



HAL
open science

Provisions and Economic Capital for Credit Losses

Dorinel Bastide, Stéphane Crépey

► **To cite this version:**

Dorinel Bastide, Stéphane Crépey. Provisions and Economic Capital for Credit Losses. 2024. hal-04389037v3

HAL Id: hal-04389037

<https://hal.science/hal-04389037v3>

Preprint submitted on 5 Dec 2024

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Provisions and Economic Capital for Credit Losses

Dorinel Bastide^{a, b}, Stéphane Crépey^c

This version: December 5, 2024

Abstract

Based on supermodularity ordering properties, we show that convex risk measures of credit losses are nondecreasing w.r.t. credit-credit and, in a wrong-way risk setup, credit-market, covariances of elliptically distributed latent factors. These results support the use of such setups for computing credit provisions and economic capital or for conducting stress test exercises and risk management analysis.

keywords. supermodular function, convex risk measure, supermodular random variable, stop loss property, elliptical distribution, credit loss.

JEL Classification. G13, G17, G21.

1 Introduction

Elliptical distributions are largely used in finance modelling, be it for credit latent variables or portfolio positions modelling (McNeil et al., 2015). They allow to generate, by means of Monte-Carlo routines, a span of possible losses. Risk indicators can then be formed so as to monitor possible future losses borne by a financial institution. In particular, regulatory bodies instruct to rely on value-at-risk and expected shortfall measures, where the last type, which falls under the remit of coherent risk measure¹, is often preferred to quantiles usage. One example can be found in Bastide, Crépey, Drapeau, and Tadese (2023), where the economic capital of a clearing member bank of a central counterparty (CCP) is based on an expected shortfall risk measure of the bank loss over one year. Such measure of loss depicts numerically a nondecreasing property w.r.t. credit-credit and credit-market dependence parameters, capturing an increase in loss given default amounts as more defaults materialize. Cousin and Laurent (2008) outlined the possible application of supermodular order for comparing CDO tranche premiums

^a *BNP Paribas Stress Testing Methodologies & Models. This article represents the opinions of the author, and it is not meant to represent the position or opinions of BNP Paribas or its members.* dorinel.2.bastide@bnpparibas.com (**corresponding author**).

^b *Laboratoire de Mathématiques et Modélisation d'Evry (LaMME), Université d'Evry/Université Paris-Saclay CNRS UMR 8071.*

^c *Université Paris Cité / LPSM CNRS UMR 8001, Paris, France*

¹see e.g. Acerbi and Tasche (2002).

w.r.t. a credit correlation parameter of their default latent variables modelled as Brownian motions. Meyer and Strulovici (2012) generalize the use of such notion, citing application to credit losses with bounded support. The supermodular order property finds its root in Marshall and Olkin (1979, Definition C.2, page 146) under the name of lattice-superadditive property. It has attracted subsequent attention with the works of Müller (1997) and Bäuerle and Müller (1998), applied to stop-loss ordering of aggregated losses. Bäuerle and Müller (2006) emphasize the role of several stochastic orders in relation with convex risk measures. In particular, multidimensional elliptically distributed random variables have the supermodular order property w.r.t. their covariance matrix coefficients (see Block and Sampson (1988, Corollary 2.3) recalled in Section B). This result will play a key role in this paper.

In Bastide, Crépey, Drapeau, and Tadese (2023), the loss takes a more complex form than what is usually found in the credit risk literature such as Cousin and Laurent (2008). This is due to a loss allocation coefficient attributed by the CCP to the surviving members. In this paper, we prove the nondecreasing property of convex risk measures w.r.t. covariance coefficients of portfolio credit losses. Our main motivation is to provide evidence of the soundness of the related approaches for computing credit provisions such as current expected credit loss (CECL, akin to the CVA in the central clearing one-period XVA setup of Bastide, Crépey, Drapeau, and Tadese (2023)), and economic capital (EC). This is important in justifying model assumptions and design, part of the model development cycle advocated by regulators and supervisors (European Central Bank, 2019). Table 1 details the two main targeted metrics in this work. Our main contribution is the identification of supermodular properties of the credit loss function for collateralized derivatives portfolios. Such property allows us to show the monotonous properties of convex risk measures applied to those portfolio credit losses with respect to covariance coefficients in elliptical models. We also outline such monotonic property for the contract value of credit multi-names derivatives.

Table 1: Metrics of interest (\cdot^0 relates to the survival probability measure of the reference bank labelled by 0).

Name	Expression	Reference
current expected credit loss (CECL)	$\mathbb{E}^0 \left(\sum_{i=1}^n f_i(X_1, \dots, X_n) g_i(Y_i) \right)$	Definition 2.1
economic capital (EC)	$\mathbb{E}S_\alpha^0 \left(\sum_{i=1}^n f_i(X_1, \dots, X_n) g_i(Y_i) \right)$	Definitions 2.2 and C.2

The paper is organized as follows. Section 2 presents the arguments based on the supermodular functions and random vectors properties that will serve to prove our main result Theorem 2.1. Results on the monotonicity of credit provisions and economic capital metrics w.r.t. covariance coefficients in elliptical models applied to exposures towards CCP are given in Section 3, with numerical illustration provided in Section 3.1. Section 4 completes the results from Cousin and Laurent (2008) regarding equity

and senior CDO tranches. Section 5 synthesizes our results. Supermodular functions, elliptical distributions and risk measures are reviewed in Sections A, B and C.

1.1 Probabilistic Setup

We consider a non-atomic probability space $(\Omega, \mathcal{A}, \mathbb{Q})$ with a probability measure \mathbb{Q} interpreted as risk-neutral in financial applications, and corresponding expectation, variance and covariance operators denoted by \mathbb{E} , Var and Cov .

We consider random vectors (where all rows are considered as column vectors) $\mathbf{X} = (\mathcal{X}_1, \dots, \mathcal{X}_m)$ and $\mathbf{X}' = (\mathcal{X}'_1, \dots, \mathcal{X}'_m)$ (for some $m \in \mathbb{N}^*$). The metrics of interest in this work are considered from a reference market participant viewpoint, namely a bank indexed by 0. In this context, it is sometimes useful to introduce a measure \mathbb{Q}^0 defined in terms of a measurable function h of a latent variable \mathcal{X}_0 of the default of the bank, such that

$$h(\mathcal{X}_0) = d\mathbb{Q}^0/d\mathbb{Q} \geq 0 \text{ and } \mathbb{E}[h(\mathcal{X}_0)] = 1. \quad (1)$$

Example 1.1. *In the setup of Bastide, Crépey, Drapeau, and Tadese (2023), even though the financial risk factors and the financial metrics are defined under \mathbb{Q} , more explicit XVA formulas arise in terms of the related bank survival probability measure \mathbb{Q}^0 .*

We index all the formerly introduced probabilistic notations by “.⁰” whenever applied in reference to \mathbb{Q}^0 .

Assumption 1.1. $\mathcal{X}_0, \mathbf{X}$ and \mathbf{X}' satisfy

$$[\mathbf{X}|\mathcal{X}_0] \leq_{sm} [\mathbf{X}'|\mathcal{X}_0],$$

i.e.

$$\mathbb{E}[f(\mathbf{X})|\mathcal{X}_0] \leq \mathbb{E}[f(\mathbf{X}')|\mathcal{X}_0] \quad (2)$$

holds for any supermodular function $f : \mathbb{R}^m \rightarrow \mathbb{R}$ such that the conditional expectations exist.

Lemma 1.1. *If $\mathcal{X}_0, \mathbf{X}$ and \mathbf{X}' satisfy Assumption 1.1, then $\mathbf{X} \leq_{sm^0} \mathbf{X}'$.*

Proof. As $h(\mathcal{X}_0) \geq 0$, for any supermodular function f on \mathbb{R}^m such that both $\mathbb{E}^0[f(\mathbf{X})]$ and $\mathbb{E}^0[f(\mathbf{X}')] exist, (1) and (2) yield$

$$\begin{aligned} \mathbb{E}^0[f(\mathbf{X})] &= \mathbb{E}[h(\mathcal{X}_0)f(\mathbf{X})] = \mathbb{E}[h(\mathcal{X}_0)\mathbb{E}[f(\mathbf{X})|\mathcal{X}_0]] \\ &\leq \mathbb{E}[h(\mathcal{X}_0)\mathbb{E}[f(\mathbf{X}')|\mathcal{X}_0]] = \mathbb{E}^0[f(\mathbf{X}')] . \square \end{aligned}$$

For a function of several real arguments, by nondecreasing we mean nondecreasing w.r.t. each of them.

Lemma 1.2. *If $\mathbf{X} \leq_{sm^0} \mathbf{Y}$, then, for any nondecreasing supermodular function $f : \mathbb{R}^m \rightarrow \mathbb{R}$, $f(\mathbf{X}) \leq_{sl^0} f(\mathbf{Y})$ where \leq_{sl^0} denotes the stop-loss order between random vectors under \mathbb{Q}^0 as per Müller (1997, Definition 2.1) recalled in Appendix A.*

Proof. For any $A \in \mathbb{R}$, the function $\varphi : \mathbb{R} \rightarrow \mathbb{R}_+$, $x \mapsto (x - A)^+$ is nondecreasing and convex. By Müller and Stoyan (2002, Theorem 3.9.3 f), page 113), $\varphi \circ f$ is nondecreasing supermodular. Hence $\mathbb{E}^0[(f(\mathbf{X}) - A)^+] \leq \mathbb{E}^0[(f(\mathbf{Y}) - A)^+]$, which yields the result by Müller (1997, Definition 2.1). \square

Elliptical setup Whenever stated in the paper, by elliptical setup, we mean that $(\mathcal{X}_0, \mathbf{X})$ and $(\mathcal{X}_0, \mathbf{X}')$ follow elliptical distributions (see Section B) under \mathbb{Q} as per

$$(\mathcal{X}_0, \mathbf{X}) = \boldsymbol{\mu} + \mathbf{A}\mathbf{Z} \text{ and } (\mathcal{X}_0, \mathbf{X}') = \boldsymbol{\mu} + \mathbf{A}'\mathbf{Z}, \quad (3)$$

for constant matrices $\mathbf{A}, \mathbf{A}' \in \mathbb{R}^{m \times k}$ of full rank with the same first row. $\mathbf{Z} = (\mathcal{Z}_0, \mathcal{Z}_1, \dots, \mathcal{Z}_k)$ follows a spherical distribution $S_{k+1}(\psi)$ (see Section B), with characteristic generator ψ . The rationale for having the same \mathcal{X}_0 in $(\mathcal{X}_0, \mathbf{X})$ and $(\mathcal{X}_0, \mathbf{X}')$ is that the point of view will be the one of a reference bank indexed by 0 (cf. Lemma B.1 and Proposition B.1). We denote by $\boldsymbol{\Gamma} = \mathbf{A}\mathbf{A}^\top$ and $\boldsymbol{\Gamma}' = \mathbf{A}'(\mathbf{A}')^\top$ the \mathbb{Q} covariance matrices of $(\mathcal{X}_0, \mathbf{X})$ and $(\mathcal{X}_0, \mathbf{X}')$, assumed to be positive semi-definite. We write $\Gamma_{ij} = \text{Cov}(\mathcal{X}_i, \mathcal{X}_j)$ for all $i, j \in 0..m$, and likewise for $\boldsymbol{\Gamma}'$. Moreover, we assume within this setup that $\Gamma_{jj} = \Gamma'_{jj}$ and $\Gamma_{0j} = \Gamma'_{0j}$, $j \in 1..m$, and $\Gamma'_{ij} \leq \Gamma_{ij}$, $i \neq j \in 1..m$. In particular, $\mathbf{X} \leq_{sm} \mathbf{X}'$, by Block and Sampson (1988, Corollary 2.3) recalled in Section B.

Example 1.2. *By Proposition B.1, the elliptical setup (3) verifies Assumption 1.1.*

2 Theoretical Results

Given measurable functions $f_i : \mathbb{R}^n \rightarrow \mathbb{R}_+$ and $g_i : \mathbb{R} \rightarrow \mathbb{R}_+$, $i \in 1..n$, let

$$\begin{aligned} \mathbb{R}^{2n} & \xrightarrow{\ell} \mathbb{R}_+ \\ (x_1, \dots, x_n, y_1, \dots, y_n) & \mapsto \sum_{i=1}^n f_i(x_1, \dots, x_n) g_i(y_i). \end{aligned} \quad (4)$$

We additionally assume that all the functions f_i are supermodular and nondecreasing on \mathbb{R}^n and all the functions g_i are nondecreasing on \mathbb{R} .

Lemma 2.1. *ℓ has increasing differences w.r.t. any pair $(x_i, y_j) \in \mathbb{R}^2$, $i, j \in 1..n$.*

Proof. Let $f_i(x_j | \mathbf{x}_{-j})$ denote the function f_i applied to x_j but keeping all other arguments $\mathbf{x}_{-j} := (x_k)_{k \neq j}$ fixed. We look at the two cases where we consider either a pair of argument $(x_i, y_i) \in \mathbb{R}^2$, $i \in 1..n$ (i.e. the pair of arguments tested for the increasing difference are part of the same term of the sum), or a pair $(x_i, y_j) \in \mathbb{R}^2$, $i \neq j \in 1..n$, and the corresponding increasing differences.

