mathcal{H})$  be the commutative subalgebra spanned by the fixed basis  $\{|j\rangle\}$ . The incoherent states  $\text{conv}\{\otimes_{k=1}^n |j_k\rangle\langle j_k|\}$  can be viewed as the state spaces of the commutative subalgebra  $\mathcal{M}^{\otimes n}$ . Indeed, this is a special case of a more general fact: given any von Neumann subalgebra  $\mathcal{N} \subseteq B(\mathcal{H})$ , the state space of  $\mathcal{F}_n = \mathcal{S}(\mathcal{N}^{\otimes n}) \subseteq \mathcal{S}(\mathcal{H}^{\otimes n})$  satisfy the above axioms, where by  $\mathcal{S}(\mathcal{N}^{\otimes n})$  we denote  $\mathcal{S}(\mathcal{H}^{\otimes n}) \cap \mathcal{N}^{\otimes n}$ . Here we note that  $\mathcal{S}(\mathcal{N}^{\otimes n})$  may contain separable states as well as entangled states  $\mathcal{S}(\mathcal{H}^{\otimes n})$ . Indeed, if the subalgebra  $\mathcal{N} \subseteq B(\mathcal{H})$  is non-commutative, then it is not hard to see that the state space  $\mathcal{S}(\mathcal{N}^{\otimes n})$  contains both separable and entangled states of the system  $\mathcal{S}(\mathcal{H}^{\otimes n})$ . This is interesting, as in the quantum coherence, one always obtains separable states as a resource.

**2.3. Subalgebra entropies.** Let  $\mathcal{H}$  be a finite dimensional Hilbert space and  $\mathcal{N} \subset B(\mathcal{H})$  be a von Neumann subalgebra (or simply, a  $*$ -subalgebra). The state space of  $\mathcal{N}$  is defined as

$$\mathcal{S}(\mathcal{N}) = \mathcal{N} \cap \mathcal{S}(\mathcal{H}).$$

Given a state  $\rho \in B(\mathcal{H})$ , we define the subalgebra entropy

$$\mathbb{D}(\rho \|\mathcal{N}) := \inf_{\sigma \in \mathcal{S}(\mathcal{N})} \mathbb{D}(\rho \|\sigma).$$

for  $\mathbb{D} = D, D_\alpha, D_\alpha^R, D_H^\epsilon, D_{\max}, D_{\min}$  correspondingly, and also the smooth Min/Max entropy w.r.t to  $\mathcal{N}$ .

$$D_{\max}^\epsilon(\rho \|\mathcal{N}) = \inf_{\rho' \sim^\epsilon \rho} D_{\max}(\rho' \|\mathcal{N}),$$

$$D_{\min}^\epsilon(\rho \|\mathcal{N}) = \sup_{\rho' \sim^\epsilon \rho} D_{\min}(\rho' \|\mathcal{N}).$$

Some of these quantities were studied in [13]. In particular, it is known that for the relative entropy  $\mathbb{D} = D$ ,

$$D(\rho \|\mathcal{N}) = D(\rho \|\mathcal{E}_{\mathcal{N}}(\rho))$$

where  $\mathcal{E}_{\mathcal{N}} : B(\mathcal{H}) \rightarrow B(\mathcal{H})$  is the unique map onto  $\mathcal{N}$  such that for all  $x \in \mathcal{N}, y \in B(\mathcal{H})$

$$\text{tr}(xy) = \text{tr}(x\mathcal{E}_{\mathcal{N}}(y)). \quad (2.4)$$

Such map  $\mathcal{E}_{\mathcal{N}}$  is called the (trace preserving) conditional expectation onto  $\mathcal{N}$ . Moreover,  $\mathcal{E}_{\mathcal{N}}$  is also unital complete positive, hence a quantum channel (CPTP map), sending identity to identity.



The Stein's lemma (Theorem 1) for hypothesis testing relative entropy  $D_H^\epsilon$  follows from the following strong asymptotic equipartition property (AEP) for  $D_{\max}^\epsilon$ .

**Theorem 4** (AEP for subalgebras). *For any  $\rho \in B(\mathcal{H})$ , subalgebra  $\mathcal{N} \subseteq B(\mathcal{H})$  and  $\epsilon \in (0, 1)$ , we have*

$$\lim_{n \rightarrow \infty} \frac{1}{n} D_{\max}^\epsilon(\rho^{\otimes n} \| \mathcal{N}^{\otimes n}) = \lim_{n \rightarrow \infty} \frac{1}{n} D_{\min}^\epsilon(\rho^{\otimes n} \| \mathcal{N}^{\otimes n}) = D(\rho \| \mathcal{N}).$$

Note that here the right hand side does not need regularization because

$$D(\rho^{\otimes n} \| \mathcal{N}^{\otimes n}) = D(\rho^{\otimes n} \| \mathcal{E}_{\mathcal{N}^{\otimes n}}(\rho^{\otimes n})) = D(\rho^{\otimes n} \| \mathcal{E}_{\mathcal{N}}^{\otimes n}(\rho^{\otimes n})) = nD(\rho \| \mathcal{N}).$$

We also emphasize that having fixed  $\epsilon > 0$  without goes to 0 is crucial in deriving the Stein's lemma.

We illustrate the above theorems with some special cases.

**Example 5.** Let  $\mathcal{N} = \mathbf{C1}$  be the trivial subalgebra. The conditional expectation onto the scalars  $\mathbf{C1} \subset B(\mathcal{H})$  is the completely depolarizing map

$$\mathcal{E}_{\mathbf{C}} : B(\mathcal{H}) \rightarrow \mathbf{C}, \mathcal{E}_{\mathbf{C}}(\rho) = \text{tr}(\rho) \frac{1}{d},$$

where  $d = \dim \mathcal{H}$ . In this setting, the subalgebra entropy are essentially the von Neumann entropy as

$$D(\rho \| \mathbf{C1}) = \log d - H(\rho).$$

Similarly,

$$D_{\max}(\rho \| \mathbf{C1}) = \log d - H_{\min}(\rho), H_{\max}(\rho \| \mathbf{C1}) = \log d - H_{1/2}(\rho).$$

Theorem 4 basically gives the AEP for von Neumann entropy,

$$\lim_{n \rightarrow \infty} \frac{1}{n} H_{\max}^\epsilon(\rho^{\otimes n}) = \lim_{n \rightarrow \infty} \frac{1}{n} H_{\min}^\epsilon(\rho^{\otimes n}) = H(\rho).$$

**Example 6.** Let  $\mathcal{H}_A$  and  $\mathcal{H}_B$  be two Hilbert spaces. The following AEP for conditional entropy was proved in [28, 27] that for a bipartite state  $\rho_{AB} \in \mathcal{S}(\mathcal{H}_A \otimes \mathcal{H}_B)$ ,

$$\lim_{n \rightarrow \infty} \frac{1}{n} H_{\max}^\epsilon(A^n | B^n)_{\rho^{\otimes n}} = \lim_{n \rightarrow \infty} \frac{1}{n} H_{\min}^\epsilon(A^n | B^n)_{\rho^{\otimes n}} = H(A|B)_\rho. \quad (2.5)$$

where the conditional entropies are defined as

$$H(A|B)_\rho = H(\rho_{AB}) - H(\rho_B),$$

$$H_{\max}(A|B)_\rho = - \inf_{\sigma \in \mathcal{S}(\mathcal{H}_B)} D_{\min}(\rho_{AB} | \mathbf{1} \otimes \sigma_B), \quad H_{\max}^\epsilon(A|B)_\rho = \min_{\rho' \sim \rho} H_{\max}(A|B)_{\rho'}$$

$$H_{\min}(A|B)_\rho = - \inf_{\sigma \in \mathcal{S}(\mathcal{H}_B)} D_{\max}(\rho_{AB} | \mathbf{1} \otimes \sigma_B), \quad H_{\min}^\epsilon(A|B)_\rho = \max_{\rho' \sim \rho} H_{\min}(A|B)_{\rho'}$$

Up to the same normalization of the previous example, this is special case of Theorem 4 by choosing  $\mathcal{N} = \mathbf{C1}_A \otimes B(\mathcal{H}_B)$  being a tensor component of  $B(\mathcal{H}_A \otimes \mathcal{H}_B)$ . In this case, the conditional expectation is the partial trace onto the first system.

