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# Provisions and Economic Capital for Credit Losses

Dorinel Bastide<sup>a, b</sup>, Stéphane Crépey<sup>c</sup>

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## Abstract

Based on supermodularity ordering properties, we show that convex risk measures of credit losses are nondecreasing w.r.t. credit-credit and, in a wrong-way risk setup, credit-market, covariances of elliptically distributed latent factors. These results support the use of such setups for computing credit provisions and economic capital or for conducting stress test exercises and risk management analysis.

**keywords.** Supermodular function, convex risk measure, supermodular random variables, stop loss property, elliptical distribution, credit loss, economic capital, current expected credit loss, CVA, CCP, CDO tranches.

**JEL Classification.** G13, G17, G21.

**Mathematics Subject Classification 2020.** 62P05 (primary), 52A21, 91G15, 91G45, 91G80 (secondary).

## 1 Introduction

Elliptical distributions are largely used in finance modelling, be it for credit latent variables or portfolio positions modelling (McNeil et al., 2015). They allow to generate, by means of Monte-Carlo routines, a span of possible losses. Risk indicators can then be formed so as to inform top management or supervisors about the monitoring of possible future losses borne by a financial institution. In particular, regulatory bodies instruct to rely on value-at-risk and expected shortfall measures, where the last type, which falls under the remit of coherent risk measure<sup>1</sup>, is often preferred to quantiles usage.

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<sup>1</sup>see e.g. Acerbi and Tasche (2002).

One example can be found in Bastide, Crépey, Drapeau, and Tadese (2023), where the economic capital of a clearing member bank of a central counterparty (CCP) is based on an expected shortfall risk measure of the bank loss over one year. Such measure of loss depicts numerically a nondecreasing property w.r.t. credit-credit and credit-market dependence parameters, capturing an increase in loss given default amounts as more defaults materialize. Cousin and Laurent (2008) outlined the possible application of supermodular order for comparing CDO tranche premiums w.r.t. a credit correlation parameter of their default latent variables modelled as Brownian motions. Meyer and Strulovici (2012) generalize the use of such notion, citing application to credit losses with bounded support. The supermodular order property finds its root in Marshall and Olkin (1979, Definition C.2, page 146) under the name of lattice-superadditive property. It has attracted subsequent attention with the works of Müller (1997) and Bäuerle and Müller (1998), applied to stop-loss ordering of aggregated losses. Bäuerle and Müller (2006) emphasize the role of several stochastic orders in relation with convex risk measures. In particular, multidimensional elliptically distributed random variables have the supermodular order property w.r.t. their covariance matrix coefficients (Block and Sampson (1988, Corollary 2.3) recalled in Section B). This result will play a key role in this paper.

In Bastide, Crépey, Drapeau, and Tadese (2023), the loss takes a more complex form than what is usually found in the credit risk literature such as Cousin and Laurent (2008). This is due to a loss allocation coefficient attributed by the CCP to the surviving members. In this paper, we prove the nondecreasing property of convex risk measures w.r.t. covariance coefficients of portfolio credit losses. Our main motivation is to provide evidence of the soundness of the related approaches for computing credit provisions such as current expected credit loss (CECL, akin to the CVA in the central clearing one-period XVA setup of Bastide, Crépey, Drapeau, and Tadese (2023)), and economic capital (EC). This is important in justifying model assumptions and design, part of the model development cycle advocated by regulators and supervisors (European Central Bank, 2019). Table 1 details the two main targeted metrics in this work.

Name	Expression	Reference
current expected credit loss (CECL)	$\mathbb{E}^0 \left( \sum_{i=1}^n f_i(X_1, \dots, X_n) g_i(Y_i) \right)$	Definition 4.4
economic capital (EC)	$\mathbb{E}\mathbb{S}_\alpha^0 \left( \sum_{i=1}^n f_i(X_1, \dots, X_n) g_i(Y_i) \right)$	Definitions 4.6 and C.2

**Table 1:** Metrics of interest ( $\cdot^0$  relates to the survival probability measure of the reference bank).

The paper is organized as follows. Section 2 introduces the strategy of proof by supermodularity arguments that will be used to establish our main result, Theorem 4.4 (leading to the CECL and EC monotonicity results of Propositions 4.5 and 4.6), in an appropriate elliptical framework with wrong-way risk. Section 3 completes the results from Cousin and Laurent (2008) regarding equity and senior CDO tranches. The properties of Section 2 are used in Section 4 for implying the monotonicity of credit

provisions and economic capital metrics w.r.t. covariance coefficients in elliptical models. Section 5 concludes. Supermodular functions, elliptical distributions and risk measures are reviewed in Sections A, B and C.

Hereafter we consider a non-atomic probability space  $(\Omega, \mathcal{A}, \mathbb{Q})$ , with corresponding expectation, variance and covariance operators denoted by  $\mathbb{E}$ ,  $\text{Var}$  and  $\text{Cov}$ . All the considered random variables are real-valued and taken in a closed linear subspace and sublattice  $\mathfrak{X}$  of  $L^1(\mathbb{Q})$  that includes the constants.

## 2 Supermodular Ordering Properties

### 2.1 Strategy of Proof

The following classical supermodularity results will be instrumental in establishing our main result Theorem 4.4.

*Müller (1997, Definition 2.6)* A random vector  $\mathbf{X} = (\mathcal{X}_1, \dots, \mathcal{X}_m)$  is said to be smaller than the random vector  $\mathbf{Y} = (\mathcal{Y}_1, \dots, \mathcal{Y}_m)$  in the supermodular ordering, written  $\mathbf{X} \leq_{sm} \mathbf{Y}$ , if  $\mathbb{E}(f(\mathbf{X})) \leq \mathbb{E}(f(\mathbf{Y}))$  holds for all the supermodular functions<sup>2</sup>  $f : \mathbb{R}^m \rightarrow \mathbb{R}$  such that the expectations exist.  $\square$

*Müller and Stoyan (2002, Definition 3.9.4, page 113)* A random vector  $\mathbf{X} = (\mathcal{X}_1, \dots, \mathcal{X}_m)$  is said to be smaller than the random vector  $\mathbf{Y} = (\mathcal{Y}_1, \dots, \mathcal{Y}_m)$  in the increasing supermodular ordering, written  $\mathbf{X} \leq_{ism} \mathbf{Y}$ , if  $\mathbb{E}(f(\mathbf{X})) \leq \mathbb{E}(f(\mathbf{Y}))$  holds for all the nondecreasing supermodular functions  $f : \mathbb{R}^m \rightarrow \mathbb{R}$  such that the expectations exist.  $\square$

An equivalent characterization of supermodular vectors is given by

*Müller and Stoyan (2002, Theorems 3.9.11 (i) and (ii), page 118)* The following statements are equivalent:

- (i)  $\mathbf{X} \leq_{sm} \mathbf{Y}$ ,
- (ii)  $\mathbf{X}$  and  $\mathbf{Y}$  have the same marginals and  $\mathbf{X} \leq_{ism} \mathbf{Y}$ .  $\square$

*Müller (1997, Theorems 3.2 (c))* If  $\mathbf{X}$ ,  $\mathbf{Y}$ ,  $\mathbf{Z}$  are random vectors such that any random vectors distributed as  $\mathbf{X}$  and  $\mathbf{Y}$  conditionally on  $\mathbf{Z} = \mathbf{z}$ , denoted by  $[\mathbf{X}|\mathbf{Z} = \mathbf{z}]$  and  $[\mathbf{Y}|\mathbf{Z} = \mathbf{z}]$ , verify  $[\mathbf{X}|\mathbf{Z} = \mathbf{z}] \leq_{sm} [\mathbf{Y}|\mathbf{Z} = \mathbf{z}]$  for all possible values of  $\mathbf{z}$ , then  $\mathbf{X} \leq_{sm} \mathbf{Y}$ .  $\square$

*Müller (1997, Definition 2.1)* For  $\mathcal{X}$  and  $\mathcal{Y}$  in  $\mathfrak{X}$ ,  $\mathcal{X}$  precedes  $\mathcal{Y}$  in stop-loss order, written  $\mathcal{X} \leq_{sl} \mathcal{Y}$ , if  $\mathbb{E}(\mathcal{X} - A)^+ \leq \mathbb{E}(\mathcal{Y} - A)^+$  holds for all real constants  $A \geq 0$ .  $\square$

Equivalently, for the same random variables  $\mathcal{X}$  and  $\mathcal{Y}$  as in Müller (1997, Definition 2.1):

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<sup>2</sup>see Appendix A.

Bäuerle and Müller (1998, Theorem 2.2 b))  $\mathcal{X} \leq_{sl} \mathcal{Y}$  if and only if  $\mathbb{E}(f(\mathcal{X})) \leq \mathbb{E}(f(\mathcal{Y}))$  holds for all the nondecreasing convex functions  $f : \mathbb{R} \rightarrow \mathbb{R}$  such that the expectations exist (e.g.  $f = id$ ).  $\square$

Müller (1997, Theorem 3.1) Let  $\mathbf{X} = (\mathcal{X}_1, \dots, \mathcal{X}_m)$  and  $\mathbf{Y} = (\mathcal{Y}_1, \dots, \mathcal{Y}_m)$  be random vectors with  $\mathbf{X} \leq_{sm} \mathbf{Y}$  and let  $S = \sum_{i=1}^m \mathcal{X}_i$ ,  $S' = \sum_{i=1}^m \mathcal{Y}_i$ . Then  $S \leq_{sl} S'$ .  $\square$

**Lemma 2.1** *If  $\mathbf{X} \leq_{sm} \mathbf{Y}$ , then, for any nondecreasing supermodular function  $f : \mathbb{R}^m \rightarrow \mathbb{R}$ ,  $f(\mathbf{X}) \leq_{sl} f(\mathbf{Y})$ .*

**Proof.** For any  $A \in \mathbb{R}$ , the function  $\varphi : \mathbb{R} \rightarrow \mathbb{R}_+$ ,  $x \mapsto (x - A)^+$  is nondecreasing and convex. By Müller and Stoyan (2002, Theorem 3.9.3 f), page 113),  $\varphi \circ f$  is nondecreasing supermodular. Hence  $\mathbb{E}((f(\mathbf{X}) - A)^+) \leq \mathbb{E}((f(\mathbf{Y}) - A)^+)$ , which yields the result by Müller (1997, Definition 2.1).  $\square$

Bäuerle and Müller (2006, Theorem 4.4) Assuming  $\rho$  a risk measure as per Definition C.1, then  $\mathcal{X} \leq_{sl} \mathcal{Y}$  implies  $\rho(\mathcal{X}) \leq \rho(\mathcal{Y})$ .  $\square$

A succinct proof of this result can be found in Bäuerle and Müller (2006) prior to its statement. We give a more detailed proof for completeness at the end of Section C. Note that the original statement postulates, instead of the law-invariance property, that the risk measure  $\rho$  has the Fatou property, that is, if  $\mathcal{X}, \mathcal{X}_1, \mathcal{X}_2, \dots$  are integrable random variables with  $\mathcal{X}_k \xrightarrow{L^1} \mathcal{X}$ , then  $\rho(\mathcal{X}) \leq \liminf_{k \rightarrow \infty} \rho(\mathcal{X}_k)$ . We recall from Kallenberg (2021, page 832) that a closed linear subspace of a Banach space is a Banach space and from Meyer-Nieberg (1991, Definition 1.2.1 i), page 12) that a sublattice of a lattice with the same meet and join operations  $\wedge$  and  $\vee$  is again a lattice. Therefore, if  $\mathfrak{X}$  is a Banach lattice (i.e. an order lattice that is a complete normed vector space, e.g.  $L^p$  space with  $p \geq 1$ ) and  $\rho : \mathfrak{X} \rightarrow \mathbb{R}$  is proper, monotonous and convex, then  $\rho$  is continuous on the interior of its domain (Ruszczynski and Shapiro, 2006, Proposition 1), thus has the Fatou property on the interior of its domain. Hence the Fatou property requirement is automatically satisfied by  $\rho$  as long as it is defined on a Banach lattice. This is the case for both expectation and expected shortfall defined on any sublattice and linear subspace of  $L^1$ .

In Section 4, we will consider financial credit losses over a period of time  $[0, T]$  of the form  $\sum_{i=1}^n f_i(\mathcal{X}_1, \dots, \mathcal{X}_n) g_i(\mathcal{Y}_i)$  for some given number  $n$  of credit obligors. The default times  $\tau_i$  of the credit obligors indexed by  $i \in 1 \dots n$  will be modelled by the latent variable  $\mathcal{X}_i$  breaching a certain threshold.  $g_i(\mathcal{Y}_i)$  represents nonnegative losses given default of obligor  $i$ . The functions  $f_i$  are nonnegative nondecreasing w.r.t. each of their arguments and have the supermodular property (when  $n > 1$ ) and the functions  $g_i$  are nonnegative and nondecreasing. We will establish in Section 4 that  $(x_1, \dots, x_n, y_1, \dots, y_n) \mapsto \sum_{i=1}^n f_i(x_1, \dots, x_n) g_i(y_i)$  is a nondecreasing supermodular function on  $\mathbb{R}^{2n}$ . Hence, by Lemma 2.1, if  $(\mathcal{X}_1, \dots, \mathcal{X}_n, \mathcal{Y}_1, \dots, \mathcal{Y}_n) \leq_{sm} (\mathcal{X}'_1, \dots, \mathcal{X}'_n, \mathcal{Y}'_1, \dots, \mathcal{Y}'_n)$ , then

$$\sum_{i=1}^n f_i(\mathcal{X}_1, \dots, \mathcal{X}_n) g_i(\mathcal{Y}_i) \leq_{sl} \sum_{i=1}^n f_i(\mathcal{X}'_1, \dots, \mathcal{X}'_n) g_i(\mathcal{Y}'_i). \quad (1)$$

Having assumed the probability space non-atomic<sup>3</sup>, Bäuerle and Müller (2006, Theorem 4.4) allows concluding that  $\rho(\sum_{i=1}^n f_i(\mathcal{X}_1, \dots, \mathcal{X}_n)g_i(\mathcal{Y}_i)) \leq \rho(\sum_{i=1}^n f_i(\mathcal{X}'_1, \dots, \mathcal{X}'_n)g_i(\mathcal{Y}'_i))$ . This is the plan for proving our main result Theorem 4.4.

The following two Lemmas 2.2 and 2.3 will be the building blocks for establishing the above-needed supermodularity property.

