Exponential stabilizability and observability at the target imply semiglobal exponential stabilizability by templated output feedback

Vincent Andrieu, Lucas Brivadis, Jean-Paul Gauthier, Ludovic Sacchelli, Ulysse Serres

Exponential stabilizability and observability at the target imply semiglobal exponential stabilizability by templated output feedback

Vincent Andrieu[a], Lucas Brivadis[b], Jean-Paul Gauthier[c], Ludovic Sacchelli[d], Ulysse Serres[a]

[a] *Univ. Lyon, Université Claude Bernard Lyon 1, CNRS, LAGEPP UMR 5007, 43 bd du 11 novembre 1918, F-69100 Villeurbanne, France (emails: vincent.andrieu@univ-lyon1.fr, ulysse.serres@univ-lyon1.fr)*

[b] *Université Paris-Saclay, CNRS, CentraleSupélec, Laboratoire des Signaux et Systèmes, 91190, Gif-sur-Yvette, France (email: lucas.brivadis@centralesupelec.fr)*

[c] *Université de Toulon, Aix Marseille Univ, CNRS, LIS, France (email: jean-paul.gauthier@univ-tln.fr)*

[d] *Inria, Université Côte d'Azur, CNRS, LJAD, MCTAO team, Sophia Antipolis, France (email: ludovic.sacchelli@inria.fr)*

Abstract

For nonlinear analytic control systems, we introduce a new paradigm for dynamic output feedback stabilization. We propose to periodically sample the usual observer based control law, and to reshape it so that it coincides with a "control template" on each time period. By choosing a control template making the system observable, we prove that this method allows to bypass the uniform observability assumption that is used in most nonlinear separation principles. We prove the genericity of control templates by adapting a universality theorem of Sussmann.

1. Introduction

Dynamic output feedback stabilization of nonlinear systems is a central problem in control theory. The most commonly accepted idea for synthesizing a control law is to split this problem into two subtasks: designing a stabilizing state feedback law, and designing an estimation procedure (an observer) to reconstruct the state from the available measurements. The dynamic output feedback scheme is formed from the combination of these two steps. This approach is used in the seminal articles by Teel and Praly [27, 28] as well as Jouan and Gauthier [15], who obtained so-called nonlinear separation principles. Their results and those they inspired require, with few exceptions discussed below, the uniform observability property:

o1. for any input, any two different states lead to different outputs,

which was proved not to be generic by Gauthier and Kupka in [13] if the output dimension is less than or equal to the input dimension. Therefore, other feedback methods have been developed to overcome non-uniform observability, often allowing time-varying feedbacks. Originally, Coron proposed this approach in [11], achieving local asymptotic stabilization through output feedback for general nonlinear systems using a periodic time-varying strategy. Assuming the target is the origin and an equilibrium of the system for null-input, Coron's results require the following two weaker notions of observability:

o2.i. given two different states, there exists an input leading to different outputs;

o2.ii. for the null-input, if the output vanishes identically, then the state is at the target.

The results from [11] are local, meaning that the basin of attraction for the output feedback loop is limited to a neighborhood of the origin whose size cannot be prefixed. Inspired by this work, Shim and Teel obtained in [24] a semiglobal result by assuming (essentially) that:

o3. there exists an input such that any two different states lead to different outputs.

Their strategy employs two periodically activated modes: in a first mode, a control making the system observable in a sufficiently short time, so that a high-gain observer can be designed to quickly estimate the state; in a second mode, the stabilizing control law based on the observer is applied. Since the input ensuring observability is fixed and applied periodically, only practical convergence is achieved, i.e., within an

arbitrarily small neighborhood of the target. More generally, modes switching hybrid methods have proven to be a powerful tool to solve non-uniform observability issues for presenting specific structures [9, 10, 17].

The first main contribution of this paper, Theorem 5, falls directly in line with the seminal efforts of [11] and [24], but we adopt a different viewpoint. Although our output feedback is periodic and realized through hybrid dynamics, we do not separate the control in two modes: each time step simultaneously allows to observe and control the system. To do so, we propose a *templated* output feedback strategy, which aims to generalize sampled-and-hold output feedback. The combination between classical nonlinear separation principles and digital controllers have been thoroughly investigated in [16], but always in the case of uniformly observable systems. In our work, on each time interval, we sample the observer-based control law value at the beginning of the interval and we modulate it into the shape of a control template, rather than holding it constant.

We pick as control template an input whose any rescaling or isometry makes the system observable, allowing the use of a high observer for an as fast as we want simultaneous state estimation and semiglobal output feedback stabilization. This control strategy is new, and helps in the context of non-uniformly observable systems, which have recently gathered attention because of their prevalence in various modern applications [1, 6, 22, 25]. Because any rescaling of a control template induces observability, a necessary condition for this method is:

o4. for the null input, any two different states lead to different outputs.

Though the existence of a control template may seem restrictive, we prove in Theorem 6, second main contribution of this paper, that it is actually equivalent to $o4$) for analytic systems. Our proof builds upon arguments used by Sussmann in 1979 for the genericity of universal inputs [26]. This result is a significant contribution in its own right that generalizes Sussmann's findings. Obviously, $o1 \Rightarrow o4 \Rightarrow o3 \Rightarrow o2.i$ and $o4 \Rightarrow o2$. Assumption $o4$ naturally excludes the case of systems unobservable at the target (see e.g. [8]), but is nonetheless generic.

Organization of the paper. The rest of the paper is divided into three parts. Section 2 is a general preliminary section that discusses the method we introduce. It contains the definitions and assumptions necessary to the paper and a full presentation of the control strategy. The section ends with the statements of the two main results of the paper Theorem 5 and Theorem 6 whose proofs are the purposes of Sections 3 and 4 respectively.

Notation. Let n, m and p be positive integers. If $N \in \mathbb{N}$, $\mathbb{R}^N$ is endowed with its canonical Euclidean canonical scalar product $\langle \cdot, \cdot \rangle$ and the induced norm $|\cdot|$; $B_{\mathbb{R}^N}(x, r)$ denotes the open ball centered at x of radius r for this norm, and $\mathbb{S}^{N-1}$ the unit sphere. If $\mathcal{I} \subset \mathbb{R}$ is an interval and $\mathcal{X} \subset \mathbb{R}^N$ is open, $PC^0(\mathcal{I}, \mathbb{R}^N)$ denotes the space of càglàd piecewise continuous functions endowed with the uniform norm $\|\cdot\|_\infty$, $C^k(\mathcal{X}, \mathbb{R}^N)$ the space of k-times continuously differentiable maps ($k \in \mathbb{N} \cup \{\infty\}$), and $C^\omega(\mathcal{X}, \mathbb{R}^N)$ the space of analytic functions If $u \in C^k(\mathcal{I}, \mathbb{R}^N)$, $u^{(k)} = \frac{\mathrm{d}^k u}{\mathrm{d}t^k}$ is a shorthand notation for its derivative of order k. For any $h \in C^1(\mathbb{R}^n, \mathbb{R}^N)$, $\frac{\partial h}{\partial x}(x)$ denotes its differential at $x \in \mathbb{R}^n$. Finally, for any set $\mathcal{S}$ in a topological space, $\mathring{\mathcal{S}}$, $\bar{\mathcal{S}}$, and $\partial\mathcal{S}$ denote its interior, closure and boundary, respectively.

2. Templated output feedback

2.1. Definitions and main assumptions

Consider an analytic control system

$$\begin{cases} \dot{x} = f(x, u) \\ y = h(x) \end{cases} \tag{1}$$

where x in $\mathbb{R}^n$ is the state of the system, u in $\mathbb{R}^p$ is the control input, y in $\mathbb{R}^m$ is the measured output and $f : \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}^n$ and $h : \mathbb{R}^n \to \mathbb{R}^m$ are analytic maps. According to the Cauchy-Lipschitz theorem, for any input $u \in PC^0(\mathbb{R}_+, \mathbb{R}^p)$ and any initial condition $x_0 \in \mathbb{R}^n$, the Cauchy problem associated to (1) admits a unique maximal solution that we denote by $t \mapsto X(x_0, t, u)$. When the dependence in x_0 and u does not

need to be emphaised, we simply write $x(t) = X(x_0, t, u)$. Moreover, we denote by $T_e(x_0, u) \in \mathbb{R}_+ \cup \{+\infty\}$ its maximal time of existence. We abuse the notation X to include inputs defined on subintervals of $\mathbb{R}$ containing 0.

Our main objective being to stabilize system (1) by means of a dynamic output feedback, we make the first of three main assumptions of the paper: the existence a priori of a stabilizing feedback.

Assumption 1. *There exists a feedback law $\lambda : \mathbb{R}^n \to \mathbb{R}^p$ that is locally Lipschitz continuous such that* (1) *in closed-loop with $u = \lambda(x)$ is locally exponentially stable (LES) at the origin, with a basin of attraction $\mathcal{X}$.*

With no loss of generality, we assume that $\lambda(0) = 0$ and $h(0) = 0$ in the following. When $\mathcal{X} = \mathbb{R}^n$, the vector field $x \mapsto f(x, \lambda(x))$ is globally asymptotically stable (GAS) and LES at the origin, but not necessarily globally exponentially stable (GES). While the historic results on output feedback stabilisation only require asymptotic stabilization by state feedback, exponential stabilisation of the origin is imposed to cope with the sampling techniques we introduce. This nuance is well explored in [16].

To stabilize (1) by means of a dynamic output feedback, we combine an observer (an online estimation algorithm of the state based on the measurement of the output) with the state-feedback law [3]. This strategy has been extensively studied under assumptions of observability for all inputs (known as uniform observability) [15, 27, 28]. Essentially, uniformly observable systems that are state feedback stabilizable are also semi-globally stabilizable by means of a dynamic output feedback. However, uniform observability assumptions are not generic [13, Chapter 3], so we aim for weaker assumptions.

