Recent developments on compactifications of stacks of shtukas

Tuan Ngo Dac, Yakov Varshavsky

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TUAN NGO DAC AND YAKOV VARSHAVSKY

Abstract. This text presents an overview of recent developments on compactifications of moduli stacks of shtukas. The aim is to explain how to tackle the problem of compactifying stacks of shtukas by two different methods: the Langton semistable reduction and the Geometric Invariant Theory.

Contents

1. Introduction

Let X be a geometrically connected smooth projective curve over a finite field $\mathbb{F}_q$ and let F denote its function field. In the seventies Drinfeld [4, 5, 6] invented several fundamental objects called *Drinfeld modules* and *shtukas* over this curve which are analogues of elliptic curves in the function field setting. In this setting, moduli spaces or stacks classifying these objects are analogues of Shimura varieties. Drinfeld [6, 7, 8] studied the moduli space of Drinfeld shtukas of rank 2 and its compactifications and proved the Langlands correspondence for GL_2 over F. Subsequently, Lafforgue [12, 13, 14] studied the moduli space of Drinfeld shtukas of arbitrary rank r and its compactifications to prove the Langlands correspondence for GL_r over F.

The notion of shtukas can be generalized for other reductive groups over F. Laumon, Rapoport and Stuhler [23] used the moduli space of

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shtukas for division algebras or $\mathcal{D}$-shtukas and proved the local Langlands conjecture for GL_r over local fields of equal characteristic. Similar moduli stacks of $\mathcal{D}$-shtukas were studied by Lafforgue [12], Lau [19, 20], and Ngo Bao-Chau [26]. For a more general reductive group G over F, stacks of G-shtukas were introduced and studied by the second author [30] (for G split) and V. Lafforgue [16, 17]. We mention that moduli stacks of G-shtukas for flat affine group schemes G of finite type over X are constructed by Arasteh Rad and Hartl in [1].

In the aforementioned works of Drinfeld and Lafforgue, a major challenge is to construct compactifications of stacks of shtukas. The main goal of this paper is to report on recent developments in this problem. This text is mainly based on the notes from the first author's talk at the BIRS-CMO workshop "Moduli, Motives and Bundles – New Trends in Algebraic Geometry" in Oaxaca, Mexico in 2022, as well as the authors' work on this topic. Details and proofs will appear in the paper [28].

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2. Stacks of GL_r-shtukas

In this section we review the definition and basic properties of stacks of GL_r-shtukas, which generalize the moduli stacks of Drinfeld shtukas studied by Drinfeld [6, 7, 8], Lafforgue [12, 13, 14, 15] (see also Varshavsky [30], V. Lafforgue [16, 17] and Arasteh Rad-Hartl [1] for a more general setting).

2.1. Notation.

Let $\mathbb{F}_q$ be a finite field with q elements. We denote by p the characteristic of $\mathbb{F}_q$ and $\overline{\mathbb{F}}_q$ be an algebraic closure of $\mathbb{F}_q$.

Let X be a geometrically connected smooth projective curve over $\mathbb{F}_q$. Denote by F its function field. We always consider schemes and stacks over $\mathbb{F}_q$. Letting Y and Z be schemes or stacks over $\mathbb{F}_q$, we use $Y \times Z$ to denote the fiber product of Y and Z over $\mathbb{F}_q$ and by $\mathrm{Frob}_Y : Y \longrightarrow Y$ the absolute Frobenius morphism of Y over $\mathbb{F}_q$. The latter is the identity on the underlying topological space and the q-th power on the structure sheaf. For any scheme S over $\mathbb{F}_q$, if $\mathcal{E}$ is a vector bundle over $X \times S$, then we write

$$\mathcal{E}^\sigma := (\mathrm{id}_X \times \mathrm{Frob}_S)^* \mathcal{E}.$$

Let r be a positive integer and $G = \mathrm{GL}_r$ the general linear group of rank r. Let $T \subset G$ be the split maximal torus of diagonal matrices and let $B \subset G$ be the Borel subgroup of upper triangular matrices. We identify the set of characters and that of cocharacters with $\mathbb{Z}^r$ as follows. A character (resp. cocharacter) of T is always of the form

$$t = (t_1, \dots, t_r) \mapsto t_1^{\lambda^1} \dots t_r^{\lambda^r}$$

(resp. $t \mapsto (t^{\lambda^1}, \dots, t^{\lambda^r})$) for some $\lambda = (\lambda^1, \dots, \lambda^r) \in \mathbb{Z}^r$, and we identify it with the element $\lambda = (\lambda^1, \dots, \lambda^r)$ of $\mathbb{Z}^r$. The natural pairing is therefore identified with the natural product $\mathbb{Z}^r \times \mathbb{Z}^r \longrightarrow \mathbb{Z}$. The set of roots Φ (resp. of positive roots Φ^+) with respect to our choice (B, T) consists of the elements of the forms $e_i - e_j$ with $1 \leq i \neq j \leq r$ (resp. with $1 \leq i < j \leq r$). The set of simple roots Δ consists of the elements of the forms $e_i - e_{i+1}$ with $1 \leq i \leq r-1$.

Let $\lambda = (\lambda^1, \dots, \lambda^r) \in \mathbb{Z}^r$ be a character (resp. a cocharacter). We say that λ is dominant if $(\lambda^1, \dots, \lambda^r)$ lies in the subset $(\mathbb{Z}^r)_+$ of decreasing sequences of integers, i.e., $\lambda^1 \geq \dots \geq \lambda^r$. We set

$$\deg \lambda := \sum_{i=1}^{r} \lambda^i.$$

Further, let $\mu = (\mu^1, \dots, \mu^r)$ be another character (resp. a cocharacter), we say that $\lambda \leq \mu$ if and only if

$$\begin{aligned} \lambda^1 &\leq \mu^1, \\ \lambda^1 + \lambda^2 &\leq \mu^1 + \mu^2, \\ &\dots \\ \lambda^1 + \dots + \lambda^{r-1} &\leq \mu^1 + \dots + \mu^{r-1}, \\ \lambda^1 + \dots + \lambda^r &= \mu^1 + \dots + \mu^r. \end{aligned}$$

2.2. **Vector bundles on curves.**

Recall that a vector bundle of rank r over X is a locally free sheaf of rank r on X. Let Bun_r be the functor which assigns to a scheme S the groupoid of vector bundles of rank r over $X \times S$, where morphisms are isomorphisms of vector bundles. This stack Bun_r is called the stack classifying vector bundles of rank r over X.

Proposition 2.1. *The stack* Bun_r *is a smooth Artin stack, locally of finite type over* $\mathbb{F}_q$.

Proof. By [22, Théorème (4.6.2.1)], Bun_r is an Artin stack locally of finite type over $\mathbb{F}_q$. By the deformation theory, Bun_r is formally smooth, and the proposition follows.

The interested reader may refer to [11, §1 and §2], [21, §2.3], [25, Theorem 2.67], and [30, Lemma 3.1] for different proofs of this proposition. These texts provide additional approaches to prove the mentioned proposition. □

Proposition 2.2. *For any integer* $d \in \mathbb{Z}$, *we denote by* Bun_r^d *the substack of* Bun_r *classifying vector bundles of rank* r *and of degree* d. *Then the connected components of* Bun_r *are exactly* Bun_r^d $(d \in \mathbb{Z})$.

Proof. See for example [30, Lemma 2.2 and Appendix A.1] where a more general statement is proved. □

2.3. **Harder-Narasimhan truncation.**

Definition 2.3. By a rational Harder-Narasimhan (HN for short) polygon, we mean an element $p = (p(0), p(1), \ldots, p(r))$ of $\mathbb{Q}^{r+1}$ such that

i) $p(0) = p(r) = 0$,

ii) p is convex, i.e., for every $1 \leq i \leq r$, we put

$$\Delta p(i) := p(i) - p(i-1)$$

and we require that $\Delta p(i) \geq \Delta p(i+1)$ for $1 \leq i < r$.