**Example 7.** For quantum coherence, one consider  $\{|j\rangle\}_{j=1}^d$  be a fixed ONB of  $\mathcal{H}$ . The commutative subalgebra  $\mathcal{I} = \text{span}\{|j\rangle\langle j|\}_{j=1}^d$  models the incoherent subalgebra and the density operators in  $\mathcal{N}$  are states with no coherence, called incoherent states. The recent progress establish a complete picture of resource theory ([31, 32, 29]). Our Theorem implies that

$$\lim_{n \rightarrow \infty} \frac{1}{n} C_{\max}^\epsilon(\rho^{\otimes n}) = \lim_{n \rightarrow \infty} \frac{1}{n} C_{\min}^\epsilon(\rho^{\otimes n}) = C(\rho). \quad (2.6)$$



where  $C(\rho) = D(\rho||\mathcal{I})$  is the relative entropy of coherence and similarly for  $C_{\max}^\epsilon$  and  $C_{\min}^\epsilon$ . Both limit holds for a fixed error parameter  $\epsilon$  without letting  $\epsilon \rightarrow 0$ , which has implications in the **strong converse** of resource theory of coherence. Such strong converse were also obtained in the recent work of Hayashi, Fang and Wang ([15])

**Example 8.** Let  $G$  be a compact group and  $U : G \rightarrow B(\mathcal{H})$  be an unitary representation. Consider  $\mathcal{N}_G := \{ x \in B(\mathcal{H}) \mid U_g x U_g^* = x, \forall g \in G \}$  be the  $G$ -invariant subalgebra. The conditional expectation onto  $\mathcal{N}_G$  is given by

$$\mathcal{E}_G(x) = \int_G U_g x U_g^* d\mu(g),$$

where  $\mu$  is the Haar measure. For any state  $\rho$ ,  $\mathcal{E}_G(\rho)$  is the projection of  $\rho$  to the  $G$ -symmetric states. The relative entropy  $D(\rho||\mathcal{N}_G) = D(\rho||\mathcal{E}_G(\rho))$  is called the relative entropy of  $G$ -asymmetry, and  $D_{\min}(\rho||\mathcal{N}_G)$  and  $D_{\max}(\rho||\mathcal{N}_G)$  are called  $G$ -asymmetry measures in [23, 22]. Our above theorems implies that for any compact group symmetry,

$$\lim_{n \rightarrow \infty} \frac{1}{n} D_{\max}^\epsilon(\rho^{\otimes n} || \mathcal{N}_G^{\otimes n}) = \lim_{n \rightarrow \infty} \frac{1}{n} D_{\min}^\epsilon(\rho^{\otimes n} || \mathcal{N}_G^{\otimes n}) = D(\rho || \mathcal{N}_G).$$

### 3. PROOF OF STEIN'S LEMMA FOR SUBALGEBRA ENTROPIES

The main result of this section connects the relative entropies for subalgebras to conditional entropies of an appropriate state. Since conditional entropies are widely used in quantum cryptography and quantum information processing tasks, our results provide an operational meaning to the relative entropies of subalgebras.

Recall that for a bipartite states  $\rho_{AB} \in \mathcal{S}(\mathcal{H}_A \otimes \mathcal{H}_B)$  sandwiched Rényi conditional entropy is defined as

for  $\alpha \in [1/2, 1) \cup (1, \infty)$  we define

$$H_\alpha(A|B)_{\rho_{AB}} = \sup_{\sigma_B \in \mathcal{S}(H_B)} -D_\alpha(\rho_{AB} || 1_A \otimes \sigma_B),$$

and

$$\lim_{\alpha \rightarrow 1} H_\alpha(A|B)_{\rho_{AB}} = H(A|B)_\rho = H(AB)_\rho - H(B)_\rho = -\text{Tr}(\rho_{AB} \log \rho_{AB}) + \text{Tr}(\rho_B \log \rho_B).$$

Our main technical tool is the following dilation theorem that connects the subalgebra entropies with conditional entropies.

**Theorem 9.** Let  $\mathcal{H}_A$  be a finite dimensional Hilbert space and  $\mathcal{N} \subset B(\mathcal{H}_A)$  be a subalgebra. Suppose  $V : \mathcal{H}_A \rightarrow \mathcal{H}_A \otimes \mathcal{H}_E$  is the Stinespring dilation isometry for the conditional expectation  $\mathcal{E} : B(\mathcal{H}_A) \rightarrow \mathcal{N}$  such that  $\mathcal{E}(x) = \text{tr}_E(VxV^*)$

i) for  $\epsilon \in [0, 1)$ ,

$$\begin{aligned} D_{\max}^\epsilon(\rho || \mathcal{N}) &= \inf_{\sigma \in B(\mathcal{H}_A)} D_{\max}^\epsilon(V\rho V^* || 1_E \otimes \sigma) = -H_{\min}^\epsilon(E|A)_{V\rho V^*}. \\ D_{\min}^\epsilon(\rho || \mathcal{N}) &= \inf_{\sigma \in B(\mathcal{H}_A)} D_{\min}^\epsilon(V\rho V^* || 1_E \otimes \sigma) = -H_{\max}^\epsilon(E|A)_{V\rho V^*}. \end{aligned}$$

ii) for  $\alpha \in [1/2, \infty]$ ,

$$D_\alpha(\rho || \mathcal{N}) = \inf_{\sigma \in B(\mathcal{H}_A)} D_\alpha(V\rho V^* || 1_E \otimes \sigma) = -H_\alpha(E|A)_{V\rho V^*}.$$

For the special case of  $\mathcal{N}$  being the maximal abelian subalgebra of  $B(\mathcal{H}_A)$ , the connection between the  $C_{\min}$  and  $C_{\max}$  relative entropy of coherence with the conditional  $H_{\max}$  and  $H_{\min}$  entropy has been obtained by Zhao *et al* [20]. Here we prove this connection  $\varepsilon$ -smoothed max- and min- entropies and for sandwich  $\alpha$ -relative entropies. This generalize the previously known results for  $\alpha = \frac{1}{2}, \infty$  or the non smoothed version  $\varepsilon = 0$ .

Combined with duality of smoothed and sandwiched conditional entropies [27], we have the following identities.

**Corollary 10.** *Let  $|\psi\rangle \in B(\mathcal{H}_A \otimes \mathcal{H}_F)$  be a purification of  $\rho$ . Then  $\xi_{EAF} = (V \otimes 1_F)|\psi\rangle\langle\psi|(V \otimes 1_F)^*$  is a purification of  $V\rho V^*$ , and*

$$D_{\max}^\varepsilon(\rho||\mathcal{N}) = -H_{\min}^\varepsilon(E|A)_\xi = H_{\max}^\varepsilon(E|F)_\xi, \varepsilon \in [0, 1] \quad (3.1)$$

$$D_{\min}^\varepsilon(\rho||\mathcal{N}) = -H_{\max}^\varepsilon(E|A)_\xi = H_{\min}^\varepsilon(E|F)_\xi, \quad (3.2)$$

$$D_\alpha(\rho||\mathcal{N}) = -H_\alpha(E|A)_\xi = H_\beta(E|F)_\xi, 2 = \frac{1}{\alpha} + \frac{1}{\beta}, \alpha \in [\frac{1}{2}, \infty] \quad (3.3)$$

It is interesting that the conditional entropies can be negative or positive but the subalgebra entropies are always non-negative.