To properly introduce notions of observability for (1), let us inductively construct the derivatives of the output as mappings of the state and the derivatives of the input. Set $H_0 = h$, and for any positive integer k, let $H_{k+1} : \mathbb{R}^n \times (\mathbb{R}^p)^k \to \mathbb{R}^m$ be such that for all $(x, \sigma) \in \mathbb{R}^n \times (\mathbb{R}^p)^k$,

$$\begin{aligned} H_{k+1}(x, \sigma_0, \sigma_1, \dots, \sigma_k) = &\frac{\partial H_k}{\partial x}(x, \sigma_0, \sigma_1, \dots, \sigma_{k-1}) f(x, \sigma_0) \\ &+ \sum_{i=0}^{k-1} \frac{\partial H_k}{\partial \sigma_i}(x, \sigma_0, \sigma_1, \dots, \sigma_{k-1}) \sigma_{i+1}. \end{aligned}$$

Let $T > 0$ and $k \in \mathbb{N}$. The k-jet of $u \in C^k([0, T], \mathbb{R}^p)$ at $t \in [0, T]$ is the $(k+1)$-tuple $u^{[k]}(t) \in (\mathbb{R}^p)^{k+1}$ given by

$$u^{[k]}(t) = \big(u(t), u^{(1)}(t), \dots, u^{(k)}(t)\big).$$

Define $\mathcal{H}_k : \mathbb{R}^n \times [0, T] \times C^k([0, T], \mathbb{R}^p) \to (\mathbb{R}^m)^{k+1}$ by

$$\mathcal{H}_k(x, t, u) = \Big(H_0(x), H_1\big(x, u(t)\big), \dots, H_k\big(x, u^{[k-1]}(t)\big)\Big).$$

Clearly, $H_j\big(x(t), u^{[j-1]}(t)\big) = \frac{\mathrm{d}^j}{\mathrm{d}t^j} h\big(X(x_0, t, u)\big) = y^{[j]}(t)$ for any $0 < j \leqslant k$, as soon as the solution to (1) initialized at x_0 and with input u is defined up to time t.

Definition 1 (Observability) (see, e.g., [13])**.** *We say that system* (1) *is:*

(i) observable *over $\mathcal{S} \subset \mathbb{R}^n$ in time T for the input $u \in PC^0([0, T], \mathbb{R}^m)$ if, for any different initial conditions $x_a, x_b \in \mathcal{S}$, there exists $t \in (0, T) \cap \big(0, \min\{T_e(x_a, u), T_e(x_b, u)\}\big)$ such that $h(X(x_a, t, u)) \neq h(X(x_b, t, u))$.*

(ii) differentially observable *over $\mathcal{S} \subset \mathbb{R}^n$ of order $k \in \mathbb{N}$ for the input $u \in C^k([0, T], \mathbb{R}^m)$ if $x \mapsto \mathcal{H}_k(x, 0, u)$ is injective over $\mathcal{S}$, and* strongly *differentially observable if moreover it is an immersion.*

Clearly, differential observability implies observability in any positive time. Also, by definition, differential (resp. strong differential) observability of some order k_0 implies differential (resp. strong differential) observability of any order $k \geqslant k_0$. Since f and h are analytic, from which follows analyticity of $x \mapsto \mathcal{H}_k(x, t, u)$ for any $(k, t, u) \in \mathbb{N} \times [0, T] \times C^k([0, T], \mathbb{R}^p)$, we also have the following lemma (whose proof can be found in Appendix) showing that, for analytic inputs, observability implies differential observability on compact sets.

Lemma 2. *If system* (1) *is observable over a compact set* $\mathcal{K} \subset \mathbb{R}^n$ *in some positive time* T *for some input* $u \in C^\omega([0,T],\mathbb{R}^m)$*, then there exists* $k \in \mathbb{N}$ *such that* (1) *is also differentially observable over* $\mathcal{K}$ *of order* k*.*

The proof of Lemma 2 is postponed to Appendix. We now give our other main assumption of the paper, regarding observability for the null-input $u = 0$. It is a twofold assumption, that expresses **o4** from the introduction rigorously.

Assumption 2.i. *The null-input makes system* (1) *observable over* $\mathcal{X}$ *in some time.*

Assumption 2.ii. *For each* $x \in \mathcal{X}$*, there exists* $k \in \mathbb{N}$ *such that* $\mathcal{H}_k(\cdot,0,0)$ *is an immersion at* x*.*

If system (1) is strongly differentially observable over $\mathcal{X}$ of some order for the null-input, then Assumptions 2.i and 2.ii clearly hold. The converse also holds in restriction to compact sets. Since strong differential observability of order $2n+1$ for the null-input is generic by [13, Chapter 4, Theorem 2.2], the weaker Assumptions 2.i and 2.ii are generic, contrarily to uniform observability.

Our aim is to obtain a nonlinear separation principle in line with [15, 27, 28] in the case where uniform observability is replaced by Assumptions 2.i and 2.ii. Doing so, we get a generic nonlinear separation principle. The price to pay is double: (i) in Assumption 1, as discussed previously, the stabilizing state-feedback law is supposed to make the system LES instead of locally asymptotically stable (LAS) as in the uniformly observable case; (ii) we allow the dynamic output feedback to have hybrid dynamics.

2.2. Hybrid dynamic output feedback design

Our strategy relies on the use of particular inputs called *control templates* which we define below.

Definition 3 (Control template)**.** *Let* $\bar\lambda > 0$*. An input* $v \in C^\infty([0,T],\mathbb{R}^p)$ *is called a* control template *of order* $q \in \mathbb{N}$ *on a compact* $\mathcal{K} \subset \mathbb{R}^n$ *if* $v(0) = (1,0,\dots,0)$ *and the map* $x \mapsto \mathcal{H}_q(x,t,\mu\mathsf{R}v)$ *is an injective immersion over* $\mathcal{K}$*, for all* $(t,\mu,\mathsf{R}) \in [0,T]\times[0,\bar\lambda]\times \mathrm{O}(p)$*.*

In Theorem 6, we state that control templates are generic under Assumptions 2.i and 2.ii. Roughly speaking, a control template is an input making the system strongly differentially observable over $\mathcal{K}$, and such that any rescaling (μ) or isometry (R) of this input preserves this observability property. The key property of control templates is given by the following lemma.

Lemma 4. *Let* $v \in C^\infty([0,T],\mathbb{R}^p)$ *be a control template of order* $q \in \mathbb{N}$ *on a compact* $\mathcal{K} \subset \mathbb{R}^n$*. There exists* $\phi \in C^0(\mathbb{R}^{m(q+1)} \times [0,T]\times[0,\bar\lambda]\times \mathrm{O}(p), \mathbb{R}^n)$ *such that for any* $x_0 \in \mathcal{K}$ *and any* $(\mu,\mathsf{R}) \times [0,\bar\lambda]\times \mathrm{O}(p)$ *the solution* (x,y) *to system* (1) *initialized at* x_0 *with input* $u = \mu\mathsf{R}v$ *satisfies*

$$x(t) = \phi\big(y^{[q]}(t), t, \mu, \mathsf{R}\big) \tag{2}$$

for all $t \in [0,T]$ *such that* $x(t) \in \mathcal{K}$*. Moreover, there exists a positive constant* L_ϕ *such that for all* $(z_a, z_b) \in (\mathbb{R}^{m(q+1)})^2$ *and all* $(t,\mu,\mathsf{R}) \in [0,T]\times[0,\bar\lambda]\times \mathrm{O}(p)$*,*

$$|\phi(z_a,t,\mu,\mathsf{R}) - \phi(z_b,t,\mu,\mathsf{R})| \leqslant L_\phi |z_a - z_b|.$$

This lemma is an immediate application of Lemma 22-(iii) to the map $(x,(t,\mu,\mathsf{R})) \mapsto \mathcal{H}_q(x,t,\mu\mathsf{R}v)$, as $\phi\big(y^{[q]}(t),t,\mu,\mathsf{R}\big) = \phi\big(\mathcal{H}_q(x(t),t,\mu\mathsf{R}v),t,\mu,\mathsf{R}\big)$.

We are now ready to design a hybrid dynamic output feedback for system (1) based on control templates. Let $\mathcal{K} \subset \mathcal{X}$ be a compact set, $\Delta > 0$ and $\theta > 0$ be observer parameters to be tuned later on.

Define $\bar\lambda = \max_{\mathcal{K}} |\lambda|$. Now we assume the existence of a control template $v \in C^\infty([0,T],\mathbb{R}^p)$ of order q on $\mathcal{K}$ and we let ϕ be as in Lemma 4. Let also $(\ell_i)_{0\leqslant i\leqslant q} \in \mathbb{R}^{q+1}$ be coefficients of a Hurwitz polynomial.

We wish to be able to select a pair (μ,R) for which $\mu\mathsf{R}v(0)$ matches a specific value, say a nominal $w \in \mathbb{R}^p$. To do so, we define, for each $w \in \mathbb{R}^p$, the nonempty set

$$\mathcal{R}(w) = \{\mathsf{R} \in \mathrm{O}(p) \colon |w|\mathsf{R}(1,0,\dots,0) = w\}.$$

We propose the following hybrid dynamic output feedback:

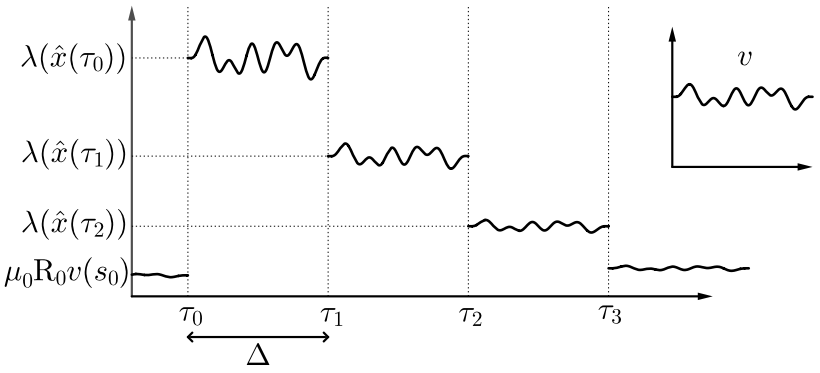

Figure 1: Trajectory of the input $u = \mu \mathsf{R} v$ applied to the system when using the templated output feedback strategy of (3). To lighten the notations, here we write $\hat{x} = \phi(z, \Delta, \mu, \mathsf{R})$. The illustration corresponds to the case of a one-dimensional input. After each jump, $u(\tau_i) = \lambda(\hat{x}(\tau_i))$. Then, over each time-interval, the input follows the shape of the control template v, rescaled by $|\lambda(\hat{x}(\tau_i))|$.