In this section, let κ be an algebraically closed field and let $\mathcal{E}$ be a vector bundle of rank r over $X \times \kappa$. For a subbundle $\mathcal{F}$ of $\mathcal{E}$, the normalized degree of $\mathcal{F}$ is defined by

$$\mathrm{ndeg}_{\mathcal{E}} \mathcal{F} := \deg \mathcal{F} - \frac{\mathrm{rk}\, \mathcal{F}}{\mathrm{rk}\, \mathcal{E}} \deg \mathcal{E}.$$

Definition 2.4. We keep the above notation. We say that $\mathcal{E}$ is semistable (resp. stable) if for every nonzero subbundle $\mathcal{F}$ of $\mathcal{E}$, we have the inequality

$$\frac{\deg \mathcal{F}}{\mathrm{rk}\, \mathcal{F}} \leq \frac{\deg \mathcal{E}}{\mathrm{rk}\, \mathcal{E}},$$

(resp. $\frac{\deg \mathcal{F}}{\mathrm{rk}\, \mathcal{F}} < \frac{\deg \mathcal{E}}{\mathrm{rk}\, \mathcal{E}}$).

In other words, $\mathcal{E}$ is semistable (resp. stable) if for every nonzero subbundle $\mathcal{F}$ of $\mathcal{E}$, we have $\mathrm{ndeg}_{\mathcal{E}} \mathcal{F} \leq 0$ (resp. $\mathrm{ndeg}_{\mathcal{E}} \mathcal{F} < 0$).

We are ready to introduce the notion of Harder–Narasimhan stratification of vector bundles.

Proposition 2.5. *There exists a canonical filtration*

$$0 =: \mathcal{E}_0 \subsetneq \mathcal{E}_1 \subsetneq \cdots \subsetneq \mathcal{E}_n := \mathcal{E}$$

such that for every $0 \leq i \leq n-1$ *we have*

1) $\frac{\deg \mathcal{E}_i}{\operatorname{rk} \mathcal{E}_i} > \frac{\deg \mathcal{E}_{i+1}}{\operatorname{rk} \mathcal{E}_{i+1}}$,
2) $\mathcal{E}_{i+1}/\mathcal{E}_i$ *is semistable.*

Remark 2.6. 1) It follows that the set of points $\{(\operatorname{rk} \mathcal{E}_i, \deg \mathcal{E}_i - \operatorname{rk} \mathcal{E}_i \frac{\deg \mathcal{E}}{\operatorname{rk} \mathcal{E}}) : 0 \leq i \leq n\}$ defines a HN polygon as introduced in Definition 2.3. This is called the Harder-Narasimhan polygon associated to the vector bundle $\mathcal{E}$. The canonical filtration is also called the Harder-Narasimhan filtration of $\mathcal{E}$.

2) Another consequence is that for every subbundle $\mathcal{F}$ of $\mathcal{E}$, the point $(\operatorname{rk} \mathcal{F}, \deg \mathcal{F} - \frac{\operatorname{rk} \mathcal{F}}{\operatorname{rk} \mathcal{E}} \deg \mathcal{E})$ lies below the Harder-Narasimhan of $\mathcal{E}$.

3) If we consider a family of vector bundles on X, then the Harder-Narasimhan polygon can only get bigger under specialization.

Definition 2.7. Let p be a HN polygon. We will say that the Harder-Narasimhan polygon of $\mathcal{E}$ is bounded by p if and only if for every subbundle $\mathcal{F}$ of $\mathcal{E}$, we have the inequality

$$\deg \mathcal{F} \leq \frac{\operatorname{rk} \mathcal{F}}{\operatorname{rk} \mathcal{E}} \deg \mathcal{E} + p(\operatorname{rk} \mathcal{F}),$$

or equivalently,

$$\operatorname{ndeg}_{\mathcal{E}} \mathcal{F} \leq p(\operatorname{rk} \mathcal{F}).$$

Proposition 2.8. *Let p be a HN polygon and $d \in \mathbb{Z}$. Then the substack $\operatorname{Bun}_r^{d,\leq p}$ of Bun_r^d is open and of finite type.*

Proof. See for example [30, Appendix A.3 and A.4]. □

2.4. **Hecke stacks.**

We introduce the Hecke stack of modifications Hecke_r following Beilinson and Drinfeld [2].

Definition 2.9. The Hecke stack Hecke_r is the stack over the category of schemes which, for every scheme S, classifies the data $E = (x, \mathcal{E}, \mathcal{E}', \varphi)$ where

i) $x \in X(S)$ called the leg of E,

ii) $\mathcal{E}$ and $\mathcal{E}'$ are vector bundles of rank r over $X \times S$,

iii) $\varphi : \mathcal{E}|_{X \times S - \Gamma_x} \xrightarrow{\sim} \mathcal{E}'|_{X \times S - \Gamma_x}$ is an isomorphism of these vector bundles outside the graph Γ_x of x.

Let κ be an algebraically closed field and let $E = (x, \mathcal{E}, \mathcal{E}', \varphi)$ be a κ-point of Hecke_r. We denote by $\mathcal{O}_x$ the completion of $\mathcal{O}_{X \times \kappa}$ at x and by F_x its fraction field. Then φ_x allows us to identify $\mathcal{E}|_{\operatorname{Spec} F_x}$ and $\mathcal{E}'|_{\operatorname{Spec} F_x}$ which are both vector spaces of dimension r over F_x. Consider two lattices $\mathcal{E}|_{\operatorname{Spec} \mathcal{O}_x}$ and $\mathcal{E}'|_{\operatorname{Spec} \mathcal{O}_x}$ in this vector space. Let ϖ_x be a uniformizing element of $\mathcal{O}_x$. By the theory of elementary divisors, there exist a basis $(e_1, e_2, \ldots, e_r)$ of $\mathcal{E}|_{\operatorname{Spec} \mathcal{O}_x}$ and a dominant coweight $\lambda = (\lambda^1, \ldots, \lambda^r) \in (\mathbb{Z}^r)_+$ such that $(\varpi_x^{\lambda^1} e_1, \ldots, \varpi_x^{\lambda^r} e_r)$ is a basis of $\mathcal{E}'|_{\operatorname{Spec} \mathcal{O}_x}$. We denote λ by $\operatorname{inv}(E)$. Remark that we always have the equality

$$\deg \mathcal{E}' = \deg \mathcal{E} + \deg \operatorname{inv}(E).$$

Definition 2.10. Let $\lambda \in (\mathbb{Z}^r)_+$ be a dominant coweight of GL_r. It is well-known (see [2]) that there is a closed reduced substack $\mathrm{Hecke}_{r,\lambda}$ of Hecke_r such that for every algebraically closed field κ, $\mathrm{Hecke}_{r,\lambda}(\kappa)$ consists of $E = (x, \mathcal{E}, \mathcal{E}', \varphi) \in \mathrm{Hecke}_r(\kappa)$ such that $\mathrm{inv}(E) \leq \lambda$.

2.5. **Iterated version of Hecke stacks.**

Notation 2.11. From now on we will fix an integer $n \geq 1$ and denote by I the finite set $\{1, \ldots, n\}$. Further, we fix a collection $\underline{\lambda} = (\lambda_i)_{i \in I}$ of dominant coweights of GL_r such that

$$\deg \underline{\lambda} := \sum_{i=1}^{n} \deg \lambda_i = 0. \tag{2.1}$$

Definition 2.12. We define the iterated Hecke stack $\mathrm{Hecke}_{r,\underline{\lambda}}$ which, to every scheme S classifies the data

$$((x_i)_{i \in I}, \mathcal{E}_0, \ldots, \mathcal{E}_n, (\varphi_i)_{i \in I})$$

(or $(x, \mathcal{E}, \varphi)$ for short) where for $i \in I$, we require that

$$(x_i, \mathcal{E}_{i-1}, \mathcal{E}_i, \varphi_i) \in \mathrm{Hecke}_{r,\lambda_i}(S).$$

It should be noted that there is a non-iterated version of these stacks that consider only one modification. This can be found in [30, Definition 2.4].

2.6. **Stacks of shtukas.**

Definition 2.13. a) For every scheme S, the stack $\mathrm{Sht}_{r,\underline{\lambda}}$ of shtukas of rank r with modifications bounded by $\underline{\lambda}$ classifies the data

$$E = ((x_i)_{i \in I}, \mathcal{E}_0, \ldots, \mathcal{E}_n, (\varphi_i)_{i \in I}, \psi)$$

(or $(x, \mathcal{E}, \varphi, \psi)$ for short) such that

i) $(x, \mathcal{E}, \varphi) \in \mathrm{Hecke}_{r,\underline{\lambda}}(S)$,

ii) $\psi : \mathcal{E}_0^{\sigma} \xrightarrow{\sim} \mathcal{E}_n$ is an isomorphism of vector bundles of rank r over $X \times S$.

b) For every integer d, we define $\mathrm{Sht}^d_{r,\underline{\lambda}}$ to be the substack of $\mathrm{Sht}_{r,\underline{\lambda}}$ classifying the data $(x, \mathcal{E}, \varphi, \psi)$ with $\deg \mathcal{E}_0 = d$.