To prove the theorem we need some results about conditional expectations, which can be of independent interests. We start with a lemma about the multiplicative domain of a unital completely positive (UCP) map. Let  $\Phi : \mathcal{A} \rightarrow \mathcal{B}$  be a UCP map between two  $C^*$ -algebra  $\mathcal{A}$  and  $\mathcal{B}$ . The multiplicative domain of  $\Phi$  is defined as below:

$$\mathcal{M}_\Phi = \{a \in \mathcal{A} : \Phi(ax) = \Phi(a)\Phi(x), \Phi(xa) = \Phi(x)\Phi(a), \forall x \in \mathcal{A}\}.$$

**Lemma 11.** *Let  $\Phi : \mathcal{A} \rightarrow B(\mathcal{H})$  be a UCP map with Stinespring dilation  $\Phi(x) = V^*\pi(x)V$ , where  $\pi : \mathcal{A} \rightarrow B(\mathcal{K})$  is a homomorphism and  $V : \mathcal{H} \rightarrow \mathcal{K}$  is a Hilbert space isometry. Then we have*

$$\mathcal{M}_\Phi = \{a \in \mathcal{A} : \pi(a)V V^* = V V^* \pi(a)\}.$$

*Proof.* Note that as  $V$  is an isometry,  $V^*V = 1$  and  $VV^* = e$  is a projection. It is well known (due to Choi [11]) that

$$\mathcal{M}_\Phi = \{a \in \mathcal{A} : \Phi(aa^*) = \Phi(a)\Phi(a)^*, \Phi(a^*a) = \Phi(a^*)\Phi(a)\}.$$

Using this characterization, one way is immediate: if  $a \in \mathcal{A}$  such that  $\pi(a)e = e\pi(a)$ , then it holds that

$$\Phi(a)\Phi(a^*) = V^*\pi(a)V V^*\pi(a^*)V = V^*V V^*\pi(a)\pi(a^*)V = \Phi(aa^*).$$

The other side  $\Phi(a^*a) = \Phi(a^*)\Phi(a)$  follows similarly, hence  $a \in \mathcal{M}_\Phi$ .

Conversely, suppose  $a \in \mathcal{M}_\Phi$ . From  $\Phi(aa^*) = \Phi(a)\Phi(a)^*$  we get

$$V^*\pi(a)\pi(a^*)V = V^*\pi(a)V V^*\pi(a^*)V.$$

Then

$$e\pi(a)\pi(a^*)e = V V^*\pi(a)\pi(a^*)V V^* = V(V^*\pi(a)V V^*\pi(a^*)V)V^* = e\pi(a)e\pi(a^*)e.$$

Hence we get

$$0 = e\pi(a)\pi(a^*)e - e\pi(a)e\pi(a^*)e = e\pi(a)(1 - e)\pi(a^*)e = e\pi(a)(1 - e)[e\pi(a)(1 - e)]^*$$

This yields,

$$e\pi(a)(1 - e) = 0 \implies e\pi(a) = e\pi(a)e. \quad (3.4)$$

Similarly, the identity  $\Phi(a^*a) = \Phi(a^*)\Phi(a)$  implies

$$e\pi(a^*) = e\pi(a^*)e. \quad (3.5)$$

Putting (3.4) and (3.5) together,

$$e\pi(a) = e\pi(a)e = (e\pi(a^*)e)^* = (e\pi(a^*))^* = \pi(a)e,$$

which completes the proof.  $\blacksquare$

We apply the above lemma to the conditional expectations. Recall that a finite von Neumann algebra  $\mathcal{M}$  is a von Neumann algebra equipped with a normal faithful finite trace  $\tau(1) < \infty$ .

**Proposition 12.** *Let  $\mathcal{N} \subseteq \mathcal{M}$  be a subalgebra of a finite von Neumann algebra  $\mathcal{M} \subset B(\mathcal{H})$  and  $\mathcal{E} : \mathcal{M} \rightarrow \mathcal{N}$  be a trace preserving conditional expectation with a Stinespring representation  $\mathcal{E}(x) = V^*\pi(x)V$ , where  $\pi : \mathcal{M} \rightarrow B(\mathcal{K})$  is a homomorphism and  $V : \mathcal{H} \rightarrow \mathcal{K}$  is an isometry. Then it holds that*

$$V\mathcal{E}(x)V^* \leq \pi(\mathcal{E}(x)), \forall x \in \mathcal{M}_+.$$

*Proof.* The multiplicative domain  $\mathcal{M}_{\mathcal{E}}$  of a trace-preserving conditional expectation  $\mathcal{E} : \mathcal{M} \rightarrow \mathcal{N}$  is exactly  $\mathcal{N}$ . Indeed, if  $a \in \mathcal{N}$ , by Kadison-Schwarz inequality

$$\mathcal{E}(aa^*) \geq \mathcal{E}(a)\mathcal{E}(a^*) = aa^*.$$

Since  $\mathcal{E}$  is trace preserving, from the faithfulness of trace, we get

$$\tau(a^*a) = \tau(\mathcal{E}(a^*)\mathcal{E}(a)) \implies \mathcal{E}(aa^*) = aa^* = \mathcal{E}(a)\mathcal{E}(a^*)$$

This shows  $\mathcal{N} \subset \mathcal{M}_{\mathcal{E}}$ . For the other side inclusion  $\mathcal{M}_{\mathcal{E}} \subset \mathcal{N}$  follows from the fact that  $\mathcal{E}$  is the projection from  $\mathcal{M}$  to  $\mathcal{N}$  with respect to the Hilbert space norm  $\|x\|_2^2 = \tau(x^*x)$ .

Now, consider  $\mathcal{E}(\cdot) = V^*\pi(\cdot)V$  be a Stinespring dilation for  $\mathcal{E}$ . From Lemma 11, it holds that

$$\pi(\mathcal{E}(x))VV^* = VV^*\pi(\mathcal{E}(x)), \forall x \in \mathcal{M}.$$

Thus for any  $x \in \mathcal{M}_+$ , we have

$$\pi(\mathcal{E}(x)) \geq VV^*\pi(\mathcal{E}(x))VV^* = V\mathcal{E}(\mathcal{E}(x))V^* = V\mathcal{E}(x)V^*.$$

where we have used that  $\mathcal{E} \circ \mathcal{E} = \mathcal{E}$ .  $\blacksquare$

**Remark 13.** For any ucp map  $\Phi : B(\mathcal{H}) \rightarrow B(\mathcal{H})$ , the Stinespring dilation can be

$$\Phi(a) = V^*(a \otimes 1)V,$$

where an isometry  $V : \mathcal{H} \rightarrow \mathcal{H} \otimes \mathcal{H}_E$  is an isometry and the homomorphism  $\pi$  is simply  $a \rightarrow a \otimes 1_E$ . In this case, we have  $\mathcal{M} = B(\mathcal{H})$  and  $\mathcal{E}(x) = V^*(x \otimes 1_E)V$ , then

$$V\mathcal{E}(x)V^* \leq \mathcal{E}(x) \otimes 1_E, \forall x \in \mathcal{M}_+.$$

We are now ready to prove our dilation theorem.

*Proof of Theorem 9.* We prove the case for  $D_{\max}^{\epsilon}$  and  $D_{\min}^{\epsilon}$ , and the argument for Sandwiched Rényi relative entropy  $D_{\alpha}$ ,  $\alpha \in (1/2, \infty)$  are similar. Indeed, the proof for  $\alpha > 1$  follows exactly like the proof for  $D_{\max}^{\epsilon}$  and the proof for  $\alpha < 1$ , follows exactly like the proof for  $D_{\min}^{\epsilon}$ . For  $\alpha = 1$  follows from the limit  $\alpha \rightarrow 1$  from the right.