**Lemma 2.2** *The supermodularity property is satisfied by any function  $\mathbb{R}^n \ni (x_1, \dots, x_n) \mapsto h(x_1, \dots, x_n) \in \mathbb{R}$  that can be written as  $h(x_1, \dots, x_n) = \sum_{i=1}^n h_i(x_i)$  for some functions  $h_1, \dots, h_n$  of single arguments. In particular, for any constants  $B_1, \dots, B_n$ , the function  $(x_1, \dots, x_n) \mapsto -\sum_{i=1}^n \mathbb{1}_{\{x_i \leq B_i\}}$  is supermodular. This function is also nondecreasing w.r.t. each of its arguments  $x_i$ .*

**Proof.** By Yildiz (2010, Corollary 1), recalled in Section A, we can focus on increasing differences. Let  $h(x_i, x_j | \mathbf{x}_{-i,j})$  denote the function  $h$  applied to  $x_i$  and  $x_j$  but keeping all other arguments  $\mathbf{x}_{-i,j} := (x_k)_{k \neq i,j}$  fixed. Fixing  $\delta, \varepsilon > 0$ , the difference  $h(x_i + \delta, x_j + \varepsilon | \mathbf{x}_{-i,j}) - h(x_i, x_j + \varepsilon | \mathbf{x}_{-i,j}) - h(x_i + \delta, x_j | \mathbf{x}_{-i,j}) + h(x_i, x_j | \mathbf{x}_{-i,j}) = h_i(x_i + \delta) + h_j(x_j + \varepsilon) - h_i(x_i) - h_j(x_j + \varepsilon) - h_i(x_i + \delta) - h_j(x_j) + h_i(x_i) + h_j(y_j)$  simplifies to 0, showing the supermodularity of  $h$ .  $\square$

**Lemma 2.3** *If  $g : \mathbb{R} \rightarrow \mathbb{R}$  and  $h : \mathbb{R} \rightarrow \mathbb{R}$  are both nondecreasing, then  $(x, y) \mapsto g(x)h(y)$  is supermodular.*

**Proof.** If  $x' \geq x$  and  $y' \geq y$ , then

$$\begin{aligned} & g(x')h(y') - g(x')h(y) - g(x)h(y') + g(x)h(y) \\ &= g(x')(h(y') - h(y)) - g(x)(h(y') - h(y)) \\ &\geq g(x)(h(y') - h(y)) - g(x)(h(y') - h(y)) = 0. \end{aligned} \tag{2}$$

Hence  $(x, y) \mapsto g(x)h(y)$  has increasing differences w.r.t. any pair  $(x, y) \in \mathbb{R}^2$ , i.e. is supermodular, by Yildiz (2010, Corollary 1).  $\square$

## 2.2 Elliptical Setup

We consider a one-period financial market model on  $(\Omega, \mathcal{A})$ , assumed arbitrage-free, with related risk-neutral probability measure  $\mathbb{Q}^*$ <sup>4</sup>. We index all the formerly introduced notation by “.” whenever applied in reference to  $\mathbb{Q} = \mathbb{Q}^*$ . We consider random vectors  $\mathbf{X} = (\mathcal{X}_0, \mathcal{X}_1, \dots, \mathcal{X}_m)$  and  $\mathbf{X}' = (\mathcal{X}_0, \mathcal{X}'_1, \dots, \mathcal{X}'_m)$  following elliptical distributions (see Section B) under  $\mathbb{Q}^*$  as per

$$\mathbf{X} = \boldsymbol{\mu} + \mathbf{A}\mathbf{Z} \text{ and } \mathbf{X}' = \boldsymbol{\mu} + \mathbf{A}'\mathbf{Z}, \tag{3}$$

<sup>3</sup>see Bäuerle and Müller (2006, Example 3.1) for a counter-example to the monotonicity property for probability space with atom(s).

<sup>4</sup>see for instance (Föllmer and Schied, 2016, Part I).

for constant matrices  $\mathbf{A}$ ,  $\mathbf{A}'$  of full rank.  $\mathbf{Z} = (Z_0, Z_1, \dots, Z_m)$  follows, under  $\mathbb{Q}^*$ , a spherical distribution  $S_n(\psi)$  (see Section B), with characteristic generator  $\psi$ . As the components 0 for both vectors  $\mathbf{X}$  and  $\mathbf{X}'$  must be the same, we also require the first row of  $\mathbf{A}$  and  $\mathbf{A}'$  to be the same. The rationale for keeping the same reference latent variable  $\mathcal{X}_0$  in  $\mathbf{X}$  and  $\mathbf{X}'$  in (3) is that the point of view will be the one of a reference bank indexed by 0 (cf. Lemma 2.4 and Proposition 2.5).

We denote by  $\mathbf{\Gamma}_{\mathbf{X}} = \mathbf{A}\mathbf{A}^\top$  and  $\mathbf{\Gamma}_{\mathbf{X}'} = \mathbf{A}'(\mathbf{A}')^\top$  the  $\mathbb{Q}^*$  covariance matrices of  $\mathbf{X}$  and  $\mathbf{X}'$ , assumed to be positive semi-definite. We write  $\mathbf{\Gamma}_{\mathbf{X}}^{ij} = \text{Cov}^*(\mathcal{X}_i, \mathcal{X}_j)$  for all  $i, j \in 0..m$ , and likewise for  $\mathbf{\Gamma}_{\mathbf{X}'}$ .

**Assumption 2.1**  $\mathbf{\Gamma}_{\mathbf{X}}^{jj} = \mathbf{\Gamma}_{\mathbf{X}'}^{jj}$  and  $\mathbf{\Gamma}_{\mathbf{X}}^{0j} = \mathbf{\Gamma}_{\mathbf{X}'}^{0j}$ ,  $j \in 1..m$ , and  $\mathbf{\Gamma}_{\mathbf{X}}^{ij} \leq \mathbf{\Gamma}_{\mathbf{X}'}^{ij}$ ,  $i \neq j \in 1..m$ .  $\square$

In particular,  $\mathbf{X}_{1:m} \leq_{sm^*} \mathbf{X}'_{1:m}$ , by Block and Sampson (1988, Corollary 2.3) recalled in Section B.

Let  $\boldsymbol{\mu}_{\mathbf{X}_{1:m}} = (\mu_{\mathcal{X}_1}, \dots, \mu_{\mathcal{X}_m})$  and  $\mathbf{\Gamma}_{\mathbf{X}_{1:m}} = \left( \mathbf{\Gamma}_{\mathbf{X}_{1:m}}^{ij} \right)_{1 \leq i, j \leq m}$  be the mean vector and the covariance matrix of  $(\mathcal{X}_1, \dots, \mathcal{X}_m)$  under  $\mathbb{Q}^*$ . We use similar notations regarding  $(\mathcal{X}'_1, \dots, \mathcal{X}'_m)$ . Our next result, Lemma 2.4, makes use of the two following results. Let  $U_{\mathbb{S}^{n-1}}$  denote the uniform distribution on the unit sphere  $\mathbb{S}^{n-1} := \{\mathbf{s} \in \mathbb{R}^n : \mathbf{s}^\top \mathbf{s} = 1\}$  in  $\mathbb{R}^n$ . A radial r.v. is a one-dimensional r.v. with values in  $[0, +\infty)$ .

*McNeil et al. (2015, Theorem 6.21, page 197)*  $\mathbf{Z}$  has a spherical distribution in  $\mathbb{R}^n$  if and only if it has a stochastic representation

$$\mathbf{Z} \stackrel{d}{=} \mathcal{R}\mathbf{S}, \quad (4)$$

where  $\mathbf{S} \sim U_{\mathbb{S}^{n-1}}$  and  $\mathcal{R} \geq 0$  is a radial r.v. independent of  $\mathbf{S}$ .  $\square$

*Fang et al. (1990, Theorem 2.18, page 45)* Let  $\mathbf{Y} \stackrel{d}{=} \boldsymbol{\mu} + \mathcal{R}\mathbf{A}\mathbf{S} \sim E_n(\boldsymbol{\mu}, \Sigma, \psi)$  (see Section B) with  $\Sigma = \mathbf{A}\mathbf{A}^\top$  positive definite. Let

$$\mathbf{Y} = \begin{pmatrix} \mathbf{Y}^{(1)} \\ \mathbf{Y}^{(2)} \end{pmatrix}, \quad \boldsymbol{\mu} = \begin{pmatrix} \boldsymbol{\mu}^{(1)} \\ \boldsymbol{\mu}^{(2)} \end{pmatrix}, \quad \Sigma = \begin{pmatrix} \Sigma^{(1,1)} & \Sigma^{(1,2)} \\ \Sigma^{(2,1)} & \Sigma^{(2,2)} \end{pmatrix}, \quad (5)$$

where  $\mathbf{Y}^{(1)}$  and  $\boldsymbol{\mu}^{(1)}$  are  $m \times 1$  vectors and  $\Sigma^{(1,1)}$  is an  $m \times m$  matrix, for some  $0 < m < n$ . Then

$$\begin{aligned} \left( \mathbf{Y}^{(1)} \middle| \mathbf{Y}^{(2)} = \mathbf{y}^{(2)} \right) &\stackrel{d}{=} \boldsymbol{\mu}_{|\mathbf{Y}^{(2)}=\mathbf{y}^{(2)}}^{(1)} + \mathcal{R}_{|\mathbf{Y}^{(2)}=\mathbf{y}^{(2)}} \mathbf{A}_{|\mathbf{Y}^{(2)}=\mathbf{y}^{(2)}}^{(1,1)} \mathbf{S}^{(m)} \\ &\sim E_m \left( \boldsymbol{\mu}_{|\mathbf{Y}^{(2)}=\mathbf{y}^{(2)}}^{(1)}, \Sigma_{|\mathbf{Y}^{(2)}=\mathbf{y}^{(2)}}^{(1,1)}, \psi_{|\mathbf{Y}^{(2)}=\mathbf{y}^{(2)}} \right), \end{aligned} \quad (6)$$

where

$$\left\{ \begin{array}{l}
\boldsymbol{\mu}_{|\mathbf{Y}^{(2)}=\mathbf{y}^{(2)}}^{(1)} = \boldsymbol{\mu}^{(1)} + \Sigma^{(1,2)}(\Sigma^{(2,2)})^{-1}(\mathbf{y}^{(2)} - \boldsymbol{\mu}^{(2)}), \\
\Sigma_{|\mathbf{Y}^{(2)}=\mathbf{x}^{(2)}}^{(1,1)} = \Sigma^{(1,1)} - \Sigma^{(1,2)}(\Sigma^{(2,2)})^{-1}\Sigma^{(2,1)} = \mathbf{A}_{|\mathbf{Y}^{(2)}=\mathbf{y}^{(2)}}^{(1,1)} \left( \mathbf{A}_{|\mathbf{Y}^{(2)}=\mathbf{y}^{(2)}}^{(1,1)} \right)^\top, \\
\mathbf{S}^{(m)} \sim U_{\mathbb{S}^{m-1}}, \\
\mathcal{R}_{|\mathbf{Y}^{(2)}=\mathbf{y}^{(2)}} \stackrel{d}{=} \left( (\mathcal{R}^2 - q(\mathbf{y}^{(2)}))^{1/2} \middle| \mathbf{y}^{(2)} = \mathbf{y}^{(2)} \right) \text{ and } \mathcal{R}_{|\mathbf{Y}^{(2)}=\mathbf{y}^{(2)}} \text{ is independent} \\
\text{of } \mathbf{S}^{(m)}, \\
q(\mathbf{y}^{(2)}) = (\mathbf{y}^{(2)} - \boldsymbol{\mu}^{(2)})^\top (\Sigma^{(2,2)})^{-1} (\mathbf{y}^{(2)} - \boldsymbol{\mu}^{(2)}), \\
\psi_{|\mathbf{Y}^{(2)}=\mathbf{y}^{(2)}} \text{ is of the form (53) for } n = m, F \text{ given as the c.d.f. of } \mathcal{R}_{|\mathbf{Y}^{(2)}=\mathbf{y}^{(2)}} \text{ and} \\
\mathbf{S} \text{ given as } \mathbf{S}^{(m)}. \quad \square
\end{array} \right. \quad (7)$$

**Lemma 2.4** *Under the elliptical form (3) satisfying Assumption 2.1 for  $\mathbf{X}$  and  $\mathbf{X}'$ , we have, under  $\mathbb{Q}^*$ ,*

$$\begin{aligned}
[\mathcal{X}_1, \dots, \mathcal{X}_m | \mathcal{X}_0] &\sim E_m(\boldsymbol{\mu}_{\mathbf{X}_{1:m} | \mathcal{X}_0}, \boldsymbol{\Gamma}_{\mathbf{X}_{1:m} | \mathcal{X}_0}, \psi | \mathcal{X}_0), \\
[\mathcal{X}'_1, \dots, \mathcal{X}'_m | \mathcal{X}_0] &\sim E_m(\boldsymbol{\mu}_{\mathbf{X}'_{1:m} | \mathcal{X}_0}, \boldsymbol{\Gamma}_{\mathbf{X}'_{1:m} | \mathcal{X}_0}, \psi' | \mathcal{X}_0),
\end{aligned} \quad (8)$$

with

$$\psi | \mathcal{X}_0 = \psi' | \mathcal{X}_0, \quad \boldsymbol{\mu}_{\mathbf{X}_{1:m} | \mathcal{X}_0} = \boldsymbol{\mu}'_{\mathbf{X}'_{1:m} | \mathcal{X}_0} \quad (9)$$

and, for any  $i, j \in 1..m$ ,

$$\boldsymbol{\Gamma}_{\mathbf{X}_{1:m} | \mathcal{X}_0}^{ij} \leq \boldsymbol{\Gamma}_{\mathbf{X}'_{1:m} | \mathcal{X}_0}^{ij}. \quad (10)$$

*Proof.* Applying (7) to  $\mathbf{Y}^{(1)} = \mathbf{X}_{1:m}$  and  $\mathbf{Y}^{(2)} = \mathcal{X}_0$  yields (8) with

$$\begin{aligned}
\boldsymbol{\mu}_{\mathbf{X}_{1:m} | \mathcal{X}_0} &= \boldsymbol{\mu}_{\mathbf{X}_{1:m}} + \frac{1}{\boldsymbol{\Gamma}_{\mathbf{X}}^{00}} \left( \boldsymbol{\Gamma}_{\mathbf{X}}^{01} \dots, \boldsymbol{\Gamma}_{\mathbf{X}}^{0m} \right)^\top (\mathcal{X}_0 - \boldsymbol{\mu}_{\mathcal{X}_0}), \\
\boldsymbol{\Gamma}_{\mathbf{X}_{1:m} | \mathcal{X}_0} &= \boldsymbol{\Gamma}_{\mathbf{X}_{1:m}} - \frac{1}{\boldsymbol{\Gamma}_{\mathbf{X}}^{00}} \sum_{j=1}^m \left( \boldsymbol{\Gamma}_{\mathbf{X}}^{0j} \right)^2
\end{aligned} \quad (11)$$

and, using Assumption 2.1 and (11),

$$\begin{aligned}
\boldsymbol{\mu}_{\mathbf{X}'_{1:m} | \mathcal{X}_0} &= \boldsymbol{\mu}_{\mathbf{X}'_{1:m}} + \frac{1}{\boldsymbol{\Gamma}_{\mathbf{X}}^{00}} \left( \boldsymbol{\Gamma}_{\mathbf{X}}^{01} \dots, \boldsymbol{\Gamma}_{\mathbf{X}}^{0m} \right)^\top (\mathcal{X}_0 - \boldsymbol{\mu}_{\mathcal{X}_0}) = \boldsymbol{\mu}_{\mathbf{X}_{1:m} | \mathcal{X}_0} \\
\boldsymbol{\Gamma}_{\mathbf{X}'_{1:m} | \mathcal{X}_0} &= \boldsymbol{\Gamma}_{\mathbf{X}'_{1:m}} - \frac{1}{\boldsymbol{\Gamma}_{\mathbf{X}}^{00}} \sum_{j=1}^m \left( \boldsymbol{\Gamma}_{\mathbf{X}}^{0j} \right)^2 \geq \boldsymbol{\Gamma}_{\mathbf{X}_{1:m} | \mathcal{X}_0}.
\end{aligned} \quad (12)$$



where  $\geq$  is meant componentwise.