Case $(x_i, y_i) \in \mathbb{R}^2$, $i \in 1..n$: the function $\mathbb{R}^2 \ni (x_i, y_i) \mapsto f_i(x_i | \mathbf{x}_{-i}) g_i(y_i)$, has the

increasing differences property by application of Lemma A.2 with $g(\cdot) = f_i(\cdot|\mathbf{x}_{-i})$ and $h(\cdot) = g_i(\cdot)$. This increasing difference property writes

$$\begin{aligned}
& f_i(x'_i|\mathbf{x}_{-i})g_i(y'_i) + \sum_{k \neq i} f_k(x'_i|\mathbf{x}_{-i})g_k(y_k) - f_i(x'_i|\mathbf{x}_{-i})g_i(y_i) - \sum_{k \neq i} f_k(x'_i|\mathbf{x}_{-i})g_k(y_k) \\
& - f_i(x_i|\mathbf{x}_{-i})g_i(y'_i) - \sum_{k \neq i} f_k(x_i|\mathbf{x}_{-i})g_k(y_k) + f_i(x_i|\mathbf{x}_{-i})g_i(y_i) + \sum_{k \neq i} f_k(x_i|\mathbf{x}_{-i})g_k(y_k) \\
= & \underbrace{f_i(x'_i|\mathbf{x}_{-i})g_i(y'_i) - f_i(x'_i|\mathbf{x}_{-i})g_i(y_i) - f_i(x_i|\mathbf{x}_{-i})g_i(y'_i) + f_i(x_i|\mathbf{x}_{-i})g_i(y_i)}_{\geq 0 \text{ by Lemma A.2}} \\
& + \underbrace{\sum_{k \neq i} f_k(x'_i|\mathbf{x}_{-i})g_k(y_k) - \sum_{k \neq i} f_k(x'_i|\mathbf{x}_{-i})g_k(y_k)}_{=0} \\
& - \underbrace{\sum_{k \neq i} f_k(x_i|\mathbf{x}_{-i})g_k(y_k) + \sum_{k \neq i} f_k(x_i|\mathbf{x}_{-i})g_k(y_k)}_{=0} \\
\geq & 0,
\end{aligned}$$

hence $(x_i, y_i) \mapsto \sum_{k=1}^n f_k(x_i|\mathbf{x}_{-i})g_k(y_k)$ has the increasing differences property.

Case $(x_i, y_j) \in \mathbb{R}^2$, $i \neq j \in 1 \dots n$: we write the increasing difference

$$\begin{aligned}
& \sum_{k \neq j} f_k(x'_i|\mathbf{x}_{-i})g_k(y_k) + f_j(x'_i|\mathbf{x}_{-i})g_j(y'_j) - \sum_{k \neq j} f_k(x'_i|\mathbf{x}_{-i})g_k(y_k) - f_j(x'_i|\mathbf{x}_{-i})g_j(y_j) \\
& - \sum_{k \neq j} f_k(x_i|\mathbf{x}_{-i})g_k(y_k) - f_j(x_i|\mathbf{x}_{-i})g_j(y'_j) + \sum_{k \neq j} f_k(x_i|\mathbf{x}_{-i})g_k(y_k) + f_j(x_i|\mathbf{x}_{-i})g_j(y_j) \\
= & f_j(x'_i|\mathbf{x}_{-i})g_j(y'_j) - f_j(x'_i|\mathbf{x}_{-i})g_j(y_j) - f_j(x_i|\mathbf{x}_{-i})g_j(y'_j) + f_j(x_i|\mathbf{x}_{-i})g_j(y_j) \geq 0,
\end{aligned}$$

by application of Lemma A.2 with $g(\cdot) = f_j(\cdot|\mathbf{x}_{-i})$ and $h(\cdot) = g_j(\cdot)$. \square

Proposition 2.1. *In the special case $f_i(x_1, \dots, x_n) \equiv f_i(x_i)$ with f_i nondecreasing on \mathbb{R} , $i \in 1 \dots n$, we have that $\mathbb{R}^{2n} \ni (x_1, \dots, x_n, y_1, \dots, y_n) \xrightarrow{\ell} \sum_{i=1}^n f_i(x_i)g_i(y_i) \in \mathbb{R}$ has increasing differences w.r.t. any pair $(x_i, y_j) \in \mathbb{R}^2$, $i, j \in 1 \dots n$.*

Proof. By Lemma A.1, ℓ has increasing differences w.r.t. any pair $(x_i, x_j) \in \mathbb{R}^2$, $i \neq j \in 1 \dots n$, as well as any pair $(y_i, y_j) \in \mathbb{R}^2$, $i \neq j$. For $i \in 1 \dots n$, $(x_i, y_i) \mapsto f_i(x_i)g_i(y_i)$, has the increasing differences property, by application of Lemma A.2 with $g(\cdot) = f_i(\cdot)$ and $h(\cdot) = g_i(\cdot)$. Hence ℓ has the increasing differences property w.r.t. any pair $(x_i, y_i) \in \mathbb{R}^2$. Finally, for $i \neq j \in 1 \dots n$, $(x_i, y_j) \mapsto f_i(x_i)g_i(y_i) + f_j(x_j)g_j(y_j)$ has the increasing differences property by Lemma A.1 with $h_i(\cdot) = f_i(\cdot)g_i(y_i)$ and $h_j(\cdot) = f_j(x_j)g_j(\cdot)$. \square

Proposition 2.2. *The function ℓ is nondecreasing supermodular.*

Proof. By Lemma A.1 with $h_i(\cdot) = f_i(x_1, \dots, x_n)g_i(\cdot)$, for any $i \in 1 \dots n$, ℓ has increasing differences with respect to any pair (y_k, y_l) , $k, l \in 1 \dots n$. By assumption and closure by

addition of the increasing differences property, ℓ has increasing differences with respect to any pair (x_k, x_l) , $k, l \in 1..n$. Finally, by Lemma 2.1, ℓ has increasing differences with respect to any pair (x_k, y_l) , $k, l \in 1..n$. Hence, by Yildiz (2010, Corollary 1)², ℓ is supermodular. \square

Hereafter we consider a one-period financial market model on (Ω, \mathcal{A}) , assumed arbitrage-free³ with risk-neutral measure \mathbb{Q} . We set $(\mathcal{X}_0, \mathbf{X}) = (X_0, X_1, \dots, X_n, Y_1, \dots, Y_n)$, with \mathcal{X}_0 re-noted X_0 , where Y_i drives a loss that obligor $i \in 1..n$ generates if it defaults, with default of each credit name $j \in 0..n$ (including the reference bank 0) driven by a latent variable X_j (for some $n > 0$). The variables X_1, \dots, X_n represent latent variables breaching a certain threshold for translating the defaults of n obligors also indexed by $1, \dots, n$. The r.v. $f_i(X_1, \dots, X_n)$ represents a loss allocation coefficient depending on the possible defaults of obligors $1, \dots, n$ while $g_i(Y_i)$ represents a nonnegative loss given default of obligor i . When $f_i(X_1, \dots, X_n) \equiv f_i(X_i)$ as in Cousin and Laurent (2008) or Meyer and Strulovici (2013), $f_i(X_i)g_i(Y_i)$ can represent the loss related to a bilateral counterparty position, i.e. a portfolio position between the reference bank 0 and its client i . As detailed in Section 3, the more general case where $f_i(X_1, \dots, X_n)$ depends on several X_j is needed to encompass the financial losses generated by clearing exposures towards a CCP. This is due to the loss allocation mechanism between surviving members (Bastide, Crépey, Drapeau, and Tadese, 2023, Section 3). It also covers the case³ of financial resolution funds (Single Resolution Board, 2021).

Theorem 2.1. *Let \mathfrak{X}^0 be a closed linear subspace and sublattice of $L^1(\mathbb{Q}^0)$, that includes the constants. If $\mathbf{X} \leq_{sm^0} \mathbf{X}'$, then, for any risk measure⁴ ρ on $\mathfrak{X}^0 \ni \ell(\mathbf{X}), \ell(\mathbf{X}')$, we have $\rho(\ell(\mathbf{X})) \leq \rho(\ell(\mathbf{X}'))$. In particular, if $\ell(\mathbf{X})$ and $\ell(\mathbf{X}')$ are \mathbb{Q}^0 integrable, then $\mathbb{E}^0[\ell(\mathbf{X})] \leq \mathbb{E}^0[\ell(\mathbf{X}')$.*

Proof. By Proposition 2.2, the function ℓ is supermodular on \mathbb{R}^{2n} . Hence, by Lemma 1.2, if $\mathbf{X} \leq_{sm^0} \mathbf{X}'$, then

$$\ell(\mathbf{X}) = \sum_{i=1}^n f_i(X_1, \dots, X_n)g_i(Y_i) \leq_{sl^0} \sum_{i=1}^n f_i(X'_1, \dots, X'_n)g_i(Y'_i) = \ell(\mathbf{X}'),$$

Having assumed the probability space non-atomic⁵, Bäuerle and Müller (2006, Theorem 4.4), given at the end of Appendix C, yields the result. \square

Remark 2.1. *By Lemma 1.1, the assumption $\mathbf{X} \leq_{sm^0} \mathbf{X}'$ in Theorem 2.1 holds, in particular, for $(\mathcal{X}_0, \mathbf{X})$ and $(\mathcal{X}_0, \mathbf{X}')$ satisfying Assumption 1.1.*

Definition 2.1. *The current expected credit loss of the reference bank 0 is $CECL(\mathcal{X}_0, \mathbf{X}) = \mathbb{E}^0[\ell(\mathbf{X})]$.*

²see Section A.

³see for instance (Föllmer and Schied, 2016, Part I).

⁴see Definition C.1.

⁵see Bäuerle and Müller (2006, Example 3.1) for a counter-example to the monotonicity property in the case of a probability space with atom.

Proposition 2.3. *If \mathcal{X}_0 , \mathbf{X} and \mathbf{X}' satisfy Assumption 1.1, then $\text{CECL}(\mathcal{X}_0, \mathbf{X}) \leq \text{CECL}(\mathcal{X}_0, \mathbf{X}')$ holds whenever $\ell(\mathbf{X}), \ell(\mathbf{X}') \in L^1(\mathbb{Q}^0)$.*

Proof. By definition of $\text{CECL}(\mathcal{X}_0, \mathbf{X})$ and application of Theorem 2.1 and Remark 2.1 to $\rho \equiv \mathbb{E}^0$. \square

Definition 2.2. *The economic capital of the reference bank 0 is $\text{EC}(\mathcal{X}_0, \mathbf{X}) = \text{ES}_\alpha^0(\ell(\mathbf{X}))$, with ES_α^0 as per Definition C.2 (applied under \mathbb{Q}^0).*

Proposition 2.4. *If \mathcal{X}_0 , \mathbf{X} and \mathbf{X}' satisfy Assumption 1.1, then $\text{EC}(\mathcal{X}_0, \mathbf{X}) \leq \text{EC}(\mathcal{X}_0, \mathbf{X}')$ holds whenever $\ell(\mathbf{X}), \ell(\mathbf{X}') \in L^1(\mathbb{Q}^0)$.*

Proof. By definition of $\text{EC}(\mathcal{X}_0, \mathbf{X})$ and application of Theorem 2.1 to $\rho \equiv \text{ES}_\alpha^0$ using Remark C.1. \square

Remark 2.2. *Wrong-way risk is the potential increase of the exposure a financial actor w.r.t. certain counterparties when their probability of default increase. A risk model should include a wrong-way risk feature in order to ensure conservative treatment. See Gregory (2015, Section 8.6.5) for more detailed explanations and Brigo, Morini, and Pallavicini (2013) for various examples of asset classes models incorporating the wrong-way risk feature. Under the elliptical model (3) and the credit loss form (4), wrong-way risk holds, in the sense that both expectations and expected shortfall measures increase when correlation between credit and portfolio exposure increases, provided that an increase of the covariance between the default latent variable X_i and the potential loss driver Y_i leads to an increase of the loss amount $g_i(Y_i)$. As shown by Propositions 2.3 and 2.4 with $\text{Cov}^0(X_i, Y_i) \leq \text{Cov}^0(X'_i, Y'_i)$ thanks to (19) in Lemma B.1 applied to the elliptical setup (3), this is indeed the case when f and g are nondecreasing (as assumed) in each of their arguments.*

3 Application to CCPs

Given real constants $\beta_1, \dots, \beta_n \geq 0$, m_1, \dots, m_n and B_1, \dots, B_n , we consider in (4)

$$\begin{aligned} \mathbb{R}^n & \xrightarrow{f_i} \mathbb{R}_+ & \text{and} & \quad \mathbb{R} \xrightarrow{g_i} \mathbb{R}_+ \\ (x_1, \dots, x_n) & \mapsto \frac{\mathbb{1}_{\{x_i \geq B_i\}}}{1 + \sum_{j=1}^n \beta_j \mathbb{1}_{\{x_j < B_j\}}} & & \quad y_i \mapsto (y_i - m_i)^+, \quad i \in 1..n. \end{aligned} \tag{5}$$

Likewise, given constants B_i , $i \in 1..n$, $\{\tau_i \leq T\} := \{X_i \geq B_i\}$ models the default event of participant $i \in 1..n$. The weights $\left(1 + \sum_{j=1}^n \beta_j \mathbb{1}_{\{X_j < B_j\}}\right)^{-1}$ represent a stylised specification of a default fund allocation in a CCP setup. Modeling the default of the reference bank 0 by $\{X_0 \geq B_0\}$ for some constant B_0 , we now take in (1)

$$h(x_0) = \frac{\mathbb{1}_{\{x_0 < B_0\}}}{1 - \gamma}, \quad \text{where } \gamma = \mathbb{Q}(X_0 \geq B_0),$$

so that \mathbb{Q}^0 is the survival measure of the reference bank associated with \mathbb{Q} (see e.g. Albanese, Crépey, Hoskinson, and Saadeddine (2021, Section 3)).

The r.v. $Y_i = \text{nom}_i \sigma_i G_i$, with $\text{nom}_i \in \mathbb{R}$, $\sigma_i > 0$ constant and G_i spherical (hence $\text{Var}(Y_i) = \text{nom}_i^2 \sigma_i^2 \text{Var}(G_i)$) represents the loss of the market participant 0 in case of the default of obligor i , collateralized by a corresponding amount m_i .

Proposition 3.1. *The function f_i in (5) is nondecreasing supermodular on \mathbb{R}^n .*

Proof. Given Yildiz (2010, Corollary 1), it is sufficient to show that f_i has increasing differences. Let $k, l \in 1, \dots, n$.