Remark 2.14. 1) Recall that we have assumed (see (2.1)) that

$$\deg \underline{\lambda} := \sum_{i=1}^{n} \deg \lambda_i = 0.$$

This condition is needed in order to get a non-empty stack $\mathrm{Sht}_{r,\underline{\lambda}}$ (see [30, Proposition 2.16 (d)]).

2) The stacks $\mathrm{Sht}_{r,\underline{\lambda}}$, $\mathrm{Sht}^d_{r,\underline{\lambda}}$ ($d \in \mathbb{Z}$) are Artin stacks locally of finite type.

3) We have a Cartesian diagram

$$\begin{array}{ccc} \mathrm{Sht}^d_{r,\underline{\lambda}} & \longrightarrow & \mathrm{Bun}_r \\ \downarrow & & \downarrow \\ \mathrm{Hecke}^d_{r,\underline{\lambda}} & \longrightarrow & \mathrm{Bun}_r \times \mathrm{Bun}_r, \end{array}$$

where

- the upper horizontal map is
$$E = ((x_i)_{i\in I}, \mathcal{E}_0, \dots, \mathcal{E}_n, (\varphi_i)_{i\in I}, \psi) \mapsto \mathcal{E}_0,$$
- the lower horizontal map is
$$((x_i)_{i\in I}, \mathcal{E}_0, \dots, \mathcal{E}_n, (\varphi_i)_{i\in I}) \mapsto (\mathcal{E}_0, \mathcal{E}_n),$$
- the left vertical map is

$$E = ((x_i)_{i\in I}, \mathcal{E}_0, \dots, \mathcal{E}_n, (\varphi_i)_{i\in I}, \psi) \mapsto ((x_i)_{i\in I}, \mathcal{E}_0, \dots, \mathcal{E}_n, (\varphi_i)_{i\in I}),$$

- the right vertical map is
$$\mathcal{E}_0 \mapsto (\mathcal{E}_0, \mathcal{E}_0^\sigma).$$

Proposition 2.15. *Let* $\mathrm{Sht}^{\leq p}_{r,\underline{\lambda}}$ *(resp.* $\mathrm{Sht}^{d,\leq p}_{r,\underline{\lambda}}$*) be the substack of* $\mathrm{Sht}_{r,\underline{\lambda}}$ *(resp.* $\mathrm{Sht}^d_{r,\underline{\lambda}}$*) classifying shtukas of rank* r *bounded by* p*, i.e., the data* $E = (x, \mathcal{E}, \varphi, \psi)$ *such that the Harder-Narasimhan polygon of the corresponding vector bundle* $\mathcal{E}_0$ *is bounded by* p*. Then it is an open substack of* $\mathrm{Sht}_{r,\underline{\lambda}}$ *and the connected components* $\mathrm{Sht}^{d,\leq p}_{r,\underline{\lambda}}$ *of this substack are of finite type.*

Proof. See [30, Proposition 2.16] where a more general result is proved. □

2.7. **Example: stacks of Drinfeld shtukas.**

When $I = \{1, 2\}$ and $\underline{\lambda} = (\lambda_1, \lambda_2)$ where $\lambda_1 = (1, 0, \dots, 0) \in \mathbb{Z}^r$ and $\lambda_2 = (0, \dots, 0, -1) \in \mathbb{Z}^r$, the corresponding stack is known as the stack Sht_r classifying Drinfeld shtukas of rank r, studied by Drinfeld [6, 7, 8] for $r = 2$ and by Lafforgue [12, 13, 14] for general r. For every scheme S, $\mathrm{Sht}_r(S)$ classifies the following data

$$\mathcal{E} \hookrightarrow \mathcal{E}' \hookleftarrow \mathcal{E}'' \xleftarrow{\sim} \mathcal{E}^\sigma$$

where $\mathcal{E}$, $\mathcal{E}'$ and $\mathcal{E}''$ are vector bundles of rank r over $X \times S$ and $\mathcal{E} \hookrightarrow \mathcal{E}'$ and $\mathcal{E}'' \hookrightarrow \mathcal{E}'$ are injective homomorphisms of vector bundles such that

- the supports of these maps are the graphs of morphisms $\infty : S \to X$ and $0 : S \to X$, respectively,
- the cokernels of these maps are the direct images of locally free of rank 1 as $\mathcal{O}_S$-modules on the graphs of ∞ and 0, respectively.

2.8. **Compactifying stacks of shtukas.**

For every HN polygon p, the substack $\mathrm{Sht}_{r,\underline{\lambda}}^{\leq p}$ (resp. $\mathrm{Sht}_{r,\underline{\lambda}}^{d,\leq p}$) is an open substack of finite type of $\mathrm{Sht}_{r,\underline{\lambda}}$ (resp. $\mathrm{Sht}_{r,\underline{\lambda}}^{d}$). However, they are not proper.

In the works of Drinfeld and Lafforgue, a major challenge lies in constructing compactifications for the corresponding substacks $\mathrm{Sht}_{r,\underline{\lambda}}^{\leq p}$ of the stack of Drinfeld shtukas Sht_{r}^{d} (as discussed in §2.7). In the general case, it would be interesting to construct compactifications for the substack $\mathrm{Sht}_{r,\underline{\lambda}}^{d,\leq p}$ of $\mathrm{Sht}_{r,\underline{\lambda}}^{d}$.

To do so, we follow the strategy of Drinfeld [8] and Lafforgue [13, 15] which can be described as follows.

- **The notion of generalized shtukas.** Drinfeld and Lafforgue start by introducing the concept of generalized shtukas, which expands that of shtukas by considering pseudo-complete homomorphisms instead of isomorphisms. The stacks classifying these generalized shtukas have a stratification, with the open stratum corresponding to $\mathrm{Sht}_{r,\underline{\lambda}}$.
- **Stability conditions.** The next and crucial step is to find stability conditions for the truncating substacks to ensure properness.

There are two methods to find stability conditions:

(1) **The semistable reduction à la Langton.** One approach is the semistable reduction à la Langton, which involves educated guessing and the use of the valuative criterion for properness. This method was successfully applied by Langton [18] to prove the properness of moduli spaces of semistable vector bundles on a smooth projective curve (even variety) over a field. We refer the reader to the works of Heinloth [9, 10] for another semistable reduction proof of this result using the affine Grassmannian.

(2) **The Geometric Invariant Theory (GIT).** Another method uses the GIT theory developed by Munford among others (see for example [24]), which is a powerful and conceptual approach. Stability conditions are derived from the Hilbert-Mumford criterion from GIT, although this is a difficult and lengthy task in many cases. Again this method was successfully applied by Seshadri [29] to prove the properness of moduli spaces of semistable vector bundles on a smooth projective curve over a field.

In what follows we explain how to employ both of these methods to tackle the problem of compactifying stacks of shtukas.

3. Generalized GL_r-shtukas

In this section we review the notion of generalized GL_r-shtukas, introduce stacks classifying these objects and present their basic properties.

3.1. **Pseudo-complete homomorphisms.**

We recall the scheme of complete homomorphisms given in [13] (see also [20, §4]) and that of pseudo-complete homomorphisms. As noticed in *loc. cit.*, this scheme is closely related to the De Concini-Procesi wonderful compactification [3] and the Vinberg semigroup [31].

We consider the immersion

$$\mathrm{GL}_r \times \mathbb{G}_m^{r-1} \to \prod_{i=1}^{r} \mathrm{End}(\wedge^i \mathbb{A}^r) \times \mathbb{A}^{r-1},$$

which sends $(g, \alpha_1, \dots, \alpha_{r-1}) \in \mathrm{GL}_r \times \mathbb{G}_m^{r-1}$ to $(u_1, \dots, u_r, \alpha_1, \dots, \alpha_{r-1})$ where

$$\begin{aligned} u_1 &= g, \\ \wedge^2 u_1 &= \alpha_1 u_2, \\ \wedge^3 u_1 &= \alpha_1^2 \alpha_2 u_3, \\ &\vdots \\ \wedge^r u_1 &= \alpha_1^{r-1} \alpha_2^{r-2} \dots \alpha_{r-1} u_r. \end{aligned}$$

We observe that the image of $\mathrm{GL}_r \times \mathbb{G}_m^{r-1}$ lies in $\prod_{i=1}^{r} \big(\mathrm{End}(\wedge^i \mathbb{A}^r) - \{0\}\big) \times \mathbb{A}^{r-1}$.