$$
\begin{cases}
\left.\begin{aligned}
&\dot{s} = 1, \quad \dot{\mu} = 0, \quad \dot{\mathsf{R}} = 0 \\
&\dot{x} = f(x, \mu \mathsf{R} v(s)) \\
&\dot{z}_0 = z_1 + \theta \ell_0 (h(x) - z_0) \\
&\dot{z}_1 = z_2 + \theta^2 \ell_1 (h(x) - z_0) \\
&\quad \vdots \\
&\dot{z}_{q-1} = z_q + \theta^q \ell_{q-1} (h(x) - z_0) \\
&\dot{z}_q = H_{q+1}\big(\phi(z, s, \mu, \mathsf{R}), \mu \mathsf{R} v^{[q]}(s)\big) \\
&\qquad\qquad + \theta^{q+1} \ell_q (h(x) - z_0)
\end{aligned}\right\} \; s \in [0, \Delta], \\
\left.\begin{aligned}
&s^+ = 0, \quad \mu^+ = \big|\lambda \circ \phi(z, \Delta, \mu, \mathsf{R})\big| \\
&\mathsf{R}^+ \in \mathcal{R}\big(\lambda \circ \phi(z, \Delta, \mu, \mathsf{R})\big) \\
&x^+ = x \\
&z^+ = \mathcal{H}_q\big(\phi(z, \Delta, \mu, \mathsf{R}), 0, \mu^+ \mathsf{R}^+ v\big)
\end{aligned}\right\} \; s = \Delta.
\end{cases}
\tag{3}
$$

Let us explain the above output feedback dynamics. The jump times are periodically triggered each time the timer s reached Δ. Over each interval of length Δ, the control law applied to the system is $u := \mu \mathsf{R} v$, making the system observable due to the definition of control templates. We employ a high-gain observer during the flow (z variable) to estimate the state thanks to the knowledge of the output $h(x)$ and input u. As usual with high-gain observers, the design suppose the knowledge of the map ϕ, whose existence is guaranteed by Lemma 4 but that can be difficult to compute in practice. Since ϕ satisfies (2) and $z(t)$ approaches $y^{[q]}(t)$, the state estimation is given by

$$
\hat{x} = \phi(z, \Delta, \mu, \mathsf{R}). \tag{4}
$$

At each jump, the scaling parameter μ and the isometry R are updated, such that at the beginning of each time period, $\mu \mathsf{R} v(0) = \lambda(\hat{x})$. Since Δ is to be chosen small enough, this guarantees that u remains close to $\lambda(\hat{x})$. Note that the z dynamics also jump, because the immersion $\mathcal{H}_q$ depends on the input u, which jumps from $\mu \mathsf{R} v(\Delta)$ to $\mu^+ \mathsf{R}^+ v(0)$. Figure 1 illustrates the trajectory of the control law $\mu \mathsf{R} v$ during the stabilization

procedure. Remark that the amplitude of the input over each time interval is proportional to its value at the beginning of the interval: this is the role of the scaling parameter μ, and it is crucial to guarantee that $u \to 0$ when $\hat{x} \to 0$.

Note that in the case where v is constant, the above closed-loop simply consists in a sample-and-hold of the dynamic output feedback based on the usual high-gain observer. Indeed, since $v(0) = (1, 0, \dots, 0)$ and due to the definition of $\mathcal{R}$, the input $u := \mu \mathsf{R} v$ is piecewise constant.

2.3. Main results

Given the existence of a control template, our first result is that the output feedback strategy proposed above successfully achieves semiglobal dynamic output feedback stabilization within the basin of attraction $\mathcal{X}$ under the assumptions of exponential stabilizability. Here, semiglobal stabilization is to be understood in the sense that for any arbitrarily large compact set of initial conditions $\mathcal{K}_0 \subset \mathcal{X}$, we can tune the hybrid dynamic output feedback to ensure convergence towards the target.

Theorem 5 (Output feedback stabilization theorem)**.** *Suppose Assumption 1 holds. Let $\mathcal{K}_0$ be a compact subset of $\mathcal{X}$. Then there exists a compact $\mathcal{K} \subset \mathcal{X}$, with $\mathcal{K}_0 \subset \mathring{\mathcal{K}}$ such that for any control template $v \in C^\infty([0,T], \mathbb{R}^p)$ on $\mathcal{K}$, there exist $\Delta^* \in (0, T]$ such that for any $\Delta \in (0, \Delta^*)$, any compact $\mathcal{Z}_0 \subset (\mathbb{R}^m)^{q+1}$, there exists $\theta^* \geqslant 1$ such that for all $\theta > \theta^*$, the set $\{0\} \times \{0\} \times [0, \Delta] \times \{0\} \times \mathrm{O}(p)$ is LES for system (3), with basin of attraction containing $\mathcal{K}_0 \times \mathcal{Z}_0 \times [0, \Delta] \times [0, \bar{\lambda}] \times \mathrm{O}(p)$.*

The proof of Theorem 5 is the goal of Section 3. In the case where the control template is constant, this result shows that sampling-and-holding the control periodically over small time intervals is sufficient to achieve a nonlinear separation principle. We exploit the same high-gain observer based strategy than the literature [15, 27, 28] that used to require uniform observability, whereas we only rely on observability at the target.

Because control templates are not necessarily constant (for example, almost all bilinear systems admit constant inputs making unobservable, as roots of the characteristic polynomial of the observability matrix [7, Theorem 2.6]), our method proposes a generalization of the sample-and-hold strategy.

After sampling the control (the input is computed at each sampling time by composing the stabilizing feedback law with the observer), we propose, instead of "holding" it constant during a time Δ, to follow the shape of the control template v from this starting point (up to a dilation μ and an isometry R that preserve observability).

The natural remaining question is the existence of control templates, and how generic they are. We propose to show their genericity with the following second main result, that is an extension of Sussmann's universality theorem [26].

Theorem 6 (Universality theorem)**.** *Suppose that Assumptions 2.i and 2.ii are satisfied. Let $\mathcal{K}$ be a compact subset of $\mathcal{X}$. Let $\bar{T} > 0$ and let $\mathcal{U}_\mathcal{K}$ be the set of inputs v in $C^\infty([0, \bar{T}], \mathbb{R}^p)$ (endowed with the compact-open topology) for which there exist $T \in (0, \bar{T}]$, q integer, such that the map $x \mapsto \mathcal{H}_q(x, t, \mu \mathsf{R} v)$ is an injective immersion over $\mathcal{K}$, for all $t \in [0, T]$, all $\mu \in [0, 1]$ and all $\mathsf{R} \in \mathrm{O}(p)$. Then*

(i) $\mathcal{U}_\mathcal{K}$ contains a countable intersection of open and dense subsets of $C^\infty([0, \bar{T}], \mathbb{R}^p)$;

(ii) the restriction of $\mathcal{U}_\mathcal{K}$ to analytic inputs, $\mathcal{U}_\mathcal{K} \cap C^\omega([0, \bar{T}], \mathbb{R}^p)$, is dense in $C^\infty([0, \bar{T}], \mathbb{R}^p)$.

The proof of Theorem 6 is the goal of Section 4. It is an extension of the universality theorem of Sussmann [26]. In short, we use the stronger assumption that the null-input makes the system strongly differentially observable on $\mathcal{K}$ (while [26] relies on differential observability only), under which we prove: (i) the immersion property in addition to the injectivity; (ii) uniformity of the genericity with respect to parameters lying in a compact set. The choice of compact-open topology for this result is completely natural and follows from [26]. Notions on this topology will be recalled when necessary. See also [12, Chapter 3] for a general reference on the topic.

With Theorem 6, control templates can be obtained in the following manner. Let v_{ref} be the constant input equal to $(\bar{\lambda} + 1, 0, \dots, 0)$. By genericity of $\mathcal{U}_\mathcal{K}$, arbitrarily choose $w \in \mathcal{U}_\mathcal{K}$ such that $\|w - v_{\mathrm{ref}}\|_\infty < 1$.

Then $|w(0)| > \bar{\lambda}$. Pick $\mathsf{R}_{\mathrm{ref}} \in \mathcal{R}(w(0))$ and set $v = \mathsf{R}_{\mathrm{ref}}^{-1} w|_{[0,T]}/|w(0)|$. Then $v(0) = (1, 0, \dots, 0)$, and by definition of $\mathcal{U}_{\mathcal{K}}$, v is a control template. Note that, as any genericity result based on transversality theory, Theorem 6 does not propose an explicit construction of the control template, but rather states that almost any choice of template must be good. In applications, one can either apply this reasoning and choose an arbitrary template, or propose an ad-hoc analysis on the system to construct a specific control template and apply Theorem 5 with this template.

By combining Theorems 5 and 6, we therefore obtain a nonlinear separation principle based on the use of control templates, under the assumptions of exponential stabilizability and a generic observability assumption.

3. Output feedback stabilization theorem

This section is devoted to the proof of Theorem 5 which we prove showing that there exist $\gamma_\star, \omega_\star > 0$ such that, for any $(x_0, z_0, s_0, \mu_0, \mathsf{R}_0) \in \mathcal{K}_0 \times \mathcal{Z}_0 \times [0, \Delta] \times [0, \bar{\lambda}] \times \mathrm{O}(p)$, the corresponding solution of (3) is complete and satisfies for all $t \geqslant 0$

$$\|(x, z, \mu)(t)\| \leqslant \gamma_\star \, \mathrm{e}^{-\omega_\star t} \, \|(x_0, z_0, \mu_0)\|.$$

First we ensure well-posedness of the closed-loop hybrid system (3) and provide a preliminary result concerning the high-gain observer convergence. Then, we prove the output feedback stabilization result.

3.1. Well-posedness

We use the framework of hybrid systems developed in [14] to define solutions of (3). Note that (3) clearly satisfies the hybrid basic conditions [14, Assumption 6.5]. Moreover, the jump times of (3) are determined by the autonomous decoupled hybrid subdynamics of s.