Denote  $e = VV^*$  for the projection in  $B(\mathcal{H}_A \otimes \mathcal{H}_E)$ . For  $D_{\max}^\varepsilon$ ,

$$\begin{aligned}
D_{\max}^\varepsilon(\rho \|\mathcal{N}) &= \inf_{\rho' \sim_\varepsilon \rho} \inf_{\sigma \in B(\mathcal{H}_A)} D_{\max}(\rho' \|\mathcal{E}(\sigma)) \\
&\stackrel{(1)}{=} \inf_{\rho' \sim_\varepsilon \rho} \inf_{\sigma \in B(\mathcal{H}_A)} D_{\max}(V\rho'V^* \| VV^*(1_E \otimes \sigma)VV^*) \\
&= \inf_{\rho' \sim_\varepsilon \rho} \inf_{\sigma \in B(\mathcal{H}_A)} D_{\max}(V\rho'V^* \| e(1_E \otimes \sigma)e) \\
&\stackrel{(2)}{=} \inf_{e\omega'e \sim_\varepsilon V\rho'V^*} \inf_{\sigma \in B(\mathcal{H}_A)} D_{\max}(e\omega'e \| e(1_E \otimes \sigma)e)
\end{aligned}$$

Here the equality (1) follows from the fact that partial isometry  $V$  preserves purified distance and  $D_{\max}$ . The equality (2) is by choosing  $e\omega'e = V\rho'V^*$ . By the fact that the restriction map  $\omega \mapsto e\omega e$  does not increasing the purified distance and  $D_{\max}$ , we have the following inequality

$$\begin{aligned}
D_{\max}^\varepsilon(\rho \|\mathcal{N}) &\leq \inf_{\omega' \sim_\varepsilon V\rho'V^*} \inf_{\sigma \in B(\mathcal{H}_A)} D_{\max}(\omega' \| 1_E \otimes \sigma) \\
&\leq \inf_{\omega' \sim_\varepsilon V\rho'V^*} \inf_{\sigma \in \mathcal{N}} D_{\max}(\omega' \| 1_E \otimes \sigma) \\
&\stackrel{(3)}{\leq} \inf_{\omega' \sim_\varepsilon V\rho'V^* \text{ and } e\omega'e = \omega} \inf_{\sigma \in \mathcal{N}} D_{\max}(\omega' \| 1_E \otimes \sigma) \\
&\stackrel{(4)}{=} \inf_{\rho' \sim_\varepsilon \rho} \inf_{\sigma \in \mathcal{N}} D_{\max}(V\rho'V^* \| 1_E \otimes \sigma) \\
&\stackrel{(5)}{\leq} \inf_{\rho' \sim_\varepsilon \rho} \inf_{\sigma \in \mathcal{N}} D_{\max}(V\rho'V^* \| V\sigma V^*) \\
&= \inf_{\rho' \sim_\varepsilon \rho} \inf_{\sigma \in \mathcal{N}} D_{\max}(\rho' \|\sigma) \\
&= D_{\max}^\varepsilon(\rho \|\mathcal{N})
\end{aligned}$$

The inequality (3) is because the infimum has one more constraint that  $\omega' = e\omega'e$  is supported on  $e$ . The equality (4) is by choosing  $\rho' = V^*\omega'V$ . The inequality (5) uses the Lemma 11 and Remark 13 that for  $\sigma \in \mathcal{N}_+$

$$[e, 1_E \otimes \sigma] = 0, \quad V\sigma V^* \leq 1_E \otimes \sigma,$$

and the fact that  $V\sigma V^* \leq 1_E \otimes \sigma$ , implies  $D_{\max}(\cdot \| 1_E \otimes \sigma) \leq D_{\max}(\cdot \| V\sigma V^*)$  ([27]).

Hence we have equality throughout, yielding

$$D_{\max}^\varepsilon(\rho \|\mathcal{N}) = \inf_{\omega' \sim_\varepsilon V\rho'V^*} \inf_{\sigma \in B(\mathcal{H}_A)} D_{\max}(\omega' \| 1_E \otimes \sigma) = -H_{\min}^\varepsilon(E|A)_{V\rho V^*}.$$

For  $D_{\min}^\varepsilon$ , we have

$$\begin{aligned}
D_{\min}^\varepsilon(\rho \|\mathcal{N}) &= \sup_{\rho' \sim_\varepsilon \rho} \inf_{\sigma \in B(\mathcal{H}_A)} D_{\min}(\rho' \|\mathcal{E}(\sigma)) \\
&= \sup_{\rho' \sim_\varepsilon \rho} \inf_{\sigma \in B(\mathcal{H}_A)} D_{\min}(V\rho'V^* \| VV^*(1_E \otimes \sigma)VV^*) \\
&= \sup_{\rho' \sim_\varepsilon \rho} \inf_{\sigma \in B(\mathcal{H}_A)} D_{\min}(V\rho'V^* \| e(1_E \otimes \sigma)e) \\
&\stackrel{(1)}{=} \sup_{\rho' \sim_\varepsilon \rho} \inf_{\sigma \in B(\mathcal{H}_A)} D_{\min}(V\rho'V^* \| 1_E \otimes \sigma)
\end{aligned}$$

Here, the equality (1) is because  $V\rho'V^*$  is supported on  $e$ , and the property

$$D_{\min}(\rho, \sigma) = -2 \log F(\rho, \sigma), \quad F(e\rho e, \sigma) = F(e\rho e, e\sigma e) \quad (3.6)$$

Then by having  $\omega' := V\rho'V^* \sim_\varepsilon V\rho V^*$ ,

$$\begin{aligned} D_{\min}^\varepsilon(\rho \| \mathcal{N}) &\leq \sup_{\omega' \sim_\varepsilon V\rho V^*} \inf_{\sigma \in B(\mathcal{H}_A)} D_{\min}(\omega' \| 1_E \otimes \sigma) \\ &\leq \sup_{\omega' \sim_\varepsilon V\rho V^*} \inf_{\sigma \in \mathcal{N}} D_{\min}(\omega' \| 1_E \otimes \sigma) \\ &\stackrel{(2)}{\leq} \sup_{\omega' \sim_\varepsilon V\rho V^*} \inf_{\sigma \in \mathcal{N}} D_{\min}(\omega' \| e(1_E \otimes \sigma)e) \\ &\stackrel{(3)}{=} \sup_{\omega' \sim_\varepsilon V\rho V^*} \inf_{\sigma \in \mathcal{N}} D_{\min}(e\omega'e \| e(1_E \otimes \sigma)e) \\ &\stackrel{(4)}{\leq} \sup_{e\omega'e \sim_\varepsilon V\rho V^*} \inf_{\sigma \in \mathcal{N}} D_{\min}(VV^*\omega'VV^* \| VV^*(1_E \otimes \sigma)VV^*) \\ &= \sup_{\rho' \sim_\varepsilon \rho} \inf_{\sigma \in \mathcal{N}} D_{\min}(V\rho'V^* \| V^*\sigma V) \\ &\stackrel{(5)}{=} \sup_{\rho' \sim_\varepsilon \rho} \inf_{\sigma \in \mathcal{N}} D_{\min}(\rho' \| \mathcal{E}(\sigma)) \\ &= D_{\min}^\varepsilon(\rho \| \mathcal{N}) \end{aligned}$$

Here, the inequality (2) uses again the  $[e, 1_E \otimes \sigma] = 0$  and hence  $e(1_E \otimes \sigma)e \leq 1_E \otimes \sigma$ . The equality (3) follows from (3.6). The inequality (4) follows from the fact that the purified distance  $d(\omega', V\rho V^*) = d(e\omega'e, V\rho V^*)$ . The equality (5) uses  $V(1_e \otimes \sigma)V^* = \mathcal{E}(\sigma) = \sigma$  for  $\sigma \in \mathcal{N}$ .