It remains to show that  $\psi_{|\mathcal{X}_0} = \psi'_{|\mathcal{X}_0}$ . For all  $x_0 \in \mathbb{R}$ , by Fang et al. (1990, Theorem 2.18, page 45), which includes (7), the radius  $\mathcal{R}_{|\mathcal{X}_0=x_0}$  of  $[\mathcal{X}_1, \dots, \mathcal{X}_m | \mathcal{X}_0 = x_0]$  is distributed like  $\left( \mathcal{R}^2 - \frac{1}{\Gamma_{\mathbf{X}}^{\mathbb{Q}^0}} (x_0 - \mu_{\mathcal{X}_0})^2 \right)^{1/2}$ , where  $\mathcal{R} := \|\mathbf{Z}\|$ , and so is the radius  $\mathcal{R}'_{|\mathcal{X}_0=x_0}$  of  $[\mathcal{X}'_1, \dots, \mathcal{X}'_m | \mathcal{X}_0 = x_0]$  (as both  $\mathbf{X}$  and  $\mathbf{X}'$  are defined based on the spherically distributed vector  $\mathbf{Z}$ ). Thus  $\mathcal{R}_{|\mathcal{X}_0} \stackrel{d}{=} \mathcal{R}'_{|\mathcal{X}_0}$ , with common c.d.f. denoted by  $F_{\mathcal{R}_{|\mathcal{X}_0}}$ . The corresponding conditional characteristic generator common to  $[\mathcal{X}_1, \dots, \mathcal{X}_m | \mathcal{X}_0]$  and  $[\mathcal{X}'_1, \dots, \mathcal{X}'_m | \mathcal{X}_0]$  is given by  $\psi_{|\mathcal{X}_0}(x) = \int_0^\infty \Omega_m(xr^2) F_{\mathcal{R}_{|\mathcal{X}_0}}(dr)$ , where  $\Omega_m(u^\top u)$  is the characteristic function of a r.v.  $\sim U_{\mathbb{S}^{m-1}}$ .  $\square$

**Proposition 2.5** *Under the assumptions of Lemma 2.4, we have:*

$$[\mathcal{X}_1, \dots, \mathcal{X}_m | \mathcal{X}_0] \leq_{sm^*} [\mathcal{X}'_1, \dots, \mathcal{X}'_m | \mathcal{X}_0]. \quad (13)$$

*Proof.* By Lemma 2.4, conditionally on  $\mathcal{X}_0$ ,  $\mathbf{X}_{1:m}$  and  $\mathbf{X}'_{1:m}$  have the same elliptical distribution under  $\mathbb{Q}^*$ , except for their covariance matrix coefficients that verify (10). Block and Sampson (1988, Corollary 2.3) recalled in Section B (here applied under  $\mathbb{Q}^*$ ) then yields the result.  $\square$

### 2.3 Preserving Supermodularity and Stop-Loss Order Properties Under a Measure Change

The metrics of interest in this work are considered from a reference market participant viewpoint, namely a bank indexed by 0. In this context, it is sometimes useful to introduce a measure  $\mathbb{Q}^0$  defined in terms of a measurable function  $h$  of a latent variable  $\mathcal{X}_0$  of the default of the bank, such that

$$h(\mathcal{X}_0) = d\mathbb{Q}^0/d\mathbb{Q}^* \geq 0 \quad \text{and} \quad \mathbb{E}^*[h(\mathcal{X}_0)] = 1. \quad (14)$$

We index all the formerly introduced notation by “.<sup>0</sup>” whenever applied in reference to  $\mathbb{Q} = \mathbb{Q}^0$ .

**Example 2.2** In the setup of Bastide, Crépey, Drapeau, and Tadese (2023), financial risk factors are specified under  $\mathbb{Q}^*$ , but explicit XVA formulas arise in terms of the related bank survival probability measure  $\mathbb{Q}^0$ .

**Assumption 2.3**  $\mathbf{X} = (\mathcal{X}_0, \mathcal{X}_1, \dots, \mathcal{X}_m)$  and  $\mathbf{X}' = (\mathcal{X}_0, \mathcal{X}'_1, \dots, \mathcal{X}'_m)$ , with components in  $\mathfrak{X}^* = L^1(\mathbb{Q}^*)$  (with the same  $\mathcal{X}_0$  in  $\mathbf{X}$  and  $\mathbf{X}'$ ), satisfy

$$[\mathcal{X}_1, \dots, \mathcal{X}_m | \mathcal{X}_0] \leq_{sm^*} [\mathcal{X}'_1, \dots, \mathcal{X}'_m | \mathcal{X}_0], \quad (15)$$

i.e.

$$\mathbb{E}^*[f(\mathcal{X}_1, \dots, \mathcal{X}_m) | \mathcal{X}_0] \leq \mathbb{E}^*[f(\mathcal{X}'_1, \dots, \mathcal{X}'_m) | \mathcal{X}_0] \quad (16)$$

holds for any supermodular function  $f : \mathbb{R}^m \rightarrow \mathbb{R}$  such that the conditional expectations exist.

**Lemma 2.6** *If  $\mathbf{X}$  and  $\mathbf{X}'$  satisfy Assumption 2.3, then  $(\mathcal{X}_1, \dots, \mathcal{X}_m) \leq_{sm^0} (\mathcal{X}'_1, \dots, \mathcal{X}'_m)$ .*

**Proof.** As  $h(\mathcal{X}_0) \geq 0$ , for any supermodular function  $f$  on  $\mathbb{R}^m$  such that both  $\mathbb{E}^0[f(\mathcal{X}_1, \dots, \mathcal{X}_m)]$  and  $\mathbb{E}^0[f(\mathcal{X}'_1, \dots, \mathcal{X}'_m)]$  exist, (14) and (16) yield

$$\begin{aligned} \mathbb{E}^0[f(\mathcal{X}_1, \dots, \mathcal{X}_m)] &= \mathbb{E}^*[h(\mathcal{X}_0)f(\mathcal{X}_1, \dots, \mathcal{X}_m)] = \mathbb{E}^*[h(\mathcal{X}_0)\mathbb{E}^*[f(\mathcal{X}_1, \dots, \mathcal{X}_m)|\mathcal{X}_0]] \\ &\leq \mathbb{E}^*[h(\mathcal{X}_0)\mathbb{E}^*[f(\mathcal{X}'_1, \dots, \mathcal{X}'_m)|\mathcal{X}_0]] = \mathbb{E}^0[f(\mathcal{X}'_1, \dots, \mathcal{X}'_m)]. \end{aligned} \quad (17)$$

Hence  $(\mathcal{X}_1, \dots, \mathcal{X}_m) \leq_{sm^0} (\mathcal{X}'_1, \dots, \mathcal{X}'_m)$ .  $\square$

In terms of applications, we start by precisizing the monotonicity result of Cousin and Laurent (2008) for equity and senior CDO tranches default leg w.r.t. the credit correlation. We then explore a more complex counterparty credit risk example pertaining to the risk management of clearing activities.

### 3 Teaser: Credit Derivatives

We analyse the monotonicity of default leg and coupon leg of synthetic equity and senior CDO tranches prices w.r.t. credit correlation. Such prices are obtained by taking the expected value under the pricing measure  $\mathbb{Q}^*$  of the loss function underlying the CDO tranche contract. The characteristics of the payoff are as follows. There are  $n$  obligors, indexed by  $i$ . All underlying CDS are assumed to mature at some common time  $T$ . For any obligor  $i$ , the default time is denoted by  $\tau_i$ , the recovery rate is  $R_i \in [0, 1]$ , the underlying notional is  $N_i \geq 0$  and the loss given default is  $L_i = (1 - R_i)N_i$ . The maximum loss is  $L_{\max} = \sum_{i=1}^n L_i$ . To simplify calculations, we assume that the payments due to the obligors defaults are only made at maturity  $T$  and the discounting rates are set to zero (nonzero discounting rates can be included as long as they are independent from the credit risk factors).

**Definition 3.1** The cumulative credit loss at time  $t \leq T$  is

$$L(t) = \sum_{i=1}^n L_i \mathbb{1}_{\{\tau_i \leq t\}}. \quad (18)$$

The default leg of an equity tranche with maturity  $T$  and detachment point  $B \in (0, L_{\max}]$  is

$$D_{\text{eq}}(T) = L(T) - (L(T) - B)^+ = \min(L(T), B). \quad (19)$$

The default leg of a senior tranche with maturity  $T$  and attachment point  $A \in [0, L_{\max})$  is

$$D_{\text{sen}}(T) = (L(T) - A)^+. \quad \square \quad (20)$$

**Definition 3.2** The payment leg consists in payments, at  $K$  regular times  $t_k$  (with  $t_K = T$ ), of a fixed spread  $s$  applied to the remaining tranche amount at risk. In the case of the equity tranche, the payment leg writes

$$P_{\text{eq}}(T) = s \frac{T}{K} \sum_{k=1}^K (B - L(t_k))^+. \quad (21)$$

In the case of the senior tranche, it writes

$$\begin{aligned} P_{\text{sen}}(T) &= s \frac{T}{K} \sum_{k=1}^K \left( L_{\max} - A - (L(t_k) - A)^+ \right) \\ &= sT(L_{\max} - A) - s \frac{T}{K} \sum_{k=1}^K (L(t_k) - A)^+. \quad \square \end{aligned} \quad (22)$$

Putting default and payment payoffs together, we obtain, for the equity tranche,

$$D_{\text{eq}}(T) - P_{\text{eq}}(T) = L(T) - (L(T) - B)^+ - s \frac{T}{K} \sum_{k=1}^K (B - L(t_k))^+, \quad (23)$$

and, for the senior tranche,

$$\begin{aligned} D_{\text{sen}}(T) - P_{\text{sen}}(T) &= \\ &= (L(T) - A)^+ - sT(L_{\max} + A) + s \frac{T}{K} \sum_{k=1}^K (L(t_k) - A)^+. \end{aligned} \quad (24)$$

Specifying  $X_i = F_i^{-1}(1 - \gamma_i(\tau_i))$ , where  $F_i$  is the c.d.f. of  $X_i$ ,  $\gamma_i$  the  $\mathbb{Q}^*$  c.d.f. of  $\tau_i$  and letting  $B_i(t) := F_i^{-1}(1 - \gamma_i(t))$ , we have  $\{\tau_i \leq t\} = \{X_i \geq B_i(t)\}$  and

$$L(t) = \sum_{i=1}^n L_i \mathbb{1}_{\{X_i \geq B_i(t)\}}. \quad (25)$$

The following result precises the outlined application for comparing CDO tranche premiums in Cousin and Laurent (2008) with heterogeneous obligors under our static setup.

**Proposition 3.1** *If  $(X_1, \dots, X_n) \sim E_n(\boldsymbol{\mu}, \boldsymbol{\Gamma}, \psi)$  and  $(X'_1, \dots, X'_n) \sim E_n(\boldsymbol{\mu}, \boldsymbol{\Gamma}', \psi)$ , with  $\boldsymbol{\Gamma} \leq \boldsymbol{\Gamma}'$  elementwise except for equal diagonal entries, then*

$$\mathbb{E}^* \left[ \left( \sum_{i=1}^n L_i \mathbb{1}_{\{X_i \geq B_i(t)\}} - A \right)^+ \right] \leq \mathbb{E}^* \left[ \left( \sum_{i=1}^n L_i \mathbb{1}_{\{X'_i \geq B_i(t)\}} - A \right)^+ \right], \quad A \in \mathbb{R}. \quad (26)$$

*That is, the price of the default leg of a senior CDO tranche is nondecreasing w.r.t.  $\rho^{cr}$ ; we also get that the price of the default leg of an equity CDO tranche is nonincreasing w.r.t.  $\rho^{cr}$ .*

**Proof.** For any  $t \in \mathbb{R}_+$ , the function

$$\begin{aligned} f_t : \quad \mathbb{R}^n &\longrightarrow \mathbb{R}_+ \\ (x_1, \dots, x_n) &\longmapsto \sum_{i=1}^n L_i \mathbb{1}_{\{x_i \geq B_i(t)\}} \end{aligned} \quad (27)$$

is nondecreasing w.r.t. each of its arguments and it is supermodular, by Lemma 2.2. Hence, due to the nondecreasing and convexity properties of  $x \mapsto (x - A)^+$ , Müller and Stoyan (2002, Theorem 3.9.3 f), page 113), recalled in Section A, implies that  $(x_1, \dots, x_n) \mapsto (f_t(x_1, \dots, x_n) - A)^+$  is also nondecreasing supermodular. Moreover, by Block and Sampson (1988, Corollary 2.3), recalled in Section B,  $(X_1, \dots, X_n) \leq_{sm^*} (X'_1, \dots, X'_n)$ . Applying Müller (1997, Definition 2.6) to  $(X_1, \dots, X_n)$ ,  $(X'_1, \dots, X'_n)$  and  $(x_1, \dots, x_n) \mapsto (f_t(x_1, \dots, x_n) - A)^+$  then yields the result for the senior tranche.

As  $\mathbb{E}^* \left[ \sum_{i=1}^n L_i \mathbb{1}_{\{X_i \geq B_i(t)\}} \right] - \mathbb{E}^* \left[ \left( \sum_{i=1}^n L_i \mathbb{1}_{\{X_i \geq B_i(t)\}} - B \right)^+ \right]$  is the price of an equity tranche default leg, where the left expectation term does not depend on  $\rho^{cr}$ , the result for the equity tranche follows.  $\square$

**Corollary 3.2** *Under the same assumptions as in Proposition 3.1, the price of the payment leg of the CDO equity (resp. senior) tranche is nondecreasing (resp. nonincreasing) w.r.t.  $\rho^{cr}$ .*

**Proof.** By call-put parity,

$$\begin{aligned} \mathbb{E}^* \left[ \left( B - \sum_{i=1}^n L_i \mathbb{1}_{\{X_i \geq B_i(t)\}} \right)^+ \right] &= \mathbb{E}^* \left[ \left( \sum_{i=1}^n L_i \mathbb{1}_{\{X'_i \geq B_i(t)\}} - B \right)^+ \right] \\ &\quad - B + \mathbb{E}^* \left[ \sum_{i=1}^n L_i \mathbb{1}_{\{X'_i \geq B_i(t)\}} \right] \end{aligned} \quad (28)$$

so that, in view of (21)-(22), as a consequence of Proposition 3.1,

$$\mathbb{E}^* \left[ \left( B - \sum_{i=1}^n L_i \mathbb{1}_{\{X_i \geq B_i(t)\}} \right)^+ \right] \leq \mathbb{E}^* \left[ \left( B - \sum_{i=1}^n L_i \mathbb{1}_{\{X'_i \geq B_i(t)\}} \right)^+ \right]. \quad \square \quad (29)$$

**Remark 3.3** For mezzanine tranches, such results do not hold. Indeed, the tranchéd loss default leg payoff function

$$D_{\text{Mezz}}(T) = (L(T) - A)^+ - (L(T) - B)^+, \quad (30)$$

where  $A, B \in (0, L_{\max})$ , is not a convex function of the cumulative loss nor is the payment leg

$$\begin{aligned} P_{\text{Mezz}}(T) &= s \frac{T}{K} \sum_{k=1}^K \left[ B - A - \left( (L(t_k) - A)^+ - (L(t_k) - B)^+ \right) \right] \\ &= sT(B - A) - s \frac{T}{K} \sum_{k=1}^K \left[ (L(t_k) - A)^+ - (L(t_k) - B)^+ \right]. \end{aligned} \quad (31)$$

Obligor id	1	2	3	4	5	6	7	8	9	10
Notional	100	105	110	115	120	100	105	110	115	120
RR (%)	30	31	32	33	34	35	36	37	38	39
$\lambda$ (%)	2	2.5	3	3.5	4	4.5	5	5.5	6	6.5

Obligor id	21	222	23	24	25	26	27	28	29	30
Notional	100	105	110	115	120	100	105	110	115	120
RR (%)	39	40	30	31	32	33	34	35	36	37
$\lambda$ (%)	12	12.5	13	13.5	14	14.5	15	15.5	16	16.5

Obligor id	11	12	13	14	15	16	17	18	19	20
Notional	100	105	110	115	120	100	105	110	115	120
RR (%)	40	30	31	32	33	34	35	36	37	38
$\lambda$ (%)	7	7.5	8	8.5	9	9.5	10	10.5	11	11.5

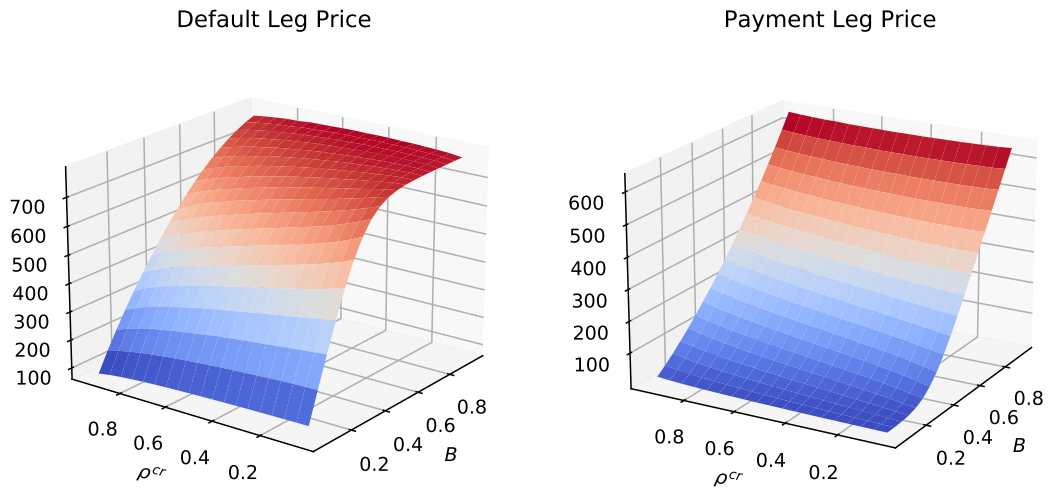
**Table 2:** CDO portfolios and obligors parameters.