Case $k, l \neq i$: Let $\mathbb{R}^2 \ni (x_k, x_l) \xrightarrow{g} \frac{\mathbb{1}_{\{x_i \geq B_i\}}}{\Lambda + \beta_k \mathbb{1}_{\{x_k < B_k\}} + \beta_l \mathbb{1}_{\{x_l < B_l\}}} \in \mathbb{R}_+$, with $\Lambda = \sum_{j \neq k, l} \beta_j \mathbb{1}_{\{x_j < B_j\}}$. For $x'_k \geq x_k$, $x'_l \geq x_l$, we form the increasing difference

$$\begin{aligned} & g(x'_k, x'_l) - g(x'_k, x_l) - g(x_k, x'_l) + g(x_k, x_l) \\ &= \frac{\mathbb{1}_{\{x_i \geq B_i\}} \beta_l (\mathbb{1}_{\{x_l < B_l\}} - \mathbb{1}_{\{x'_l < B_l\}})}{\text{denom}_1} - \frac{\mathbb{1}_{\{x_i \geq B_i\}} \beta_l (\mathbb{1}_{\{x_l < B_l\}} - \mathbb{1}_{\{x'_l < B_l\}})}{\text{denom}_2}, \end{aligned} \quad (6)$$

with $\text{denom}_1 = (\Lambda + \beta_k \mathbb{1}_{\{x'_k < B_k\}} + \beta_l \mathbb{1}_{\{x'_l < B_l\}})(\Lambda + \beta_k \mathbb{1}_{\{x'_k < B_k\}} + \beta_l \mathbb{1}_{\{x_l < B_l\}})$ and $\text{denom}_2 = (\Lambda + \beta_k \mathbb{1}_{\{x_k < B_k\}} + \beta_l \mathbb{1}_{\{x'_l < B_l\}})(\Lambda + \beta_k \mathbb{1}_{\{x_k < B_k\}} + \beta_l \mathbb{1}_{\{x_l < B_l\}})$. If $x_l \leq x'_l < B_l$ or $B_l \leq x_l \leq x'_l$, then the increasing difference (6) is zero as the numerators of both terms are zero. If $x_l < B_l \leq x'_l$, then both numerators in (6) equal $\mathbb{1}_{\{x_i \geq B_i\}} \beta_l$. In this case: (i) if $x_k \leq x'_k < B_k$, then both denominators in (6) equal $(\Lambda + \beta_k)(\Lambda + \beta_k + \beta_l)$ and the increasing difference (6) is zero; (ii) if $B_k \leq x_k \leq x'_k$, then both denominators in (6) equal $\Lambda(\Lambda + \beta_l)$ and the increasing difference (6) is zero; If $x_k < B_k \leq x'_k$, then the increasing difference in (6) writes equivalently

$$\frac{\mathbb{1}_{\{x_i \geq B_i\}} \beta_l}{\text{denom}_1 \text{denom}_2} (\text{denom}_2 - \text{denom}_1),$$

with $\text{denom}_2 = (\Lambda + \beta_k + \beta_l)(\Lambda + \beta_k) \geq \Lambda(\Lambda + \beta_l) = \text{denom}_1$ as $\beta_j \geq 0$ for all $j \in 1..n$. Hence the increasing difference (6) is nonnegative.

Case $k = i$ and $l \neq i$: Let $\mathbb{R}^2 \ni (x_i, x_l) \xrightarrow{g} \frac{\mathbb{1}_{\{x_i \geq B_i\}}}{\Lambda + \beta_i \mathbb{1}_{\{x_i < B_i\}} + \beta_l \mathbb{1}_{\{x_l < B_l\}}} \in \mathbb{R}_+$, with $\Lambda = \sum_{j \neq i, l} \beta_j \mathbb{1}_{\{x_j < B_j\}}$. For $x'_i \geq x_i$, $x'_l \geq x_l$, we form the increasing difference

$$\begin{aligned} & g(x'_i, x'_l) - g(x'_i, x_l) - g(x_i, x'_l) + g(x_i, x_l) \\ &= \frac{\mathbb{1}_{\{x'_i \geq B_i\}} \beta_l (\mathbb{1}_{\{x_l < B_l\}} - \mathbb{1}_{\{x'_l < B_l\}})}{\text{denom}_1} - \frac{\mathbb{1}_{\{x_i \geq B_i\}} \beta_l (\mathbb{1}_{\{x_l < B_l\}} - \mathbb{1}_{\{x'_l < B_l\}})}{\text{denom}_2}, \end{aligned} \quad (7)$$

with $\text{denom}_1 = (\Lambda + \beta_i \mathbb{1}_{\{x'_i < B_i\}} + \beta_l \mathbb{1}_{\{x'_l < B_l\}})(\Lambda + \beta_i \mathbb{1}_{\{x'_i < B_i\}} + \beta_l \mathbb{1}_{\{x_l < B_l\}})$ and $\text{denom}_2 = (\Lambda + \beta_i \mathbb{1}_{\{x_i < B_i\}} + \beta_l \mathbb{1}_{\{x'_l < B_l\}})(\Lambda + \beta_i \mathbb{1}_{\{x_i < B_i\}} + \beta_l \mathbb{1}_{\{x_l < B_l\}})$. If $x_l \leq x'_l < B_l$ or $B_l \leq x_l \leq x'_l$, then the increasing difference (7) is zero as the numerators of both terms are zero. If $x_l < B_l \leq x'_l$, then the numerator of the first terms in (7) is $\mathbb{1}_{\{x'_i \geq B_i\}}$ and $\mathbb{1}_{\{x_i \geq B_i\}}$ for the second term. In this case: (i) if $x_i \leq x'_i < B_i$, then both numerators in (7) are zero and the increasing difference (7) is zero; (ii) if $B_i \leq x_i \leq x'_i$, then both

numerators in (7) are equal to $\mathbb{1}_{\{x_i \geq B_i\}}\beta_l$ and both denominators are equal $\Lambda(\Lambda + \beta_l)$ so the increasing difference (7) is zero; If $x_i < B_i \leq x'_i$, then the second term in (7) is zero and the first term is nonnegative. Hence the increasing difference (7) is nonnegative. \square

Hence the general results of Section 2 apply to f_i and g_i as per (5). In particular, for such f_i and g_i , Propositions 2.3 and 2.4 ensure the nondecreasing property of both CECL and EC w.r.t. covariance coefficients of $(\mathcal{X}_0, \mathbf{X})$ and $(\mathcal{X}_0, \mathbf{X}')$ under the elliptical setup (3).

3.1 CCP Use Case Design

We consider a CCP service with 20 members, labeled by $i \in 0..n = 19$, trading for cleared clients (see Bastide, Crépey, Drapeau, and Tadese (2023, Sections 2 & 5)). Each member faces one client. The corresponding financial network is depicted in Figure 1.

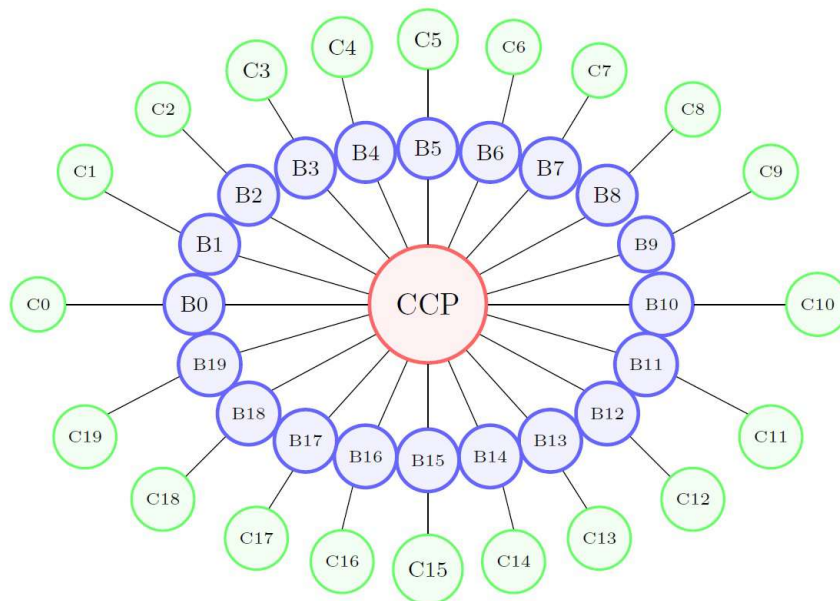


Figure 1: Financial network composed of a CCP, its 20 members (labeled by B) and one cleared client per member.

All clients are assumed to be risk-free. For any member i , its posted initial margin (IM) to the CCP is calculated based on the idea of a variation margin (VM) call not fulfilled over a slippage time period Δ_s at a confidence level $\alpha \in (1/2, 1)$. More precisely, the IM relies on a VaR metric under the risk-neutral measure \mathbb{Q} at the confidence level α , denoted by VaR_α applied to the non-coverage of VM call on the cleared portfolio. The latter is modelled as a scaled Student t-distribution \mathcal{S}_ν with ν degrees of freedom, with c.d.f. S_ν . The scaling reflects both Δ_s , the portfolio nominal size nom_i and its

standard deviation σ_i . Namely,

$$\text{IM}_i = \text{VaR}_\alpha(\text{nom}_i \sigma_i \sqrt{\Delta_s} \mathcal{S}_\nu) = |\text{nom}_i| \sigma_i \sqrt{\Delta_s} \mathcal{S}_\nu^{-1}(\alpha).$$

A (funded) default fund pooled between its clearing members (see Bastide, Crépey, Drapeau, and Tadese (2023, Section 5)) is calculated at the CCP level as

$$\text{Cover2} = \text{SLOIM}_{(0)} + \text{SLOIM}_{(1)}, \quad (8)$$

for the two largest stressed losses over IM (SLOIM_i) among members, identified with subscripts (0) and (1). SLOIM_i is calculated as a value-at-risk $\text{VaR}_{\alpha'}$ at a confidence level $\alpha' > \alpha$ of the loss over IM, i.e.

$$\text{SLOIM}_i = \text{VaR}_{\alpha'}(\text{nom}_i \sigma_i \sqrt{\Delta_s} \mathcal{S}_\nu - \text{IM}_i) = |\text{nom}_i| \sigma_i \sqrt{\Delta_s} \left(\mathcal{S}_\nu^{-1}(\alpha') - \mathcal{S}_\nu^{-1}(\alpha) \right).$$

The total amount (8) is then allocated between the clearing members to define their (funded) default fund contributions as $\text{DF}_i = \frac{\text{SLOIM}_i}{\sum_j \text{SLOIM}_j} \times \text{Cover2}$. Finally, the loss function of the reference member 0 with default fund contribution DF_0 is of the form (5) with $\beta_j = \frac{\text{DF}_j}{\text{DF}_0}$ and $m_i = \text{IM}_i + \text{DF}_i$, i.e.,

$$\ell(\mathbf{X}) = \sum_{i=1}^n \frac{1}{1 + \sum_{j=1}^n \frac{\text{DF}_j}{\text{DF}_0} \mathbb{1}_{\{X_j < B_j\}}} \mathbb{1}_{\{X_i \geq B_i\}} \times (Y_i - \text{IM}_i - \text{DF}_i)^+. \quad (9)$$

Let $\text{sgn}(x) = 1$ if $x > 0$, 0 if $x = 0$, -1 otherwise. An elliptical model is specified under \mathbb{Q} as

$$\begin{cases} Y_i = \text{nom}_i \sigma_i \sqrt{\Delta_l} \sqrt{\mathcal{K}} \left(\sqrt{\rho^{mkt}} \mathcal{E} + \sqrt{\rho^{wvr}} \mathcal{W}_i + \sqrt{1 - \rho^{mkt} - \rho^{wvr}} \mathcal{E}_i \right) \\ X_i = \sqrt{\mathcal{K}} \left(\sqrt{\rho^{cr}} \mathcal{T} + \text{sgn}(\text{nom}_i) \sqrt{\rho^{wvr}} \mathcal{W}_i + \sqrt{1 - \rho^{cr} - \rho^{wvr}} \mathcal{T}_i \right) \end{cases} \quad (10)$$

for any $i \in 0..n$, where \mathcal{T} , \mathcal{T}_i , \mathcal{E} , \mathcal{E}_i and \mathcal{W}_i are i.i.d. random variables following normal distributions and \mathcal{K}/ν follows an inverse-chi-squared distribution degree of freedom ν , independent from any other considered r.v.. Hence all X_i 's and Y_i 's follow Student's t-distribution with degree of freedom ν . Δ_l is the period accounting for the time taken by the CCP to novate or liquidate its portfolios in case of defaults (practically, $\Delta_l > \Delta_s$ by a few business days). T represents the final maturity of the CCP portfolios. $B_i = \mathcal{S}_\nu^{-1}(1 - \text{DP}_i(T))$ where $\text{DP}_i(T)$ is the default probability over the period $[0, T]$ defined from a constant default intensity λ_i given for each member i in Table 2 so that $\text{DP}_i(T) = 1 - e^{-\lambda_i T}$ which can be obtained from their 1-year \mathbb{Q} default probability $\text{DP}_i(1Y)$, inferred either from the agency ratings or the CDS quotes when available, as $\lambda_i = -\ln(1 - \text{DP}_i(1Y))$. The model is well defined if and only if $0 < \rho^{wvr} < \min(1 - \rho^{cr}, 1 - \rho^{mkt})$. Note that $\text{Cov}(X_i, Y_i) = \text{nom}_i \sigma_i \sqrt{\Delta_l} \sqrt{\rho^{wvr}} \text{sgn}(\text{nom}_i) \sqrt{\rho^{wvr}} = |\text{nom}_i| \sigma_i \sqrt{\Delta_l} \rho^{wvr} \geq 0$, hence increasing ρ^{wvr} leads to an increase of $\text{Cov}(X_i, Y_i)$.

The participants and portfolios input parameters are detailed in Table 2, where cm is the identifier of the clearing member, λ is the one year \mathbb{Q} default intensity of the member

expressed in basis points, size represents the overall portfolio size of the member detained within the CCP, and vol is the annual volatility used for the portfolio variations. The portfolios listed in Table 2 relate to the members towards the CCP (which are mirroring the portfolios between the members and their clients). The sizes of the CCP portfolios of members sum up to 0, in line with the CCP clearing condition (without proprietary trades).