Consequently, the jump times $(\tau_i)_{i \in \mathbb{N}}$ are given by $\tau_i = \Delta - s_0 + i\Delta$ for $i \in \mathbb{N}$. Thus, any solution $(x, z, s, \mu, \mathsf{R}) : E \to \mathbb{R}^n \times \mathbb{R}^{m(q+1)} \times [0, \Delta] \times \mathbb{R}_+ \times \mathrm{O}(p)$ to (3) is a hybrid arc defined on a hybrid time domain $E \subset \mathbb{R}_+ \times \mathbb{N}$ of the form $E = \bigcup_{i=-1}^{I-1} [0, T_e) \cap [\tau_i, \tau_{i+1}] \times \{i\}$ (with $\tau_{-1} := 0$) where either $T_e = +\infty$ and $I = +\infty$ (complete trajectory) or $T_e \in \mathbb{R}_+^*$, $I \in \mathbb{N}^*$, and $\tau_{I-1} < T_e \leqslant \tau_I$ (non-complete trajectory). Since the flow map of (3) is singled-valued and locally Lipschitz continuous, the Cauchy problem associated to the flow map admits a unique maximal solution of class C^1. Thus, for any parameters $\Delta > 0$, $(\ell_i)_{0 \leqslant i \leqslant q} \in \mathbb{R}^{q+1}$ and $\theta > 0$, and for eachcolorred/every initial condition $(x_0, z_0, s_0, \mu_0, \mathsf{R}_0) \in \mathbb{R}^n \times \mathbb{R}^{m(q+1)} \times [0, \Delta] \times \mathbb{R}_+ \times \mathrm{O}(p)$, (3) admits a unique maximal solution $(x, z, s, \mu, \mathsf{R}) : E \to \mathbb{R}^n \times \mathbb{R}^{m(q+1)} \times [0, \Delta] \times \mathbb{R}_+ \times \mathrm{O}(p)$ such that $t \mapsto (x, z, s, \mu, \mathsf{R})(t, i)$ is C^1 for each i. We denote by $T_e \in \mathbb{R}_+ \cup \{+\infty\}$ the time of existence of such solutions. Moreover, if $(x, z, s, \mu, \mathsf{R})$ remains bounded, then the trajectory is complete, i.e. $E = \bigcup_{i \geqslant -1} [\tau_i, \tau_{i+1}] \times \{i\}$ and $T_e = \infty$.

In order to shorten the notations and enhance the readibility, we write $(x_t, z_t, s_t, \mu_t, \mathsf{R}_t) := (x, z, s, \mu, \mathsf{R})(t, i)$ for all $(t, i) \in E$ such that $\tau_i < t \leqslant \tau_{i+1}$ and $(x_{\tau_i}^+, z_{\tau_i}^+, s_{\tau_i}^+, \mu_{\tau_i}^+, \mathsf{R}_{\tau_i}^+) := (x, z, s, \mu, \mathsf{R})(\tau_i, i)$.

3.2. Convergence of the estimation error

Before we start, let us give $\mathcal{K}$. According to the converse Lyapunov function theorem [20, Théorème 2.348 and Remarque 2.350], there exists a proper function $V \in C^\infty(\mathcal{X}, \mathbb{R}_+)$ such that for any compact set $\mathcal{L} \subset \mathcal{X}$, three positive constants $(c_i)_{1 \leqslant i \leqslant 3}$ exist, satisfying

$$\begin{aligned} &\frac{\partial V}{\partial x}(\xi) f\big(\xi, \lambda(\xi)\big) \leqslant -c_1 |\xi|^2, \\ &|\xi|^2 \leqslant V(\xi) \leqslant c_2 |\xi|^2, \qquad \left| \frac{\partial V}{\partial x}(\xi) \right| \leqslant c_3 |\xi|. \end{aligned} \tag{5}$$

for all $\xi \in \mathcal{L}$. For any $R \geqslant 0$, $\mathcal{D}(R) := \{x \in \mathbb{R}^n : V(x) \leqslant R\}$ is compact because V is proper, and $\mathcal{D}(R) \subset \mathring{\mathcal{D}}(R')$ as soon as $R' > R$. Now, let $R_2 > R_1 > R_0 > 0$ be such that $\mathcal{K}_0 \subset \mathcal{D}(R_0)$, and choose $\mathcal{L} = \mathcal{K} = \mathcal{D}(R_2)$, which fixes the c_i's in (5) as well as $\bar{\lambda}$, ϕ and L_ϕ from Section 2.2.

Let us introduce once and for all the different constants that appear in the subsequent proofs. Recall that for a given maximal solution, T_e denotes its time of existence. Let $t^* = \inf\big\{t \in [0, T_e)\colon x_t \notin \mathcal{D}(R_2)\big\}$. Finally, fix Δ_0 such that the solution to (3) satisfies $x_t \in \mathcal{D}(R_1)$ for all $(x_0, z_0, s_0, \mu_0, \mathsf{R}_0) \in \mathcal{K}_0 \times (\mathbb{R}^m)^{q+1} \times [0, 3\Delta_0] \times [0, \bar{\lambda}] \times \mathrm{O}(p)$. Such a Δ_0 always exists since the distance between $\mathcal{K}_0$ and $\partial\mathcal{D}(R_1)$ is strictly positive, and f with inputs smaller than $\bar{\lambda}$ is bounded on $\mathcal{D}(R_2)$.

We can now investigate the exponential stability of the estimation error defined, for all $t < T_e$, by

$$\varepsilon_t = z_t - \mathcal{H}_q(x_t, s_t, \mu_t \mathsf{R}_t v). \tag{6}$$

In view of the definition of control templates in combination with Lemma 4, the following lemma follows directly from the usual results on high-gain observers.

Lemma 7. *Suppose assumptions of Theorem 5 hold. Let $\Delta \in (0, \Delta_0]$ Then, according to [5, Theorem 4.1] (or [13, Chapter 6.2]), there exist $\theta_1^*, \gamma_1 \geqslant 1$ and $\omega_1 > 0$ such that for all $\theta > \theta_1^*$, the solution of (3) satisfies for any $i \geqslant 0$, $t \in (\tau_i, \tau_{i+1}] \cap [0, t^*)$,*

$$|\varepsilon_t| \leqslant \gamma_1 \theta^q \, \mathrm{e}^{-\theta\omega_1(t-\tau_i)} \, |\varepsilon_{\tau_i}^+|. \tag{7}$$

The above lemma gives an estimate of the error over a single window $[\tau_i, \tau_{i+1})$. Now we give the result over $[0, t^*)$. Let $L_{\mathcal{H}_q}$ denote the Lipschitz constant of $(x, s, \mu, \mathsf{R}) \mapsto \mathcal{H}_q(x, s, \mu \mathsf{R} v(s))$ over $\mathcal{D}(R_2) \times [0, \Delta] \times [0, \bar{\lambda}] \times \mathrm{O}(p)$, L_f the Lipschitz constant of f over $\mathcal{D}(R_2) \times B_{\mathbb{R}^p}(0, \bar{\lambda}\|v\|_\infty)$, and L_λ the Lipschitz constant of λ over $\mathcal{D}(R_2)$.

Lemma 8. *Let $\Delta \in [0, \Delta_0]$. Under the assumptions of Theorem 5, there exists $\theta_2^* \geqslant 1$ such that for any $\theta \geqslant \theta_2^*$, there exist $\gamma_{\mathcal{E}}(\theta, \Delta) \geqslant 1$ and $\omega_{\mathcal{E}}(\theta) > 0$ satisfying*

$$|\varepsilon_t| \leqslant \sqrt{\gamma_{\mathcal{E}}(\theta, \Delta)} \, \mathrm{e}^{-\frac{\omega_{\mathcal{E}}(\theta)}{2} t} \, |(x_0, z_0, \mu_0)|, \tag{8}$$

for all $t \in [0, t^*)$. Moreover, for any $t > \Delta$,

$$\lim_{\theta \to +\infty} \omega_{\mathcal{E}}(\theta) = +\infty \text{ and } \lim_{\theta \to +\infty} \gamma_{\mathcal{E}}(\theta, \Delta) \, \mathrm{e}^{-\omega_{\mathcal{E}}(\theta) t} = 0. \tag{9}$$

Proof. For any $i \geqslant 0$ such that $\tau_i < t^*$, we have $s_{\tau_i}^+ = 0$. Consequently, according system (3) and Definition (6)

$$\begin{aligned}
\big|\varepsilon_{\tau_i}^+\big| &= \big|z_{\tau_i}^+ - \mathcal{H}_q(x_{\tau_i}^+, s_{\tau_i}^+, \mu_{\tau_i}^+ \mathsf{R}_{\tau_i}^+ v)\big| \\
&= \big|z_{\tau_i}^+ - \mathcal{H}_q(x_{\tau_i}, 0, \mu_{\tau_i}^+ \mathsf{R}_{\tau_i}^+ v)\big| \\
&= \big|\mathcal{H}_q\big(\phi(z_{\tau_i}, \Delta, \mu_{\tau_i}, \mathsf{R}_{\tau_i}), 0, \mu_{\tau_i}^+ \mathsf{R}_{\tau_i}^+ v\big) - \mathcal{H}_q(x_{\tau_i}, 0, \mu_{\tau_i}^+ \mathsf{R}_{\tau_i}^+ v)\big| \\
&\leqslant L_{\mathcal{H}_q} \big|\phi(z_{\tau_i}, \Delta, \mu_{\tau_i}, \mathsf{R}_{\tau_i}) - x_{\tau_i}\big|.
\end{aligned}$$

Moreover, since $x_{\tau_i} = \phi \circ \mathcal{H}_q(x_{\tau_i}, \Delta, \mu_{\tau_i} \mathsf{R}_{\tau_i} v)$, setting $L = \max\{L_{\mathcal{H}_q} L_\phi, 1\}$, we get

$$|\varepsilon_{\tau_i}^+| \leqslant L_{\mathcal{H}_q} L_\phi |z_{\tau_i} - \mathcal{H}_q(x_{\tau_i}, \Delta, \mu_{\tau_i} \mathsf{R}_{\tau_i} v)| \leqslant L |\varepsilon_{\tau_i}|. \tag{10}$$

On the other hand, if $\theta \geqslant \theta_1^*$ (with θ_1 as in Lemma 7), according to (7), for any $i \in \{-1, \dots, I-1\}$, we get

$$|\varepsilon_t| \leqslant \gamma_1 \theta^q \, \mathrm{e}^{-\theta\omega_1(t-\tau_i)} \, |\varepsilon_{\tau_i}^+|, \quad \forall t \in (\tau_i, \tau_{i+1}] \cap (0, t^*). \tag{11}$$

In particular, (10) and (11) yield for all $i \geqslant 1$,

$$|\varepsilon_{\tau_i}| \leqslant \gamma_1 L \theta^q \, \mathrm{e}^{-\theta\omega_1 \Delta} \, |\varepsilon_{\tau_{i-1}}|. \tag{12}$$

Let $t \in (0, t^*)$. There exists $i \geqslant -1$ such that $\tau_i < t \leqslant \tau_{i+1}$. With (11) and (12), an immediate induction yields

$$|\varepsilon_t| \leqslant \gamma_1 L \theta^q \, \mathrm{e}^{-\theta\omega_1(t-\tau_i)} \left(\gamma_1 L \theta^q \, \mathrm{e}^{-\theta\omega_1 \Delta}\right)^i |\varepsilon_{\tau_0}| \tag{13}$$