### 3.1 Numerical Results

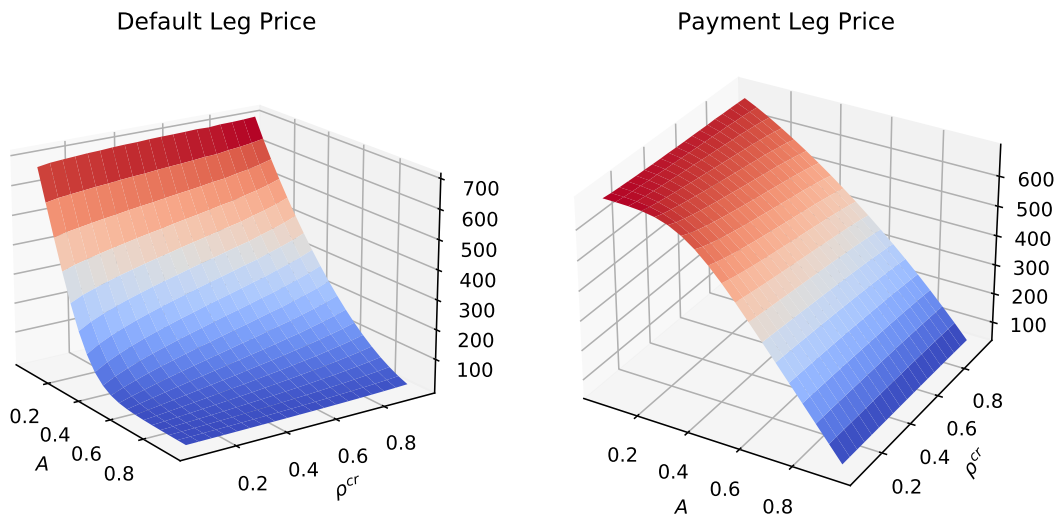
The results are illustrated in Figure 1 for the equity tranches, varying detachment point from 5% to 95% with 5% steps, i.e. considering the tranches from  $[0, 5\%]$  to  $[0, 95\%]$ . Figure 2 illustrates the results for the senior tranches, varying attachment point from 5% to 95% with 5% steps, i.e. considering the tranches from  $[5\%, 100\%]$  to  $[95\%, 100\%]$ . The correlation  $\rho^{cr}$  is varied from 5% to 95% with 5% step for both tranche types. The parameters of the underlying obligors and CDSs are detailed in Table 2, where values have been assigned arbitrarily to ensure heterogeneity of the various obligors. The CDO tranche spread has been set to 10% with a single coupon paid at a maturity of 5 years. The monotonicity patterns are observed for both tranches, with incremental prices between two consecutive credit correlation steps being nonpositive for the CDO equity tranche default leg prices and the CDO senior tranche payment leg prices. Incremental prices between two consecutive credit correlation steps are nonnegative for both the CDO equity tranche payment leg prices and the CDO senior tranche default leg prices. These results are in line with Proposition 3.1 and Corollary 3.2. Also, incremental prices between two attachment point steps are nonnegative for both the CDO equity tranche default leg and payment leg prices, as expected from (29). The incremental prices between two attachment point steps are nonpositive for the CDO senior tranche default leg and payment leg prices, as expected from (26).

## 4 Main Results: Counterparty Credit Risk

We assume the setup of Sections 2.2 and 2.3 regarding  $\mathbf{X} = (\mathcal{X}_0, \mathcal{X}_1, \dots, \mathcal{X}_m) = (X_0, X_1, \dots, X_n, Y_1, \dots, Y_n)$  and  $\mathbf{X}' = (\mathcal{X}'_0, \mathcal{X}'_1, \dots, \mathcal{X}'_m) = (X_0, X'_1, \dots, X'_n, Y'_1, \dots, Y'_n)$



**Figure 1:** Default leg and Payment leg prices of CDO equity tranches w.r.t. latent variable credit correlation  $\rho^{cr}$  and detachment point  $B$ .



**Figure 2:** Default leg and Payment leg prices of CDO senior tranches w.r.t. latent variable credit correlation  $\rho^{cr}$  and attachment point  $A$ . Note that the axes are different from Figure 1, for a better readability.

(for some  $n > 0$ ). Let

$$\mathcal{L} = \sum_{i=1}^n f_i(X_1, \dots, X_n) g_i(Y_i) \in \mathfrak{X}^0 = L^1(\mathbb{Q}^0), \quad (32)$$

where  $Y_i \in \mathfrak{X}^*$  drives a loss that obligor  $i \in 1..n$  generates if it defaults, with default of each credit name  $j \in 0..n$  (including the reference bank 0 as in Section 2.3) driven by a latent variable  $X_j \in \mathfrak{X}^*$ . The  $f_i : \mathbb{R}^n \rightarrow \mathbb{R}_+$  are measurable supermodular (when  $n > 1$ ) functions nondecreasing w.r.t. each of their arguments, and the  $g_i : \mathbb{R} \rightarrow \mathbb{R}_+$  are measurable nondecreasing functions. When  $f_i(X_1, \dots, X_n) = f_i(X_i)$  as in Cousin and Laurent (2008) or Meyer and Strulovici (2013),  $f_i(X_i)g_i(Y_i)$  can represent the loss related to a bilateral counterparty position, i.e. a portfolio position between the reference bank 0 and its client  $i$ . As detailed in Section 4.1, the more general case where  $f_i(X_1, \dots, X_n)$  depends on several  $X_j$  encompasses the financial losses generated by clearing exposures towards a central counterparty (CCP). This is due to the loss allocation coefficients attributed to each surviving member (Bastide, Crépey, Drapeau, and Tadese, 2023). It also covers the case of financial resolution funds (Single Resolution Board, 2021). As outlined in Section 2.1, if we can establish that  $\mathbb{R}^{2n} \ni (x_1, \dots, x_n, y_1, \dots, y_n) \mapsto \sum_{i=1}^n f_i(x_1, \dots, x_n) g_i(y_i) \in \mathbb{R}^+$  is supermodular, then we can conclude that  $\rho(\mathcal{L}) \leq \rho(\mathcal{L}')$  holds for any risk measure  $\rho$  on  $\mathfrak{X}^0$ , with  $\mathcal{L}' = \sum_{i=1}^n f_i(X'_1, \dots, X'_n) g_i(Y'_i) \in \mathfrak{X}^0$ .

**Remark 4.1** Wrong-way risk is the potential increase of the exposure a financial actor w.r.t. certain counterparties when their probability of default increase. A risk model should include a wrong-way risk feature in order to ensure conservative treatment. See Gregory (2015, Section 8.6.5) for more detailed explanations and Brigo, Morini, and Pallavicini (2013) for various examples of asset classes models incorporating the wrong-way risk feature. Under the elliptical model (3) and the credit loss form (32), wrong-way risk holds provided that an increase of the covariance between the default latent variable  $X_i$  and the potential loss driver  $Y_i$  leads to an increase of the loss amount  $g_i(Y_i)$ . This is the case when  $f$  and  $g$  are nondecreasing (as assumed) in each of their arguments.

**Lemma 4.1** *If  $f_i : \mathbb{R}^n \rightarrow \mathbb{R}$  is a supermodular function nondecreasing w.r.t. each of its arguments and  $g_i : \mathbb{R} \rightarrow \mathbb{R}$  is a nondecreasing function,  $i \in 1..n$ , then  $\mathbb{R}^{2n} \ni (x_1, \dots, x_n, y_1, \dots, y_n) \mapsto \sum_{i=1}^n f(x_1, \dots, x_n) g_i(y_i) \in \mathbb{R}$  has increasing differences w.r.t. any pair  $(x_i, y_j) \in \mathbb{R}^2$ ,  $i, j \in 1..n$ .*

**Proof.** Let  $f_i(x_j | \mathbf{x}_{-j})$  denote the function  $f_i$  applied to  $x_j$  but keeping all other arguments  $\mathbf{x}_{-j} := (x_k)_{k \neq j}$  fixed. We look at the two cases where we consider either a pair of argument  $(x_i, y_i) \in \mathbb{R}^2$ ,  $i \in 1..n$  (i.e. the pair of arguments tested for the increasing difference are part of the same term of the sum), or a pair  $(x_i, y_j) \in \mathbb{R}^2$ ,  $i \neq j \in 1..n$ , and the corresponding increasing differences.

*Case  $(x_i, y_i) \in \mathbb{R}^2$ ,  $i \in 1..n$ :* the function  $\mathbb{R}^2 \ni (x_i, y_i) \mapsto f_i(x_i | \mathbf{x}_{-i}) g_i(y_i)$ , has the increasing differences property by application of Lemma 2.3 with  $g(\cdot) = f_i(\cdot | \mathbf{x}_{-i})$  and

$h(\cdot) = g_i(\cdot)$ . The increasing difference writes

$$\begin{aligned}
& f_i(x'_i|\mathbf{x}_{-i})g_i(y'_i) + \sum_{k \neq i} f_k(x'_i|\mathbf{x}_{-i})g_k(y_k) - f_i(x'_i|\mathbf{x}_{-i})g_i(y_i) - \sum_{k \neq i} f_k(x'_i|\mathbf{x}_{-i})g_k(y_k) \\
& - f_i(x_i|\mathbf{x}_{-i})g_i(y'_i) - \sum_{k \neq i} f_k(x_i|\mathbf{x}_{-i})g_k(y_k) + f_i(x_i|\mathbf{x}_{-i})g_i(y_i) + \sum_{k \neq i} f_k(x_i|\mathbf{x}_{-i})g_k(y_k) \\
& = \underbrace{f_i(x'_i|\mathbf{x}_{-i})g_i(y'_i) - f_i(x'_i|\mathbf{x}_{-i})g_i(y_i) - f_i(x_i|\mathbf{x}_{-i})g_i(y'_i) + f_i(x_i|\mathbf{x}_{-i})g_i(y_i)}_{\geq 0 \text{ by Lemma 2.3}} \\
& + \underbrace{\sum_{k \neq i} f_k(x'_i|\mathbf{x}_{-i})g_k(y_k) - \sum_{k \neq i} f_k(x'_i|\mathbf{x}_{-i})g_k(y_k)}_{=0} \\
& - \underbrace{\sum_{k \neq i} f_k(x_i|\mathbf{x}_{-i})g_k(y_k) + \sum_{k \neq i} f_k(x_i|\mathbf{x}_{-i})g_k(y_k)}_{=0} \\
& \geq 0,
\end{aligned}$$

hence  $(x_i, y_i) \mapsto \sum_{k=1}^n f_k(x_i|\mathbf{x}_{-i})g_k(y_k)$  has the increasing differences property.

Case  $(x_i, y_j) \in \mathbb{R}^2$ ,  $i \neq j \in 1..n$ : we write the increasing difference

$$\begin{aligned}
& \sum_{k \neq j} f_k(x'_i|\mathbf{x}_{-i})g_k(y_k) + f_j(x'_i|\mathbf{x}_{-i})g_j(y'_j) - \sum_{k \neq j} f_k(x'_i|\mathbf{x}_{-i})g_k(y_k) - f_j(x'_i|\mathbf{x}_{-i})g_j(y_j) \\
& - \sum_{k \neq j} f_k(x_i|\mathbf{x}_{-i})g_k(y_k) - f_j(x_i|\mathbf{x}_{-i})g_j(y'_j) + \sum_{k \neq j} f_k(x_i|\mathbf{x}_{-i})g_k(y_k) + f_j(x_i|\mathbf{x}_{-i})g_j(y_j) \\
& = f_j(x'_i|\mathbf{x}_{-i})g_j(y'_j) - f_j(x'_i|\mathbf{x}_{-i})g_j(y_j) - f_j(x_i|\mathbf{x}_{-i})g_j(y'_j) + f_j(x_i|\mathbf{x}_{-i})g_j(y_j) \geq 0,
\end{aligned}$$

by application of Lemma 2.3 with  $g(\cdot) = f_j(\cdot|\mathbf{x}_{-i})$  and  $h(\cdot) = g_j(\cdot)$ .  $\square$

**Proposition 4.2** *If  $f_i : \mathbb{R} \rightarrow \mathbb{R}$  is a nondecreasing function and  $g_i : \mathbb{R} \rightarrow \mathbb{R}$  is a nondecreasing function,  $i \in 1..n$ , then  $\mathbb{R}^{2n} \ni (x_1, \dots, x_n, y_1, \dots, y_n) \mapsto \sum_{i=1}^n f_i(x_i)g_i(y_i) \in \mathbb{R}$  has increasing differences w.r.t. any pair  $(x_i, y_j) \in \mathbb{R}^2$ ,  $i, j \in 1..n$ .*

**Proof.** By Lemma 2.2,  $\lambda$  has increasing differences w.r.t. any pair  $(x_i, x_j) \in \mathbb{R}^2$ ,  $i \neq j \in 1..n$ , as well as any pair  $(y_i, y_j) \in \mathbb{R}^2$ ,  $i \neq j$ . For  $i \in 1..n$ ,  $(x_i, y_i) \mapsto f_i(x_i)g_i(y_i)$ , has the increasing differences property by application of Lemma 2.3 with  $g(\cdot) = f_i(\cdot)$  and  $h(\cdot) = g_i(\cdot)$ . Hence,  $\lambda$  has the increasing differences property for any pair  $(x_i, y_i) \in \mathbb{R}^2$ . Finally, for  $i \neq j \in 1..n$ ,  $(x_i, y_j) \mapsto f_i(x_i)g_i(y_i) + f_i(x_j)g_j(y_j)$  has the increasing differences property by Lemma 2.2 with  $h_i(\cdot) = f_i(\cdot)g_i(y_i)$  and  $h_j(\cdot) = f_j(x_j)g_j(\cdot)$ . Hence,  $\lambda$  has the increasing differences property for any pair  $(x_i, y_j) \in \mathbb{R}^2$ ,  $i \neq j \in 1..n$ .  $\square$

**Proposition 4.3** *Let  $f_i : \mathbb{R}^n \rightarrow \mathbb{R}$  be a nondecreasing supermodular function and  $g_i : \mathbb{R} \rightarrow \mathbb{R}$  be a nondecreasing function,  $i \in 1..n$ . Then the function*

$$\mathbb{R}^{2n} \ni (x_1, \dots, x_n, y_1, \dots, y_n) \mapsto \sum_{i=1}^n f_i(x_1, \dots, x_n)g_i(y_i) \in \mathbb{R} \quad (33)$$



is supermodular.