Hence we have equality throughout, yielding

$$D_{\min}^\varepsilon(\rho \| \mathcal{N}) = \inf_{\omega' \sim_\varepsilon V\rho V^*} \inf_{\sigma \in B(\mathcal{H}_A)} D_{\min}(\omega' \| 1_E \otimes \sigma) = -H_{\max}^\varepsilon(E|A)_{V\rho V^*}. \quad \blacksquare$$

Now we are ready to state and prove the following main theorem of this section.

**Theorem 14.** *For any subalgebra  $\mathcal{N} \subseteq B(\mathcal{H})$  and a  $\rho \in B(\mathcal{H})$ , and for any  $\varepsilon \in (0, 1)$ , we have*

$$\lim_{n \rightarrow \infty} \frac{1}{n} D_{\max}^\varepsilon(\rho^{\otimes n} \| \mathcal{N}^{\otimes n}) = \lim_{n \rightarrow \infty} \frac{1}{n} D_{\min}^\varepsilon(\rho^{\otimes n} \| \mathcal{N}^{\otimes n}) = \lim_{n \rightarrow \infty} \frac{1}{n} D_H^\varepsilon(\rho^{\otimes n} \| \mathcal{N}^{\otimes n}) = D(\rho \| \mathcal{N}).$$

*Proof.* Let  $\mathcal{E} : B(\mathcal{H}) \rightarrow \mathcal{N}$  be the unique trace preserving conditional expectation. It is known that

$$D(\rho \| \mathcal{N}) = D(\rho \| \mathcal{E}(\rho)).$$

Thus, one direction follows from the AEP of two states:

$$D(\rho \| \mathcal{N}) = D(\rho \| \mathcal{E}(\rho)) = \lim_{n \rightarrow \infty} \frac{1}{n} D_{\max}^\varepsilon(\rho^{\otimes n} \| \mathcal{E}(\rho)^{\otimes n}) \geq \lim_{n \rightarrow \infty} \frac{1}{n} D_{\max}^\varepsilon(\rho^{\otimes n} \| \mathcal{N}^{\otimes n}).$$

Similarly for  $D_{\min}^\varepsilon$  and  $D_H^\varepsilon$ . For the other direction, suppose  $\mathcal{E}$  as a unital completely positive map has Stinespring dilation

$$\mathcal{E}(x) = V^*(x \otimes 1_E)V,$$

then by Theorem 9 we have the identity for  $\rho^{\otimes n}$ ,

$$D_{\max}^\varepsilon(\rho^{\otimes n} \| \mathcal{N}^{\otimes n}) = -H_{\min}^\varepsilon(E^n | A^n)_{(V\rho V^*)^{\otimes n}}$$

Now, by the strong AEP of conditional  $H_{\min}$  entropy (e.g. [27, Corollary 6.24] ),

$$\lim_{n \rightarrow \infty} \frac{1}{n} D_{\max}^{\varepsilon}(\rho^{\otimes n} \| \mathcal{N}^{\otimes n}) = \lim_{n \rightarrow \infty} -\frac{1}{n} H_{\min}^{\varepsilon}(E^n | A^n)_{(V\rho V^*)^{\otimes n}} = -H(E|A)_{(V\rho V^*)} = D(\rho \| \mathcal{N}),$$

where the last equality is again from the duality Theorem 9 for standard relative entropy  $D$  ( $\alpha = 1$  in the  $D_{\alpha}$  case). So we have proven

$$\lim_{n \rightarrow \infty} \frac{1}{n} D_{\max}^{\varepsilon}(\rho^{\otimes n} \| \mathcal{N}^{\otimes n}) = D(\rho \| \mathcal{N}).$$

and the argument for  $D_{\min}^{\varepsilon}$  is similar.

For hypothesis testing relative entropy, we use the following relation (see [1]) between  $D_{\max}^{\varepsilon}$  and  $D_H^{\varepsilon}$ , for  $\varepsilon \in (0, 1)$ ,

$$D_{\max}^{\sqrt{1-\varepsilon}}(\rho \| \sigma) \leq D_H^{\varepsilon}(\rho \| \sigma) + \log\left(\frac{1}{1-\varepsilon}\right). \quad (3.7)$$

Now taking minimum over all states in  $\mathcal{N}$  we have

$$D_{\max}^{\sqrt{1-\varepsilon}}(\rho \| \mathcal{N}) \leq D_H^{\varepsilon}(\rho \| \mathcal{N}) + \log\left(\frac{1}{1-\varepsilon}\right).$$

In terms of regularization, for any  $\varepsilon \in (0, 1)$

$$\lim_{n \rightarrow \infty} \frac{1}{n} D_H^{\varepsilon}(\rho^{\otimes n} \| \mathcal{N}^{\otimes n}) \geq \lim_{n \rightarrow \infty} \frac{1}{n} (D_{\max}^{\sqrt{1-\varepsilon}}(\rho^{\otimes n} \| \mathcal{N}^{\otimes n}) - \log\left(\frac{1}{\varepsilon}\right)) = D(\rho \| \mathcal{N})$$

That completes the proof. ■

**Remark 15.** We note that having the strong AEP

$$\lim_{n \rightarrow \infty} \frac{1}{n} D_{\max}^{\varepsilon}(\rho^{\otimes n} \| \mathcal{N}^{\otimes n}) = D(\rho \| \mathcal{N})$$

without letting  $\varepsilon \rightarrow 0$  is crucial in the argument to derive the (strong) Stein's Lemma

$$\lim_{n \rightarrow \infty} \frac{1}{n} D_H^{\varepsilon}(\rho^{\otimes n} \| \mathcal{N}^{\otimes n}) = D(\rho \| \mathcal{N}),$$

also without  $\varepsilon \rightarrow 0$ . This is because in the inequality (3.7), if  $\varepsilon \rightarrow 0$ , the smoothing parameter  $\sqrt{1-\varepsilon} \rightarrow 1$ .

#### 4. RESOURCE THEORY OF SUBALGEBRA COHERENCE

**4.1. Generalized coherence.** The Stein lemma and AEP above suggests that one can consider a resource theory with free states coming from any subalgebra. Let  $\mathcal{H}$  be a Hilbert space of dimension  $d$  and  $\mathcal{N} \subseteq B(\mathcal{H})$  be a von Neumann subalgebra. Before introducing the generalized coherence we introduce the notation of Pimsner-Popa index of a von Neumann subalgebra ([25]) which was motivated by Jone's index theory for subfactors ([19]).

**Definition 16.** Let  $\mathcal{E}_{\mathcal{N}} : B(\mathcal{H}) \rightarrow \mathcal{N}$  be the trace preserving conditional expectation. The **Pimsner-Popa index** for the inclusion  $\mathcal{N} \subseteq B(\mathcal{H})$  is defined as

$$\lambda(B(\mathcal{H}) : \mathcal{N}) = \max\{\lambda > 0 \mid \lambda x \leq \mathcal{E}_{\mathcal{N}}(x), \forall x \in \mathcal{B}(\mathcal{H})_+\},$$

Here  $B(\mathcal{H})_+$  denotes the set of all positive semidefinite operators on  $\mathcal{H}$ .

This subalgebra index was mainly introduced for the study of subfactor theory but here we restrict ourselves to the finite dimensional setting. By the structure theory of finite-dimensional unital  $*$ -algebras we know that  $\mathcal{N}$  is unitarily equivalent to the direct sum of matrix algebras:

$$\mathcal{N} = \bigoplus_k (1_{m_k} \otimes \mathcal{M}_{n_k}(\mathbb{C})),$$

for some unique choice of positive integers  $m_k, n_k$  with  $d = \sum_k m_k n_k$ . The following formula was obtained in [25, Theorem 6.1]

$$\lambda(B(\mathcal{H}) : \mathcal{N})^{-1} = \sum_k \min\{m_k, n_k\} n_k.$$

Note that the Pimsner-Popa index is always a fraction:  $\lambda(B(\mathcal{H}) : \mathcal{N}) \leq 1$ , and  $\lambda(B(\mathcal{H}) : \mathcal{N}) = 1$  if and only if  $\mathcal{N} = B(\mathcal{H})$ .