Now let $\theta_2^*(\Delta) \geqslant \theta_1^*$ be such $\mathrm{e}^{\frac{\theta}{2}\omega_1\Delta} \geqslant \gamma_1 L\theta^q$ for all $\theta \geqslant \theta_2^*$. Then, having in mind that $\tau_i = \tau_0 + i\Delta$, inequality (13) yields for all $t \in (0, t^*)$ and all $\theta \geqslant \theta_2^*$

$$|\varepsilon_t| \leqslant \mathrm{e}^{\frac{\theta}{2}\omega_1\Delta}\,\mathrm{e}^{-\frac{\theta}{2}\omega_1(t-\tau_0)}\,|\varepsilon_{\tau_0}|.$$

If $\tau_0 = 0$, we are done. Otherwise, since $\varepsilon^+_{\tau_{-1}} = \varepsilon_0$ and $L \geqslant 1$, (11) yields $|\varepsilon_{\tau_0}| \leqslant \gamma_1 L\theta^q\,\mathrm{e}^{-\theta\omega_1\tau_0}\,|\varepsilon_0|$. In any case,

$$\begin{aligned}|\varepsilon_t| &\leqslant \gamma_1 L\theta^q\,\mathrm{e}^{\frac{\theta}{2}\omega_1\Delta}\,\mathrm{e}^{-\frac{\theta}{2}\omega_1 t}\,|\varepsilon_0| \\ &\leqslant \mathrm{e}^{\theta\omega_1\Delta}\,\mathrm{e}^{-\frac{\theta}{2}\omega_1 t}\,|\varepsilon_0|. \end{aligned} \tag{14}$$

Finally, using $\mathcal{H}_q(0, s_0, 0) = 0$, we get by triangular inequality

$$\begin{aligned}|\varepsilon_0| &\leqslant |z_0| + |\mathcal{H}_q(x_0, s_0, \mu_0\mathsf{R}_0 v) - \mathcal{H}_q(0, s_0, 0)| \\ &\leqslant |z_0| + L_{\mathcal{H}_q}(|x_0| + \|v\|_\infty|\mu_0|),\end{aligned}$$

which, together with (14) proves that ε remains bounded over $[0, t^*)$, that (8) is satisfied for a suitable choice of $(\gamma_{\mathcal{E}}(\theta, \Delta), \omega_{\mathcal{E}}(\theta))$, and that (9) holds. ■

3.3. Stabilization of the system at the target

Before investigating the exponential stability of the system's state, we provide a natural intermediary result on templated output feedback. Namely, after time τ_0, we can bound the gap between the state feedback and templated output feedback at for all time in the window $[\tau_i, \tau_{i+1})$ by reducing Δ. We also wish to relate this bound to the size of the state, in order to relate it to the Lyapunov function V. What is crucial here is that rather than keeping memory of $|x_{\tau_i}|$, we can actually recover $|x_t|$. Some elements of the proof being reminiscent (or similar) of the proof of Lemma (8), so they may be exposed more succinctly.

Lemma 9. *Suppose assumptions of Theorem 5 hold. Then there exist $\Delta^*_{\mathrm{gap}} \in (0, \Delta_0]$, $\beta \in \mathbb{R}_+$ and a class $\mathcal{K}$ function α such that for any $\Delta \in (0, \Delta^*_{\mathrm{gap}})$ and any $i \geqslant 0$, $t \in (\tau_i, \tau_{i+1}] \cap [0, t^*)$,*

$$|\mu_t\mathsf{R}_t v(s_t) - \lambda(x_t)| \leqslant \alpha(\Delta)|x_t| + \beta|\varepsilon_{\tau_i}|. \tag{15}$$

Proof. We only sketch the main ideas which we divide in the four following steps.

Step 1: We show that there exist $\beta_1 \in \mathbb{R}_+$ and a class $\mathcal{K}$ function α_1 such that

$$\big|\mu_t\mathsf{R}_t v(s_t) - \lambda(x_t)\big| \leqslant \beta_1|\varepsilon_{\tau_i}| + \beta_1 \sup_{\sigma\in[\tau_i,t]} |x_t - x_\sigma| + \alpha_1(\Delta)|x_t|. \tag{16}$$

Indeed, for any $\tilde{\mathsf{R}}_t \in \mathcal{R}\big(\lambda(x_t)\big)$, [1] and any $\tilde{\mathsf{R}}_i \in \mathrm{O}(p)$, we have (since $\mu_t = \mu_{\tau_i} = |\lambda(\hat{x}_{\tau_i})|$, with $\hat{x}$ defined by (4))

$$\begin{aligned}\big|\mu_t\mathsf{R}_t v(s_t) - \lambda(x_t)\big| &= \big|\mu_t\mathsf{R}_t v(s_t) - |\lambda(x_t)|\tilde{\mathsf{R}}_t v(0)\big| \\ &\leqslant \big||\lambda(\hat{x}_{\tau_i})|\mathsf{R}_t v(s_t) - |\lambda(x_{\tau_i})|\tilde{\mathsf{R}}_i v(s_t)\big| \\ &\quad + \big||\lambda(x_{\tau_i})|\tilde{\mathsf{R}}_i v(s_t) - |\lambda(x_t)|\tilde{\mathsf{R}}_t v(s_t)\big| \\ &\quad + \big||\lambda(x_t)|\tilde{\mathsf{R}}_t v(s_t) - |\lambda(x_t)|\tilde{\mathsf{R}}_t v(0)\big|. \end{aligned} \tag{17}$$

Using Lemma 23, we choose $\tilde{\mathsf{R}}_i \in \mathcal{R}(\lambda(x_{\tau_i}))$ such that

$$\big\|\mu_t\mathsf{R}_t - |\lambda(x_{\tau_i})|\tilde{\mathsf{R}}_i\big\| \leqslant \big|\lambda(\hat{x}_{\tau_i}) - \lambda(x_{\tau_i})\big|$$

[1] Recall that $\tilde{\mathsf{R}}_t \in \mathcal{R}\big(\lambda(x_t)\big) \iff \lambda(x_t) = |\lambda(x_t)|\tilde{\mathsf{R}}_t v(0)$.

and, $\tilde{\mathsf{R}}_t \in \mathcal{R}(\lambda(x_t))$, so that

$$\big\||\lambda(x_{\tau_i})|\tilde{\mathsf{R}}_i - |\lambda(x_t)|\tilde{\mathsf{R}}_t\big\| \leqslant \big|\lambda(x_{\tau_i}) - \lambda(x_t)\big|.$$

Inequality (16) is then obtained bounding from above each member of the right handside of (17).

Step 2: Using (16) and Grönwall's inequality (performing computations similar to those in [16, Appendix D]) there exists a class $\mathcal{K}$ function α_2 such that

$$\sup_{\sigma\in[\tau_i,t]} |x_t - x_\sigma| \leqslant \alpha_2(\Delta)\bigg(|\varepsilon_{\tau_i}| + \sup_{\sigma\in[\tau_i,t]} |x_\sigma|\bigg). \tag{18}$$

Step 3: Inequalities $|x_\sigma| \leqslant |x_t| + |x_t - x_\sigma|$ and (18) yield $\sup_{\sigma\in[\tau_i,t]}|x_\sigma| \leqslant |x_t| + \alpha_2(\Delta)\big(|\varepsilon_{\tau_i}| + \sup_{\sigma\in[\tau_i,t]}|x_\sigma|\big)$, implying that, there exists $\beta_2 \in \mathbb{R}_+$ such that for all $\Delta \leqslant \Delta^*_{\text{gap}}$ (with Δ^*_{gap} satisfying $\alpha_2(\Delta^*_{\text{gap}}) \leqslant 1$)

$$\sup_{\sigma\in[\tau_i,t]} |x_\sigma| \leqslant \beta_2\left(|x_t| + |\varepsilon_{\tau_i}|\right). \tag{19}$$

Step 4: Finally, inequality (15) follows after substitutying $\sup_{\sigma\in[\tau_i,t]}|x_t - x_\sigma|$ in (16) with its bound obtained in (18) and $\sup_{\sigma\in[\tau_i,t]}|x_\sigma|$ with its bound obtained in (19). ■

Now that the gap in control is bounded, we can prove exponential stability of the state.

Lemma 10. *Suppose assumptions of Theorem 5 hold. Then there exists $\Delta^* \in (0, \Delta_0]$ such that for any $\Delta \in (0, \Delta^*)$, there exists $\theta^* \geqslant 1$, such that for any $\theta > \theta^*$, there exist $\gamma_{\mathcal{X}}(\theta,\Delta) \geqslant 1$ and $\omega_{\mathcal{X}}(\theta,\Delta) > 0$ such that*

$$\forall t \in [0, t^*), \quad |x_t| \leqslant \gamma_{\mathcal{X}}\, \mathrm{e}^{-\omega_{\mathcal{X}} t}\, |(x_0, z_0, \mu_0)|. \tag{20}$$

Proof. For any $i \geqslant 0$ and any $t \in (\tau_i, \tau_{i+1}] \cap [0, t^*)$, we get with (5), Lemma 9 and Young's inequality

$$\begin{aligned}
\frac{\mathrm{d}V(x_t)}{\mathrm{d}t} + c_1|x_t|^2 &\leqslant \frac{\partial V}{\partial x}(x_t)\big|f\big(x_t, \mu\mathsf{R}v(s_t)\big) - f\big(x_t, \lambda(x_t)\big)\big| \\
&\leqslant c_3 L_f |x_t|\big|\mu_t \mathsf{R}_t v(s_t) - \lambda(x_t)\big| \\
&\leqslant c_3 L_f \left(\alpha(\Delta)|x_t|^2 + \beta|x_t||\varepsilon_{\tau_i}|\right) \\
&\leqslant c_3 L_f \left[\alpha(\Delta)|x_t|^2 + \beta\left(\frac{\delta|x_t|^2}{2} + \frac{|\varepsilon_{\tau_i}|^2}{2\delta}\right)\right].
\end{aligned} \tag{21}$$