Definition 3.1. a) We define the scheme $V^0_{\mathrm{GL}_r}$ of complete homomorphisms to be the schematic closure of $\mathrm{GL}_r \times \mathbb{G}_m^{r-1}$ in $\prod_{i=1}^{r} \big(\mathrm{End}(\wedge^i \mathbb{A}^r) - \{0\}\big) \times \mathbb{A}^{r-1}$.

b) We define the scheme V_{GL_r} of pseudo-complete homomorphisms to be the schematic closure of $\mathrm{GL}_r \times \mathbb{G}_m^{r-1}$ in $\prod_{i=1}^{r} \mathrm{End}(\wedge^i \mathbb{A}^r) \times \mathbb{A}^{r-1}$.

Let $\mathfrak{r} = (r_1, \dots, r_{k-1})$ be a strictly increasing sequence of integers between 0 and r, i.e., $0 < r_1 < \dots < r_{k-1} < r$. Put $r_0 = 0$ and $r_k = r$. Denote by $V^0_{\mathrm{GL}_r, \mathfrak{r}}$ the locally closed subscheme of $V^0_{\mathrm{GL}_r}$ defined by requiring that $\alpha_i = 0$ for every $i \in \mathfrak{r}$ and that α_i is invertible for every $i \notin \mathfrak{r}$. Then $V^0_{\mathrm{GL}_r, \mathfrak{r}}$ admits a modular interpretation (see [13, §1, Proposition 1]).

3.2. **Stacks of generalized shtukas.**

Definition 3.2. We define the stack of generalized shtukas $\mathrm{GenSht}_{r,\underline{\lambda}}$ which for every scheme S, classifies the data

$$(x_1, \dots, x_n, \mathcal{E}_0, \dots, \mathcal{E}_n, \varphi_1, \dots, \varphi_n, \mathcal{L}_1, \dots, \mathcal{L}_{r-1}, l_1, \dots, l_{r-1}, u_1, \dots, u_r)$$

(or $(x, \mathcal{E}, \varphi, \mathcal{L}, l, u)$ for short) where

i) $(x, \mathcal{E}, \varphi)$ is an object of $\mathrm{Hecke}_{r,\underline{\lambda}}(S)$,

ii) for every $1 \leq i \leq r-1$, $\mathcal{L}_i$ is a line bundle over S and l_i is a global section of $\mathcal{L}_i$, i.e., $(\mathcal{L}_i, l_i) \in [\mathbb{A}^1/\mathbb{G}_m](S)$,

iii) u is a pseudo-complete homomorphism $\mathcal{E}_0^\sigma \Rightarrow \mathcal{E}_n$ over $(\mathcal{L}^{\otimes(q-1)}, l^{q-1})$, i.e., for every $1 \leq i \leq r$,

$$u_i : \big(\wedge^i \mathcal{E}_0 \otimes \bigotimes_{1\leq j<i} \mathcal{L}_j^{i-j}\big)^\sigma \longrightarrow \wedge^i \mathcal{E}_n \otimes \bigotimes_{1\leq j<i} \mathcal{L}_j^{i-j}$$

is a homomorphism of vector bundles such that the following conditions hold:

a) For any choice of trivialization of $\mathcal{E}_0$, $\mathcal{E}_n$ and $\mathcal{L}_i$ locally on $X \times S$, the family $(u_1, \ldots, u_r, l_1, \ldots, l_{r-1})$ lies in V_{GL_r}.

b) If we identify $\mathcal{E}_0$ and $\mathcal{E}_n$ outside the union of the graphs of x_j $(j \in I)$, then for all $1 \leq i \leq r$, u_i can be iterated and we require that none of u_i are nilpotent at any geometric point of S.

For every integer d, let $\mathrm{GenSht}^d_{r,\underline{\lambda}}$ be the substack of $\mathrm{GenSht}_{r,\underline{\lambda}}$ classifying the data $(x, \mathcal{E}, \varphi, \mathcal{L}, l, u)$ with $\deg \mathcal{E}_0 = d$.

3.3. **Stratification.**

The stratification of $[\mathbb{A}^1/\mathbb{G}_m]^{r-1}$ induces a stratification of $\mathrm{GenSht}_{r,\underline{\lambda}}$ as follows. Let $\mathfrak{r} = (r_1, \ldots, r_{k-1})$ be an increasing sequence of integers between 0 and r. We put $r_0 = 0$. Let $\mathrm{GenSht}_{r,\underline{\lambda},\mathfrak{r}}$ be the locally closed substack of $\mathrm{GenSht}_{r,\underline{\lambda}}$ such that $l_i = 0$ for every $i \in \mathfrak{r}$ and l_i is invertible for every $i \notin \mathfrak{r}$.

Let κ be an algebraically closed field. Let $E = (x, \mathcal{E}, \varphi, \mathcal{L}, l, u)$ be a κ-point of $\mathrm{GenSht}_{r,\underline{\lambda},\mathfrak{r}}$. We will say that E is of type $\mathfrak{r}$. We choose a trivialization of line bundles $\mathcal{L}_{r_i}$ over κ. Then the pseudo-complete homomorphism $u : \mathcal{E}_0^\sigma \Rightarrow \mathcal{E}_n$ over $(\mathcal{L}^{\otimes(q-1)}, l^{q-1})$ gives rise to the data consisting of

i) a decreasing filtration $\overline{\mathcal{E}}_{0,*} = (\overline{\mathcal{E}}_{0,0} = \mathcal{E}_0^\sigma \supsetneq \overline{\mathcal{E}}_{0,1} \supsetneq \cdots \supsetneq \overline{\mathcal{E}}_{0,k-1} \supsetneq \overline{\mathcal{E}}_{0,k} = 0)$ by subbundles $\overline{\mathcal{E}}_{0,i}$ of $\mathcal{E}_0^\sigma$ of rank $r - r_i$,

ii) an increasing filtration $\mathcal{E}_{n,*} = (0 = \mathcal{E}_{n,0} \subsetneq \mathcal{E}_{n,1} \subsetneq \cdots \subsetneq \mathcal{E}_{n,k-1} \subsetneq \mathcal{E}_{n,k} = \mathcal{E}_n)$ by subbundles $\mathcal{E}_{n,i}$ of $\mathcal{E}_n$ of rank r_i,

iii) for every $1 \leq i \leq k$, an isomorphism over the generic point

$$v_i : \overline{\mathcal{E}}_{0,i-1}/\overline{\mathcal{E}}_{0,i} \to \mathcal{E}_{n,i}/\mathcal{E}_{n,i-1}$$

of vector bundles.

Remark 3.3. 1) For all $1 \leq i \leq k$ we always have an injection

$$\det\big(\overline{\mathcal{E}}_0/\overline{\mathcal{E}}_{0,i}\big) \hookrightarrow \det \mathcal{E}_{n,i}$$

of vector bundles of the same rank.

2) Since $\mathcal{E}_n$ is obtained from $\mathcal{E}_0$ by a chain of modifications, the increasing filtration $\mathcal{E}_{n,*}$ induces an increasing filtration

$$\mathcal{E}_{0,*} = (0 = \mathcal{E}_{0,0} \subsetneq \mathcal{E}_{0,1} \subsetneq \cdots \subsetneq \mathcal{E}_{0,k-1} \subsetneq \mathcal{E}_{0,k} = \mathcal{E}_0)$$

by subbundles $\mathcal{E}_{0,i}$ of $\mathcal{E}_0$ of rank r_i. The nilpotent conditions of u_i are equivalent to the fact that for every $1 \leq i \leq k$, we have

$$\overline{\mathcal{E}}_{0,i} \cap \mathcal{E}_{0,i}^\sigma = 0.$$

4. Stability conditions for Drinfeld shtukas

In this section we work with Drinfeld shtukas given as in §2.7 and review the stability conditions introduced by Drinfeld [8] for $r = 2$ and Lafforgue [13] for general r.

4.1. **Notation.**

Recall that $I = \{1, 2\}$ and $\underline{\lambda} = (\lambda_1, \lambda_2)$ where $\lambda_1 = (1, 0, \dots, 0) \in \mathbb{Z}^r$ and $\lambda_2 = (0, \dots, 0, -1) \in \mathbb{Z}^r$. We will use the notation of Lafforgue: a Drinfeld shtuka of rank r is represented by the data

$$(\mathcal{E} \hookrightarrow \mathcal{E}' \hookleftarrow \mathcal{E}'' \xleftarrow{\sim} \mathcal{E}^\sigma),$$

and a generalized Drinfeld shtuka of rank r is represented by the data

$$E = (\mathcal{E} \hookrightarrow \mathcal{E}' \hookleftarrow \mathcal{E}'' \Leftarrow \mathcal{E}^\sigma, (\mathcal{L}, l)).$$

We will drop the subscriptions $\underline{\lambda}$ and denote the stacks classifying Drinfeld shtukas of rank r and generalized Drinfeld shtukas of rank r by Sht_r and GenSht_r respectively.