For example, for  $\mathcal{N}$  being the diagonal subalgebra,  $\lambda(B(\mathcal{H}) : \mathcal{N}) = \frac{1}{d}$ ; if  $B(\mathcal{H}) = \mathcal{M}_m(\mathbb{C}) \otimes \mathcal{M}_n(\mathbb{C})$  and  $\mathcal{N} = \mathcal{M}_m(\mathbb{C}) \otimes 1_n$ , then  $\lambda(B(\mathcal{H}) : \mathcal{N}) = \frac{1}{\min\{m, n\}n}$ .

We say a quantum state  $\rho \in S(\mathcal{H})$  is **incoherent with respect to  $\mathcal{N}$**  (or in short  $\mathcal{N}$ -incoherent) if  $\rho \in \mathcal{N}$ . We denote these states as  $\mathcal{S}(\mathcal{N})$ . The subalgebra entropies discussed in previous section can be viewed as  $\mathcal{N}$ -coherence measure. Indeed, for  $\mathbb{D} = D_{\max}, D_{\min}$  and  $D_\alpha, \alpha \in (1/2, \infty)$ ,  $\mathbb{D}(\rho || \mathcal{N}) = 0$  if and only if  $\rho \in \mathcal{N}$ . It is proved in [13] that these  $\mathcal{N}$ -coherence measures share a same picture for maximal coherence

$$\max_{\rho \in \mathcal{S}(\mathcal{N})} D_\alpha(\rho || \mathcal{N}) = \max_{\rho \in \mathcal{S}(\mathcal{N})} D_\alpha(\rho || \mathcal{E}_{\mathcal{N}}(\rho)) = \log \lambda(B(\mathcal{H}) : \mathcal{N})^{-1}.$$

The key observation was implicitly obtained in the Pimsner and Popa's work that for any finite dimensional inclusion  $\mathcal{N} \subset B(\mathcal{H})$ , there exists a projection  $e$  such that

$$\mathcal{E}_{\mathcal{N}}(e_{\mathcal{N}}) = \lambda(B(\mathcal{H}) : \mathcal{N})f,$$

where  $f$  is another projection. Then the flat state  $\rho_e = \frac{e}{\text{tr}(e)}$  attain the maximal  $D_\alpha(\cdot || \mathcal{N})$  for all  $\alpha \in [1/2, \infty]$ .

For example, when  $\mathcal{N} = \text{span}\{|j\rangle\langle j| : j = 1, \dots, d\}$  being the diagonal subalgebra, the **maximally coherent state**  $e = |\phi\rangle\langle\phi|$  can be a rank one projection with  $|\phi\rangle = \frac{1}{\sqrt{d}} \sum_{i=1}^d |i\rangle$  and the dephasing map gives

$$\mathcal{E}_{\mathcal{N}}(|\phi\rangle\langle\phi|) = \sum_j |\langle j|\phi\rangle|^2 |j\rangle\langle j| = \frac{1}{d} 1.$$

Motivated by this, we call the flat state  $\rho_e$  attaining the maximal subalgebra entropies the **maximally  $\mathcal{N}$ -coherent state**. We note that in general, its conditional expectation  $\mathcal{E}_{\mathcal{N}}(e_{\mathcal{N}}) = \lambda(B(\mathcal{H}) : \mathcal{N})f$  is only multiple of a projection, not necessarily multiple of the identity matrix. For example, if  $B(\mathcal{H}) = \mathcal{M}_m(\mathbb{C}) \otimes \mathcal{M}_n(\mathbb{C})$  and  $\mathcal{N} = \mathcal{M}_m(\mathbb{C}) \otimes 1_n$  with  $m > n$ , then the **maximally  $\mathcal{N}$ -coherent state** can be  $e = |\psi\rangle\langle\psi|$  with  $|\psi\rangle = \frac{1}{\sqrt{n}} \sum_{i=1}^n |i\rangle|i\rangle$ , and  $\mathcal{E}_{\mathcal{N}}(e) = \frac{1}{n} \sum_i |i\rangle\langle i|$  is a flat state but not full rank.

**4.2. Incoherent operations.** We consider each quantum system  $\mathcal{H}$  is equipped with a distinguished subalgebra  $\mathcal{N} \subset B(\mathcal{H})$ . In other words, each object is the inclusion  $\mathcal{N} \subset B(\mathcal{H})$  of a subalgebra. For two such systems  $\mathcal{N} \subset B(\mathcal{H})$  and  $\mathcal{M} \subset B(\mathcal{K})$ , we write  $\mathcal{E}_{\mathcal{N}}$  and  $\mathcal{E}_{\mathcal{M}}$  for the conditional expectations respectively.

**Definition 17.** Let  $\Phi : B(\mathcal{H}) \rightarrow B(\mathcal{K})$  be a quantum channel.



- i) we say  $\Phi : B(\mathcal{H}) \rightarrow B(\mathcal{K})$  is called  $\mathcal{N} - \mathcal{M}$  **maximally incoherent operation (MIO)** if  $\Phi(\mathcal{S}(\mathcal{N})) \subset \mathcal{S}(\mathcal{M})$ . Equivalently,  $\Phi$  is MIO if  $\Phi \circ \mathcal{E}_{\mathcal{N}} = \mathcal{E}_{\mathcal{M}} \circ \Phi \circ \mathcal{E}_{\mathcal{N}}$ .
- ii) we say  $\Phi : B(\mathcal{H}) \rightarrow B(\mathcal{K})$  is called  $\mathcal{N} - \mathcal{M}$  **dephasing-covariant incoherent operation (DIO)** if  $\Phi \circ \mathcal{E}_{\mathcal{N}} = \mathcal{E}_{\mathcal{M}} \circ \Phi$ .

We will also simply call  $\Phi$  with condition MIO operations and DIO operations if the algebra is clear/fixed from the context.

One immediate property of MIO operations is the monotonicity

$$D_{\alpha}(\rho || \mathcal{N}) \geq D_{\alpha}(\Phi(\rho) || \mathcal{M})$$

for any  $\alpha \in [1/2, \infty]$ . This follows from the monotonicity of  $D_{\alpha}$  under quantum channel and the fact  $\Phi(\mathcal{S}(\mathcal{N})) \subset \mathcal{S}(\mathcal{M})$ . In general, all DIO operations are MIO. For example, the conditional expectation  $\mathcal{E}_{\mathcal{N}} : B(\mathcal{H}) \rightarrow \mathcal{N}$  as the analog of completely dephasing map is both  $\mathcal{N}$ -MIO and  $\mathcal{N}$ -DIO. Here we do not discuss the analog of **incoherent operations (IO)** and **strictly incoherent operations (SIO)**, although one can define these concepts in the general subalgebra setting.

**4.3. Dilution w.r.t subalgebra coherence.** For each inclusion  $\mathcal{N} \subset B(\mathcal{H})$ , we fix an index achieving projection  $e_{\mathcal{N}}$  such that

$$\mathcal{E}_{\mathcal{N}}(e_{\mathcal{N}}) = \lambda(B(\mathcal{H}) : \mathcal{N})f,$$

where  $f \in \mathcal{N}$  is another projection. We emphasize that  $e_{\mathcal{N}}$  is always a projection outside of  $\mathcal{N}$ , but not in  $\mathcal{N}$ .

We introduce the notion of subalgebra coherence cost as follows.