Pick $\delta > 0$ and $\Delta^* \in [0, \Delta^*_{\text{gap}}]$ (with Δ^*_{gap} as in Lemma 9) such that $c_3 L_f\big(\alpha(\Delta^*) + \delta\beta\big) \leqslant c_1$. Then, with inequalities (8) and (5), inequality (21) yields for all $\Delta < \Delta^*$

$$\frac{\mathrm{d}V(x_t)}{\mathrm{d}t} \leqslant -\omega_V(\Delta)V(x_t) + \beta'\gamma_{\mathcal{E}}\, \mathrm{e}^{-\omega_{\mathcal{E}} t}\, |(x_0, z_0, \mu_0)|^2, \tag{22}$$

where $\omega_V(\Delta) = \big[c_1 - c_3 L_f\big(\alpha(\Delta) + \delta\beta\big)/2\big]/c_2$ and $\beta' = c_3 L_f \beta/\delta$. Let $\theta_3^* \geqslant \theta_2^*$ (with θ_2^* as in Lemma 8) be such that $\omega_{\mathcal{E}}(\theta_3^*, \Delta) \geqslant \omega_V(\Delta)$. Then, applying Grönwall inequality to (22) yields, for all $\theta > \theta_3^*$ and all $t \geqslant 2\Delta$,

$$\begin{aligned}
V(x_t) &\leqslant \mathrm{e}^{-\omega_V(t-2\Delta)}\, V(x_{2\Delta}) \\
&\quad + \frac{\beta'}{2}\gamma_{\mathcal{E}}\, \mathrm{e}^{-\omega_{\mathcal{E}} 2\Delta}\, \frac{\mathrm{e}^{-\omega_V(t-2\Delta)} - \mathrm{e}^{-\omega_{\mathcal{E}}(t-2\Delta)}}{\omega_{\mathcal{E}} - \omega_V} |(x_0, z_0, \mu_0)|^2 \\
&\leqslant R_1 + \beta' \frac{\gamma_{\mathcal{E}}\, \mathrm{e}^{-2\Delta\omega_{\mathcal{E}}}}{\omega_{\mathcal{E}} - \omega_V} |(x_0, z_0, \mu_0)|^2,
\end{aligned} \tag{23}$$

the latter following from our choice of Δ_0. Using (9), we pick $\theta^* \geqslant \theta_3^*$ so that for all $\theta > \theta^*$,

$$R_1 + \beta' \frac{\gamma_{\mathcal{E}}(\theta,\Delta)\, \mathrm{e}^{-2\Delta\omega_{\mathcal{E}}(\theta)}}{\omega_{\mathcal{E}}(\theta) - \omega_V} \sup_{\mathcal{K}_0\times\mathcal{Z}_0\times[0,\bar{\lambda}]} |(x_0, z_0, \mu_0)|^2 \leqslant R_2,$$

which, together with (23), implies $t^* = T_e$ and (20) for a suitable choice of $\gamma_{\mathcal{X}}$ and $\omega_{\mathcal{X}}$. ■

With Lemmas 8 and 10, we are now able to conclude the proof of Theorem 5.

Proof of Theorem 5. Suppose $\Delta \in (0, \Delta^*)$ and $\theta > \theta^*$ (with Δ^* and θ^* as in Lemma 10). Clearly, $(s_t, \mu_t, \mathsf{R}_t)$ remains bounded in $[0, \Delta] \times [0, \bar{\lambda}] \times \mathrm{O}(p)$. Because, according to Lemmas 8 and 10, ε_t and x_t were, so is z_t. Thus $T_e = +\infty$. Moreover,

$$\begin{aligned}\mu_t &\leqslant \big|\lambda\big(\phi(z_t, \Delta, \mu_t, \mathsf{R}_t)\big) - \lambda(x_t)\big| + \big|\lambda(x_t)\big| \\ &\leqslant L_\lambda L_\phi |z_t - \mathcal{H}_q(x_t, \Delta, \mu_t \mathsf{R}_t v)| + L_\lambda |x_t| \\ &= L_\lambda L_\phi |\varepsilon_t| + L_\lambda |x_t|,\end{aligned}$$

showing that μ_t is exponentially stable at 0 since ε_t and x_t are. The exponential stability of z_t toward 0 follows from

$$\begin{aligned}|z_t| &\leqslant |\varepsilon_t| + |\mathcal{H}_q(x_t, s_t, \mu_t \mathsf{R}_t v)| \\ &\leqslant |\varepsilon_t| + |\mathcal{H}_q(x_t, \Delta, \mu_t \mathsf{R}_t v) - \mathcal{H}_q(0, \Delta, 0)| \\ &\leqslant |\varepsilon_t| + L_{\mathcal{H}_q}(|x_t| + \|v\|_\infty |\mu_t|),\end{aligned}$$

which concludes the proof of Theorem 5. ■

4. Universality theorem

Introduced by Sussmann in [26], universal inputs are inputs for which the resulting output of a system such as (1) allows to distinguish between any two states that may be distinguished. In other words, v is a universal input if, for any two points $x_a, x_b \in \mathbb{R}^n$ and some input u for which $t \mapsto h(X(x_a, t, u))$ and $t \mapsto h(X(x_b, t, u))$ do not coincide, then $t \mapsto h(X(x_a, t, v))$ and $t \mapsto h(X(x_b, t, v))$ do not coincide either. In [26], it is proved that for analytic systems there always exist universal analytic inputs, and that universality is a generic property. In this section, we lift arguments from [26] to provide a proof of our own universality theorem, Theorem 6. It is a stronger result that follows from stronger assumptions (especially Assumption 2.ii, that the system is strongly differentially observable for the null-input). Naturally, assumptions from [26] hold under ours.

We assume in the rest of the section that $\bar{T} > 0$ is a fixed time frame for which we discuss the properties of inputs in $C^\infty([0, \bar{T}], \mathbb{R}^p)$ endowed with the compact-open topology.

Definition 11. *Let k be a positive integer. We say that a $(k-1)$-jet $\sigma \in (\mathbb{R}^p)^k$* distinguishes *a pair (x_a, x_b) in $\mathbb{R}^n \times \mathbb{R}^n$, if there exists an integer $0 \leqslant j \leqslant k$ such that $H_j(x_a, \sigma) \neq H_j(x_b, \sigma)$. By extension, an input $u \in C^\infty([0, \bar{T}], \mathbb{R}^p)$ is said to* distinguish *(x_a, x_b), if there exist a positive integer k and a time $t \in [0, \bar{T}]$ such that $\mathcal{H}_k(x_a, t, u) \neq \mathcal{H}_k(x_b, t, u)$. That is, the $(k-1)$-jet of u at t distinguishes (x_a, x_b). Finally, a pair (x_a, x_b) is said to be* distinguishable *if there exists a jet that distinguishes it.*

Let $\mathsf{D} = \{(x, x) \colon x \in \mathbb{R}^n\}$ denote the diagonal of the state space. No (x, x) pair can be distinguished, but under Assumption 2.i, any pair in $(\mathbb{R}^n \times \mathbb{R}^n) \setminus \mathsf{D}$ is distinguishable. We also define for any $\eta > 0$, a neighborhood of the diagonal

$$\mathsf{D}_\eta = \{(x_a, x_b) \in \mathbb{R}^n \times \mathbb{R}^n \colon |x_a - x_b| \leqslant \eta\}. \tag{24}$$

Over the whole section, K denotes a subset of $\mathbb{R}^n \times \mathbb{R}^n$. For such a K, we denote by $\underline{\mathsf{K}}$ its projection onto $\mathbb{R}^n$ with respect to both factors, that is:

$$\underline{\mathsf{K}} = \{x \in \mathbb{R}^n \colon (\{x\} \times \mathbb{R}^n \cup \mathbb{R}^n \times \{x\}) \cap \mathsf{K} \neq \varnothing\}. \tag{25}$$

Naturally, if K is compact, $\underline{\mathsf{K}}$ is also compact.

Definition 12. *The input $u \in C^\infty([0,\bar{T}],\mathbb{R}^p)$ is said to be universal over $\mathsf{K} \subset \mathbb{R}^n \times \mathbb{R}^n$ if it distinguishes any distinguishable pair in K. Following Sussmann, we call $Z(\bar{T},\mathsf{K})$ the set of inputs that are universal over K.*

In addition, we call $Z'(\bar{T},\mathsf{K})$ the subset of inputs $v \in C^\infty([0,\bar{T}],\mathbb{R}^p)$ for which there exists $(T,q) \in (0,\bar{T}] \times \mathbb{N}$ such that for all $(t,\mu,\mathsf{R}) \in [0,T] \times [0,1] \times \mathrm{O}(p)$, the map $(x_a,x_b) \mapsto \mathcal{H}_q(x_a,t,\mu\mathsf{R}v) - \mathcal{H}_q(x_b,t,\mu\mathsf{R}v)$ never vanishes over $\mathsf{K} \setminus \mathsf{D}$ and $x \mapsto \mathcal{H}_q(x,t,\mu\mathsf{R}v)$ is an immersion on $\underline{\mathsf{K}}$.

Remark 13. *For any $u \in C^\infty([0,\bar{T}],\mathbb{R}^p)$, $k \in \mathbb{N}$, $\varepsilon_0,\dots,\varepsilon_k$ positive constants, we define*

$$V(u,\varepsilon_0,\dots,\varepsilon_k) = \left\{ v \colon \|v^{(j)} - u^{(j)}\|_\infty < \varepsilon_j, \forall j \leqslant k \right\}. \tag{26}$$

The family of these sets is a fundamental system of neighborhoods of u in the compact-open topology.