Remark 3.1. *The random variables (10) follow Student t -distributions that are continuous. Therefore, 0 is the only possible atom of the nonnegative credit loss (9). Hence, by Acerbi and Tasche (2002, Corollary 5.3), Definition C.2 is equivalent to $\mathbb{E}\mathcal{S}_\alpha(\mathcal{X}) = \mathbb{E}(\mathcal{X}|\mathcal{X} \geq \text{Var}_\alpha(\mathcal{X}))$ (Acerbi and Tasche, 2002, Eqn. (3.7)) whenever $\text{Var}_\alpha(\mathcal{X}) > 0$, i.e. for $\alpha \in (\frac{1}{2}, 1)$ sufficiently close to 1 so that $\text{Var}_\alpha(\mathcal{X}) > 0$. In our numerical illustration with $\alpha = 99.75\%$, this is indeed the case.*

Table 2: Member characteristics and CCP portfolio parameters, ordered by decreasing member $|\text{size}|$.

cm	0	1	2	3	4	5	6	7	8	9
λ (bps)	50	60	70	80	90	200	190	180	170	160
size	-242	184	139	105	-80	-61	-46	35	26	-20
vol (%)	20	21	22	23	24	25	26	27	28	29

cm	10	11	12	13	14	15	16	17	18	19
λ (bps)	150	140	130	120	110	100	90	80	70	60
size	-15	-11	-9	-6	5	-4	-3	2	2	-1
vol (%)	30	31	32	33	34	35	36	37	38	39

3.2 Numerical Results

The parameters of the CECL and EC calculations are summarized in Table 3. The confidence level at 97% for SLOIM in DF calibration allows for a ratio of default fund over initial margin of about 10% in our calculations, a ratio (of this level or less) often observed in practice. Note that the chosen period length of $T = 5$ years covers the bulk (if not the final maturity) of most realistic CCP portfolios.

Figures 2, 3 and 4 show the results of CECL and EC calculated for the members 0, 5, and 10, each under their survival risk measure (i.e. letting them in turn play the role of the reference bank indexed by 0 in the above). In each figure, the credit-credit correlation ρ^{cr} and ρ^{wcr} is varied between 5% and 95%, using 5% step. The same nondecreasing pattern is observed for all three members, with nonnegative incremental CECL and EC between two consecutive credit-credit and credit-market correlation steps, in line with Propositions 2.3 and 2.4 applied under the specific case (10) of the elliptical setup (3). The market-market correlation ρ^{mkt} has been kept constant with value 4%.

Table 3: CECL and EC calculation configuration.

One-period length T	5 years
Liquidation period at default Δ_l	5 days
Portfolio variations correlation ρ^{mkt}	4%
Credit factors correlation ρ^{cr}	5%-95%
Correlation between credit factors and portfolio variations ρ_i^{wvr}	5%-95%
IM covering period (MPoR) Δ_s	2 days
IM quantile level	95%
SLOIM calculation for DF Cover-2	VaR 97%
DF allocation rule	based on IM
Quantile level used for clearing members EC calculation	99.75%
Number of Monte-Carlo simulation (for CECL and EC computations)	1M
Number of batches (for EC computations)	100

The results of the centered EC, i.e. $EC - CECL$, are also provided for each of these 3 members in Figure 5. As $CECL \ll EC$ holds for all three members, the monotonicity is also observed for this centered version of EC.

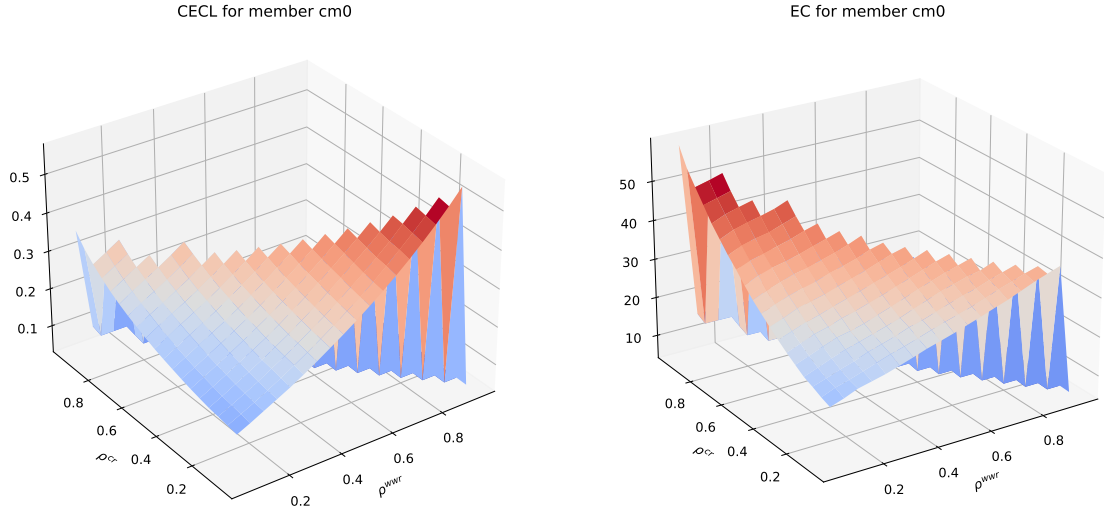


Figure 2: Member 0 CECL and EC w.r.t. credit factors correlation ρ^{cr} and credit and portfolio variation factors correlation ρ^{wvr} .

Remark 3.2. In our example, $\text{Cov}(Y_i, Y_j) = \text{nom}_i \text{nom}_j \sigma_i \sigma_j \frac{\nu}{\nu-2} \rho^{mkt}$. Hence, depending on the sign of $\text{nom}_i \text{nom}_j$, increasing ρ^{mkt} either increases or decreases $\text{Cov}(Y_i, Y_j)$. Thus, we cannot hope to observe a monotonous behaviour of EC or CECL w.r.t. ρ^{mkt} .

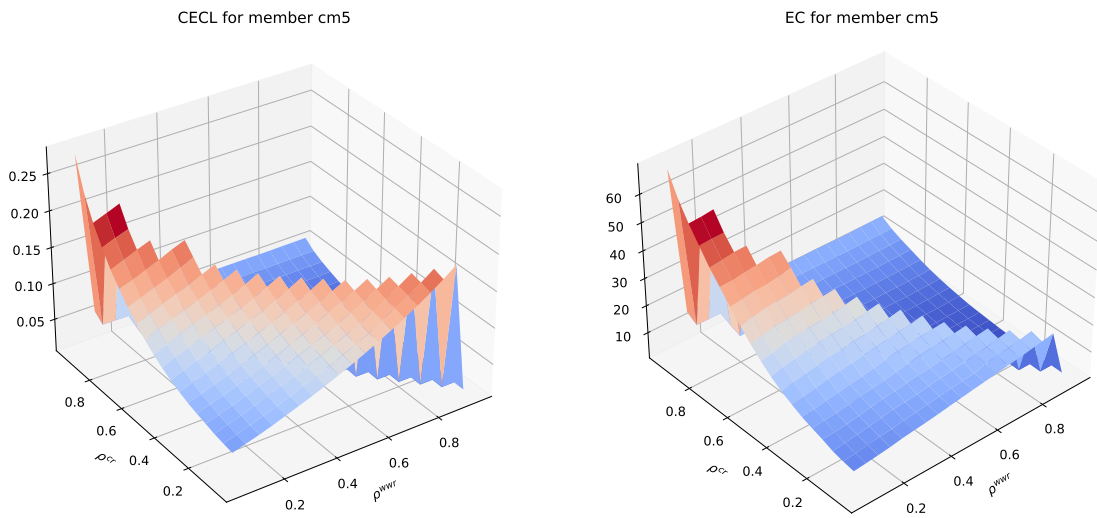


Figure 3: Member 5 CECL and EC w.r.t. credit factors correlation ρ^{cr} and credit and portfolio variation factors correlation ρ^{wvr} .

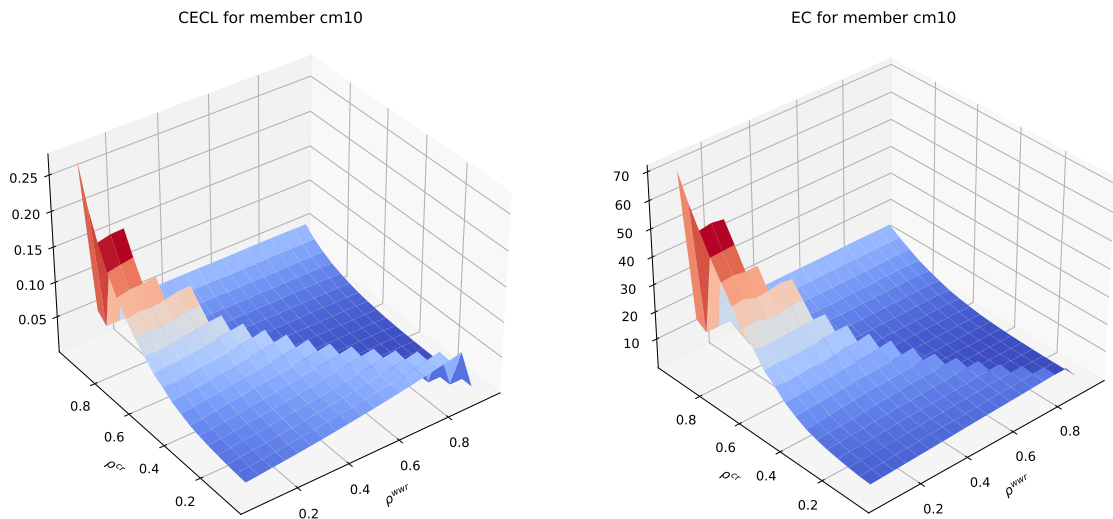


Figure 4: Member 10 CECL and EC w.r.t. credit factors correlation ρ^{cr} and credit and portfolio variation factors correlation ρ^{wvr} .

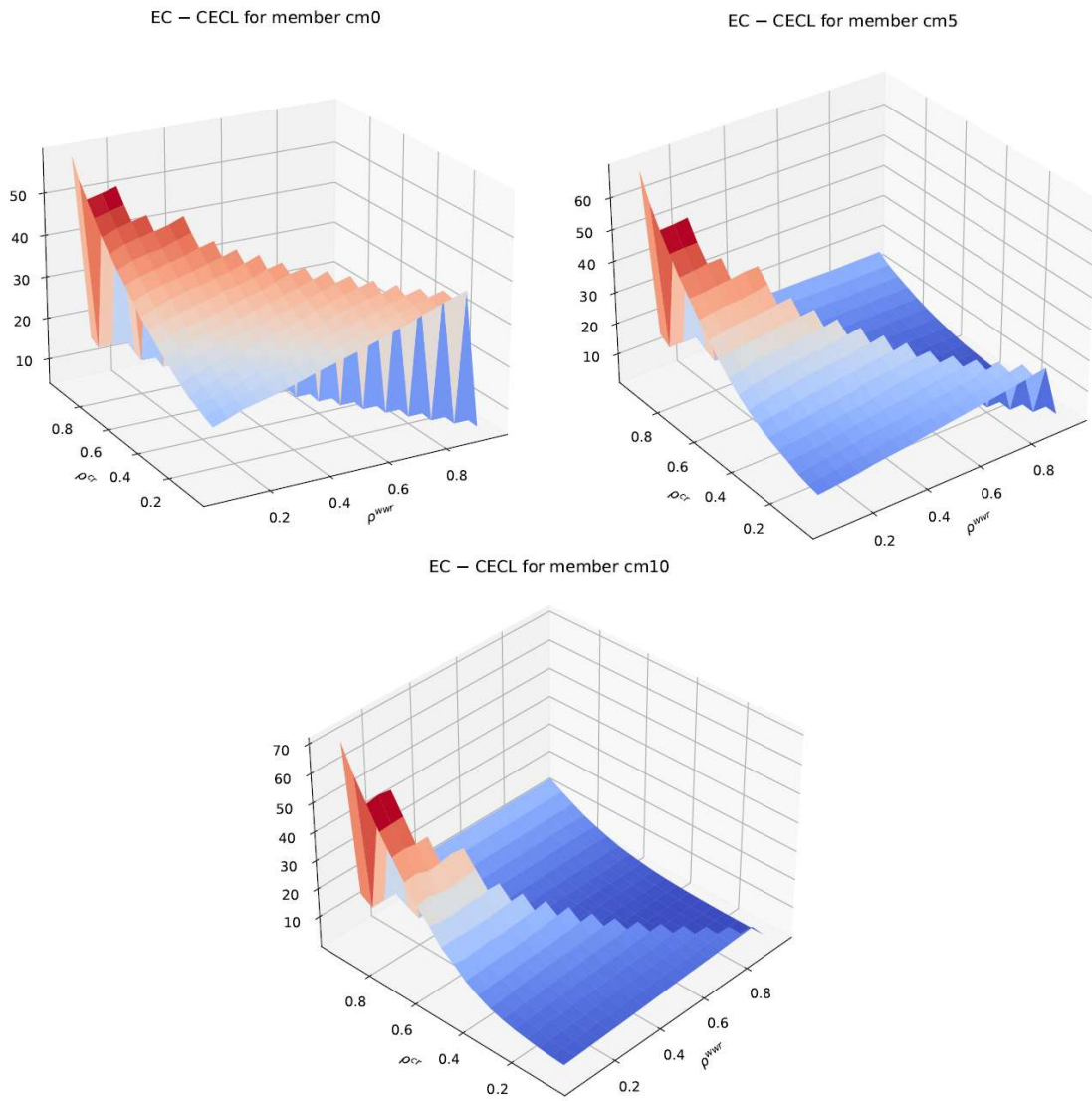


Figure 5: Members 0, 5 and 10 EC - CECL w.r.t. credit factors correlation ρ^{cr} and credit and portfolio variation factors correlation ρ^{wvr} .

4 Credit Derivatives Supplement

We analyse the monotonicity of default leg and coupon leg of synthetic equity and senior CDO tranches prices w.r.t. credit correlation, also using supermodular properties. Such prices are obtained by taking the expected value under \mathbb{Q} , assumed to be, in this section, the risk-neutral pricing measure of the loss function underlying the CDO tranche contract. The characteristics of the payoff are as follows. There are n obligors, indexed by i . All underlying CDS are assumed to mature at some common time T . For any obligor i , the default time is denoted by τ_i , the deterministic recovery rate is $R_i \in [0, 1]$, the underlying notional is $N_i \geq 0$ and the loss given default is $L_i = (1 - R_i)N_i$. The maximum loss is $L_{\max} = \sum_{i=1}^n L_i$. To simplify calculations, we assume that the payments due to the obligors defaults are only made at maturity T and the discounting rates are set to zero (nonzero discounting rates can be included as long as they are independent from the credit risk factors).