**Proof.** By Lemma 2.2 with  $h_i(\cdot) = f_i(x_1, \dots, x_n)g_i(\cdot)$ , for any  $i \in 1..n$ ,  $\Lambda$  has increasing differences with respect to any pair  $(y_k, y_l)$ ,  $k, l \in 1..n$ . By assumption and closure by addition of the increasing differences property,  $\Lambda$  has increasing differences with respect to any pair  $(x_k, x_l)$ ,  $k, l \in 1..n$ . Finally, by Lemma 4.1,  $\Lambda$  has increasing differences with respect to any pair  $(x_k, y_l)$ ,  $k, l \in 1..n$ . Hence, by Yildiz (2010, Corollary 1)<sup>5</sup>,  $\Lambda$  is supermodular.  $\square$

**Theorem 4.4** Assume  $\mathcal{L}$  and  $\mathcal{L}'$  defined by (32) applied respectively to  $(X_1, \dots, X_n, Y_1, \dots, Y_n)$  and  $(X'_1, \dots, X'_n, Y'_1, \dots, Y'_n)$  s.t.  $(X_1, \dots, X_n, Y_1, \dots, Y_n) \leq_{sm^0} (X'_1, \dots, X'_n, Y'_1, \dots, Y'_n)$ . Then, for any risk measure<sup>6</sup>  $\rho$  on  $\mathfrak{X}^0 \ni \mathcal{L}, \mathcal{L}'$ , we have  $\rho(\mathcal{L}) \leq \rho(\mathcal{L}')$ . In particular, if  $\mathfrak{X}^0 = L^1(\mathbb{Q}^0)$  and that  $\mathcal{L}$  and  $\mathcal{L}'$  are  $\mathbb{Q}^0$  integrable, then  $\mathbb{E}^0[\mathcal{L}] \leq \mathbb{E}^0[\mathcal{L}']$ .

**Proof.** By Proposition 4.3, the function  $(x_1, \dots, x_n, y_1, \dots, y_n) \mapsto \sum_{i=1}^n f_i(x_1, \dots, x_n)g_i(y_i)$  is supermodular on  $\mathbb{R}^{2n}$ . The result  $\rho(\mathcal{L}) \leq \rho(\mathcal{L}')$  then follows the way outlined in Section 2.  $\square$

**Remark 4.2** By Lemma 2.6, the conclusion of Theorem 4.4 holds, in particular, for  $\mathbf{X} = (X_0, X_1, \dots, X_n, Y_1, \dots, Y_n)$  and  $\mathbf{X}' = (X_0, X'_1, \dots, X'_n, Y'_1, \dots, Y'_n)$  (see beginning of Section 4) satisfying Assumption 2.3, e.g. (by Proposition 2.5) in the  $\mathbb{Q}^*$  elliptical setup (3) under Assumption 2.1.

**Remark 4.3** An analogous monotonicity result does not hold for  $\check{\rho} = \rho(\cdot - \mathbb{E}(\cdot))$  (see Definition C.1). For instance, if  $\rho = \mathbb{E}\mathbb{S}_\alpha$ ,  $\mathcal{U}_1$  is uniform on  $[0, 1]$  and  $\mathcal{U}_2$  is uniform on  $[1, \theta]$  for some  $\theta > 1$ , then  $\mathcal{U}_1 \leq \mathcal{U}_2$  a.s.,  $\mathbb{E}\mathbb{S}_\alpha(\mathcal{U}_1) = \frac{1+\alpha}{2} < \mathbb{E}\mathbb{S}_\alpha(\mathcal{U}_2) = 1 + (\theta - 1)\frac{1+\alpha}{2}$ , but  $\mathbb{E}\mathbb{S}_\alpha(\mathcal{U}_1) - \mathbb{E}(\mathcal{U}_1) = \frac{\alpha}{2} > \mathbb{E}\mathbb{S}_\alpha(\mathcal{U}_2) - \mathbb{E}(\mathcal{U}_2) = (\theta - 1)\frac{\alpha}{2}$  for  $1 < \theta < 2$ .

We now take in (14)

$$h(x_0) = \frac{\mathbb{1}_{\{x_0 < B_0\}}}{1 - \gamma}, \text{ where } \gamma = \mathbb{Q}^*(X_0 \geq B_0), \quad (34)$$

so that  $\mathbb{Q}^0$  is the survival measure of the reference bank associated with  $\mathbb{Q}^*$  (Albanese, Crépey, Hoskinson, and Saadeddine, 2021, Section 3).

**Definition 4.4** The current expected credit loss CECL of the reference bank 0 is  $\text{CECL} = \mathbb{E}^0[\mathcal{L}]$ .

Similarly, we define  $\text{CECL}' = \mathbb{E}^0[\mathcal{L}']$ .

**Remark 4.5** From a financial application point of view, the CECL that considers the lifetime for all related asset is preferred to the IFRS9 expected credit loss (ECL) that segments into stage 1 and stage 2. The CECL allows for a 1-year maturity assumption instead of the full lifetime of the assets considered for the ECL<sup>7</sup>.

<sup>5</sup>see Section A.

<sup>6</sup>see Definition C.1.

<sup>7</sup>see IFRS (2022, Article 5.3.3) and European Systemic Risk Board (2019).

**Proposition 4.5** *If  $\mathcal{L}$  and  $\mathcal{L}'$  are defined by (32) for  $\mathbf{X}$  and  $\mathbf{X}'$  satisfying Assumption 2.3, e.g. (by Proposition 2.5) in the elliptical setup (3) under Assumption 2.1, then  $\text{CECL} \leq \text{CECL}'$  whenever  $\mathcal{L}, \mathcal{L}' \in L^1(\mathbb{Q}^0)$ .*

**Proof.** By definition of CECL and application of Theorem 4.4 and Remark 4.2 to  $\rho = \mathbb{E}^0$ .  $\square$

**Definition 4.6** The economic capital of the reference bank 0 is  $\text{EC} = \mathbb{ES}_\alpha^0(\mathcal{L})$ , with  $\mathbb{ES}_\alpha^0$  as per Definition C.2 assumed under  $\mathbb{Q}^0$  and  $\mathcal{L} \geq 0$  given by (32).

Similarly, let  $\text{EC}' = \mathbb{ES}_\alpha^0(\mathcal{L}')$ .

**Proposition 4.6** *If  $\mathcal{L}$  and  $\mathcal{L}'$  are defined by (32) for  $\mathbf{X}$  and  $\mathbf{X}'$  satisfying Assumption 2.3, e.g. (by Proposition 2.5) in the elliptical setup (3) under Assumption 2.1, then  $\text{EC} \leq \text{EC}'$  whenever  $\mathcal{L}, \mathcal{L}' \in L^1(\mathbb{Q}^0)$ .*

**Proof.** By definition of EC and application of Theorem 4.4 and Remark 4.3 to  $\rho = \mathbb{ES}_\alpha^0$ .  $\square$

**Remark 4.7** The Definition C.2 of  $\mathbb{ES}_\alpha^0$  ensures its domain is all  $L^1(\mathbb{Q}^0)$ , thus is in particular proper. Acerbi and Tasche (2002, Proposition 3.1) outlines  $\mathbb{ES}_\alpha^0$  is subadditive and positively homogeneous (therefore convex) as well as monotonous (in our case where we consider loss variables, for  $\mathcal{X} \leq 0$  we have  $\rho(\mathcal{X}) \leq 0$  and using subadditivity we get for  $\mathcal{X} \leq \mathcal{Y}$ ,  $\rho(\mathcal{X}) = \rho(\mathcal{X} - \mathcal{Y} + \mathcal{Y}) \leq \rho(\mathcal{X} - \mathcal{Y}) + \rho(\mathcal{Y}) \leq \rho(\mathcal{Y})$ ). Hence it verifies the assumptions of (Ruszczynski and Shapiro, 2006, Proposition 1). Therefore  $\mathbb{ES}_\alpha^0$  is continuous on  $L^1(\mathbb{Q}^0)$  and Bäuerle and Müller (2006, Theorem 4.4) applies to  $\rho = \mathbb{ES}_\alpha^0$ .

#### 4.1 CCP Case Study

We denote  $\mathbf{Y} = \mathbf{X}_{n+1:2n} = (Y_1, \dots, Y_n)$  and  $\mathbf{Y}' = \mathbf{X}'_{n+1:2n} = (Y'_1, \dots, Y'_n)$ . Given real constants  $\beta_1, \dots, \beta_n \geq 0$ ,  $m_1, \dots, m_n$  and  $B_1, \dots, B_n$ , we consider the credit losses

$$\mathcal{L}(\mathbf{X}_{1:n}, \mathbf{Y}) = \sum_{i=1}^n \frac{1}{1 + \sum_{j=1}^n \beta_j \mathbb{1}_{\{X_j < B_j\}}} \mathbb{1}_{\{X_i \geq B_i\}} \times (Y_i - m_i)^+. \quad (35)$$

$\mathbf{X}_{0:n} = (X_0, X_1, \dots, X_n)$  drives the default events, namely,  $\{\tau_i \leq T\} = \{X_i \geq B_i\}$  models the default event of participant  $i \in 0..n$ , e.g. its liability return taken as the latent factor breaching a certain threshold.  $Y_i = \text{nom}_i \sigma_i G_i$ , with  $\text{nom}_i \in \mathbb{R}$ ,  $\sigma_i > 0$  and  $G_i$  spherical (hence  $\text{Var}^*(Y_i) = \text{nom}_i^2 \sigma_i^2 \text{Var}^*(G_i)$ ) represents the loss of the market participant 0 in case of the default of obligors indexed by  $i$ , collateralized by a corresponding amount  $m_i$ . The weights  $\left(1 + \sum_{j=1}^n \beta_j \mathbb{1}_{\{X_j < B_j\}}\right)^{-1}$  represent a stylised specification of a default fund allocation in a central counterparty (CCP) setup, with then  $\beta_j = \text{DF}_j / \text{DF}$  (see Section 4.2), or of a liability size allocation in a single resolution fund (SRF) setup.

**Lemma 4.7** For any  $i \in 1..n$ , the function  $\mathbb{R}^n \ni (x_1, \dots, x_n) \xrightarrow{f_i} \frac{\mathbb{1}_{\{x_i \geq B_i\}}}{1 + \sum_{j=1}^n \beta_j \mathbb{1}_{\{x_j < B_j\}}} \in \mathbb{R}_+$  is nondecreasing supermodular.

*Proof.* By Yildiz (2010, Corollary 1), it is sufficient to show that  $f_i$  has increasing differences. Let  $k, l \in 1, \dots, n$ .

*Case  $k, l \neq i$ :* Let  $\mathbb{R}^2 \ni (x_k, x_l) \xrightarrow{g} \frac{\mathbb{1}_{\{x_i \geq B_i\}}}{\Lambda + \beta_k \mathbb{1}_{\{x_k < B_k\}} + \beta_l \mathbb{1}_{\{x_l < B_l\}}} \in \mathbb{R}_+$ , with  $\Lambda = \sum_{j \neq k, l} \beta_j \mathbb{1}_{\{x_j < B_j\}}$ . For  $x'_k \geq x_k, x'_l \geq x_l$ , we form the increasing difference

$$\begin{aligned} & g(x'_k, x'_l) - g(x'_k, x_l) - g(x_k, x'_l) + g(x_k, x_l) \\ &= \frac{\mathbb{1}_{\{x_i \geq B_i\}} \beta_l (\mathbb{1}_{\{x_l < B_l\}} - \mathbb{1}_{\{x'_l < B_l\}})}{\text{denom}_1} - \frac{\mathbb{1}_{\{x_i \geq B_i\}} \beta_l (\mathbb{1}_{\{x_l < B_l\}} - \mathbb{1}_{\{x'_l < B_l\}})}{\text{denom}_2}, \end{aligned} \quad (36)$$

with  $\text{denom}_1 = (\Lambda + \beta_k \mathbb{1}_{\{x'_k < B_k\}} + \beta_l \mathbb{1}_{\{x'_l < B_l\}})(\Lambda + \beta_k \mathbb{1}_{\{x'_k < B_k\}} + \beta_l \mathbb{1}_{\{x_l < B_l\}})$  and  $\text{denom}_2 = (\Lambda + \beta_k \mathbb{1}_{\{x_k < B_k\}} + \beta_l \mathbb{1}_{\{x'_l < B_l\}})(\Lambda + \beta_k \mathbb{1}_{\{x_k < B_k\}} + \beta_l \mathbb{1}_{\{x_l < B_l\}})$ . If  $x_l \leq x'_l < B_l$  or  $B_l \leq x_l \leq x'_l$ , then the increasing difference (36) is zero as the numerators of both terms are zero. If  $x_l < B_l \leq x'_l$ , then both numerators in (36) equal  $\mathbb{1}_{\{x_i \geq B_i\}} \beta_l$ . In this case: (i) if  $x_k \leq x'_k < B_k$ , then both denominators in (36) equal  $(\Lambda + \beta_k)(\Lambda + \beta_k + \beta_l)$  and the increasing difference (36) is zero; (ii) if  $B_k \leq x_k \leq x'_k$ , then both denominators in (36) equal  $\Lambda(\Lambda + \beta_l)$  and the increasing difference (36) is zero; If  $x_k < B_k \leq x'_k$ , then the increasing difference in (36) writes equivalently

$$\frac{\Lambda \beta_l}{\text{denom}_1 \text{denom}_2} (\text{denom}_2 - \text{denom}_1), \quad (37)$$

with  $\text{denom}_2 = (\Lambda + \beta_k + \beta_l)(\Lambda + \beta_k) \geq \Lambda(\Lambda + \beta_l) = \text{denom}_1$  as  $\beta_j \geq 0$  for all  $j \in 1..n$ . Hence the increasing difference (36) is nonnegative.