Lemma 14. *$Z'(\bar{T},\mathsf{K})$ is open if $\mathsf{K} \subset \mathbb{R}^n \times \mathbb{R}^n$ is compact.*

Proof. Fix any $v \in Z'(\bar{T},\mathsf{K})$, $T > 0$ and q integer as in Definition 12. By construction (see Section 2.1), $\mathcal{H}_q$ is an analytic function of the $(q-1)$-jets of v, and therefore is continuous with respect to $(x,t,\mu,\mathsf{R},u) \subset \mathbb{R}^n \times [0,T] \times [0,1] \times \mathrm{O}(p) \times C^\infty([0,\bar{T}],\mathbb{R}^p)$. Applying Lemma 22, items (i)-(ii), with $\Pi = [0,T] \times [0,1] \times \mathrm{O}(p) \times C^\infty([0,\bar{T}],\mathbb{R}^p)$ and $F_\pi(x) = \mathcal{H}_q(x,t,\mu\mathsf{R}u)$ for $\pi = (t,\mu,\mathsf{R},u)$ yields the existence of

$$\rho(u) = \min_{\pi \in [0,T]\times[0,1]\times\mathrm{O}(p)\times\{u\}} \min\left\{\rho_2(\pi), \rho_1(\pi,\eta_2(\pi))\right\}$$

continuous with respect to u and such that

$$|\mathcal{H}_q(x_a,t,\mu\mathsf{R}u) - \mathcal{H}_q(x_b,t,\mu\mathsf{R}u)| \geqslant \rho(u)|x_a - x_b|.$$

Since $v \in Z'(\bar{T},\mathsf{K})$, $\rho(v) > 0$ and there exist $k \in \mathbb{N}$, $k \geqslant q-1$, $\varepsilon_0,\dots,\varepsilon_k > 0$ such that if $u \in V(v,\varepsilon_0,\dots,\varepsilon_k)$ (as in (26)), then $\rho(u) > 0$. Consequently, for any $u \in V(v,\varepsilon_0,\dots,\varepsilon_k)$, $t \in [0,T]$, $\mu \in [0,1]$, $\mathsf{R} \in \mathrm{O}(p)$, $(x_a,x_b) \mapsto \mathcal{H}_q(x_b,t,\mu\mathsf{R}v) - \mathcal{H}_q(x_a,t,\mu\mathsf{R}u)$ never vanishes over $\mathsf{K} \setminus \mathsf{D}$. Up to reducing $\varepsilon_0,\dots,\varepsilon_k$, we also recover the injectivity of the Jacobian over $\underline{\mathsf{K}}$, which concludes the openness argument. ∎

Remark 15. *In [26, Theorem 2.2], the set $Z(\bar{T},\mathsf{K})$ of universal inputs over K could not be proved to be open because of the issue at the diagonal. Even if, for a given v, the map $G_v : (x_a,x_b) \mapsto \mathcal{H}_q(x_b,0,v) - \mathcal{H}_q(x_a,0,v)$ vanishes only when $x_a = x_b$ in K, there may exists arbitrarily small perturbations δv such that $G_{v+\delta v}$ has nontrivial vanishing points off the diagonal (as with the Whitney pleat for instance [4]). The injectivity requirement on the Jacobian of $\mathcal{H}_q$ is crucial to avoid that difficulty.*

Following a similar reasoning to the proof of [26, Theorem 2.2], which states that, under analicity assumption, the set of inputs that distinguish any distinguishable pair is dense in $C^\infty([0,\bar{T}],\mathbb{R}^p)$, we are able, *mutatis mutandis* (but not without efforts), to prove Theorem 6.

To the universality of inputs provided by [26, Theorem 2.2], we add the strong differential observability (see Definition 1-*(ii)*) as well as uniformity with respect to parameters μ and R. To do so, we amend the intermediary lemmas and theorems leading to [26, Theorem 2.2] (namely [26, Lemma 3.1 to Theorem 3.9]). While the parameters' side will be achieved by a refinement on the proof of [26, Theorem 3.9], the immersion property will require a finer approach. For shortness of presentation, some elements remain black boxes.

Let us introduce some notation. Set $\Gamma_0 = \mathbb{R}^n \times \mathbb{R}^n$ and $\Gamma_k = \mathbb{R}^n \times \mathbb{R}^n \times (\mathbb{R}^p)^k$. If $j \leqslant k$, we set $\mu_{kj} : \Gamma_k \to \Gamma_j$ to be the trivial projection, and since the map H_j can be extended to $(\mathbb{R}^p)^k$, we can slightly abuse notation and write $H_j(x,\sigma)$ even when $\sigma \in (\mathbb{R}^p)^k$. For each nonnegative integer k, consider the sets

$$A_k = \{(x_a,x_b,\sigma) \in \Gamma_k \colon \forall j \leqslant k,\ H_j(x_a,\sigma) = H_j(x_b,\sigma)\},$$

$$\tilde{A}_k = \left\{(x,\sigma) \in \mathbb{R}^n \times (\mathbb{R}^p)^k \colon \mathrm{rk}\left(\frac{\partial H_j}{\partial x}(x,\sigma)\right)_{0\leqslant j\leqslant k} < n\right\},$$

$$A'_k = \left\{(x_a,x_b,\sigma) \in \Gamma_k \colon \exists\, c \in \{a,b\},\ (x_c,\sigma) \in \tilde{A}_k\right\}.$$

The set $A_k \cup A'_k$ denotes the sets of jets of bad inputs, observability-wise, as seen in the next statement, which is an immediate counterpart of [26, Lemmas 3.6 and 3.7].

Lemma 16 (Bad jets lemma)**.** *Let $k \in \mathbb{N}$. Then,*

1. *$A_k \cup A'_k$ is an analytic subset of Γ_k.*
2. *If $j \leqslant k$, $\mu_{kj}(A_k \cup A'_k) \subset A_j \cup A'_j$.*
3. *$(x_a, x_b, \sigma) \in A_k$ if and only if σ fails to distinguih the pair (x_a, x_b).*
4. *$(x_a, x_b, \sigma) \in A'_k$ if and only if at x_a or x_b the map $x \mapsto (H_0(x,\sigma), \dots, H_k(x,\sigma))$ fails to be an immersion.*

The technical heart of the proof is probably the following result, showing that the amount of space that the sets $A_k \cup A'_k$ occupy within Γ_k becomes smaller as k grows. (Meaning that it is more difficult for a jet of high order to not distinguish between two points). In a later proof, this will allow to invoke Sard's lemma type arguments to show that jets having sufficiently high order should be universal (at least over a compact). Let us recall the notions of dimension and codimension used in [26]. If A is a subset of a smooth manifold E, its dimension is the maximal dimension of any smooth submanifold of E included in A, and is denoted by $\dim A$. The codimension of A is $\operatorname{codim}_E A = \dim E - \dim A$.

Lemma 17. *Let $\mathcal{E} = (E_k)_{k \geqslant 0} \subset \Gamma_k$ be a family such that:*

1. *E_0 is contained in the set of distinguishable pairs,*
2. *E_k is an open subset of Γ_k,*
3. *$\mu_{jk}(E_j) \subset E_k$ when $j \geqslant k$,*
4. *$(A_k \cup A'_k) \cap E_k$ is a union of a finite number of connected analytic submanifolds of Γ_k.*

Set $c'_{\mathcal{E}}(k) = \operatorname{codim}_{E_k}((A_k \cup A'_k) \cap E_k)$. Then, under assumption 2.ii, $\lim_{k\to\infty} c'_{\mathcal{E}}(k) = +\infty$.

In order for Lemma 17 to hold (employing exactly the same proof as Sussmann's), we need to ensure that if $(x_a, x_b, \sigma) \in A_k \cup A'_k$, then there exists $j \geqslant k$ and τ such that $\mu_{jk}(x_a, x_b, \tau) = (x_a, x_b, \sigma)$ and $(x_a, x_b, \tau) \notin A_j \cup A'_j$. For the sets A_k, this requirement is established in [26, Lemma 3.5]. Regarding the sets A'_k, employing the same proof technique as in [26, Lemma 3.5], we are able to establish the following.

Lemma 18 (Jet extension lemma)**.** *Suppose Assumption 2.ii holds. Let $x \in \mathbb{R}^n$, let k be a nonnegative integer, and let $\sigma \in (\mathbb{R}^p)^k$ such that $(x, \sigma) \in \tilde{A}_k$. Then there exist $j \geqslant k$ and a $(j-1)$-jet τ which coincides with σ up to order $k-1$ and such that $(x, \tau) \notin \tilde{A}_j$.*

Proof. We need to prove that there exist $j \geqslant k$ and $w \in C^\infty(\mathbb{R}_+, \mathbb{R}^p)$, with $w^{[j-1]}(0) = \tau$ and $w^{[k-1]}(0) = \sigma$, that satisfy: for all $\xi \in \mathbb{S}^{n-1}$, there exists $0 \leqslant i \leqslant j$ such that

$$\left\langle \left.\frac{\mathrm{d}^i}{\mathrm{d}t^i}\right|_{t=0} \frac{\partial h}{\partial x}(X(x,t,w)), \xi \right\rangle \neq 0. \tag{27}$$

That is, the map $\tilde{x} \mapsto (H_0(\tilde{x},\tau), \dots, H_j(\tilde{x},\tau))$ is an immersion at x, thus implying that $(x,\tau) \notin \tilde{A}_j$ by Lemma 16. Let $u \in C^\infty(\mathbb{R}_+, \mathbb{R}^p)$ be such that $u^{[k-1]}(0) = \sigma$. Let $T \in (0, T_e(x,u))$. Set $x' = X(x,T,u)$, $\bar{v} = 2|u(T)|$, and let r be a positive constant. For any $v \in C^0\big(\mathbb{R}_+, B_{\mathbb{R}^p}(0,\bar{v})\big)$, set $t^* = \inf\{t > 0 \colon |X(x',t,v) - x'| > r\}$. If $t^* < \infty$, then

$$\begin{aligned} r = |X(x',t^*,v) - x'| &\leqslant \int_0^{t^*} |f(X(x',s,v),v(s))| \mathrm{d}s \\ &\leqslant t^* \kappa, \qquad \text{where } \kappa = \sup_{B_{\mathbb{R}^n}(x',r)\times B_{\mathbb{R}^p}(0,\bar{v})} |f|. \end{aligned}$$

Thus, $\eta := r/\kappa \leqslant t^*$ and by construction, we have $X\,(x', [0,\eta], v) \subset B_{\mathbb{R}^n}(x', r)$, for all $v \in C^0\big(\mathbb{R}_+, B_{\mathbb{R}^p}(0,\bar{v})\big)$.

Fix T' positive such that $T' < \inf_{B_{\mathbb{R}^n}(x',r)} T_e(\cdot,0)$ (such a T' exists since $T_e(\cdot,0)$ is lower semi-continuous). Let now $K \subset \mathbb{R}^n$ be a compact subset containing $X\left(x,[0,T],u\right) \cup X\left(B_{\mathbb{R}^n}(x',r),[0,T'],0\right)$ in its interior. By Assumption 2.ii, since K is compact, there exists $k_0 \geqslant k$ such that $\mathcal{H}_{k_0}(\cdot,0,0)$ is an immersion over K. Indeed one such k_0 exists for all element of K by assumption. This property being open and $\mathcal{H}_{k_0}(\cdot,0,0)$ continuously differentiable, we get a uniform k_0 over K by a finite open cover argument.

Let $u_\eta \in C^{k_0-1}([0,\eta],B_{\mathbb{R}^p}(0,\bar{v}))$ be such that $u_\eta^{[k_0-1]}(0) = u^{[k_0-1]}(T)$ and $u_\eta^{[k_0-1]}(\eta) = 0$. Now we consider $v \in C^{k_0-1}([0,T+\eta+T'],\mathbb{R}^p)$, a (k_0-1)-continuously differentiable extension of the input u:

$$v(t) = \begin{cases} u(t) & \text{if } t \in [0,T) \\ u_\eta(t-T) & \text{if } t \in [T,T+\eta) \\ 0 & \text{if } t \in [T+\eta,T+\eta+T'] \end{cases}$$

By construction, $X(x,t,v)$ is well defined for all $t \in [0,T+\eta+T']$ and belong to $\mathring{K}$.