Definition 4.1. *The cumulative credit loss at time $t \leq T$ is*

$$L(t) = \sum_{i=1}^n L_i \mathbb{1}_{\{\tau_i \leq t\}}.$$

The default leg of an equity tranche with maturity T and detachment point $B \in (0, L_{\max}]$ is

$$D_{eq}(T) = L(T) - (L(T) - B)^+ = \min(L(T), B).$$

The default leg of a senior tranche with maturity T and attachment point $A \in [0, L_{\max})$ is

$$D_{sen}(T) = (L(T) - A)^+. \quad \square$$

Definition 4.2. *The payment leg consists in payments, at K regular times t_k (with $t_K = T$), of a fixed spread s applied to the remaining tranche amount at risk. In the case of the equity tranche, the payment leg writes*

$$P_{eq}(T) = s \frac{T}{K} \sum_{k=1}^K (B - L(t_k))^+. \quad (11)$$

In the case of the senior tranche, it writes

$$\begin{aligned} P_{sen}(T) &= s \frac{T}{K} \sum_{k=1}^K \left(L_{\max} - A - (L(t_k) - A)^+ \right) \\ &= sT(L_{\max} - A) - s \frac{T}{K} \sum_{k=1}^K (L(t_k) - A)^+. \quad \square \end{aligned} \quad (12)$$

Putting default and payment payoffs together, we obtain, for the equity tranche,

$$D_{\text{eq}}(T) - P_{\text{eq}}(T) = L(T) - (L(T) - B)^+ - s \frac{T}{K} \sum_{k=1}^K (B - L(t_k))^+,$$

and, for the senior tranche,

$$D_{\text{sen}}(T) - P_{\text{sen}}(T) = (L(T) - A)^+ - sT(L_{\max} + A) + s \frac{T}{K} \sum_{k=1}^K (L(t_k) - A)^+.$$

Specifying $X_i = F_i^{-1}(1 - \gamma_i(\tau_i))$, where F_i is the invertible c.d.f. of X_i , γ_i the \mathbb{Q} c.d.f. of τ_i and letting $B_i(t) := F_i^{-1}(1 - \gamma_i(t))$, we have $\{\tau_i \leq t\} = \{X_i \geq B_i(t)\}$ and

$$L(t) = \sum_{i=1}^n L_i \mathbb{1}_{\{X_i \geq B_i(t)\}}.$$

The following result precises the outlined application for comparing CDO tranche premiums in Cousin and Laurent (2008) with heterogeneous obligors under our static setup.

Proposition 4.1. *If $(X_1, \dots, X_n) \sim E_n(\boldsymbol{\mu}, \boldsymbol{\Gamma}, \psi)$ and $(X'_1, \dots, X'_n) \sim E_n(\boldsymbol{\mu}, \boldsymbol{\Gamma}', \psi)$, with $\boldsymbol{\Gamma} \leq \boldsymbol{\Gamma}'$ elementwise except for equal diagonal entries, then*

$$\mathbb{E} \left[\left(\sum_{i=1}^n L_i \mathbb{1}_{\{X_i \geq B_i(t)\}} - A \right)^+ \right] \leq \mathbb{E} \left[\left(\sum_{i=1}^n L_i \mathbb{1}_{\{X'_i \geq B_i(t)\}} - A \right)^+ \right], \quad A \in \mathbb{R}. \quad (13)$$

That is, the price of the default leg of a senior CDO tranche is nondecreasing w.r.t. the homogeneous correlation coefficient between the X_i 's noted ρ^{cr} ; we also get that the price of the default leg of an equity CDO tranche is nonincreasing w.r.t. ρ^{cr} .

Proof. For any $t \in \mathbb{R}_+$, the function

$$f_t : \quad \mathbb{R}^n \quad \longrightarrow \quad \mathbb{R}_+ \\ (x_1, \dots, x_n) \longmapsto \sum_{i=1}^n L_i \mathbb{1}_{\{x_i \geq B_i(t)\}}$$

is nondecreasing on \mathbb{R}^n and it is supermodular, by Lemma A.1. Hence, due to the nondecreasing and convexity properties of $x \mapsto (x - A)^+$, Müller and Stoyan (2002, Theorem 3.9.3 f), page 113), recalled in Section A, implies that $(x_1, \dots, x_n) \mapsto (f_t(x_1, \dots, x_n) - A)^+$ is also nondecreasing supermodular. Moreover, by Block and Sampson (1988, Corollary 2.3), recalled in Section B, $(X_1, \dots, X_n) \leq_{sm} (X'_1, \dots, X'_n)$. Applying Müller (1997, Definition 2.6) to (X_1, \dots, X_n) , (X'_1, \dots, X'_n) and $(x_1, \dots, x_n) \mapsto (f_t(x_1, \dots, x_n) - A)^+$ then yields the result for the senior tranche.

As $\mathbb{E} \left[\sum_{i=1}^n L_i \mathbb{1}_{\{X_i \geq B_i(t)\}} \right] - \mathbb{E} \left[\left(\sum_{i=1}^n L_i \mathbb{1}_{\{X_i \geq B_i(t)\}} - B \right)^+ \right]$ is the price of an equity tranche default leg, where the left expectation term does not depend on ρ^{cr} , the result for the equity tranche follows. \square

Corollary 4.1. *Under the assumptions of Proposition 4.1, the price of the payment leg of the CDO equity (resp. senior) tranche is nondecreasing (resp. nonincreasing) w.r.t. ρ^{cr} .*

Proof. By call-put parity,

$$\mathbb{E} \left[\left(B - \sum_{i=1}^n L_i \mathbb{1}_{\{X_i \geq B_i(t)\}} \right)^+ \right] = \mathbb{E} \left[\left(\sum_{i=1}^n L_i \mathbb{1}_{\{X'_i \geq B_i(t)\}} - B \right)^+ \right] \\ - B + \mathbb{E} \left[\sum_{i=1}^n L_i \mathbb{1}_{\{X'_i \geq B_i(t)\}} \right]$$

so that, in view of (11)-(12), as a consequence of Proposition 4.1,

$$\mathbb{E} \left[\left(B - \sum_{i=1}^n L_i \mathbb{1}_{\{X_i \geq B_i(t)\}} \right)^+ \right] \leq \mathbb{E} \left[\left(B - \sum_{i=1}^n L_i \mathbb{1}_{\{X'_i \geq B_i(t)\}} \right)^+ \right]. \quad \square \quad (14)$$

Remark 4.1. *For mezzanine tranches, such results do not hold. Indeed, the tranchéd loss default leg payoff function*

$$D_{Mezz}(T) = (L(T) - A)^+ - (L(T) - B)^+,$$

where $A, B \in (0, L_{\max})$, is not a convex function of the cumulative loss, nor is the payment leg

$$P_{Mezz}(T) = s \frac{T}{K} \sum_{k=1}^K \left[B - A - \left((L(t_k) - A)^+ - (L(t_k) - B)^+ \right) \right] \\ = sT(B - A) - s \frac{T}{K} \sum_{k=1}^K \left[(L(t_k) - A)^+ - (L(t_k) - B)^+ \right].$$

The results are illustrated in Figure 6 for the equity tranches, varying detachment point from 5% to 95% with 5% steps, i.e. considering the tranches from [0, 5%] to [0, 95%]. Figure 7 illustrates the results for the senior tranches, varying attachment point from 5% to 95% with 5% steps, i.e. considering the tranches from [5%, 100%] to [95%, 100%]. The correlation ρ^{cr} is varied from 5% to 95% with 5% step for both tranche types. The parameters of the underlying obligors and CDSs are detailed in Table 4, where values have been assigned arbitrarily to ensure heterogeneity of the various obligors. The CDO tranche spread has been set to 10% with a single coupon paid at a maturity of 5 years. The monotonicity patterns are observed for both tranches, with incremental prices between two consecutive credit correlation steps being nonpositive for the CDO equity tranche default leg prices and the CDO senior tranche payment leg prices. Incremental prices between two consecutive credit correlation steps are nonnegative for both the CDO equity tranche payment leg prices and the CDO senior tranche default leg prices. These results are in line with Proposition 4.1 and Corollary 4.1. Also, incremental

prices between two attachment point steps are nonnegative for both the CDO equity tranche default leg and payment leg prices, as expected from (14). The incremental prices between two attachment point steps are nonpositive for the CDO senior tranche default leg and payment leg prices, as expected from (13).

Table 4: CDO portfolios and obligors parameters.

Obligor id	1	2	3	4	5	6	7	8	9	10
Notional	100	105	110	115	120	100	105	110	115	120
RR (%)	30	31	32	33	34	35	36	37	38	39
λ (%)	2	2.5	3	3.5	4	4.5	5	5.5	6	6.5
Obligor id	11	12	13	14	15	16	17	18	19	20
Notional	100	105	110	115	120	100	105	110	115	120
RR (%)	40	30	31	32	33	34	35	36	37	38
λ (%)	7	7.5	8	8.5	9	9.5	10	10.5	11	11.5
Obligor id	21	22	23	24	25	26	27	28	29	30
Notional	100	105	110	115	120	100	105	110	115	120
RR (%)	39	40	30	31	32	33	34	35	36	37
λ (%)	12	12.5	13	13.5	14	14.5	15	15.5	16	16.5

Remark 4.2. *These results can be extended to stochastic recovery rates, correlated to default latent variables to capture wrong-way risk, in the spirit of Frey et al. (2001); Frey and McNeil (2003); Pykhtin (2003) and Chabaane et al. (2004).*

5 Synthesis of the Results

The main mathematical results of the paper are summed up in Table 5. In a nutshell, if a participant uses a convex risk measure to assess its credit risk depicted as an aggregation of nonnegative losses driven by elliptically distributed factors, then the measure increases with the covariance coefficients between these factors. These results and their numerical illustrations support the use of such elliptical factor models for both risk management and regulatory credit provision and capital requirement purposes.

References

Acerbi, C. and D. Tasche (2002). On the coherence of expected shortfall. *Journal of Banking & Finance* 26, 1487–1503.

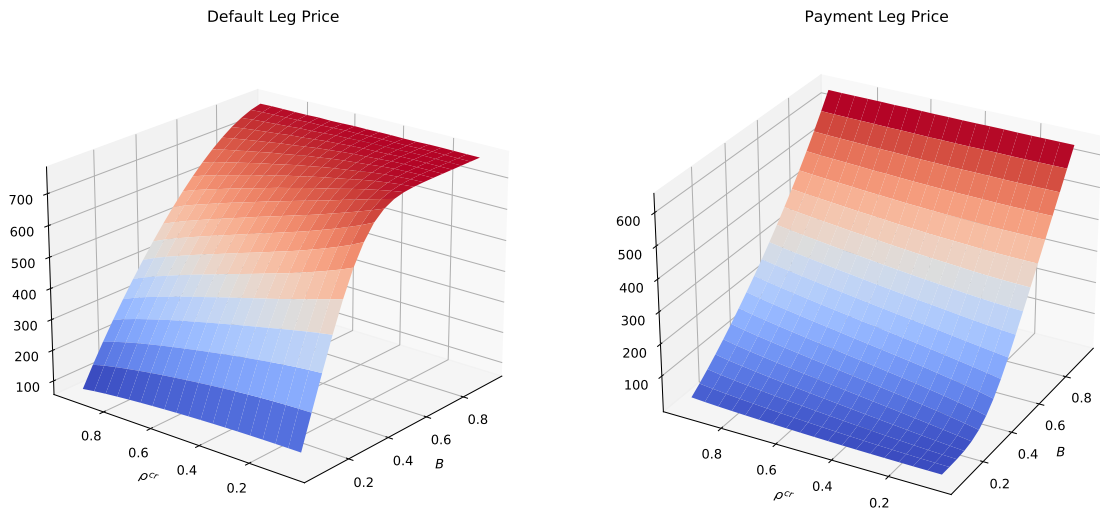


Figure 6: Default leg and Payment leg prices of CDO equity tranches w.r.t. latent variable credit correlation ρ^{cr} and detachment point B .

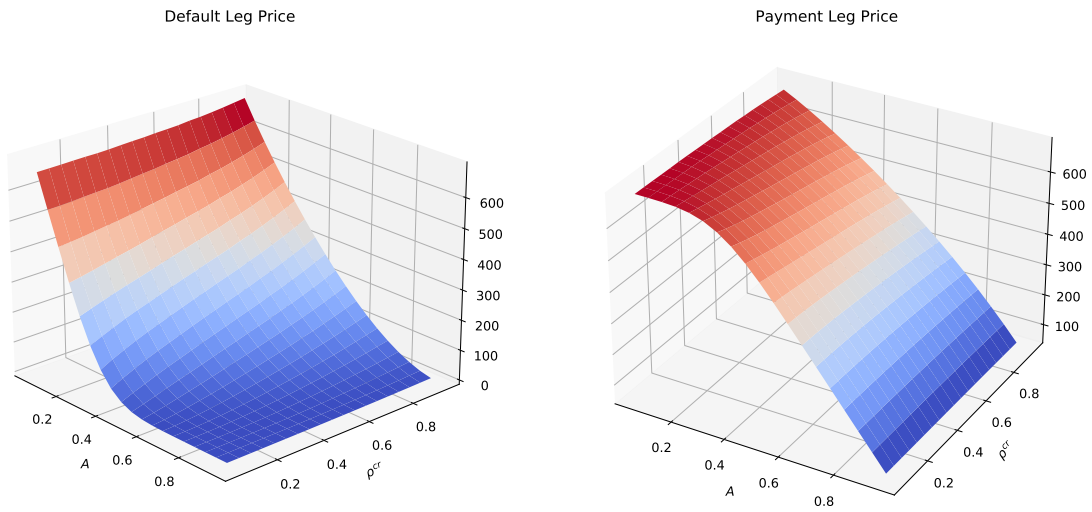


Figure 7: Default leg and Payment leg prices of CDO senior tranches w.r.t. latent variable credit correlation ρ^{cr} and attachment point A . Note that the axes are different from Figure 6, for a better readability.

Table 5: Main theoretical and applied results of the paper (with risk neutral measure \mathbb{Q} , reference participant labelled by 0, related survival measure \mathbb{Q}^0).