**Definition 18.** i) Let  $\rho$  be a state in  $B(\mathcal{H})$  with the subalgebra  $\mathcal{N} \subseteq B(\mathcal{H})$ . For another subalgebra  $\mathcal{M} \subseteq B(\mathcal{K})$  and Hilbert space  $\mathcal{K}$  with the index  $\lambda(B(\mathcal{K}) : \mathcal{M})$  (which we write  $\lambda_{\mathcal{M}}$  for short), we define the minimum dilution fidelity

$$G_{\mathcal{N}}(\rho, \lambda_{\mathcal{M}}) := \inf_{\Phi \in \mathcal{O}} F(\rho, \Phi(e_{\mathcal{M}})),$$

where  $\mathcal{O} \in \{MIO, DIO\}$  denote a class of  $\mathcal{M} - \mathcal{N}$  incoherent operations  $\Phi : B(\mathcal{K}) \rightarrow B(\mathcal{H})$ . Given  $\epsilon \in [0, 1]$ , we define the **one-shot  $\epsilon$ - $\mathcal{N}$ -coherent cost** under operation  $\mathcal{O}$  as

$$C_{\mathcal{N}, \mathcal{O}}^{\epsilon}(\rho) := \inf_{\mathcal{M} \subseteq B(\mathcal{K})} \{\log \lambda_{\mathcal{M}}^{-1} | G(\rho, \lambda_{\mathcal{M}}) \geq 1 - \epsilon\},$$

where the infimum is over all Hilbert spaces  $\mathcal{K}$  and all subalgebras of  $\mathcal{M} \subseteq B(\mathcal{K})$ . The **asymptotic  $\mathcal{N}$ -coherence cost** is defined as

$$C_{\mathcal{N}, \mathcal{O}}^{\infty}(\rho) = \lim_{\epsilon \rightarrow 0} \lim_{n \rightarrow \infty} \frac{1}{n} C_{\mathcal{N}, \mathcal{O}}^{\epsilon}(\rho^{\otimes n}).$$

The next theorem gives an operational meaning of subalgebra relative entropy  $D(\cdot || \mathcal{N})$  as  $\mathcal{N}$ -coherence cost.

**Theorem 19.** For a given  $\rho \in B(\mathcal{H})$  and  $\epsilon \geq 0$ , let  $\mathcal{N} \subseteq B(\mathcal{H})$  be a subalgebra. Then

$$D_{\max}^{\epsilon}(\rho || \mathcal{N}) \leq C_{\mathcal{N}, MIO}^{\epsilon}(\rho) \leq D_{\max}^{\epsilon}(\rho || \mathcal{N}) + 1.$$

As an consequence, we have

$$C_{\mathcal{N}, MIO}^{\infty}(\rho) = D(\rho || \mathcal{N}).$$

*Proof.* The proof is similar to the case of coherence [31]. Suppose  $\log \lambda_{\mathcal{M}}^{-1} = C_{\mathcal{N}, MIO}^{\epsilon}(\rho)$  and hence there exists an MIO  $\Phi$  such that  $F(\rho, \Phi(e_{\mathcal{M}})) \geq 1 - \epsilon$ . If we let  $\rho' = \Phi(e_{\mathcal{M}})$ , then we have

$$\begin{aligned} D_{\max}^{\epsilon}(\rho \| \mathcal{N}) &\leq D_{\max}(\rho' \| \mathcal{N}) \\ &= \inf_{\sigma \in \mathcal{S}(\mathcal{N})} D_{\max}(\Phi(e_{\mathcal{M}}) \| \sigma) \\ &\leq D_{\max}(\Phi(e_{\mathcal{M}}) \| \Phi \circ \mathcal{E}_{\mathcal{M}}(e_{\mathcal{M}})) \\ &\leq D_{\max}(e_{\mathcal{M}} \| \mathcal{E}_{\mathcal{M}}(e_{\mathcal{M}})) \\ &= \log(\lambda_{\mathcal{M}}) \end{aligned}$$

The last equality follows from the fact the index is achieved by  $D_{\max}(e_{\mathcal{M}} \| \mathcal{E}_{\mathcal{M}}(e_{\mathcal{M}}))$ , that is,  $\mathcal{E}_{\mathcal{M}}(e_{\mathcal{M}}) = \lambda_{\mathcal{M}} f$ , for some projection  $f \geq e$ .

The other direction, let

$$D_{\max}^{\epsilon}(\rho \| \mathcal{N}) = D_{\max}(\rho' \| \sigma) = \log \mu ,$$

for a  $\rho'$  with  $F(\rho, \rho') \geq 1 - \epsilon$  and  $\sigma \in \mathcal{N}$ . Let  $n = \lceil \mu \rceil$ . By definition,  $n\sigma \geq \rho'$ . Now let  $\mathcal{K} = \mathbb{C}^n$  and  $\mathcal{M} \subseteq B(\mathcal{K})$  be the diagonal subalgebra of  $B(\mathcal{K})$ . Take the maximally coherent state  $e_{\mathcal{M}} = \frac{1}{n} \sum_{i=1}^n E_{i,i}$  satisfying  $\mathcal{E}_{\mathcal{M}}(e_{\mathcal{M}}) = \frac{1}{n} \mathbb{1}$ . Define the following map

$$\Phi(x) = \frac{n}{n-1} (1 - \text{tr}(e_{\mathcal{M}} x)) (\sigma - \frac{1}{n} \rho') + \text{tr}(e_{\mathcal{M}} x) \rho'.$$

As  $n\sigma \geq \rho'$ , the above map is completely positive, and send every state in  $B(\mathcal{K})$  to a state in  $B(\mathcal{H})$ , hence trace-preserving. Also, because  $\text{tr}(e_{\mathcal{M}} \gamma) = \frac{1}{n}$ , for any state  $\gamma \in \mathcal{M}$ , it follows that  $\Phi(\mathcal{S}(\mathcal{M})) \in \mathcal{S}(\mathcal{N})$ , which makes  $\Phi$  an  $\mathcal{M} - \mathcal{N}$  MIO operation. Note also that  $\Phi(e_{\mathcal{M}}) = \rho'$ . Hence we have

$$C_{\mathcal{N}, MIO}^{\epsilon}(\rho) \leq \log n \leq \log(1 + \mu) \leq 1 + \log \mu = 1 + D_{\max}^{\epsilon}(\rho \| \mathcal{N}).$$

This completes the proof of the first assertion of the theorem. The asymptotic result follows from the AEP Theorem 14.  $\blacksquare$

**Remark 20.** a) Thanks to the strong AEP, we do not need a vanishing  $\epsilon \rightarrow 0$  in the definition of the asymptotic coherence cost  $C_{\mathcal{N}, MIO}^{\infty}$ . Instead, we obtain a **strong converse** that for any positive fidelity error  $\epsilon$ ,  $C_{\mathcal{N}, MIO}^{\epsilon, \infty} := \lim_{n \rightarrow \infty} \frac{1}{n} C_{MIO}^{\epsilon}(\rho^{\otimes n}) = D(\rho \| \mathcal{N})$ .

b) In the proof of achievability, we use the standard diagonal subalgebra coherence to dilute the general coherent state  $\rho$  and our construction shows this is optimal. Similar construction can be made by using the subalgebra  $\mathcal{N} = \mathcal{M}_n(\mathbb{C}) \otimes 1_n$  inside  $B(\mathcal{H}) = \mathcal{M}_n(\mathbb{C}) \otimes \mathcal{M}_n(\mathbb{C})$ , then the **maximally  $\mathcal{N}$ -coherent state** is actually the maximally entangled state  $\psi = \frac{1}{n} \sum_{i,j=1}^n |i\rangle\langle j| \otimes |i\rangle\langle j|$ , and  $\mathcal{E}_{\mathcal{N}}(e) = \frac{1}{n} \mathbb{1}$  is a multiple of identity operator.