*Case  $k = i$  and  $l \neq i$ :* Let  $\mathbb{R}^2 \ni (x_i, x_l) \xrightarrow{g} \frac{\mathbb{1}_{\{x_i \geq B_i\}}}{\Lambda + \beta_i \mathbb{1}_{\{x_i < B_i\}} + \beta_l \mathbb{1}_{\{x_l < B_l\}}} \in \mathbb{R}_+$ , with  $\Lambda = \sum_{j \neq i, l} \beta_j \mathbb{1}_{\{x_j < B_j\}}$ . For  $x'_i \geq x_i, x'_l \geq x_l$ , we form the increasing difference

$$\begin{aligned} & g(x'_i, x'_l) - g(x'_i, x_l) - g(x_i, x'_l) + g(x_i, x_l) \\ &= \frac{\mathbb{1}_{\{x'_i \geq B_i\}} \beta_l (\mathbb{1}_{\{x_l < B_l\}} - \mathbb{1}_{\{x'_l < B_l\}})}{\text{denom}_1} - \frac{\mathbb{1}_{\{x_i \geq B_i\}} \beta_l (\mathbb{1}_{\{x_l < B_l\}} - \mathbb{1}_{\{x'_l < B_l\}})}{\text{denom}_2}, \end{aligned} \quad (38)$$

with  $\text{denom}_1 = (\Lambda + \beta_i \mathbb{1}_{\{x'_i < B_i\}} + \beta_l \mathbb{1}_{\{x'_l < B_l\}})(\Lambda + \beta_i \mathbb{1}_{\{x'_i < B_i\}} + \beta_l \mathbb{1}_{\{x_l < B_l\}})$  and  $\text{denom}_2 = (\Lambda + \beta_i \mathbb{1}_{\{x_i < B_i\}} + \beta_l \mathbb{1}_{\{x'_l < B_l\}})(\Lambda + \beta_i \mathbb{1}_{\{x_i < B_i\}} + \beta_l \mathbb{1}_{\{x_l < B_l\}})$ . If  $x_l \leq x'_l < B_l$  or  $B_l \leq x_l \leq x'_l$ , then the increasing difference (38) is zero as the numerators of both terms are zero. If  $x_l < B_l \leq x'_l$ , then the numerator of the first terms in (38) is  $\mathbb{1}_{\{x'_i \geq B_i\}}$  and  $\mathbb{1}_{\{x_i \geq B_i\}}$  for the second term. In this case: (i) if  $x_i \leq x'_i < B_i$ , then both numerators in (38) are zero and the increasing difference (38) is zero; (ii) if  $B_i \leq x_i \leq x'_i$ , then both numerators in (38) are equal to  $\mathbb{1}_{\{x_i \geq B_i\}} \beta_l$  and both denominators are equal  $\Lambda(\Lambda + \beta_l)$  so the increasing difference (38) is zero; If  $x_i < B_i \leq x'_i$ , then the second term in (38) is zero and the first term is nonnegative. Hence the increasing difference (38) is nonnegative.  $\square$

**Proposition 4.8** *The function*

$$\mathbb{R}^{2n} \ni (x_1, \dots, x_n, y_1, \dots, y_n) \xrightarrow{f} \sum_{i=1}^n \frac{1}{1 + \sum_{j=1}^n \beta_j \mathbb{1}_{\{x_j < B_j\}}} \mathbb{1}_{\{x_i \geq B_i\}} \times (y_i - m_i)^+ \in \mathbb{R}_+ \quad (39)$$

*is nondecreasing supermodular.*

**Proof.** Let  $f_i(x_1, \dots, x_n) = \frac{\mathbb{1}_{\{x_i \geq B_i\}}}{1 + \sum_{j=1}^n \beta_j \mathbb{1}_{\{x_j < B_j\}}}$  and  $g_i(y_i) = (y_i - m_i)^+$  in (33). By Lemma 4.7,  $f_i$  has increasing differences with respect to any pair  $(x_k, x_l) \in \mathbb{R}^2$ ,  $k, l \in 1..n$  for any  $i \in 1..n$ . Hence  $f$  verifies all assumptions of Proposition 4.3, thus it is supermodular.  $\square$

**Corollary 4.9** *If  $\mathcal{L}(\mathbf{X}_{1:n}, \mathbf{Y}), \mathcal{L}(\mathbf{X}'_{1:n}, \mathbf{Y}') \in L^1(\mathbb{Q}^0)$ , then,*

$$\mathbb{E}^0 [\mathcal{L}(\mathbf{X}_{1:n}, \mathbf{Y})] \leq \mathbb{E}^0 [\mathcal{L}(\mathbf{X}'_{1:n}, \mathbf{Y}')], \quad (40)$$

*i.e. CECL is nondecreasing w.r.t.  $\Gamma_{\mathbf{X}}^{ij}$  for each  $i, j \in 1..2n$ .*

**Proof.** By Proposition 2.5 and Lemma 2.6,  $(X_1, \dots, X_n, Y_1, \dots, Y_n) \leq_{sm^0} (X'_1, \dots, X'_n, Y'_1, \dots, Y'_n)$ . Applying Proposition 4.5 with  $f$  defined by (39), which is supermodular thanks to Proposition 4.8, then yields the result.  $\square$

Similarly (with Proposition 4.6 instead of Proposition 4.5 in the above argument)

**Corollary 4.10** *If  $\mathcal{L}(\mathbf{X}_{1:n}, \mathbf{Y}), \mathcal{L}(\mathbf{X}'_{1:n}, \mathbf{Y}') \in L^1(\mathbb{Q}^0)$ , then,*

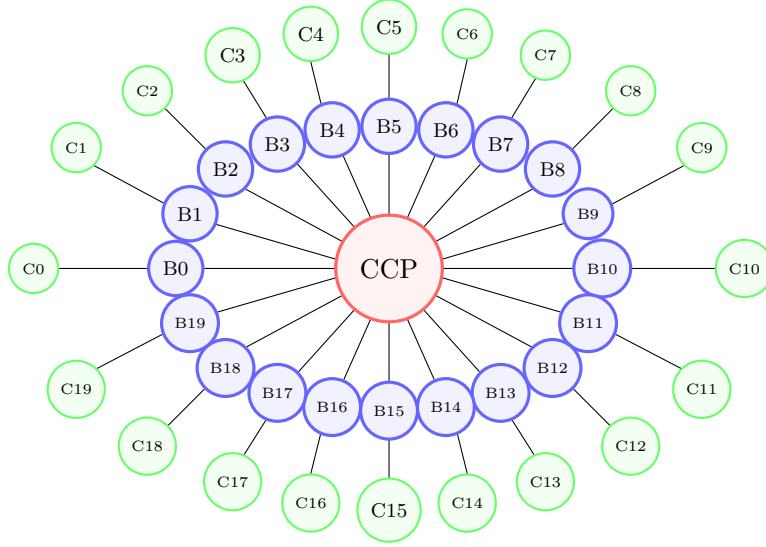
$$\mathbb{ES}_\alpha^0(\mathcal{L}(\mathbf{X}_{1:n}, \mathbf{Y})) \leq \mathbb{ES}_\alpha^0(\mathcal{L}(\mathbf{X}'_{1:n}, \mathbf{Y}')), \quad (41)$$

*i.e. EC is nondecreasing w.r.t.  $\Gamma_{\mathbf{X}}^{ij}$  for each  $i, j \in 1..2n$ .  $\square$*

## 4.2 Numerical Setup

We consider a CCP service with 20 members, labeled by  $i \in 0..n = 19$ , trading for cleared clients (i.e. without bilateral or centrally cleared proprietary trading). Each member faces one client. The corresponding financial network is depicted in Figure 3. All clients are assumed to be risk-free. For any member  $i$ , its posted initial margin (IM) to the CCP is calculated based on the idea of a variation margin (VM) call not fulfilled over a slippage time period  $\Delta_s$  at a confidence level  $\alpha \in (1/2, 1)$ . Such IM uses a VaR metric under the member survival measure applied to the non-coverage of VM call on the cleared portfolio. The latter follows a scaled Student t-distribution  $\mathcal{S}_\nu$  with  $\nu$  degrees of freedom, with c.d.f.  $S_\nu$ , and where such scaling reflects both  $\Delta_s$ , the portfolio nominal size, denoted  $\text{nom}_i$ , and its standard deviation, denoted  $\sigma_i$ . Namely,

$$\text{IM}_i = \text{VaR}^*(\text{nom}_i \sigma_i \sqrt{\Delta_s} \mathcal{S}_\nu) = |\text{nom}_i| \sigma_i \sqrt{\Delta_s} S_\nu^{-1}(\alpha). \quad (42)$$



**Figure 3:** Financial network composed of one CCP service, its 20 members (labeled by B) and one cleared client per member

The default fund is calculated at the CCP level as

$$\text{Cover2} = \text{SLOIM}_{(0)} + \text{SLOIM}_{(1)}, \quad (43)$$

for the two largest stressed losses over IM ( $\text{SLOIM}_i$ ) among members, identified with subscripts (0) and (1).  $\text{SLOIM}_i$  is calculated as the value-at-risk  $\text{VaR}^{\alpha'}$  at a confidence level  $\alpha' > \alpha$  of the loss over IM, i.e.

$$\text{SLOIM}_i = \text{VaR}^{\alpha'}(\text{nom}_i \sigma_i \sqrt{\Delta_s} \mathcal{S}_\nu - \text{IM}_i) = |\text{nom}_i| \sigma_i \sqrt{\Delta_s} (S_\nu^{-1}(\alpha') - S_\nu^{-1}(\alpha)). \quad (44)$$

The total amount (43) is then allocated between the clearing members to define their (funded) default fund contributions as  $\text{DF}_i = \frac{\text{SLOIM}_i}{\sum_j \text{SLOIM}_j} \times \text{Cover2}$ . Finally, the loss function of the reference member 0 with default fund contribution  $\text{DF}_0$  is

$$\begin{aligned} \mathcal{L}(\mathbf{X}, \mathbf{Y}) &= \sum_{i=1}^n \frac{\text{DF}_0}{1 + \sum_{j=1}^n \text{DF}_j \mathbf{1}_{\{X_j < B_j\}}} \mathbf{1}_{\{X_i \geq B_i\}} \times (Y_i - \text{IM}_i - \text{DF}_i)^+ \\ &= \sum_{i=1}^n \frac{1}{1 + \sum_{j=1}^n \frac{\text{DF}_j}{\text{DF}_0} \mathbf{1}_{\{X_j < B_j\}}} \mathbf{1}_{\{X_i \geq B_i\}} \times (Y_i - \text{IM}_i - \text{DF}_i)^+. \end{aligned} \quad (45)$$

Let  $\text{sgn}(x) = 1$  if  $x > 0$ ,  $0$  if  $x = 0$ ,  $-1$  otherwise. An elliptical model is specified under  $\mathbb{Q}^*$  as

$$\begin{cases} Y_i = \text{nom}_i \sigma_i \sqrt{\Delta_l} \left( \sqrt{\rho^{mkt}} \mathcal{E} + \sqrt{\rho^{wwr}} \mathcal{W}_i + \sqrt{1 - \rho^{mkt} - \rho^{wwr}} \mathcal{E}_i \right) \\ X_i = \sqrt{\rho^{cr}} \mathcal{T} + \text{sgn}(\text{nom}_i) \sqrt{\rho^{wwr}} \mathcal{W}_i + \sqrt{1 - \rho^{cr} - \rho^{wwr}} \mathcal{T}_i \end{cases} \quad (46)$$

for any  $i \in 0..n$ , where  $\mathcal{T}$ ,  $\mathcal{T}_i$ ,  $\mathcal{E}$ ,  $\mathcal{E}_i$  and  $\mathcal{W}_i$  are i.i.d. random variables following centered Student t-distributions of degree 3.  $\Delta_l$  is the period accounting for the time taken by the CCP to novate or liquidate its portfolios in case of defaults (practically,  $\Delta_l > \Delta_s$  by a few business days).  $T$  represents the final maturity of the clearing members portfolios, assumed to be the same for all members.  $B_i = \mathcal{S}_\nu^{-1}(1 - \text{DP}_i(T))$  where  $\text{DP}_i(T)$  is the default probability over the period  $[0, T]$  defined from a constant default intensity  $\lambda_i$  given for each member  $i$  in Table 3 (which can be obtained from their 1-year  $\mathbb{Q}^*$  default probability  $\text{DP}_i(1Y)$ , inferred either from the agency ratings or the CDS quotes when available, as  $\lambda_i = -\ln(1 - \text{DP}_i(1Y))$  so that  $\text{DP}_i(T) = 1 - e^{-\lambda_i T}$ ). The model is well defined if and only if  $0 < \rho^{wwr} < \min(1 - \rho^{cr}, 1 - \rho^{mkt})$ . Also, note that  $\text{Cov}^*(X_i, Y_i) = \text{nom}_i \sigma_i \sqrt{\Delta_l} \sqrt{\rho^{wwr}} \text{sgn}(\text{nom}_i) \sqrt{\rho^{wwr}} = |\text{nom}_i| \sigma_i \sqrt{\Delta_l} \rho^{wwr} \geq 0$ , hence increasing  $\rho^{wwr}$  leads to an increase of  $\text{Cov}^*(X_i, Y_i)$ .

The participants and portfolios parameter inputs are detailed in Table 3, where cm is the identifier of the clearing member,  $\lambda$  is the one year  $\mathbb{Q}^*$  default intensity of the member expressed in basis points, size represents the overall portfolio size of the member detained within the CCP, and vol is the annual volatility used for the portfolio variations. The portfolios listed in Table 3 relate to the members towards the CCP (which are mirroring the portfolios between the members and their clients). The sizes of the CCP portfolios of members sum up to 0, in line with the CCP clearing condition (without proprietary trades).

**Remark 4.8** The random variables (46) follow Student t-distributions that are continuous. Therefore, 0 is the only possible atom of the nonnegative credit loss (45). Hence, by Acerbi and Tasche (2002, Corollary 5.3), Definition C.2 is equivalent to  $\mathbb{E}\mathcal{S}_\alpha(\mathcal{X}) = \mathbb{E}(\mathcal{X} | \mathcal{X} \geq \text{VaR}_\alpha(\mathcal{X}))$  (Acerbi and Tasche, 2002, Eqn. (3.7)) whenever  $\text{VaR}_\alpha(\mathcal{X}) > 0$ , i.e. for  $\alpha \in (\frac{1}{2}, 1)$  sufficiently close to 1 so that  $\text{VaR}_\alpha(\mathcal{X}) > 0$ . In our numerical illustration with  $\alpha = 99.75\%$ , this is indeed the case.

### 4.3 Numerical Results

The parameters of the CECL and EC calculations are summarized in Table 4. The confidence level at 97% for SLOIM in DF calibration allows for a ratio of default fund over initial margin of about 10% in our calculations, a ratio (of this level or less) often observed in practice. Note that the chosen period length of  $T = 5$  years covers the bulk (if not the final maturity) of most realistic CCP portfolios.

Figures 4, 5 and 6 show the results of CECL and EC calculated for the members 0, 5, and 10, each under their survival risk measure (i.e. letting them in turn play the

cm	0	1	2	3	4	5	6	7	8	9
$\lambda$ (bps)	50	60	70	80	90	200	190	180	170	160
size	-242	184	139	105	-80	-61	-46	35	26	-20
vol (%)	20	21	22	23	24	25	26	27	28	29

cm	10	11	12	13	14	15	16	17	18	19
$\lambda$ (bps)	150	140	130	120	110	100	90	80	70	60
size	-15	-11	-9	-6	5	-4	-3	2	2	-1
vol (%)	30	31	32	33	34	35	36	37	38	39

**Table 3:** Member characteristics and CCP portfolio parameters, ordered by decreasing member  $|\text{size}|$ .

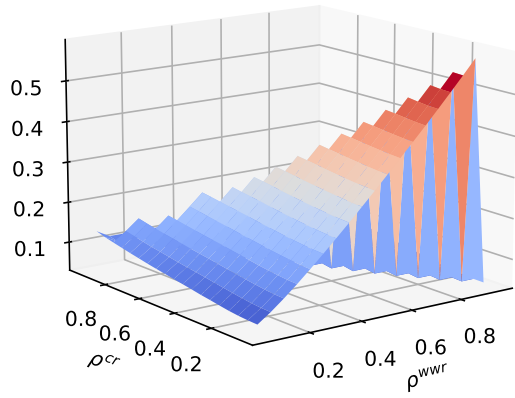
One-period length $T$	5 years
Liquidation period at default $\Delta_l$	5 days
Portfolio variations correlation $\rho^{mkt}$	30%
Credit factors correlation $\rho^{cr}$	20%
Correlation between credit factors and portfolio variations $\rho_i^{wvr}$	20%
IM covering period (MPoR) $\Delta_s$	2 days
IM quantile level	95%
SLOIM calculation for DF Cover-2	VaR 97%
DF allocation rule	based on IM
Quantile level used for clearing members EC calculation	99.75%
Number of Monte-Carlo simulation (for CECL and EC computations)	10M
Number of batches (for EC computations)	100

**Table 4:** CECL and EC calculation configuration

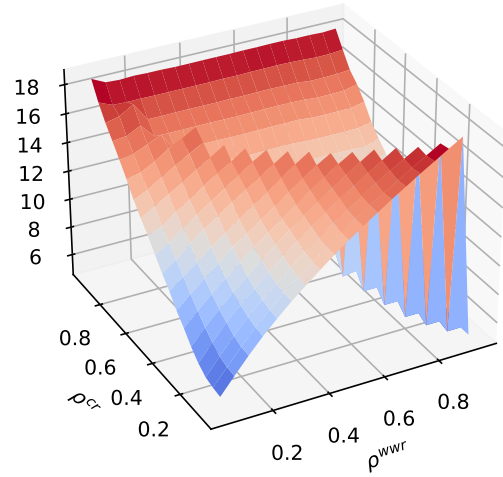
role of the reference bank indexed by 0 in previous sections). In each figure, the credit-credit correlation  $\rho^{cr}$  and  $\rho^{wvr}$  is varied between 5% and 95%, using 5% step. The same nondecreasing pattern is observed for all three members, with nonnegative incremental CECL and EC between two consecutive credit-credit and credit-market correlation steps, in line with Corollaries 4.9 and 4.10. The market-market correlation  $\rho^{mkt}$  has been kept constant with value 4%. The results of the centered EC, i.e.  $EC - CECL$ , are also provided for each of these 3 members in Figure 7. As  $CECL \ll EC$  holds for all three members, despite Remark 4.3, the monotonicity is also observed for this centered version of EC.