Proposition 2.2	Let $f_i : \mathbb{R}^n \rightarrow \mathbb{R}$ be nondecreasing supermodular functions, $g_i : \mathbb{R} \rightarrow \mathbb{R}$ be nonnegative nondecreasing functions, $i \in 1..n$. Then the function $(x_1, \dots, x_n, y_1, \dots, y_n) \mapsto \sum_{i=1}^n f_i(x_1, \dots, x_n)g_i(y_i)$ is supermodular on \mathbb{R}^{2n} .
Theorem 2.1	If $\mathbf{X} \leq_{sm^0} \mathbf{X}'$ and ρ is a risk measure ^a on \mathfrak{X}^0 , then $\rho(\ell(\mathbf{X})) \leq \rho(\ell(\mathbf{X}'))$ holds whenever $\ell(\mathbf{X}), \ell(\mathbf{X}') \in \mathfrak{X}^0$.
Proposition 4.1 and Corollary 4.1	The price of the default leg of an equity (resp. senior) tranche is nonincreasing (resp. nondecreasing) w.r.t. the credit correlation ρ^{cr} . The price of the payment leg of the CDO equity (resp. senior) tranche is nonincreasing (resp. nondecreasing) w.r.t. ρ^{cr} .
Proposition 2.3	Under Assumption 1.1, $\text{CECL}(\mathcal{X}_0, \mathbf{X}) \leq \text{CECL}(\mathcal{X}_0, \mathbf{X}')$, in particular in the elliptical setup (3).
Proposition 2.4	Under Assumption 1.1, $\text{EC}(\mathcal{X}_0, \mathbf{X}) \leq \text{EC}(\mathcal{X}_0, \mathbf{X}')$, in particular in the elliptical setup (3).

^asee Definition C.1.

- Albanese, C., S. Crépey, R. Hoskinson, and B. Saadeddine (2021). XVA analysis from the balance sheet. *Quantitative Finance* 21(1), 99–123.
- Bastide, D., S. Crépey, S. Drapeau, and M. Tadese (2023). Derivatives risks as costs in a one-period network model. *Frontiers of Mathematical Finance* 2(3), 283–312.
- Bäuerle, N. and A. Müller (1998). Modelling and comparing dependencies in multivariate risk portfolios. *ASTIN Bulletin* 28(1), 59–76.
- Bäuerle, N. and A. Müller (2006). Stochastic orders and risk measures: consistency and bounds. *Insurance: Mathematics and Economics* 38(1), 132–148.
- Block, H. W. and A. R. Sampson (1988). Conditionally ordered distributions. *Journal of Multivariate Analysis* 27, 91–104.
- Brigo, D., M. Morini, and A. Pallavicini (2013). *Counterparty Credit Risk, Collateral and Funding: With Pricing Cases for All Asset Classes*. Wiley.
- Chabaane, A., J.-P. Laurent, and J. Salomon (2004, 03). Double impact: Credit risk assessment and collateral value. *Revue Finance* 25.
- Cousin, A. and J.-P. Laurent (2008). Comparison results for exchangeable credit risk portfolios. *Insurance: Mathematics and Economics* 42(3), 1118–1127.
- European Central Bank (October 2019). ECB guide to internal models. Available at https://www.bankingsupervision.europa.eu/ecb/pub/pdf/ssm.guidetointernalmodels_consolidated_201910~97fd49fb08.en.pdf. Accessed on October 14, 2021.
- Fang, K.-T., S. Kotz, and K. W. Ng (1990). *Symmetric Multivariate and Related Distributions*. CRC Press, New York.
- Föllmer, H. and A. Schied (2016). *Stochastic Finance: An Introduction in Discrete Time* (4th ed.). De Gruyter Graduate, Berlin, Germany.
- Frey, R. and A. McNeil (2003, 07). Dependent defaults in models of portfolio credit risk. *Journal of Risk* 6.
- Frey, R., A. McNeil, and M. Nyfeler (2001, 01). Copulas and credit models. *Risk* 14, 111–114.
- Gregory, J. (2015). *The xVA challenge: counterparty credit risk, funding, collateral and capital*. Wiley.
- Kallenberg, O. (2021). *Foundations of Modern Probability* (3 ed.). Switzerland: Springer Nature.
- Marshall, A. W. and I. Olkin (1979). *Inequalities: Theory of Majorization and Its Applications*. Academic Press, New York.

- McNeil, A. J., R. Frey, and P. Embrechts (2015). *Quantitative Risk Management: Concepts, Techniques and Tools* (revised ed.). Princeton Series in Finance.
- Meyer, M. A. and B. Strulovici (2013, May). The Supermodular Stochastic Ordering. CEPR Discussion Papers 9486.
- Meyer, M. A. and B. Strulovici (July 2012). Increasing interdependence of multivariate distributions. *Journal of Economic Theory* 147(4), 91–104.
- Meyer-Nieberg, P. (1991). *Banach Lattices*. Universitext. Springer-Verlag.
- Müller, A. (1997). Stop-loss order for portfolios of dependent risks. *Insurance: Mathematics and Economics* 21, 219–223.
- Müller, A. and M. Scarsini (2000). Some remarks on the supermodular order. *Journal of Multivariate Analysis* 73, 107–119.
- Müller, A. and D. Stoyan (2002). *Comparison Methods for Stochastic Models and Risks*. Wiley Series in Probability and Statistics.
- Pflug, G. C. and W. Römisch (2007). *Modeling, Measuring and Managing Risk*. World Scientific Publishing Co. Pte. Ltd., Singapore.
- Pykhtin, M. (2003, 01). Unexpected recovery risk. *Risk* 16, 74–78.
- Ruszczynski, A. and A. Shapiro (2006). Optimization of convex risk functions. *Mathematics of Operations Research* 31(3), 433–452.
- Shaked, M. and J. G. Shanthikumar (1997). Supermodular stochastic orders and positive dependence of random vectors. *Journal of Multivariate Analysis* 61(MV971656), 86–101.
- Shapiro, A. (2013, February). On Kusuoka Representation of Law Invariant Risk Measures. *Mathematics of Operations Research* 38(1), 142–152.
- Single Resolution Board (2021, 01). The Single Resolution Fund. Technical report, Single Resolution Board. Available on <https://www.srb.europa.eu/en/content/single-resolution-fund>. Accessed on September 8, 2022.
- Sterneman, A. and F. van Perlo-ten Kleij (2005, September). Spherical distributions: Schoenberg (1938) revisited. *Expositiones Mathematicae* 23(3), 281–287.
- Yildiz, M. (2010, April). Lecture notes on supermodular games. Available at <https://dspace.mit.edu/handle/1721.1/106995/>. Accessed on December 22, 2021.

A Supermodular Functions

Müller (1997, Definitions 2.4 and 2.6) A function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is said to be supermodular if

$$\begin{aligned} f(x_1, \dots, x_i + \varepsilon, \dots, x_j + \delta, \dots, x_n) - f(x_1, \dots, x_i + \varepsilon, \dots, x_j, \dots, x_n) \geq \\ f(x_1, \dots, x_i, \dots, x_j + \delta, \dots, x_n) - f(x_1, \dots, x_i, \dots, x_j, \dots, x_n) \end{aligned}$$

holds for all $x \in \mathbb{R}^n$, $\varepsilon, \delta > 0$ and $1 \leq i < j \leq n$. \square

Müller and Scarsini (2000, Theorem 2.2 (a)) For functions twice differentiable on \mathbb{R}^d , the supermodularity is equivalent to the nonnegativity of its second derivatives. \square

More general definitions can be found in Meyer-Nieberg (1991) and Yildiz (2010).

Meyer-Nieberg (1991, Section 1.1, page 1) A partially ordered set (\mathfrak{S}, \geq) is said to be a lattice if and only if any two elements x, y have a greatest common minorant, denoted $x \wedge y$, and a least common majorant, denoted $x \vee y$. \square

Yildiz (2010, Definition 6, page 6) A function $f : \mathfrak{S} \rightarrow \mathbb{R}$ is said to be supermodular on a lattice (\mathfrak{S}, \geq) if

$$f(x \vee y) + f(x \wedge y) \geq f(x) + f(y)$$

holds for all $x, y \in \mathfrak{S}$. \square

For a family of lattices $(\mathfrak{S}_1, \leq), \dots, (\mathfrak{S}_n, \leq)$, let $\mathfrak{S} = \mathfrak{S}_1 \times \dots \times \mathfrak{S}_n$ be endowed with the coordinate-wise order $(x_1, \dots, x_n) \leq (y_1, \dots, y_n)$ if and only if $\forall i, x_i \leq y_i$. This order makes $\mathfrak{S} = \mathfrak{S}_1 \times \dots \times \mathfrak{S}_n$ a lattice. For $x \in \mathfrak{S}$ and any i and j , define $\mathbf{x}_{-(i,j)} = (x_k)_{k \neq i,j}$. For any function $f : \mathfrak{S} \rightarrow \mathbb{R}$, define $f(\cdot | \mathbf{x}_{-(i,j)}) : \mathfrak{S}_i \times \mathfrak{S}_j \rightarrow \mathbb{R}$ as the restriction of f to vectors with entries other than i and j fixed at $\mathbf{x}_{-(i,j)}$ (Yildiz, 2010, Section 2.4).

Yildiz (2010, remark after Definition 7, page 7) A function $f : \mathfrak{S} = \mathfrak{S}_1 \times \dots \times \mathfrak{S}_n \rightarrow \mathbb{R}$ is said to be pairwise supermodular if

$$f((x_i, x_j) \vee (x'_i, x'_j) | \mathbf{x}_{-(i,j)}) + f((x_i, x_j) \wedge (x'_i, x'_j) | \mathbf{x}_{-(i,j)}) \geq f(x_i, x_j | \mathbf{x}_{-(i,j)}) + f(x'_i, x'_j | \mathbf{x}_{-(i,j)})$$

holds for all $x_1, \dots, x_n \in \mathfrak{S}_1 \times \dots \times \mathfrak{S}_n$. \square

Yildiz (2010, Definition 7) A function $f : \mathfrak{S} = \mathfrak{S}_1 \times \dots \times \mathfrak{S}_n \rightarrow \mathbb{R}$ is said to have increasing differences if

$$\begin{aligned} [x_i \geq x_{i'}, x_j \geq x_{j'}] \implies \\ f(x_{i'}, x_{j'} | \mathbf{x}_{-(i,j)}) - f(x_{i'}, x_j | \mathbf{x}_{-(i,j)}) \geq f(x_i, x_{j'} | \mathbf{x}_{-(i,j)}) - f(x_i, x_j | \mathbf{x}_{-(i,j)}) \end{aligned}$$

holds for any $\mathbf{x} = (x_1, \dots, x_n)$, i, j, i', j' . \square

If the partial order is a total order, increasing differences and supermodularity are equivalent. In particular:

Yildiz (2010, Corollary 1) For any $f : \mathbb{R}^n \rightarrow \mathbb{R}$, the following are equivalent:

1. f is supermodular;
2. f has increasing differences;
3. f is pairwise supermodular. \square

Müller and Stoyan (2002, Theorem 3.9.3 f), page 113) If $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is nondecreasing and supermodular and $\phi : \mathbb{R} \rightarrow \mathbb{R}$ is nondecreasing and convex, then $\phi \circ f : \mathbb{R}^n \rightarrow \mathbb{R}$ is nondecreasing supermodular. \square

The following two Lemmas A.1 and A.2 will be the building blocks for establishing our main result Theorem 2.1.

Lemma A.1. *The supermodularity property is satisfied by any function $\mathbb{R}^n \ni (x_1, \dots, x_n) \mapsto h(x_1, \dots, x_n) \in \mathbb{R}$ that can be written as $h(x_1, \dots, x_n) = \sum_{i=1}^n h_i(x_i)$ for some functions h_1, \dots, h_n of single arguments. In particular, for any constants B_1, \dots, B_n , the function $(x_1, \dots, x_n) \mapsto \sum_{i=1}^n \mathbb{1}_{\{x_i \geq B_i\}}$ is supermodular. This function is also nondecreasing on \mathbb{R}^n .*

Proof. By Yildiz (2010, Corollary 1), recalled in Section A, we can focus on increasing differences. Let $h(x_i, x_j | \mathbf{x}_{-i,j})$ denote the function h applied to x_i and x_j but keeping all other arguments $\mathbf{x}_{-i,j} := (x_k)_{k \neq i,j}$ fixed. Fixing $\delta, \varepsilon > 0$, the difference $h(x_i + \delta, x_j + \varepsilon | \mathbf{x}_{-i,j}) - h(x_i, x_j + \varepsilon | \mathbf{x}_{-i,j}) - h(x_i + \delta, x_j | \mathbf{x}_{-i,j}) + h(x_i, x_j | \mathbf{x}_{-i,j}) = h_i(x_i + \delta) + h_j(x_j + \varepsilon) - h_i(x_i) - h_j(x_j + \varepsilon) - h_i(x_i + \delta) - h_j(x_j) + h_i(x_i) + h_j(x_j)$ simplifies to 0, showing the supermodularity of h . \square

Lemma A.2. *If $g : \mathbb{R} \rightarrow \mathbb{R}$ and $h : \mathbb{R} \rightarrow \mathbb{R}$ are both nondecreasing, then $(x, y) \mapsto g(x)h(y)$ is supermodular.*

Proof. If $x' \geq x$ and $y' \geq y$, then

$$\begin{aligned} & g(x')h(y') - g(x')h(y) - g(x)h(y') + g(x)h(y) \\ &= g(x')(h(y') - h(y)) - g(x)(h(y') - h(y)) \\ &\geq g(x)(h(y') - h(y)) - g(x)(h(y') - h(y)) = 0. \end{aligned}$$

Hence $(x, y) \mapsto g(x)h(y)$ has increasing differences w.r.t. any pair $(x, y) \in \mathbb{R}^2$, i.e. is supermodular, by Yildiz (2010, Corollary 1). \square

The following classical results on supermodular random vectors will be instrumental in establishing our main result Theorem 2.1.