A similar result for DIO operations also holds. For that we introduce another definition of subalgebra max relative entropy:

$$\begin{aligned} D_{\max, \mathcal{E}_{\mathcal{N}}}(\rho) &:= D_{\max}(\rho \| \mathcal{E}_{\mathcal{N}}(\rho)) = \log \min\{\lambda > 0 \mid \rho \leq \lambda \mathcal{E}_{\mathcal{N}}(\rho)\} \\ D_{\max, \mathcal{E}_{\mathcal{N}}}^{\epsilon}(\rho) &= \inf\{D_{\max, \mathcal{E}_{\mathcal{N}}}(\rho') \mid F(\rho, \rho') \geq 1 - \epsilon\} \end{aligned}$$

It is clear from the definition that  $D_{\max, \mathcal{E}_{\mathcal{N}}}(\rho) \geq D_{\max}(\rho \| \mathcal{N})$  and  $D_{\max, \mathcal{E}_{\mathcal{N}}}^{\epsilon}(\rho) \geq D_{\max}^{\epsilon}(\rho \| \mathcal{N})$ , and both  $D_{\max, \mathcal{E}_{\mathcal{N}}}$  and  $D_{\max, \mathcal{E}_{\mathcal{N}}}^{\epsilon}$  are upper bounded by  $\log \lambda(B(\mathcal{H}) : \mathcal{N})^{-1}$ .

**Theorem 21.** Let  $\mathcal{N} \subseteq B(\mathcal{H})$  be a subalgebra and  $\mathcal{E}_{\mathcal{N}} : B(\mathcal{H}) \rightarrow \mathcal{N}$  be the trace preserving conditional expectation onto  $\mathcal{N}$ . Then for any  $\rho \in B(\mathcal{H})$  and  $\epsilon > 0$ , we have

$$D_{\max, \mathcal{E}_{\mathcal{N}}}^{\epsilon}(\rho) \leq C_{DIO}^{\epsilon}(\rho) \leq D_{\max, \mathcal{E}_{\mathcal{N}}}^{\epsilon}(\rho) + 1. \quad (4.1)$$

*Proof.* The direction  $D_{\max, \mathcal{E}_{\mathcal{N}}}^{\epsilon}(\rho) \leq C_{DIO}^{\epsilon}(\rho)$  follows similar to the proof of the Theorem 19. Indeed, if we take  $C_{\mathcal{N}, DIO}^{\epsilon}(\rho) = \log \lambda_{\mathcal{M}}^{-1}$  and suppose there exists a  $\mathcal{M}$ - $\mathcal{N}$  DIO operation  $\Phi$  such that  $F(\rho, \Phi(e_{\mathcal{M}})) \geq 1 - \epsilon$ . Let  $\rho' = \Phi(e_{\mathcal{M}})$ . Then

$$\begin{aligned} D_{\max, \mathcal{E}_{\mathcal{N}}}^{\epsilon}(\rho) &\leq D_{\max, \mathcal{E}_{\mathcal{N}}}(\rho') \\ &= D_{\max}(\rho' \| \mathcal{E}_{\mathcal{N}}(\rho')) \\ &= D_{\max}(\Phi(e_{\mathcal{M}}) \| \mathcal{E}_{\mathcal{N}}(\Phi(e_{\mathcal{M}}))) \\ &\stackrel{(1)}{=} D_{\max}(\Phi(e_{\mathcal{M}}) \| \Phi(\mathcal{E}_{\mathcal{M}}(e_{\mathcal{M}}))) \\ &\leq D_{\max}(e_{\mathcal{M}} \| \mathcal{E}_{\mathcal{M}}(e_{\mathcal{M}})) \\ &\leq \log \lambda_{\mathcal{M}} \end{aligned}$$

where the equality (1) uses the definition of DIO  $\Phi \circ \mathcal{E}_{\mathcal{M}} = \mathcal{E}_{\mathcal{N}} \circ \Phi$ .

For the upper bound, we again let  $\rho'$  be the state where  $D_{\max, \mathcal{E}_{\mathcal{N}}}^{\epsilon}(\rho)$  is attained such that  $D_{\max, \mathcal{E}_{\mathcal{N}}}^{\epsilon}(\rho) = D_{\max, \mathcal{E}_{\mathcal{N}}}(\rho') = \log \lambda$ , i.e.  $\rho' \leq \lambda \mathcal{E}_{\mathcal{M}}(\rho')$ . Let  $n = \lceil \lambda \rceil$ . And let  $\mathcal{K} = \mathbb{C}^n$  and consider the diagonal subalgebra  $\mathcal{M} \subseteq B(\mathcal{K})$  with  $e_{\mathcal{M}}$  being the usual maximally coherent state and define the map

$$\Phi(x) = \frac{n}{n-1} [1 - \text{tr}(e_{\mathcal{M}}x)] (\mathcal{E}_{\mathcal{N}}(\rho') - \frac{1}{n}\rho') + \text{tr}(e_{\mathcal{M}}x)\rho'.$$

As  $\rho' \leq \lambda \mathcal{E}_{\mathcal{M}}(\rho')$ , the above map is completely positive. To prove that it is a DIO we compute, for any state  $\gamma \in B(\mathcal{K})$ ,

$$\Phi(\mathcal{E}_{\mathcal{M}}(\gamma)) = \frac{n}{n-1} (1 - \frac{1}{n}) (\mathcal{E}_{\mathcal{N}}(\rho') - \frac{1}{n}\rho') + \frac{1}{n}\rho' = \mathcal{E}_{\mathcal{N}}(\rho').$$

The factor  $(1 - \frac{1}{n})$  in the above calculation comes from the fact that  $\text{tr}(e_{\mathcal{M}}\gamma) = \frac{1}{n}$ , for all state  $\gamma \in \mathcal{M}$ . On the other hand, for a general state  $\sigma \in \mathcal{S}(\mathcal{K})$

$$\mathcal{E}_{\mathcal{N}}(\Phi(\sigma)) = \frac{n}{n-1} [1 - \text{tr}(e_{\mathcal{M}}\sigma)] (\mathcal{E}_{\mathcal{N}}(\rho') - \frac{1}{n}\mathcal{E}_{\mathcal{N}}(\rho')) + \text{tr}(e_{\mathcal{M}}\sigma)\mathcal{E}_{\mathcal{N}}(\rho') = \mathcal{E}_{\mathcal{N}}(\rho').$$

Hence  $\mathcal{E}_{\mathcal{N}} \circ \Phi = \Phi \circ \mathcal{E}_{\mathcal{M}}$ , which shows  $\Phi$  is  $\mathcal{M}$ - $\mathcal{N}$  DIO.

Finally, it follows that  $\Phi(e_{\mathcal{M}}) = \rho'$  and hence we have

$$C_{DIO}^{\epsilon}(\rho) \leq \log n \leq 1 + \log \lambda = 1 + D_{\max, \mathcal{E}_{\mathcal{N}}}^{\epsilon}(\rho') = D_{\max, \mathcal{E}_{\mathcal{N}}}^{\epsilon}(\rho) + 1. \quad \blacksquare$$

## 5. OUTLOOK

In this paper we established that the state space of subalgebras provide a big class of free states for which the generalized Stien's lemma holds. This motivated us to look deeper into the resource theory of coherence in this general set up. We provided an operational meaning to the subalgebra entropy in the context of resource dilution. Our result can be used in the analyses of quantum information processing tasks that exploit coherence as resources, such as quantum key distribution and random number generation. For future directions, we would like to focus on the coherence distillation in particular, and establish the asymptotic coherence distillation results. Also, our

approach allows us to extend the framework of the current work in the paradigm of infinite dimensional setting, which we leave it as a future endeavour. Indeed, since the index theory for subalgebras has been a widely popular topic in operator algebras, we expect to extend some of our results in a more general context.

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