**Remark 4.9** In our example,  $\text{Cov}^*(Y_i, Y_j) = \text{nom}_i \text{nom}_j \sigma_i \sigma_j \frac{\nu}{\nu-2} \rho^{mkt}$ . Hence, depending on the sign of  $\text{nom}_i \text{nom}_j$ , increasing  $\rho^{mkt}$  either increases or decreases  $\text{Cov}^*(Y_i, Y_j)$ . Thus, we cannot hope to observe a monotonous behaviour of EC or CECL w.r.t.  $\rho^{mkt}$ .

CECL for member cm0

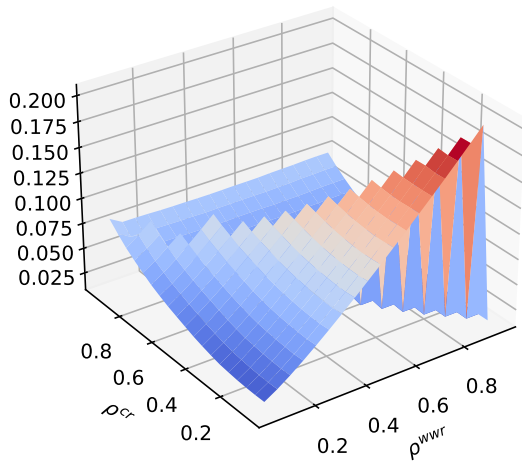


EC for member cm0

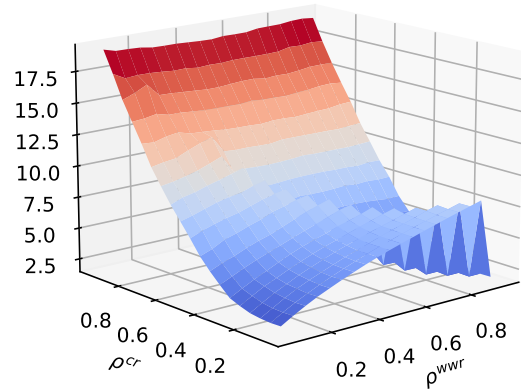


**Figure 4:** Member 0 CECL and EC w.r.t. credit factors correlation  $\rho^{cr}$  and credit and portfolio variation factors correlation  $\rho^{wvr}$ .

CECL for member cm5

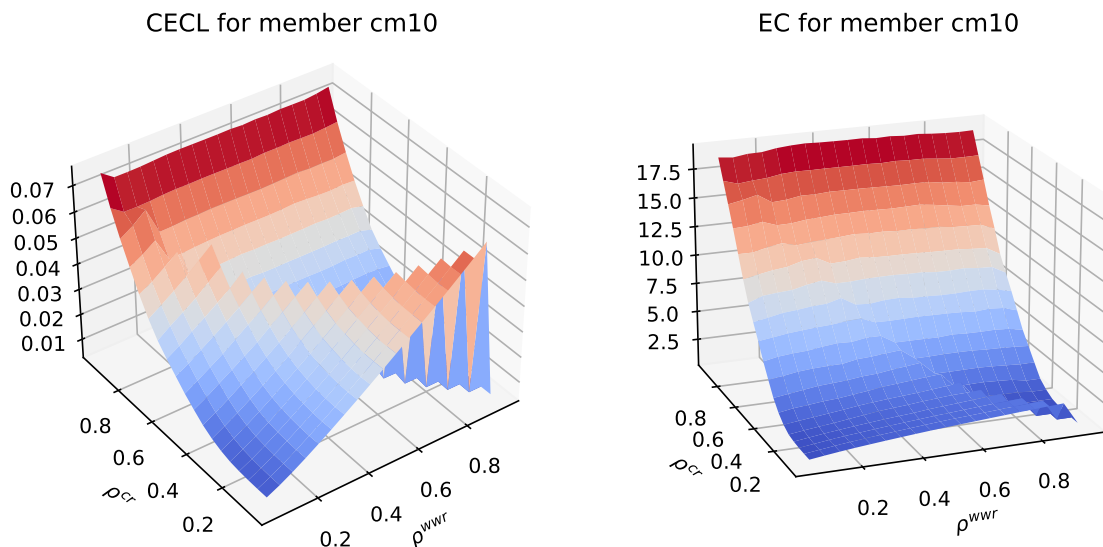


EC for member cm5



**Figure 5:** Member 5 CECL and EC w.r.t. credit factors correlation  $\rho^{cr}$  and credit and portfolio variation factors correlation  $\rho^{wvr}$ .





**Figure 6:** Member 10 CECL and EC w.r.t. credit factors correlation  $\rho^{cr}$  and credit and portfolio variation factors correlation  $\rho^{wvr}$ .

## 5 Conclusion

The main mathematical results of the paper are summed up in Table 5. In a nutshell, if a participant uses a convex risk measure to assess its credit risk depicted as an aggregation of nonnegative losses driven by elliptically distributed factors, then the measure increases with the covariance coefficients between these factors. These results and their numerical illustrations support the use of such elliptical factor models for both risk management and regulatory credit provision and capital requirement purposes.

## A Supermodular Functions

*Müller (1997, Definitions 2.4 and 2.6)* A function  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  is said to be supermodular if

$$f(x_1, \dots, x_i + \varepsilon, \dots, x_j + \delta, \dots, x_n) - f(x_1, \dots, x_i + \varepsilon, \dots, x_j, \dots, x_n) \geq f(x_1, \dots, x_i, \dots, x_j + \delta, \dots, x_n) - f(x_1, \dots, x_i, \dots, x_j, \dots, x_n) \quad (47)$$

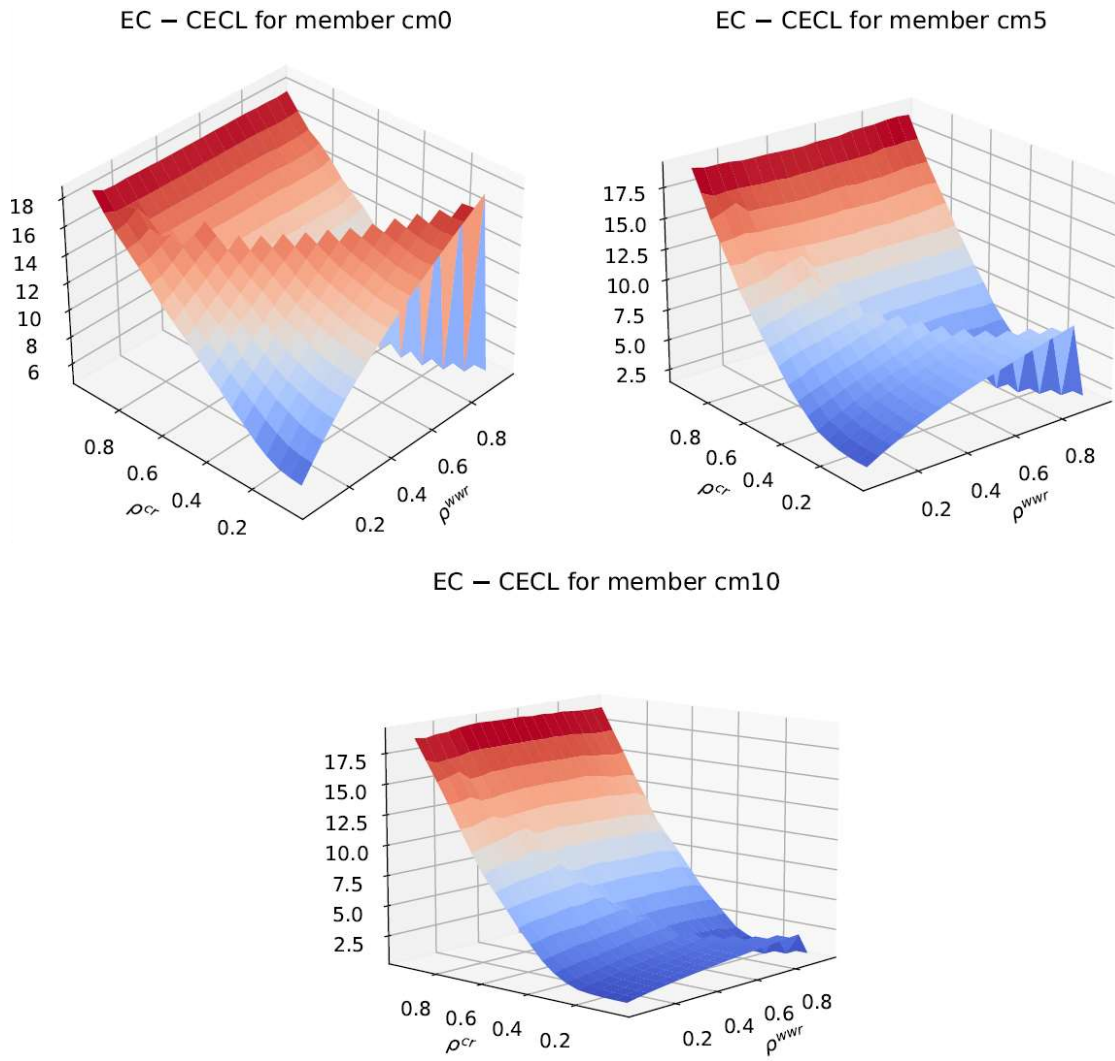
holds for all  $x \in \mathbb{R}^n$ ,  $\varepsilon, \delta > 0$  and  $1 \leq i < j \leq n$ .  $\square$

*Müller and Scarsini (2000, Theorem 2.2 (a))* For functions twice differentiable on  $\mathbb{R}^d$ , the supermodularity is equivalent to the nonnegativity of its second derivatives.  $\square$

More general definitions can be found in Meyer-Nieberg (1991) and Yildiz (2010).

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<sup>8</sup>see Definition C.1.



**Figure 7:** Members 0, 5 and 10 EC – CECL w.r.t. credit factors correlation  $\rho^{cr}$  and credit and portfolio variation factors correlation  $\rho^{wvr}$ .

Proposition 4.3	Let $f_i : \mathbb{R}^n \rightarrow \mathbb{R}$ be nondecreasing supermodular functions, $g_i : \mathbb{R} \rightarrow \mathbb{R}$ be nonnegative nondecreasing functions, $i \in 1..n$ . Then the function $(x_1, \dots, x_n, y_1, \dots, y_n) \mapsto \sum_{i=1}^n f_i(x_1, \dots, x_n)g_i(y_i)$ is supermodular on $\mathbb{R}^{2n}$ .
Theorem 4.4	If $(\mathbf{X}_{1:n}, \mathbf{Y})^\top \leq_{sm^0} (\mathbf{X}'_{1:n}, \mathbf{Y}')^\top$ and $\rho$ is a risk measure <sup>8</sup> on $\mathfrak{X}^0$ , then $\rho(\mathcal{L}) \leq \rho(\mathcal{L}')$ holds for any $\mathcal{L}, \mathcal{L}' \in \mathfrak{X}^0$ of the form (32) applied to $(\mathbf{X}_{1:n}, \mathbf{Y})^\top$ and $(\mathbf{X}'_{1:n}, \mathbf{Y}')^\top$ .
Proposition 3.1 and Corollary 3.2	The price of the default leg of an equity (resp. senior) tranche is nonincreasing (resp. nondecreasing) w.r.t. the credit correlation $\rho^{cr}$ . The price of the payment leg of the CDO equity (resp. senior) tranche is nonincreasing (resp. nondecreasing) w.r.t. $\rho^{cr}$ .
Proposition 4.5	Under Assumption 2.1 on $(X_0, X_1, \dots, X_n, Y_1, \dots, Y_n)$ and $(X_0, X'_1, \dots, X'_n, Y'_1, \dots, Y'_n)$ in the roles of $\mathbf{X}$ and $\mathbf{X}'$ there, CECL is a nondecreasing function of the $\mathbb{Q}^*$ covariance coefficients.
Proposition 4.6	Under Assumption 2.1 on $(X_0, X_1, \dots, X_n, Y_1, \dots, Y_n)$ and $(X_0, X'_1, \dots, X'_n, Y'_1, \dots, Y'_n)$ in the roles of $\mathbf{X}$ and $\mathbf{X}'$ there, EC is a nondecreasing function of the $\mathbb{Q}^*$ covariance coefficients.

**Table 5:** Main theoretical and applied results of the paper (with risk neutral measure  $\mathbb{Q}^*$ , reference participant labelled by 0, related survival measure  $\mathbb{Q}^0$ ).

*Meyer-Nieberg (1991, Section 1.1, page 1)* A partially ordered set  $(\mathfrak{S}, \geq)$  is said to be a lattice if and only if any two elements  $x, y$  have a greatest common minorant, denoted  $x \wedge y$ , and a least common majorant, denoted  $x \vee y$ .  $\square$

*Yildiz (2010, Definition 6, page 6)* A function  $f : \mathfrak{S} \rightarrow \mathbb{R}$  is said to be supermodular on a lattice  $(\mathfrak{S}, \geq)$  if

$$f(x \vee y) + f(x \wedge y) \geq f(x) + f(y) \quad (48)$$

holds for all  $x, y \in \mathfrak{S}$ .  $\square$

For a family of lattices  $(\mathfrak{S}_1, \leq), \dots, (\mathfrak{S}_n, \leq)$ , let  $\mathfrak{S} = \mathfrak{S}_1 \times \dots \times \mathfrak{S}_n$  be endowed with the coordinate-wise order  $(x_1, \dots, x_n) \leq (y_1, \dots, y_n)$  if and only if  $\forall i, x_i \leq y_i$ . This order makes  $\mathfrak{S} = \mathfrak{S}_1 \times \dots \times \mathfrak{S}_n$  a lattice. For  $x \in \mathfrak{S}$  and any  $i$  and  $j$ , define  $\mathbf{x}_{-(i,j)} = (x_k)_{k \neq i,j}$ . For any function  $f : \mathfrak{S} \rightarrow \mathbb{R}$ , define  $f(\cdot | \mathbf{x}_{-(i,j)}) : \mathfrak{S}_i \times \mathfrak{S}_j \rightarrow \mathbb{R}$  as the restriction of  $f$  to vectors with entries other than  $i$  and  $j$  fixed at  $\mathbf{x}_{-(i,j)}$  (Yildiz, 2010, Section 2.4).