4.2. **Stratification and stability conditions.**

4.2.1. *Stratification.* Let $\mathfrak{r} = (r_1, \dots, r_{k-1})$ be an increasing sequence of integers between 0 and r. We put $r_0 = 0$. Let κ be an algebraically closed field. Let

$$E = (\mathcal{E} \hookrightarrow \mathcal{E}' \hookleftarrow \mathcal{E}'' \Leftarrow \mathcal{E}^\sigma, (\mathcal{L}, l))$$

be a κ-point of $\mathrm{GenSht}_{r,\mathfrak{r}}$, so it is of type $\mathfrak{r}$. We choose a trivialization of line bundles $\mathcal{L}_{r_i}$ over κ and will drop $(\mathcal{L}, l)$ in the data of E. Then the pseudo-complete homomorphism $\mathcal{E}^\sigma \Rightarrow \mathcal{E}''$ gives rise to

i) a decreasing filtration $\overline{\mathcal{E}}_* = (\overline{\mathcal{E}} = \mathcal{E}^\sigma \supsetneq \overline{\mathcal{E}}_1 \supsetneq \cdots \supsetneq \overline{\mathcal{E}}_{k-1} \supsetneq \overline{\mathcal{E}}_k = 0)$ by subbundles $\overline{\mathcal{E}}_i$ of $\mathcal{E}^\sigma$ of rank $r - r_i$,

ii) an increasing filtration $\mathcal{E}''_* = (0 = \mathcal{E}''_0 \subsetneq \mathcal{E}''_1 \subsetneq \cdots \subsetneq \mathcal{E}''_{k-1} \subsetneq \mathcal{E}''_k = \mathcal{E}'')$ by subbundles $\mathcal{E}''_i$ of $\mathcal{E}''$ of rank r_i,

iii) for every $1 \leq i \leq k$, an isomorphism over the generic point

$$v_i : \overline{\mathcal{E}}_{i-1}/\overline{\mathcal{E}}_i \to \mathcal{E}''_i/\mathcal{E}''_{i-1}$$

of vector bundles.

As before, we denote by $\mathcal{E}_*$ and $\mathcal{E}'_*$ the increasing filtration of $\mathcal{E}$ and $\mathcal{E}'$ respectively induced by $\mathcal{E}''_*$. Recall that for every $1 \leq i \leq k$, we have $\overline{\mathcal{E}}_{0,i} \cap \mathcal{E}^\sigma_{0,i} = 0$.

4.2.2. *Iterated shtukas.* Following Lafforgue [13, §1, Part c)], we introduce the notion of iterated shtukas.

Definition 4.1. Let E be a generalized Drinfeld shtuka of rank r and of type $\mathfrak{r}$ over κ as in §4.2.1. We say that E is an iterated shtuka if the following conditions hold:

(1) For every $1 \leq i \leq k$, the isomorphism over the generic point
$$v_i : \overline{\mathcal{E}}_{i-1}/\overline{\mathcal{E}}_i \to \mathcal{E}''_i/\mathcal{E}''_{i-1}$$
extends to an isomorphism of vector bundles.
(2) For every $1 \leq i \leq k$, the injection $\mathcal{E}''_i \hookrightarrow \mathcal{E}'_i$ is an isomorphism.
(3) For every $1 \leq i \leq k$, the composition $\mathcal{E}'_i \to \mathcal{E}' \to \mathcal{E}'/\mathcal{E}$ is surjective.
(4) For every $1 \leq i < k$, we have $\overline{\mathcal{E}}_i + \mathcal{E}^\sigma_{i+1} = \mathcal{E}^\sigma$.

The stack classifying iterated (Drinfeld) shtukas of rank r will be denoted by $\overline{\mathrm{Sht}}_r$.

Remark 4.2. It is important to note that for iterated shtukas, the maps v_i are isomorphisms rather than isomorphisms over the generic point. This distinction arises from the specific choice of $\underline{\lambda}$ for Drinfeld shtukas and has significant implications for the geometric properties of the stack of Drinfeld shtukas and its compactifications.

4.2.3. *Good objects of an iterated shtuka.* We keep the above notation and suppose that $E = (\mathcal{E} \hookrightarrow \mathcal{E}' \hookleftarrow \mathcal{E}'' \Leftarrow \mathcal{E}^\sigma)$ is an iterated shtuka of rank r and of type $\mathfrak{r}$ over κ. By definition, a good object of E consists of a pair $(\mathcal{F}, \mathcal{F}')$ of subbundles of $\mathcal{E}$ and $\mathcal{E}'$ respectively such that

- $\mathcal{F} \hookrightarrow \mathcal{F}'$ and $\operatorname{rk}\mathcal{F} = \operatorname{rk}\mathcal{F}'$;
- there exists some $1 \leq i \leq k$ such that $\mathcal{E}_{i-1} \subsetneq \mathcal{F} \subseteq \mathcal{E}_i$ and $\mathcal{E}'_{i-1} \subsetneq \mathcal{F}' \subseteq \mathcal{E}'_i$;
- the injection $\overline{\mathcal{E}}_{i-1}/\overline{\mathcal{E}}_i \xrightarrow{\sim} \mathcal{E}''_i/\mathcal{E}''_{i-1} \hookrightarrow \mathcal{E}'_i/\mathcal{E}'_{i-1}$ sends the quotient $\mathcal{F}^\sigma \cap \overline{\mathcal{E}}_{i-1}/\mathcal{F}^\sigma \cap \overline{\mathcal{E}}_i$ to $\mathcal{F}' \cap \mathcal{E}'_i/\mathcal{F}' \cap \mathcal{E}'_{i-1}$.

Let $(\mathcal{F}, \mathcal{F}')$ be a good object of E. Then we say that

- it is of type I if either $i = 1$ or $\mathcal{F}^\sigma + \overline{\mathcal{E}}_{i-1} = \mathcal{E}^\sigma$;
- it is of type II if $i > 1$ and $\mathcal{F}^\sigma + \overline{\mathcal{E}}_{i-1} \subsetneq \mathcal{E}^\sigma$.

It should be noted that if $r = 2$ which is the case studied by Drinfeld, every good object of E is of type I.

4.2.4. *Stability conditions.* Let p be a HN polygon. We define $\overline{\mathrm{Sht}}^{\leq p}_{r,\mathfrak{r}}$ to be the substack of $\overline{\mathrm{Sht}}_{r,\mathfrak{r}}$ such that E belongs to $\overline{\mathrm{Sht}}^{\leq p}_{r,\mathfrak{r}}(\kappa)$ if the following conditions hold:

- For all $1 \leq i \leq k$,
$$p(r_i) - 1 < \deg \mathcal{E}_i - \frac{\operatorname{rk}\mathcal{E}_i}{\operatorname{rk}\mathcal{E}} \deg \mathcal{E} \leq p(r_i).$$
- For all good objects $(\mathcal{F}, \mathcal{F}')$ of E which is of type I,
$$\deg \mathcal{F} - \frac{\operatorname{rk}\mathcal{F}}{\operatorname{rk}\mathcal{E}} \deg \mathcal{E} \leq p(\operatorname{rk}\mathcal{F}).$$
- For all good objects $(\mathcal{F}, \mathcal{F}')$ of E which is of type II,
$$\deg \mathcal{F} - \frac{\operatorname{rk}\mathcal{F}}{\operatorname{rk}\mathcal{E}} \deg \mathcal{E} \leq p(\operatorname{rk}\mathcal{F}) - 1.$$

4.3. **Compactifications and Langton semistable reduction.**
The main theorem of Lafforgue in [13] which generalizes the work of Drinfeld [8] reads (see [13, Théorème 12, page 1015]):

Theorem 4.3. *1) There exists a unique open substack* $\overline{\mathrm{Sht}}_r^{\leq p}$ *of* $\overline{\mathrm{Sht}}_r$ *whose trace in each stratum* $\overline{\mathrm{Sht}}_{r,\mathfrak{r}}$ *is* $\overline{\mathrm{Sht}}_{r,\mathfrak{r}}^{\leq p}$.