Müller (1997, Definition 2.6) A random vector $\mathbf{X} = (\mathcal{X}_1, \dots, \mathcal{X}_m)$ is said to be smaller than the random vector $\mathbf{Y} = (\mathcal{Y}_1, \dots, \mathcal{Y}_m)$ in the supermodular ordering, written $\mathbf{X} \leq_{sm} \mathbf{Y}$, if $\mathbb{E}(f(\mathbf{X})) \leq \mathbb{E}(f(\mathbf{Y}))$ holds for all the supermodular functions⁶ $f : \mathbb{R}^m \rightarrow \mathbb{R}$ such that the expectations exist. \square

⁶see Appendix A.

Müller and Stoyan (2002, Definition 3.9.4, page 113) A random vector $\mathbf{X} = (\mathcal{X}_1, \dots, \mathcal{X}_m)$ is said to be smaller than the random vector $\mathbf{Y} = (\mathcal{Y}_1, \dots, \mathcal{Y}_m)$ in the increasing supermodular ordering, written $\mathbf{X} \leq_{ism} \mathbf{Y}$, if $\mathbb{E}(f(\mathbf{X})) \leq \mathbb{E}(f(\mathbf{Y}))$ holds for all the nondecreasing supermodular functions $f : \mathbb{R}^m \rightarrow \mathbb{R}$ such that the expectations exist. \square

An equivalent characterization of supermodular vectors is given by

Müller and Stoyan (2002, Theorems 3.9.11 (i) and (ii), page 118) The following statements are equivalent:

1. $\mathbf{X} \leq_{sm} \mathbf{Y}$,
2. \mathbf{X} and \mathbf{Y} have the same marginals and $\mathbf{X} \leq_{ism} \mathbf{Y}$. \square

Müller (1997, Theorems 3.2 (c)) If \mathbf{X} , \mathbf{Y} , \mathbf{Z} are random vectors such that any random vectors distributed as \mathbf{X} and \mathbf{Y} conditionally on $\mathbf{Z} = \mathbf{z}$, denoted by $[\mathbf{X}|\mathbf{Z} = \mathbf{z}]$ and $[\mathbf{Y}|\mathbf{Z} = \mathbf{z}]$, verify $[\mathbf{X}|\mathbf{Z} = \mathbf{z}] \leq_{sm} [\mathbf{Y}|\mathbf{Z} = \mathbf{z}]$ for all possible values of \mathbf{z} , then $\mathbf{X} \leq_{sm} \mathbf{Y}$. \square

Müller (1997, Definition 2.1) For \mathcal{X} and \mathcal{Y} in $L^1(\mathbb{Q})$, \mathcal{X} precedes \mathcal{Y} in stop-loss order, written $\mathcal{X} \leq_{sl} \mathcal{Y}$, if $\mathbb{E}(\mathcal{X} - A)^+ \leq \mathbb{E}(\mathcal{Y} - A)^+$ holds for all real constants $A \geq 0$. \square

Bäuerle and Müller (1998, Theorem 2.2 b)) For any \mathcal{X} and \mathcal{Y} , $\mathcal{X} \leq_{sl} \mathcal{Y}$ if and only if $\mathbb{E}(f(\mathcal{X})) \leq \mathbb{E}(f(\mathcal{Y}))$ holds for all the nondecreasing convex functions $f : \mathbb{R} \rightarrow \mathbb{R}$ such that the expectations exist (e.g. $f = id$). \square

Remark A.1. The former equivalent condition is also known as the second-order stochastic dominance, according to Meyer and Strulovici (2012, paragraph 8.1.2, page 267).

B Elliptical Distributions

McNeil et al. (2015, Definition 6.17, page 196) A random vector $\mathbf{Z} = (\mathcal{Z}_1, \dots, \mathcal{Z}_n)^\top$ has a spherical distribution in \mathbb{R}^n if, for every orthogonal map $\mathbf{A} \in \mathbb{R}^{n \times n}$ (i.e. $\mathbf{A}\mathbf{A}^\top = \mathbf{A}^\top\mathbf{A} = I_n$),

$$\mathbf{AZ} \stackrel{d}{=} \mathbf{Z}. \quad \square$$

McNeil et al. (2015, Theorem 6.18, page 196) The following are equivalent.

- (1) \mathbf{Z} is spherical in \mathbb{R}^n .
- (2) There exists a function $\psi : \mathbb{R}_+ \rightarrow \mathbb{C}$ such that, for all $\mathbf{u} = (u_1, \dots, u_n)^\top \in \mathbb{R}^n$, the characteristic function of \mathbf{Z} is

$$\mathbb{E}\left(e^{i\mathbf{u}^\top \mathbf{Z}}\right) = \psi(\mathbf{u}^\top \mathbf{u}) = \psi(u_1^2 + \dots + u_n^2).$$

(3) For every $\mathbf{u} \in \mathbb{R}^n$, $\mathbf{u}^\top \mathbf{Z} \stackrel{d}{=} \|\mathbf{u}\| \mathcal{Z}_1$. \square

ψ is then called the characteristic generator of \mathbf{Z} and the notation $\mathbf{Z} \sim S_n(\psi)$ is used (see Fang, Kotz, and Ng (1990) and McNeil, Frey, and Embrechts (2015)). We denote by $\mathbb{S}^{n-1} := \{\mathbf{s} \in \mathbb{R}^n : \mathbf{s}^\top \mathbf{s} = 1\}$ the unit sphere in \mathbb{R}^n , and by $U_{\mathbb{S}^{n-1}}$, the uniform distribution on \mathbb{S}^{n-1} which has a spherical distribution in \mathbb{R}^n (see Fang, Kotz, and Ng (1990, Example 2.1, page 28 & Theorem 3.1, page 70)).

Fang et al. (1990, Theorem 2.2, page 29) A function ψ is a generator of an n -dimensional elliptical r.v. if and only if it can be written as

$$\psi(x) = \int_0^\infty \Omega_n(xr^2)F(dr), \quad (15)$$

where $F(\cdot)$ is some c.d.f. over \mathbb{R}^+ and $\Omega_n(\mathbf{u}^\top \mathbf{u})$ is the characteristic generator of a random vector $\mathbf{S} \sim U_{\mathbb{S}^{n-1}}$, namely (Steerneman and van Perlo-ten Kleij, 2005, Eqn. (2))

$$\Omega_n(\mathbf{u}^\top \mathbf{u}) = \mathbb{E} \left(e^{i\mathbf{u}^\top \mathbf{S}} \right) = \frac{\Gamma(m/2)}{\sqrt{\pi}\Gamma((m-1)/2)} \int_{-1}^1 e^{i\mathbf{u}^\top \mathbf{u}t} (1-t^2)^{(m-3)/2} dt \mathbf{u} \in \mathbb{R}^n. \quad \square$$

McNeil et al. (2015, Theorem 6.21, page 197) \mathbf{Z} has a spherical distribution in \mathbb{R}^n if and only if it has a stochastic representation

$$\mathbf{Z} \stackrel{d}{=} \mathcal{R}\mathbf{S},$$

where $\mathbf{S} \sim U_{\mathbb{S}^{n-1}}$ and $\mathcal{R} \geq 0$ is a radial (i.e. a nonnegative scalar) r.v. independent of \mathbf{S} . \square

McNeil et al. (2015, Definition 6.25, page 200) $\mathbf{X} = (\mathcal{X}_1, \dots, \mathcal{X}_n)$ is said to have an elliptical distribution in \mathbb{R}^n with parameters $\boldsymbol{\mu}$, Σ , ψ , where Σ is an $n \times n$ square semi-positive definite matrix, if

$$\mathbb{E} \left(e^{i\mathbf{u}^\top (\mathbf{X} - \boldsymbol{\mu})} \right) = \psi(\mathbf{u}^\top \Sigma \mathbf{u}), \quad \mathbf{u} \in \mathbb{R}^n.$$

We then write $\mathbf{X} \sim E_n(\boldsymbol{\mu}, \Sigma, \psi)$ ⁷. \square

McNeil et al. (2015, Proposition 6.27, page 200) $\mathbf{X} \sim E_n(\boldsymbol{\mu}, \Sigma, \psi)$ if and only if there exist \mathbf{S} , \mathcal{R} and \mathbf{A} satisfying

$$\mathbf{X} \stackrel{d}{=} \boldsymbol{\mu} + \mathcal{R}\mathbf{A}\mathbf{S},$$

where $\mathbf{S} \sim U_{\mathbb{S}^{k-1}}$, \mathcal{R} is a radial r.v. independent of \mathbf{S} , and \mathbf{A} in $\mathbb{R}^{n \times k}$ satisfies $\mathbf{A}\mathbf{A}^\top = \Sigma$. \square

⁷see Fang, Kotz, and Ng (1990) and McNeil, Frey, and Embrechts (2015).

Remark B.1. As outlined in McNeil, Frey, and Embrechts (2015, Eqn. (6.41), page 201)), for Σ positive definite,

$$\mathbf{X} \sim E_n(\boldsymbol{\mu}, \Sigma, \psi) \iff \Sigma^{-1/2}(\mathbf{X} - \boldsymbol{\mu}) \sim S_n(\psi). \quad (16)$$

Following McNeil, Frey, and Embrechts (2015, Eqn. (6.42), page 201)), for an elliptical variate $\mathbf{X} \sim E_n(\boldsymbol{\mu}, \Sigma, \psi)$, if Σ has full rank n , then, by McNeil et al. (2015, Corollary 6.22, page 198) and (16),

$$\left(\sqrt{(\mathbf{X} - \boldsymbol{\mu})^\top \Sigma^{-1}(\mathbf{X} - \boldsymbol{\mu})}, \frac{\Sigma^{-1/2}(\mathbf{X} - \boldsymbol{\mu})}{\sqrt{(\mathbf{X} - \boldsymbol{\mu})^\top \Sigma^{-1}(\mathbf{X} - \boldsymbol{\mu})}} \right) \stackrel{d}{=} (\mathcal{R}, \mathbf{S}). \quad \square$$

Fang et al. (1990, Theorem 2.18, page 45) Let $\mathbf{Y} \stackrel{d}{=} \boldsymbol{\mu} + \mathcal{R}\mathbf{A}\mathbf{S} \sim E_n(\boldsymbol{\mu}, \Sigma, \psi)$ (see Section B) with $\Sigma = \mathbf{A}\mathbf{A}^\top$ positive definite. Let

$$\mathbf{Y} = \begin{pmatrix} \mathbf{Y}^{(1)} \\ \mathbf{Y}^{(2)} \end{pmatrix}, \quad \boldsymbol{\mu} = \begin{pmatrix} \boldsymbol{\mu}^{(1)} \\ \boldsymbol{\mu}^{(2)} \end{pmatrix}, \quad \Sigma = \begin{pmatrix} \Sigma^{(1,1)} & \Sigma^{(1,2)} \\ \Sigma^{(2,1)} & \Sigma^{(2,2)} \end{pmatrix},$$

where $\mathbf{Y}^{(1)}$ and $\boldsymbol{\mu}^{(1)}$ are $m \times 1$ vectors and $\Sigma^{(1,1)}$ is an $m \times m$ matrix, for some $0 < m < n$. Then

$$\begin{aligned} \left(\mathbf{Y}^{(1)} \mid \mathbf{Y}^{(2)} = \mathbf{y}^{(2)} \right) &\stackrel{d}{=} \boldsymbol{\mu}_{|\mathbf{Y}^{(2)}=\mathbf{y}^{(2)}}^{(1)} + \mathcal{R}_{|\mathbf{Y}^{(2)}=\mathbf{y}^{(2)}} \mathbf{A}_{|\mathbf{Y}^{(2)}=\mathbf{y}^{(2)}}^{(1,1)} \mathbf{S}^{(m)} \\ &\sim E_m \left(\boldsymbol{\mu}_{|\mathbf{Y}^{(2)}=\mathbf{y}^{(2)}}^{(1)}, \Sigma_{|\mathbf{Y}^{(2)}=\mathbf{y}^{(2)}}^{(1,1)}, \psi_{|\mathbf{Y}^{(2)}=\mathbf{y}^{(2)}} \right), \end{aligned}$$

where

$$\left\{ \begin{array}{l} \boldsymbol{\mu}_{|\mathbf{Y}^{(2)}=\mathbf{y}^{(2)}}^{(1)} = \boldsymbol{\mu}^{(1)} + \Sigma^{(1,2)} (\Sigma^{(2,2)})^{-1} (\mathbf{y}^{(2)} - \boldsymbol{\mu}^{(2)}), \\ \Sigma_{|\mathbf{Y}^{(2)}=\mathbf{y}^{(2)}}^{(1,1)} = \Sigma^{(1,1)} - \Sigma^{(1,2)} (\Sigma^{(2,2)})^{-1} \Sigma^{(2,1)} = \mathbf{A}_{|\mathbf{Y}^{(2)}=\mathbf{y}^{(2)}}^{(1,1)} \left(\mathbf{A}_{|\mathbf{Y}^{(2)}=\mathbf{y}^{(2)}}^{(1,1)} \right)^\top, \\ \mathbf{S}^{(m)} \sim U_{\mathbb{S}^{m-1}}, \\ \mathcal{R}_{|\mathbf{Y}^{(2)}=\mathbf{y}^{(2)}} \stackrel{d}{=} \left((\mathcal{R}^2 - q(\mathbf{y}^{(2)}))^{1/2} \mid \mathbf{y}^{(2)} = \mathbf{y}^{(2)} \right) \text{ and } \mathcal{R}_{|\mathbf{Y}^{(2)}=\mathbf{y}^{(2)}} \text{ is independent} \\ \text{of } \mathbf{S}^{(m)}, \\ q(\mathbf{y}^{(2)}) = (\mathbf{y}^{(2)} - \boldsymbol{\mu}^{(2)})^\top (\Sigma^{(2,2)})^{-1} (\mathbf{y}^{(2)} - \boldsymbol{\mu}^{(2)}), \\ \psi_{|\mathbf{Y}^{(2)}=\mathbf{y}^{(2)}} \text{ is of the form (15) for } n = m, F \text{ given as the c.d.f. of } \mathcal{R}_{|\mathbf{Y}^{(2)}=\mathbf{y}^{(2)}} \text{ and} \\ \mathbf{S} \text{ given as } \mathbf{S}^{(m)}. \quad \square \end{array} \right. \quad (17)$$

Block and Sampson (1988, Corollary 2.3) Let $\mathbf{X} \sim E_n(\boldsymbol{\mu}, \Sigma, \psi)$ and $h : \mathbb{R}^n \rightarrow \mathbb{R}$ be a supermodular, bounded and right-continuous function. Then $\mathbb{E}(h(\mathbf{X}))$ is nondecreasing in the off-diagonal elements of Σ . \square

The extension of Block and Sampson (1988, Corollary 2.3) to all supermodular functions follows from Müller and Scarsini (2000, Theorem 3.3 and Theorem 3.4). Hence, if \mathbf{X} , \mathbf{X}' are defined as per (3), then $\mathbf{X} \leq_{sm} \mathbf{X}'$. As a direct consequence outlined in Shaked and Shanthikumar (1997) after their Definition 2.1, “it follows that the family of multivariate normal distributions (more generally, the family of elliptically contoured distributions) is increasing in the supermodular stochastic order as the correlations increase.”