*Yildiz (2010, remark after Definition 7, page 7)* A function  $f : \mathfrak{S} = \mathfrak{S}_1 \times \dots \times \mathfrak{S}_n \rightarrow \mathbb{R}$  is said to be pairwise supermodular if

$$\begin{aligned} f((x_i, x_j) \vee (x'_i, x'_j) | \mathbf{x}_{-(i,j)}) + f((x_i, x_j) \wedge (x'_i, x'_j) | \mathbf{x}_{-(i,j)}) \\ \geq f(x_i, x_j | \mathbf{x}_{-(i,j)}) + f(x'_i, x'_j | \mathbf{x}_{-(i,j)}) \end{aligned} \quad (49)$$

holds for all  $x_1, \dots, x_n \in \mathfrak{S}_1 \times \dots \times \mathfrak{S}_n$ .  $\square$

*Yildiz (2010, Definition 7)* A function  $f : \mathfrak{S} = \mathfrak{S}_1 \times \dots \times \mathfrak{S}_n \rightarrow \mathbb{R}$  is said to have

increasing differences if

$$\begin{aligned} [x_i \geq x_{i'}, x_j \geq x_{j'}] \implies \\ f(x_{i'}, x_{j'} | \mathbf{x}_{-(i,j)}) - f(x_{i'}, x_j | \mathbf{x}_{-(i,j)}) \geq f(x_i, x_{j'} | \mathbf{x}_{-(i,j)}) - f(x_i, x_j | \mathbf{x}_{-(i,j)}) \end{aligned} \quad (50)$$

holds for any  $\mathbf{x} = (x_1, \dots, x_n)$ ,  $i, j, i', j'$ .  $\square$

If the partial order is a total order, increasing differences and supermodularity in the sense of Definition A are equivalent. In particular:

*Yildiz (2010, Corollary 1)* For any  $f : \mathbb{R}^n \rightarrow \mathbb{R}$ , the following are equivalent:

1.  $f$  is supermodular;
2.  $f$  has increasing differences;
3.  $f$  is pairwise supermodular.  $\square$

*Müller and Stoyan (2002, Theorem 3.9.3 f), page 113)* If  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  is nondecreasing and supermodular and  $\phi : \mathbb{R} \rightarrow \mathbb{R}$  is nondecreasing and convex, then  $\phi \circ f : \mathbb{R}^n \rightarrow \mathbb{R}$  is nondecreasing supermodular.  $\square$

## B Elliptical Distributions

*McNeil et al. (2015, Definition 6.17, page 196)* A random vector  $\mathbf{Z} = (Z_1, \dots, Z_n)^\top$  has a spherical distribution in  $\mathbb{R}^n$  if, for every orthogonal map  $\mathbf{A} \in \mathbb{R}^{n \times n}$  ( $\mathbf{A}\mathbf{A}^\top = \mathbf{A}^\top\mathbf{A} = \mathbf{I}_n$ ),

$$\mathbf{AZ} \stackrel{d}{=} \mathbf{Z}. \quad \square \quad (51)$$

*McNeil et al. (2015, Theorem 6.18, page 196)* The following are equivalent.

- (1)  $\mathbf{Z}$  is spherical in  $\mathbb{R}^n$ .
- (2) There exists a function  $\psi : \mathbb{R}_+ \rightarrow \mathbb{C}$  such that, for all  $\mathbf{u} = (u_1, \dots, u_n)^\top \in \mathbb{R}^n$ , the characteristic function of  $\mathbf{Z}$  is

$$\mathbb{E} \left( e^{i\mathbf{u}^\top \mathbf{Z}} \right) = \psi(\mathbf{u}^\top \mathbf{u}) = \psi(u_1^2 + \dots + u_n^2). \quad (52)$$

- (3) For every  $\mathbf{u} \in \mathbb{R}^n$ ,  $\mathbf{u}^\top \mathbf{Z} \stackrel{d}{=} \|\mathbf{u}\| Z_1$ .  $\square$

$\psi$  is called the characteristic generator of  $\mathbf{Z}$  and the notation  $\mathbf{Z} \sim S_n(\psi)$  is used (see Fang, Kotz, and Ng (1990) and McNeil, Frey, and Embrechts (2015)). We denote by  $\mathbb{S}^{n-1} := \{\mathbf{s} \in \mathbb{R}^n : \mathbf{s}^\top \mathbf{s} = 1\}$  the unit sphere in  $\mathbb{R}^n$ , and by  $U_{\mathbb{S}^{n-1}}$  the uniform distribution on  $\mathbb{S}^{n-1}$ .

Fang et al. (1990, Theorem 2.2, page 29) A function  $\psi$  is a generator of an  $n$ -dimensional elliptical r.v. if and only if it can be written as

$$\psi(x) = \int_0^\infty \Omega_n(xr^2)F(dr), \quad (53)$$

where  $F(\cdot)$  is some c.d.f. over  $\mathbb{R}^+$  and  $\Omega_n(\mathbf{u}^\top \mathbf{u})$  is the characteristic generator of a random vector  $\mathbf{S} \sim U_{\mathbb{S}^{n-1}}$ , namely (Steerneman and van Perlo-ten Kleij, 2005, Eqn. (2))

$$\Omega_n(\mathbf{u}^\top \mathbf{u}) = \mathbb{E} \left( e^{i\mathbf{u}^\top \mathbf{S}} \right) = \frac{\Gamma(m/2)}{\sqrt{\pi}\Gamma((m-1)/2)} \int_{-1}^1 e^{i\mathbf{u}^\top \mathbf{u}t} (1-t^2)^{(m-3)/2} dt, \quad \mathbf{u} \in \mathbb{R}^n. \quad \square \quad (54)$$

McNeil et al. (2015, Definition 6.25, page 200)  $\mathbf{X} = (\mathcal{X}_1, \dots, \mathcal{X}_n)$  is said to have an elliptical distribution in  $\mathbb{R}^n$  with parameters  $\boldsymbol{\mu}$ ,  $\Sigma$ ,  $\psi$ , where  $\Sigma$  is an  $n \times n$  square semi-positive definite matrix, if

$$\mathbb{E} \left( e^{i\mathbf{u}^\top (\mathbf{X} - \boldsymbol{\mu})} \right) = \psi(\mathbf{u}^\top \Sigma \mathbf{u}), \quad \mathbf{u} \in \mathbb{R}^n. \quad (55)$$

We then write  $\mathbf{X} \sim E_n(\boldsymbol{\mu}, \Sigma, \psi)$ <sup>9</sup>.  $\square$

McNeil et al. (2015, Proposition 6.27, page 200)  $\mathbf{X} \sim E_n(\boldsymbol{\mu}, \Sigma, \psi)$  if and only if there exist  $\mathbf{S}$ ,  $\mathcal{R}$  and  $\mathbf{A}$  satisfying

$$\mathbf{X} \stackrel{d}{=} \boldsymbol{\mu} + \mathcal{R}\mathbf{A}\mathbf{S},$$

where  $\mathbf{S} \sim U_{\mathbb{S}^{k-1}}$ ,  $\mathcal{R}$  is a radial r.v. independent of  $\mathbf{S}$ , and  $\mathbf{A}$  in  $\mathbb{R}^{n \times k}$  satisfies  $\mathbf{A}\mathbf{A}^\top = \Sigma$ .  $\square$

**Remark B.1** As outlined in McNeil, Frey, and Embrechts (2015, Eqn. (6.41), page 201)), for  $\Sigma$  positive definite,

$$\mathbf{X} \sim E_n(\boldsymbol{\mu}, \Sigma, \psi) \iff \Sigma^{-1/2}(\mathbf{X} - \boldsymbol{\mu}) \sim S_n(\psi). \quad (56)$$

Following McNeil, Frey, and Embrechts (2015, Eqn. (6.42), page 201)), for an elliptical variate  $\mathbf{X} \sim E_n(\boldsymbol{\mu}, \Sigma, \psi)$ , if  $\Sigma$  has full rank  $n$ , then, by McNeil et al. (2015, Corollary 6.22, page 198) and (56),

$$\left( \sqrt{(\mathbf{X} - \boldsymbol{\mu})^\top \Sigma^{-1}(\mathbf{X} - \boldsymbol{\mu})}, \frac{\Sigma^{-1/2}(\mathbf{X} - \boldsymbol{\mu})}{\sqrt{(\mathbf{X} - \boldsymbol{\mu})^\top \Sigma^{-1}(\mathbf{X} - \boldsymbol{\mu})}} \right) \stackrel{d}{=} (\mathcal{R}, \mathbf{S}). \quad \square \quad (57)$$

Block and Sampson (1988, Corollary 2.3) Let  $\mathbf{X} \sim E_n(\boldsymbol{\mu}, \Sigma, \psi)$  and  $h : \mathbb{R}^n \rightarrow \mathbb{R}$  be a supermodular, bounded and right-continuous function. Then  $\mathbb{E}(h(\mathbf{X}))$  is nondecreasing in the off-diagonal elements of  $\Sigma$ .  $\square$

The extension of Block and Sampson (1988, Corollary 2.3) to all supermodular functions follows from Müller and Scarsini (2000, Theorem 3.3 and Theorem 3.4). Hence, if  $\mathbf{X}$ ,

<sup>9</sup>see Fang, Kotz, and Ng (1990) and McNeil, Frey, and Embrechts (2015).

$\mathbf{X}'$  verify Assumption 2.1 under some probability measure  $\mathbb{Q}$ , then  $\mathbf{X} \leq_{sm} \mathbf{X}'$ . As a direct consequence outlined in Shaked and Shanthikumar (1997) after their Definition 2.1, “it follows that the family of multivariate normal distributions (more generally, the family of elliptically contoured distributions) is increasing in the supermodular stochastic order as the correlations increase.”

## C Risk Measures

The following definition of a risk measure relaxes the definition of a convex risk measure in Shapiro (2013, Introduction (A3)) by not requiring the translation-equivariance property.

**Definition C.1** A risk measure is a function  $\rho : \mathfrak{X} \rightarrow \mathbb{R}$  satisfying, for  $\mathcal{X}, \mathcal{Y} \in \mathfrak{X}$ :

- a) properness:  $\rho(\mathcal{X}) > -\infty$  and  $\text{dom } \rho := \{\mathcal{X} \in \mathfrak{X}; \rho(\mathcal{X}) < +\infty\} \neq \emptyset$ ;
- b) law-invariance: if  $\mathcal{X} \stackrel{d}{=} \mathcal{Y}$  then  $\rho(\mathcal{X}) = \rho(\mathcal{Y})$ ;
- c) monotonicity: if  $\mathcal{X} \leq \mathcal{Y}$  then  $\rho(\mathcal{X}) \leq \rho(\mathcal{Y})$ ;
- d) convexity:  $\rho(\lambda\mathcal{X} + (1-\lambda)\mathcal{Y}) \leq \lambda\rho(\mathcal{X}) + (1-\lambda)\rho(\mathcal{Y})$ ,  $\lambda \in (0, 1)$ .

$(-\rho(-\cdot))$  is named an acceptability functional<sup>10</sup>. We denote  $\check{\rho} = \rho(\cdot - \mathbb{E}(\cdot))$ . If  $\rho$  is translation-equivariant i.e.  $\rho(\mathcal{X} + c) = \rho(\mathcal{X}) + c$ ,  $c \in \mathbb{R}$  (see Pflug and Römisch (2007, Definition 2.2 (i), page 29)), then  $\check{\rho}(-\cdot)$  is known as a deviation risk functional<sup>11</sup> and  $\rho(-\cdot)$  a risk capital functional<sup>12</sup>.  $\square$

**Definition C.2** The expected shortfall (ES)<sup>13</sup> at the confidence level (quantile)  $\alpha \in (\frac{1}{2}, 1)$  of a loss  $\mathcal{X} \in \mathfrak{X} = L^1(\mathbb{Q})$  is  $\text{ES}_\alpha(\mathcal{X}) = (1-\alpha)^{-1} \left( \mathbb{E}(\mathcal{X} \mathbf{1}_{\{\mathcal{X} \geq \text{VaR}_\alpha(\mathcal{X})\}}) + \text{VaR}_\alpha(\mathcal{X})(\mathbb{Q}(\mathcal{X} < \text{VaR}_\alpha(\mathcal{X})) - \alpha) \right)$ , with  $\text{VaR}_\alpha(\mathcal{X}) = \inf \{x \in \mathbb{R} : \mathbb{Q}(\mathcal{X} \leq x) > \alpha\}$ .  $\square$

ES is a standing example of a translation-equivariant risk measure.

We conclude this appendix by a detailed proof of Bäuerle and Müller (2006, Theorem 4.4) stated in Section 2.1. The proof relies on the definition of stochastic order:

*Bäuerle and Müller (1998, Definition 2.1a)* For  $\mathcal{X}$  and  $\mathcal{Y}$  in  $\mathfrak{X}$  with respective c.d.f.  $F_\mathcal{X}$  and  $F_\mathcal{Y}$ ,  $\mathcal{X}$  precedes  $\mathcal{Y}$  in stochastic order, written  $\mathcal{X} \leq_{st} \mathcal{Y}$ , if  $F_\mathcal{X}(x) \geq F_\mathcal{Y}(x)$ ,  $x \in \mathbb{R}$ .  $\square$

**Proof of Bäuerle and Müller (2006, Theorem 4.4) (see Section 2.1).** If  $\mathcal{X} \leq_{sl} \mathcal{Y}$ , then, by Müller and Stoyan (2002, Theorem 1.5.14, page 22), there exists a

<sup>10</sup>see Pflug and Römisch (2007, Definition 2.17, pages 35-36).

<sup>11</sup>see Pflug and Römisch (2007, Definition 2.21, page 37).

<sup>12</sup>see Pflug and Römisch (2007, Definition 2.25, pages 38-39).

<sup>13</sup>see Acerbi and Tasche (2002, Definition 2.6).

r.v.  $\mathcal{Z} \in \mathfrak{X}$  s.t.  $\mathcal{X} \leq_{st} \mathcal{Z} \leq_{cx} \mathcal{Y}$ , where  $\leq_{cx}$  is the convex order. By Müller and Stoyan (2002, Theorem 1.2.4, page 3), there exist r.v.  $\mathcal{X}'$  and  $\mathcal{Z}'$  on a modified probability space  $(\Omega', \mathcal{A}', \mathbb{Q}')$ , with same respective laws as  $\mathcal{X}$  and  $\mathcal{Z}$ , such that  $\mathcal{X}' \leq \mathcal{Z}'$  holds with certainty i.e.,  $\forall \omega \in \Omega', \mathcal{X}'(\omega) \leq \mathcal{Z}'(\omega)$  where  $\leq$  is the partial order<sup>14</sup> on  $\mathbb{R}^m$  if  $\mathcal{X}$  and  $\mathcal{Z}$  take value in  $\mathbb{R}^m$ . The law-invariance and monotonicity of  $\rho$  yield

$$\rho(\mathcal{X}) = \rho(\mathcal{X}') \leq \rho(\mathcal{Z}') = \rho(\mathcal{Z}). \quad (58)$$

From Bäuerle and Müller (2006, Theorem 4.3), which requires the convexity and law-invariance of  $\rho$ , we also have  $\rho(\mathcal{Z}) \leq \rho(\mathcal{Y})$ , hence  $\rho(\mathcal{X}) \leq \rho(\mathcal{Y})$ .  $\square$

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<sup>14</sup>see Müller and Stoyan (2002, Remark 1.2.5, page 3).

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