2) Suppose that p *is sufficiently convex. Then the morphism*

$$\overline{\mathrm{Sht}}_r^{\leq p} \to X \times X$$

is proper.

The proof of the theorem for the properness of the moduli stack of shtukas is based on the semistable reduction technique introduced by Langton [18]. The valuative criterion for properness is used in the proof, and the main steps for the existence part are explained below. For a more detailed proof, we refer the reader to [13, 14].

- **Step 1: Extend a shtuka on the generic fiber.**
  We first focus on the generic fiber with a shtuka E bounded by p that we have to degenerate, with its φ-space structure, and introduces the notion of an iterated lattice in this φ-space. Drinfeld showed that there are infinitely many iterated lattices M, that each induces a degeneration $E(M)$ of the shtuka and that our task is to show that one has the properties dictated by the chosen polygon p.
- **Step 2: Find a modification of a generalized shtuka.**
  We then introduce the procedure of elementary transformations to move from one iterated lattice to another. If the special fiber E^M does not belong to the substack $\overline{\mathrm{Sht}}_r^{\leq p}$, the procedure can be applied to find a suitable modification $E(M')$ of E, corresponding to an elementary transformation M' of M.
- **Step 3: Show that after a finite number of elementary transformations, we get a generalized shtuka bounded by p.**
  In Langton's original paper [18] and in various works using his semistable reduction, a quantity $\beta(M) \in \mathbb{Z}$ is associated to each lattice M. This quantity has the property that $\beta(M) \leq 0$ if and only if the lattice M is the desired one. The proof shows that after a suitable modification, the new lattice M' is "less stable", meaning $\beta(M') \leq \beta(M)$. After a finite number of modifications, a strict inequality must be obtained, thus completing the proof.
  However, defining the quantity $\beta(M)$ is a difficult task, and the proof described in [13, 14] is much more intricate than the simplified explanation presented here.

5. Stability conditions for GL_r-shtukas

In the study of stacks $\mathrm{GenSht}_{r,\underline{\lambda}}$ that classify GL_r-shtukas with arbitrary modifications, finding stability criteria is very challenging even with educated guessing. However, the application of the GIT machinery, as discussed in [27], provides a means to obtain stability conditions for these stacks. The use of GIT allows stability conditions to be deduced from the Hilbert-Mumford criterion, although this is still a difficult and lengthy task.

5.1. **Stratification and stability conditions.**

5.1.1. *Stratification.* We use the notation of §3.3. Let $\mathfrak{r} = (r_1, \dots, r_{k-1})$ be an increasing sequence of integers between 0 and r. We put $r_0 = 0$. Let κ be an algebraically closed field. Let $E = (x, \mathcal{E}, \varphi, \mathcal{L}, l, u)$ be a κ-point of $\mathrm{GenSht}_{r,\underline{\lambda},\mathfrak{r}}$. We choose a trivialization of line bundles $\mathcal{L}_{r_i}$ over κ. Recall that we have

i) a decreasing filtration $\overline{\mathcal{E}}_{0,*} = (\overline{\mathcal{E}}_{0,0} = \mathcal{E}_0^\sigma \supsetneq \overline{\mathcal{E}}_{0,1} \supsetneq \cdots \supsetneq \overline{\mathcal{E}}_{0,k-1} \supsetneq \overline{\mathcal{E}}_{0,k} = 0)$ by subbundles $\overline{\mathcal{E}}_{0,i}$ of $\mathcal{E}_0^\sigma$ of rank $r - r_i$,

ii) an increasing filtration $\mathcal{E}_{n,*} = (0 = \mathcal{E}_{n,0} \subsetneq \mathcal{E}_{n,1} \subsetneq \cdots \subsetneq \mathcal{E}_{n,k-1} \subsetneq \mathcal{E}_{n,k} = \mathcal{E}_n)$ by subbundles $\mathcal{E}_{n,i}$ of $\mathcal{E}_n$ of rank r_i,

iii) for every $1 \leq i \leq k$, an isomorphism over the generic point

$$v_i : \overline{\mathcal{E}}_{0,i-1}/\overline{\mathcal{E}}_{0,i} \hookrightarrow \mathcal{E}_{n,i}/\mathcal{E}_{n,i-1}$$

of vector bundles.

The increasing filtration $\mathcal{E}_{n,*}$ induces an increasing filtration

$$\mathcal{E}_{0,*} = (0 = \mathcal{E}_{0,0} \subsetneq \mathcal{E}_{0,1} \subsetneq \cdots \subsetneq \mathcal{E}_{0,k-1} \subsetneq \mathcal{E}_{0,k} = \mathcal{E}_0)$$

by subbundles $\mathcal{E}_{0,i}$ of $\mathcal{E}_0$ of rank r_i. It is required that for every $1 \leq i \leq k$, we have

$$\overline{\mathcal{E}}_{0,i} \cap \mathcal{E}_{0,i}^\sigma = 0.$$

Definition 5.1. In the above notation, for every HN polygon p and every subbundle $\mathcal{F}$ of $\mathcal{E}_0$, we define

$$p_E(\mathcal{F}) := \sum_{i=1}^{k} \big(\Delta p(r_{i-1}+1) + \cdots + \Delta p(r_{i-1} + \mathrm{rk}(\mathcal{F}^\sigma \cap \overline{\mathcal{E}}_{0,i-1}/\mathcal{F}^\sigma \cap \overline{\mathcal{E}}_{0,i}))\big).$$

The p-defect of $\mathcal{F}$ is defined by

$$\begin{aligned} \mathrm{def}_p \mathcal{F} &:= \mathrm{ndeg}_{\mathcal{E}_0} \mathcal{F} - p_E(\mathcal{F}) \\ &= \deg \mathcal{F} - \frac{\mathrm{rk}\, \mathcal{F}}{r} \deg \mathcal{E}_0 - p_E(\mathcal{F}). \end{aligned}$$

Remark 5.2. In our recent work [28], the notion of p-defect is introduced and is shown to play a central role in that paper. The details of this concept and its significance will be explained in the subsequent section of the paper.

5.1.2. *Stability conditions.* Let p be a HN polygon as in Definition 2.3.

We define $\mathrm{GenSht}_{r,\underline{\lambda}}^{\leq p}$ to be the largest substack of $\mathrm{GenSht}_{r,\underline{\lambda}}$ such that a geometric point $E = (x, \mathcal{E}, \varphi, \mathcal{L}, l, u)$ of $\mathrm{GenSht}_{r,\underline{\lambda}}$ lies in $\mathrm{GenSht}_{r,\underline{\lambda}}^{\leq p}$ if and only if for every subbundle $\mathcal{F}$ of $\mathcal{E}_0$, we have the inequality

$$\mathrm{ndeg}_{\mathcal{E}_0} \mathcal{F} \leq p_E(\mathcal{F})$$

where we recall that $\mathrm{ndeg}_{\mathcal{E}_0} \mathcal{F} = \deg \mathcal{F} - \frac{\mathrm{rk}\,\mathcal{F}}{r} \deg \mathcal{E}_0$. In this case, we will say that the generalized shtuka $(x, \mathcal{E}, \varphi, \mathcal{L}, l, u)$ is bounded by p. We we also denote by $\mathrm{GenSht}_{r,\underline{\lambda},\mathfrak{r}}^{\leq p}$ (resp. $\mathrm{GenSht}_{r,\underline{\lambda}}^{d,\leq p}$) the intersection of $\mathrm{GenSht}_{r,\underline{\lambda}}^{\leq p}$ with $\mathrm{GenSht}_{r,\underline{\lambda},\mathfrak{r}}$ (resp. $\mathrm{GenSht}_{r,\underline{\lambda}}^{d}$).

Proposition 5.3. *The stack* $\mathrm{GenSht}_{r,\underline{\lambda}}^{\leq p}$ *is an open substack of* $\mathrm{GenSht}_{r,\underline{\lambda}}$. *Moreover, it contains* $\mathrm{Sht}_{r,\underline{\lambda}}^{\leq p}$ *as an open substack.*

5.2. Compactifications and GIT.

Definition 5.4. Let d be an integer and let p be a HN polygon. We will say that p is d-generic if for every integer $0 < k < r$ and for every sequence of integers $(r_1, \ldots, r_k)$ such that $0 < r_1 < \cdots < r_k \leq r$, the number

$$\frac{k}{r} d + \Delta p(r_1) + \cdots + \Delta p(r_k)$$

is not an integer.