B.1 Preserving Elliptical Distributions with Conditioning

Let $\boldsymbol{\mu} = (\mu_1, \dots, \mu_m)$ and $\boldsymbol{\Gamma}^{\mathbf{X}} = (\Gamma_{ij})_{1 \leq i, j \leq m}$ be the mean vector and the covariance matrix of $(\mathcal{X}_1, \dots, \mathcal{X}_m)$ under \mathbb{Q} . We use similar notations $\boldsymbol{\mu}'$ and $\boldsymbol{\Gamma}^{\mathbf{X}'}$ regarding $(\mathcal{X}'_1, \dots, \mathcal{X}'_m)$.

Lemma B.1. *Under the elliptical model setup (3) for $(\mathcal{X}_0, \mathbf{X})$ and $(\mathcal{X}_0, \mathbf{X}')$, we have*

$$\begin{aligned} [\mathbf{X}|\mathcal{X}_0] &\sim E_m(\boldsymbol{\mu}_{|0}, \boldsymbol{\Gamma}_{|0}, \psi_{|0}), \\ [\mathbf{X}'|\mathcal{X}_0] &\sim E_m(\boldsymbol{\mu}'_{|0}, \boldsymbol{\Gamma}'_{|0}, \psi'_{|0}), \end{aligned} \tag{18}$$

with

$$\psi_{|0} = \psi'_{|0}, \quad \boldsymbol{\mu}_{|0} = \boldsymbol{\mu}'_{|0}$$

and, for any $i, j \in 1..m$,

$$\Gamma_{|0} \leq \Gamma'_{|0}, \tag{19}$$

where \leq is meant componentwise.

Proof. Applying (17) to $\mathbf{Y}^{(1)} = \mathbf{X}$ and $\mathbf{Y}^{(2)} = \mathcal{X}_0$ yields (18) with

$$\begin{aligned} \boldsymbol{\mu}_{|0} &= \boldsymbol{\mu} + \frac{1}{\Gamma_{00}} (\Gamma_{01}, \dots, \Gamma_{0m})^\top (\mathcal{X}_0 - \mu_0), \\ \boldsymbol{\Gamma}_{|0} &= \boldsymbol{\Gamma}^{\mathbf{X}} - \frac{1}{\Gamma_{00}} (\Gamma_{01}, \dots, \Gamma_{0m})^\top (\Gamma_{01}, \dots, \Gamma_{0m}) \end{aligned} \tag{20}$$

and, using (20),

$$\begin{aligned} \boldsymbol{\mu}'_{|0} &= \boldsymbol{\mu} + \frac{1}{\Gamma_{00}} (\Gamma_{01}, \dots, \Gamma_{0m})^\top (\mathcal{X}_0 - \mu_0) = \boldsymbol{\mu}_{|0} \\ \boldsymbol{\Gamma}'_{|0} &= \boldsymbol{\Gamma}^{\mathbf{X}'} - \frac{1}{\Gamma_{00}} (\Gamma_{01}, \dots, \Gamma_{0m})^\top (\Gamma_{01}, \dots, \Gamma_{0m}) \geq \boldsymbol{\Gamma}_{|0}. \end{aligned}$$

where \geq is meant componentwise.

It remains to show that $\psi_{|0} = \psi'_{|0}$. For all $x_0 \in \mathbb{R}$, by Fang et al. (1990, Theorem 2.18, page 45), which includes (17), the radius $\mathcal{R}_{|\mathcal{X}_0=x_0}$ of $[\mathbf{X}|\mathcal{X}_0 = x_0]$ is distributed like $\left(\mathcal{R}^2 - \frac{1}{\Gamma_{00}} (x_0 - \mu_{\mathcal{X}_0})^2 \right)^{1/2}$, where $\mathcal{R} := \|\mathbf{Z}\|$, and so is the radius $\mathcal{R}'_{|\mathcal{X}_0=x_0}$

of $[\mathbf{X}'|\mathcal{X}_0 = x_0]$ (as both $(\mathcal{X}_0, \mathbf{X})$ and $(\mathcal{X}_0, \mathbf{X}')$ are defined based on the spherically distributed vector \mathbf{Z}). Thus $\mathcal{R}_{|\mathcal{X}_0} \stackrel{d}{=} \mathcal{R}'_{|\mathcal{X}_0}$, with common c.d.f. denoted by $F_{\mathcal{R}_{|\mathcal{X}_0}}$. The corresponding conditional characteristic generator common to $[\mathbf{X}|\mathcal{X}_0]$ and $[\mathbf{X}'|\mathcal{X}_0]$ is given by $\psi_{|\mathcal{X}_0}(x) = \int_0^\infty \Omega_m(xr^2)F_{\mathcal{R}_{|\mathcal{X}_0}}(dr)$, where $\Omega_m(u^\top u)$ is the characteristic function of a random variable that is uniformly distributed on the unit sphere of \mathbb{R}^m . \square

Proposition B.1. *Under the assumptions of Lemma B.1, we have:*

$$[\mathbf{X}|\mathcal{X}_0] \leq_{sm} [\mathbf{X}'|\mathcal{X}_0].$$

Proof. By Lemma B.1, conditionally on \mathcal{X}_0 , \mathbf{X} and \mathbf{X}' have the same elliptical distribution under \mathbb{Q} , except for their covariance matrix coefficients that verify (19). Block and Sampson (1988, Corollary 2.3) recalled in Section B (here applied under \mathbb{Q}) then yields the result. \square

C Risk Measures

The following definition of a risk measure relaxes the definition of a convex risk measure in Shapiro (2013, Introduction (A3)) by not requiring the translation-equivariance property.

Definition C.1. *Let \mathfrak{X} be a closed linear subspace and sublattice of $L^1(\mathbb{Q})$, that includes the constants. A risk measure on \mathfrak{X} is a function $\rho : \mathfrak{X} \rightarrow \mathbb{R}$ satisfying, for $\mathcal{X}, \mathcal{Y} \in \mathfrak{X}$:*

A properness: $\rho(\mathcal{X}) > -\infty$ and $\text{dom } \rho := \{\mathcal{X} \in \mathfrak{X}; \rho(\mathcal{X}) < +\infty\} \neq \emptyset$;

B law-invariance: if $\mathcal{X} \stackrel{d}{=} \mathcal{Y}$ then $\rho(\mathcal{X}) = \rho(\mathcal{Y})$;

C monotonicity: if $\mathcal{X} \leq \mathcal{Y}$ then $\rho(\mathcal{X}) \leq \rho(\mathcal{Y})$;

D convexity: $\rho(\lambda\mathcal{X} + (1-\lambda)\mathcal{Y}) \leq \lambda\rho(\mathcal{X}) + (1-\lambda)\rho(\mathcal{Y})$, $\lambda \in (0, 1)$. \square

Certain risk measures are also translation-equivariant (i.e. $\rho(\mathcal{X} + c) = \rho(\mathcal{X}) + c$, $c \in \mathbb{R}$ see (Pflug and Römisch, 2007, Definition 2.2 (i), page 29)), such as:

Definition C.2. *The expected shortfall (ES)⁸ at the confidence level (quantile) $\alpha \in (\frac{1}{2}, 1)$ of a loss $\mathcal{X} \in \mathfrak{X} = L^1(\mathbb{Q})$ is $\mathbb{ES}_\alpha(\mathcal{X}) = (1-\alpha)^{-1} \left(\mathbb{E}(\mathcal{X} \mathbf{1}_{\{\mathcal{X} \geq \text{VaR}_\alpha(\mathcal{X})\}}) + \text{VaR}_\alpha(\mathcal{X})(\mathbb{Q}(\mathcal{X} < \text{VaR}_\alpha(\mathcal{X})) - \alpha) \right)$, with $\text{VaR}_\alpha(\mathcal{X}) = \inf \{x \in \mathbb{R} : \mathbb{Q}(\mathcal{X} \leq x) > \alpha\}$. \square*

Remark C.1. *The domain of \mathbb{ES}_α^0 is all $L^1(\mathbb{Q}^0)$, thus \mathbb{ES}_α^0 is in particular proper. Acerbi and Tasche (2002, Proposition 3.1) outlines that \mathbb{ES}_α^0 is subadditive and positively homogeneous (therefore convex) as well as monotonous (in our case where we consider loss variables, for $\mathcal{X} \leq 0$ we have $\rho(\mathcal{X}) \leq 0$ and using subadditivity we obtain for $\mathcal{X} \leq \mathcal{Y}$ that $\rho(\mathcal{X}) = \rho(\mathcal{X} - \mathcal{Y} + \mathcal{Y}) \leq \rho(\mathcal{X} - \mathcal{Y}) + \rho(\mathcal{Y}) \leq \rho(\mathcal{Y})$). Hence \mathbb{ES}_α^0 is a risk measure in the sense Definition C.*

⁸see Acerbi and Tasche (2002, Definition 2.6).

We conclude this appendix by a detailed proof of Bäuerle and Müller (2006, Theorem 4.4). The proof relies on the definition of stochastic order:

Bäuerle and Müller (1998, Definition 2.1 a)) For \mathcal{X} and \mathcal{Y} with respective c.d.f. $F_{\mathcal{X}}$ and $F_{\mathcal{Y}}$, \mathcal{X} precedes \mathcal{Y} in stochastic order, written $\mathcal{X} \leq_{st} \mathcal{Y}$, if $F_{\mathcal{X}}(x) \geq F_{\mathcal{Y}}(x)$, $x \in \mathbb{R}$. \square

Remark C.2. *The former order definition is also known as the first-order stochastic dominance, according to Meyer and Strulovici (2012, paragraph 8.1.2, page 266).*

Bäuerle and Müller (2006, Theorem 4.4) Assuming ρ a risk measure as per Definition C.1, then, for any \mathcal{X} and \mathcal{Y} in \mathfrak{X} , $\mathcal{X} \leq_{st} \mathcal{Y}$ implies $\rho(\mathcal{X}) \leq \rho(\mathcal{Y})$. \square

Proof. A succinct proof of this result can be found in Bäuerle and Müller (2006) prior to its statement, but, for completeness, we give a more detailed proof hereafter. If $\mathcal{X} \leq_{st} \mathcal{Y}$, then, by Müller and Stoyan (2002, Theorem 1.5.14, page 22), there exists a r.v. $\mathcal{Z} \in \mathfrak{X}$ s.t. $\mathcal{X} \leq_{st} \mathcal{Z} \leq_{cx} \mathcal{Y}$, where \leq_{cx} is the convex order. By Müller and Stoyan (2002, Theorem 1.2.4, page 3), there exist r.v. \mathcal{X}' and \mathcal{Z}' on a modified probability space $(\Omega', \mathcal{A}', \mathbb{Q}')$, with same respective laws as \mathcal{X} and \mathcal{Z} , such that $\mathcal{X}' \leq \mathcal{Z}'$ holds with certainty i.e., $\forall \omega \in \Omega'$, $\mathcal{X}'(\omega) \leq \mathcal{Z}'(\omega)$ where \leq is the partial order⁹ on \mathbb{R}^m if \mathcal{X} and \mathcal{Z} take value in \mathbb{R}^m . The law-invariance and monotonicity of ρ yield

$$\rho(\mathcal{X}) = \rho(\mathcal{X}') \leq \rho(\mathcal{Z}') = \rho(\mathcal{Z}).$$

From Bäuerle and Müller (2006, Theorem 4.3), which requires the convexity and law-invariance of ρ , we also have $\rho(\mathcal{Z}) \leq \rho(\mathcal{Y})$, hence $\rho(\mathcal{X}) \leq \rho(\mathcal{Y})$. \square

Remark C.3. *Note that the original statement postulates, instead of the law-invariance property, that the risk measure ρ has the Fatou property, that is, if $\mathcal{X}, \mathcal{X}_1, \mathcal{X}_2, \dots$ are integrable random variables with $\mathcal{X}_k \xrightarrow{L^1} \mathcal{X}$, then $\rho(\mathcal{X}) \leq \liminf_{k \rightarrow \infty} \rho(\mathcal{X}_k)$. We recall from Kallenberg (2021, page 832) that a closed linear subspace of a Banach space is a Banach space and from Meyer-Nieberg (1991, Definition 1.2.1 i), page 12) that a sublattice of a lattice with the same meet and join operations \wedge and \vee is again a lattice. Therefore, if \mathfrak{X} is a Banach lattice (i.e. an order lattice that is a complete normed vector space, e.g. L^p space with $p \geq 1$) and $\rho : \mathfrak{X} \rightarrow \mathbb{R}$ is proper, monotonous and convex, then ρ is continuous on the interior of its domain (Ruszczynski and Shapiro, 2006, Proposition 1), thus has the Fatou property on the interior of its domain. Hence the Fatou property requirement is automatically satisfied by ρ as long as it is defined on a Banach lattice. This is the case for both expectation and expected shortfall defined on any sublattice and linear subspace of L^1 .*

⁹see Müller and Stoyan (2002, Remark 1.2.5, page 3).

Acknowledgements

We thank Mekonnen Tadese, postdoctoral researcher at Université Paris Cité / LPSM, for useful references and discussions.

We thank the two anonymous reviewers for their meticulous in-depth review and valuable suggestions that helped improve the quality of the paper.

Funding

This research benefited from the support of the *Chair Stress Test, RISK Management and Financial Steering*, led by the French École polytechnique and its Foundation and sponsored by BNP Paribas.