In [27] the first author uses the Hilbert-Mumford criterion from GIT to prove the following result (see [27, Théorème VII.3.3]):

Theorem 5.5. *Let d be an integer and p be a d-generic, sufficiently convex HN polygon. Suppose that q which is the cardinal of the base finite field $\mathbb{F}_q$ is sufficiently big. Then the morphism*

$$\mathrm{GenSht}_{r,\underline{\lambda}}^{d,\leq p} \longrightarrow X^n$$

is proper.

The proof of this theorem regarding the properness of the mentioned stack is known to be long and involves complicated calculations. The interested reader can refer to [27] for a detailed explanation and the complete proof of the theorem.

5.3. Relations with the works of Drinfeld and Lafforgue.

We work again with Drinfeld shtukas, i.e., $I = \{1, 2\}$ and $\underline{\lambda} = (\lambda_1, \lambda_2)$ where $\lambda_1 = (1, 0, \ldots, 0) \in \mathbb{Z}^r$ and $\lambda_2 = (0, \ldots, 0, -1) \in \mathbb{Z}^r$. Let d be an integer and p be a d-generic, sufficiently convex HN polygon. For all $0 < k < r$ we set

$$q(k) := \left\lfloor p(k) - \frac{k}{r} d \right\rfloor + \frac{k}{r} d,$$

where $\lfloor . \rfloor$ denotes the floor function. In particular, $q(k) < p(k) < q(k) + 1$ as p is d-generic.

In [27] it is also shown that the family of compactifications constructed by Lafforgue as presented in §4 lies in the family of compactifications given in the previous section if the polygon p is well chosen. More precisely, it is proved that (see [27, Théorème VI.1.1]):

Theorem 5.6. *We keep the above notation. We suppose that $\{p(k) - q(k)\}_{0<k<r}$ is a strictly decreasing sequence. Let $E = (\mathcal{E} \hookrightarrow \mathcal{E}' \hookleftarrow \mathcal{E}'' \Leftarrow \mathcal{E}^\sigma, (\mathcal{L}, l)) \in \mathrm{GenSht}_{r,\mathfrak{r}}(\kappa)$ be a generalized Drinfeld shtuka over an algebraically closed field κ. Then the following conditions are equivalent:*

(1) For every subbundle $\mathcal{F}$ of $\mathcal{E}$, we have the inequality

$$\deg \mathcal{F} - \frac{\mathrm{rk}\,\mathcal{F}}{r} \deg \mathcal{E} \leq p_E(\mathcal{F})$$

where we recall that $p_E(\mathcal{F})$ is given by

$$p_E(\mathcal{F}) = \sum_{i=1}^{k} \big(\Delta p(r_{i-1}+1) + \cdots + \Delta p(r_{i-1} + \mathrm{rk}(\mathcal{F}^\sigma \cap \overline{\mathcal{E}}_{i-1}/\mathcal{F}^\sigma \cap \overline{\mathcal{E}}_i))\big).$$

(2) E belongs to $\overline{\mathrm{Sht}}^{\leq p}_{r,\mathfrak{r}}$ (see §4.2.4).

Remark 5.7. It is important to note that stability conditions implies that E verifies the list of conditions for iterated shtukas listed in Definition 4.1. This implication follows from the specific choice of $\underline{\lambda}$ for Drinfeld shtukas.

6. Langton semistable reduction for GL_r-shtukas

In this section, we provide a concise overview of our recent work where we present a proof of the previous theorem using the semistable reduction technique inspired by Langton. Further, we improve Theorem 5.5 by removing the technical condition that q is sufficiently big.

Theorem 6.1. *Let d be an integer and let p be a d-generic, sufficiently convex HN polygon. Then the morphism*

$$\mathrm{GenSht}^{d,\leq p}_{r,\underline{\lambda}} \longrightarrow X^n$$

is proper.

We sketch some ideas of the proof. Details and proofs will be given in [28]. The key concept used in our proof is the notion of the p-defect of a subbundle, as defined in Definition 5.1. Notably, even for stacks of Drinfeld shtukas, our proof offers a more direct and conceptual approach compared to the works of Lafforgue [13, 14]. Additionally, the semistable reduction method allows us to immediately extend our results to construct compactifications of stacks of shtukas for division algebras or $\mathcal{D}$-shtukas, thus addressing a problem previously investigated by Lafforgue [13], Lau [19, 20], and Ngo Bao-Chau [26].

6.1. **Setup.**

Let A be a discrete valuation ring (DVR) whose quotient field is denoted by K and whose residue field is κ which is supposed to be algebraically closed. Let π be a uniformizer and v the valuation of A.

We consider the regular surface $X \otimes A$. Denote by A_X the local ring in $X \otimes A$ of the generic point of the special fiber $X \otimes \kappa$. It is a DVR. The quotient field K_{A_X} of A_X is the function field of the generic fiber $X \otimes K$ and the residue field κ_{A_X} of A_X is the function field of the special fiber $X \otimes \kappa$. The element π is still a uniformizer of A_X. Denote by $\widehat{A}_X$ (resp. $\widehat{K}_{A_X}$, $\widehat{\kappa}_{A_X}$) be the completion of A_X (resp. K_{A_X}, κ_{A_X}).

To prove the result we check the valuative criterion for properness. We will sketch a proof of the existence part in a particular case, where the K-point lies in the open stratum. Let $E = (x, \mathcal{E}, \varphi, \mathcal{L}, l, u)$ be a shtuka in $\mathrm{Sht}^{d,\leq p}_{r,,\underline{\lambda}}$ over $\operatorname{Spec} K$ and we want to extend it and get a generalized shtuka in $\mathrm{GenSht}^{d,\leq p}_{r,,\underline{\lambda}}$ over $\operatorname{Spec} O_{K'}$ after we replace K with a finite extension K'.

6.2. **Step 1: Extend a shtuka on the generic fiber.**

6.2.1. *Lattice and extension of vector bundles.* Since $X \otimes A$ is a regular surface, the following result is well-known (see for example [18, §3, Proposition 6]):

Proposition 6.2. *The functor which to every vector bundle on $X \otimes A$ associates its restriction $\mathcal{E}$ on the generic fiber $X \otimes K$ and its fiber M on $\operatorname{Spec} A_X$, is an equivalence of categories between the category of vector bundles of rank r on $X \otimes A$ and that of pairs $(\mathcal{E}, M)$ where*

i) $\mathcal{E}$ is a vector bundle of rank r on the generic curve $X \otimes K$,

ii) M is an A_X-lattice in the generic fiber V of $\mathcal{E}$, which is a vector space of dimension r over K_{A_X}.

The above functor admits a quasi-inverse functor denoted by

$$(\mathcal{E}, M) \mapsto \mathcal{E}(M).$$

We denote by $\mathcal{E}^M$ the induced vector bundle on the special fiber $X \otimes \kappa$ whose generic fiber is $V^M = M/\pi M$.

6.2.2. *Iterated lattices.* Recall that $E = (x, \mathcal{E}, \varphi, \mathcal{L}, l, u)$ be a shtuka in $\mathrm{Sht}^{d,\leq p}_{r,,\underline{\lambda}}$ over $\operatorname{Spec} K$. Let V be the generic fiber of $\mathcal{E}_0$. It is a vector space of dimension r over the field K_{A_X}. Then the morphism φ appearing in E induces an isomorphism $\varphi : V^\sigma \xrightarrow{\sim} V$ viewed as a semilinear map $\varphi : V \longrightarrow V$. We define $\widehat{V} := V \otimes_{K_{A_X}} \widehat{K}_{A_X}$ and denote by $\widehat{\varphi}$ the induced semilinear map $\widehat{\varphi} : \widehat{V} \longrightarrow \widehat{V}$. Then the map $M \mapsto M \otimes_{A_X} \widehat{A}_X$ is a bijection between A_X-lattices in V and $\widehat{A}_X$-lattices in $\widehat{V}$.

Definition 6.3. Let M be an A_X-lattice in V which is stable by φ.

a) We say that M is a *non-degenerated lattice* (with respect to φ) if $A_X\varphi(M) = M$. Otherwise, we say that M is *degenerated*.

b) We say that M is an *admissible lattice* if the reduction map

$$\overline{\varphi} : M/\pi M \longrightarrow M/\pi M$$

is not nilpotent.

c) We say that M is an *iterated lattice* if there exist non-negative integers $d_1, d_2, \ldots, d_{r-1}$ such that, if for every $1 \leq i \leq r$, we set

$$u_s = \Big(\prod_{1 \leq i < s} \pi^{d_i(s-i)} \Big)^{-(q-1)} \wedge^s \varphi,$$

then each u_s induces a well-define map

$$u_s : \wedge^s M \longrightarrow \wedge^s M$$

and moreover the reduction $\overline{u}_s : \wedge^s V^M \longrightarrow \wedge^s V^M$ is not nilpotent.

We define the type $\mathfrak{r}$ of the iterated lattice M the sequence of indices $(r_1, \ldots, r_{k-1})$ such that $0 < r_1 < \cdots < r_{k-1} < r$ and that for every $1 \leq i < k$, $r_i \in \mathfrak{r}$ if and only if $d_{r_i} \geq 1$. We put $r_0 = 0$ and $r_k = r$.

Drinfeld showed that among the set of lattices in V which are stable by φ, there is a unique maximal one. Further, if a lattice is non-degenerated, then it is maximal. By [8] (see also [13, §2, Part a)]), after some finite extension of K, the new maximal lattice is admissible. Then we deduce that after some finite extension of K, there exists an iterated lattice in V (see [13, §2, Proposition 4]).

6.2.3. *Extension of shtukas.* We observe (see [13, §2, Part b)]) that to extend a shtuka E over K to a generalized shtuka (resp. a shtuka) over A is equivalent to choose an iterated lattice M (resp. a non-degenerated lattice) of V. By the previous discussion there is at most one non-degenerated lattice. Thus we get

Proposition 6.4. *The stack* $\mathrm{Sht}_{r,\underline{\lambda}}$ *is separated.*

Notation 6.5. Let M be an iterated lattice of V. We will denote the corresponding generalized shtuka over A by $E(M)$ and its restriction to the special fiber $X \times \kappa$ by E^M. We consider the generalized shtuka $E(M)$ over A and the generalized shtuka E^M over the residue field κ. Then $\mathfrak{r}$ is the type of E^M, and we get

i) a strictly decreasing filtration $\overline{V}^M = \overline{V}_0^M \supsetneq \overline{V}_1^M \supsetneq \cdots \supsetneq \overline{V}_{k-1}^M \supsetneq \overline{V}_k^M = 0$ by subspaces of $\overline{V}^M = (V^M)^\sigma$ of codimension r_1,...,r_{k-1},

ii) a strictly increasing filtration $0 \subsetneq V_1^M \subsetneq \cdots \subsetneq V_{k-1}^M \subsetneq V_k^M = V^M$ by subspaces of V^M of dimension r_1,...,r_{k-1},

iii) these filtrations are equipped with isomorphisms between successive quotients

$$v_i : \overline{V}_{i-1}^M / \overline{V}_i^M \xrightarrow{\sim} V_i^M / V_{i-1}^M.$$

6.3. **Step 2: Find a modification of a generalized shtuka.**

6.3.1. *Modified lattices.* Let M be a lattice of V and W a subspace of V^M. Following Langton [18] we define the new lattice M' of V by

$$M' := \ker(M \twoheadrightarrow V^M \twoheadrightarrow V^M/W)$$

and we write $M' = \mathrm{Mod}_W(M)$.

We denote by W' the subspace of $V^{M'}$ given by $W' = \ker(V^{M'} \longrightarrow W)$. Then we have two short exact sequences (see [18, §4]):

$$0 \longrightarrow \mathcal{E}^{M'}_{0,W'} \longrightarrow \mathcal{E}^{M'}_0 \longrightarrow \mathcal{E}^{M}_{0,W} \longrightarrow 0$$
$$0 \longrightarrow \mathcal{E}^{M}_{0,W} \longrightarrow \mathcal{E}^{M}_0 \longrightarrow \mathcal{E}^{M'}_{0,W'} \longrightarrow 0.$$

6.3.2. *Modified iterated lattices.* We suppose further that M is an iterated lattice in V. Let W be a subspace of V^M and $M' = \mathrm{Mod}_W(M)$. In general, this modified lattice M' is not necessarily an iterated lattice.

We now present an important class of subspaces of V^M such that the corresponding modified lattice is always an iterated one. Letting W a subspace of V^M, we say that W is *very good* if there exists i such that $V^M_{i-1} \subsetneq W \subset V^M_i$ and the isomorphism $v_i : \overline{V}^M_{i-1}/\overline{V}^M_i \xrightarrow{\sim} V^M_i/V^M_{i-1}$ induces an isomorphism $W^\sigma \cap \overline{V}^M_{i-1} \xrightarrow{\sim} W/V^M_{i-1}$.

Lemma 6.6. *Let W be a very good subspace of V^M. Then the modified lattice $M' := \mathrm{Mod}_W(M)$ is an iterated lattice of V.*

6.3.3. *Defect and modification.* For the rest of this section we are in the situation of Notation 6.5. Recall that the notion of p-defect for subbundles of $\mathcal{E}^M_0$ is given as in Definition 5.1. We define

$$\mathrm{def}_p M := \max_{\mathcal{F}} \{\mathrm{def}_p \mathcal{F}\}$$

where $\mathcal{F}$ runs through the set of subbundles of $\mathcal{E}^M_0$. Observe that $\mathrm{def}_p M \geq 0$ and $\mathrm{def}_p M = 0$ if and only if E^M lies in $\mathrm{GenSht}^{d,\leq p}_{r,\underline{\lambda}}(\kappa)$.

In [28] we prove

Proposition 6.7. *There exists a unique subbundle $\mathcal{F}_{\max}$ of $\mathcal{E}^M_0$ such that $\mathrm{def}_p \mathcal{F}_{\max}$ is maximal.*

Suppose that E^M does not belong to $\mathrm{GenSht}^{d,\leq p}_{r,\underline{\lambda}}(\kappa)$, i.e., $\mathrm{def}_p M > 0$. Let W be the generic fiber of $\mathcal{F}_{\max}$. It is a proper subspace of V^M. Then we show that $M' := \mathrm{Mod}_W(M)$ is still an iterated lattice of V. Actually, one needs to replace K by a finite extension, but we will ignore this subtlety. We will call $E(M')$ the modification of the generalized shtuka $E(M)$.

6.4. **Step 3: Show that the modified shtuka is less stable.**

We can prove that the modified shtuka $E^{M'}$ has smaller p-defect than E^M. More precisely,

Proposition 6.8. *a) We keep the above notation. Then we always have an inequality*

$$def_p M' \leq def_p M.$$

b) Suppose that $def_p M' = def_p M$. Let $\mathcal{F}'$ be the unique subbundle of $\mathcal{E}_0^{M'}$ of maximal p-defect, then the morphism $\mathcal{E}_0^{M'} \longrightarrow \mathcal{E}_0^{M}$ induces an isomorphism $\mathcal{F}' \xrightarrow{\sim} \mathcal{F}$.

To conclude, we must show that after a finite number of modifications, the modified shtuka has strictly smaller p-defect. The idea of the proof is as follows. Suppose that this is not the case. Then Proposition 6.8 implies that the transformation Mod_W can be repeated infinitely.

For every integer $n \geq 0$, we define $M^{(n)} := \mathrm{Mod}_W^n(M)$. Denote by $\mathcal{F}^{(n)}$ be the unique subbundle of $\mathcal{E}_0^{M^{(n)}}$ of maximal p-defect. In particular, we have

$$\mathrm{def}_p \mathcal{F}^{(n)} = \mathrm{def}_p M^{(n)} = \mathrm{def}_p M > 0.$$

Further, the morphism $\mathcal{E}_0^{M^{(n)}} \longrightarrow \mathcal{E}_0^{M^{(n-1)}}$ induces an isomorphism $\mathcal{F}^{(n)} \xrightarrow{\sim} \mathcal{F}^{(n-1)}$ for all $n \geq 1$. Repeating the argument of [13, §2, Proposition 14] (see also [18, Lemma 2]), we see that there exists a subbundle $\mathcal{F}_K$ of $\mathcal{E}_0$ over the generic fiber $X \times K$ such that $\mathrm{rk}\, \mathcal{F}_K = \mathrm{rk}\, \mathcal{F}^{(n)}$ and $\deg \mathcal{F}_K = \deg \mathcal{F}^{(n)}$. Thus we obtain

$$\mathrm{ndeg}_{\mathcal{E}_0} \mathcal{F}_K - p(\mathrm{rk}\, \mathcal{F}_K) = \mathrm{def}_p \mathcal{F}^{(n)} = \mathrm{def}_p M > 0,$$

which contradicts the fact that E belongs to $\mathrm{Sht}_{r,\underline{\lambda}}^{d,\leq p}(K)$.

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