We then write $v(t) = P(t) + t^{k_0}Q(t)$, where $P(t)$ is the Taylor polynomial of degree k_0-1 of v at 0. In fact, $P^{[k-1]}(0) = \sigma$. Using Weierstrass approximation theorem of order k_0-1, we can find a sequence of polynomials Q_ℓ such that for all $0 \leqslant i \leqslant k_0-1$, $Q_\ell^{(i)}$ converges uniformly towards $Q^{(i)}$ as $\ell \to \infty$. As such, $v_\ell = P(t) + t^{k_0}Q_\ell(t)$ is a sequence of polynomials with $v_\ell^{(i)}$ converging uniformly towards $v^{(i)}$ $(i \leqslant k_0-1)$ while ${v_\ell}^{[k-1]}(0) = \sigma$.

Since $v_\ell^{(i)}$ tends uniformly to 0 on $[T+\eta,T+\eta+T']$, we have, for ℓ large enough, that $X(x,t,v_\ell) \in \mathring{K}$, for all $t \in [0,T+\eta+T']$. Also, because being an immersion is an open property, and $\mathcal{H}_{k_0}$ is continuous, $\mathcal{H}_{k_0}(\cdot,t,v_\ell)$ is an immersion at x for all $t \in [T+\eta,T+\eta+T']$ and all ℓ large enough. Fix one such ℓ. This allows to conclude that for all $\xi \in \mathbb{S}^{n-1}$, there exists $0 \leqslant i \leqslant k_0$ such that

$$\left\langle \left.\frac{\mathrm{d}^i}{\mathrm{d}t^i}\right|_{t=T+\eta} \frac{\partial h}{\partial x}(X(x,t,v_\ell)),\xi \right\rangle \neq 0,$$

implying that the analytic (recall v_ℓ is actually a polynomial) map $t \mapsto \left\langle \frac{\partial h}{\partial x}(X(x,t,v_\ell)),\xi \right\rangle$ is non-zero. Hence for any $\xi \in \mathbb{S}^{n-1}$, there exists $i(\xi) \geqslant 0$ such that (27) holds with $i = i(\xi)$. By continuity of the scalar product, we again get an open property, and are able to use compactness of $\mathbb{S}^{n-1}$ to conclude that there exists $j \geqslant k_0$ such that $i(\xi) < j$ for all $\xi \in \mathbb{S}^{n-1}$. In conclusion, $\tau = {v_\ell}^{[j-1]}(0)$ satisfies $(x,\tau) \notin \tilde{A}_j$. ■

We now have the necessary elements to prove Lemma 17.

Proof of Lemma 17. The proof goes exactly the same as the proof of [26, Lemma 3.8]. We just need to substitute [26, Lemmas 3.7, 3.6] with Lemma 16, and [26, Lemmas 3.5] with [26, Lemma 3.5] plus Lemma 18. ■

Now we propose an original refinement of [26, Theorem 3.9] Sussmann's proof that includes the contraction coefficient μ and the action of the orthogonal group $\mathrm{O}(p)$ defined by: $\mathsf{R} \cdot \sigma = (\mathsf{R}\sigma_1,\ldots,\mathsf{R}\sigma_k)$ for any $(\mathsf{R},\sigma) \in \mathrm{O}(p) \times (\mathbb{R}^p)^k$ and any $k \geqslant 1$.

Proposition 19. *Let Assumptions 2.i and 2.ii hold. Let K be a compact subset of $\mathbb{R}^n \times \mathbb{R}^n \setminus \mathsf{D}$. Let $(B_\ell)_{\ell \geqslant 0}$ be a sequence of open balls of positive radii and centered at $0 \in \mathbb{R}^p$. Then there exist $k > 0$ and an open and dense subset W of $B_0 \times B_1 \times \cdots \times B_{k-1}$ such that, whenever $(x_a,x_b,\sigma,\mu,\mathsf{R}) \in \mathsf{K} \times W \times [0,1] \times \mathrm{O}(p)$, then $(x_a,x_b,\mu\mathsf{R}\cdot\sigma) \notin A_k \cup A'_k$.*

Proof. We follow the reasoning of Sussmann's Theorem 3.9. The first step is to restrict the search by stipulating that the result only needs to hold true for some neighborhood in $\mathbb{R}^n \times \mathbb{R}^n \setminus \mathsf{D}$ of any pair (x_a,x_b) such that $x_a \neq x_b$. Then the statement holds on any compact subset K.

On the one hand, by Assumptions 2.i and 2.ii, there exists $k' \geqslant 0$ such that, $(x_a,x_b,0,\ldots,0) \notin A_k \cup A'_k$ for all $k \geqslant k'$. Because $A_k \cup A'_k$ is closed, it follows that for any $k > k'$, there exists $\varepsilon_k > 0$ such that, taking $E_0 = B_{\mathbb{R}^n}(x_a,\varepsilon_k) \times B_{\mathbb{R}^n}(x_b,\varepsilon_k)$, $(A_k \cup A'_k) \cap \left[E_0 \times B_{\mathbb{R}^{pk}}(0,\varepsilon_k)\right] = \varnothing$ and $\bar{E}_0 \subset \mathbb{R}^n \times \mathbb{R}^n \setminus \mathsf{D}$.

For $k \geqslant 1$, set $E_k = E_0 \times B_0 \times \cdots \times B_{k-1}$. Then the induced family $\mathcal{E} = (E_\ell)_{\ell \geqslant 0}$ satisfies conditions of Lemma 17 (condition 1 follows from Assumption 2.i, conditions 2 and 3 are immediate, and condition 4 also holds since the sets $E_0, (B_\ell)_{\ell \geqslant 0}$ were picked analytic).

On the other hand, by Lemma 17, there exists $k'' \geqslant k'$ such that $c'_{\mathcal{E}}(k) > 2n + 1 + p(p-1)/2$ for all $k \geqslant k''$. Fix $k > k''$ and let $\varsigma : \Gamma_k \times [0, +\infty) \times \mathrm{O}(p) \to \mathbb{R}^{pk}$ be the surjective analytic map given by $\varsigma(x_a, x_b, \sigma, \mu, \mathsf{R}) = \mu \mathsf{R} \cdot \sigma$. The cone $L = \varsigma\big(\overline{(A_k \cup A'_k) \cap E_k} \times [0, +\infty) \times \mathrm{O}(p)\big)$ is a semianalytic set of positive codimension. By Sard's Lemma, L has zero measure, thus $\mathbb{R}^{pk} \setminus L$ is dense.

$$W = \left[B_{\mathbb{R}^{pk}}(0, \varepsilon_k) \cup (\mathbb{R}^{pk} \setminus L)\right] \cap \left[B_0 \times \cdots \times B_{k-1}\right].$$

Then W is clearly open and dense in $B_0 \times \cdots \times B_{k-1}$. Now let $(x'_a, x'_b, \mu, \mathsf{R}) \in E_0 \times [0,1] \times \mathrm{O}(p)$ and $\sigma \in W$. Let us prove that $(x'_a, x'_b, \mu \mathsf{R} \cdot \sigma) \notin (A_k \cup A'_k) \cap E_k$. If $\sigma \in B_{\mathbb{R}^{pk}}(0, \varepsilon_k)$, then $\mu \mathsf{R} \cdot \sigma \in B_{\mathbb{R}^{pk}}(0, \varepsilon_k)$ also, and $(x'_a, x'_b, \mu \mathsf{R} \cdot \sigma) \notin (A_k \cup A'_k) \cap E_k$. If $\sigma \notin B_{\mathbb{R}^{pk}}(0, \varepsilon_k)$, either $\mu = 0$, and $(x'_a, x'_b, 0) \notin A_k \cup A'_k$ by assumption, or $\mu \neq 0$, which we now assume. Clearly, $\mu \mathsf{R} \cdot \sigma \in B_0 \times \cdots \times B_{k-1}$. By construction of L, if $\mu \mathsf{R} \cdot \sigma \in L$ then $\sigma \in \mu^{-1} \mathsf{R}^{-1} \cdot L = L$. Since, $\sigma \in \mathbb{R}^{pk} \setminus L$, this means that $\mu \mathsf{R} \sigma \in W$ also, and thus $(x'_a, x'_b, 0) \notin A_k \cup A'_k$ again. ■

Like Sussmann's [26], the last step of the proof is to move from spaces of jets to spaces of functions. Let us first show that we can safely move from a good jet (one that avoids a critical set $A_k \cup A'_k$, see Lemma 16) to a good input (one that belongs to a $Z'(\bar{T}, \mathsf{K})$, see Definition 12).

Lemma 20. *Let K be a compact subset of $\mathbb{R}^n \times \mathbb{R}^n \setminus \mathsf{D}$. Let $k \in \mathbb{N}$ and let $\sigma \in (\mathbb{R}^p)^k$ be a $(k-1)$-jet such that for all $(x_a, x_b, \mu, \mathsf{R}) \in \mathsf{K} \times [0,1] \times \mathrm{O}(p)$, $(x_a, x_b, \mu \mathsf{R} \cdot \sigma) \notin A_k \cup A'_k$. Then any $u \in C^\infty([0, T], \mathbb{R}^p)$ with $u^{[k-1]}(0) = \sigma$ having its $(k-1)$-jet at 0 given by σ belongs to $Z'(\bar{T}, \mathsf{K})$.*

Proof. Let $\sigma(t) = u^{[k-1]}(t)$. It describes a smooth trajectory in the space $(\mathbb{R}^p)^k$ such that $\sigma(0) = \sigma$. Assume by contradiction that $u \notin Z'(\bar{T}, \mathsf{K})$. By points 3-4 of Lemma 16, it follows that for any $N \in \mathbb{N}$, there exist $t^N \in (0, \bar{T}/N]$, $(x_a^N, x_b^N, \mu^N, \mathsf{R}^N) \in \mathsf{K} \times [0,1] \times \mathrm{O}(p)$ such that $(x_a^N, x_b^N, \mu^N \mathsf{R}^N \cdot \sigma(t^N)) \in A_k \cup A'_k$. The sequence $(x_a^N, x_b^N, \mu^N, \mathsf{R}^N)$ converges up to extraction as $N \to \infty$, since it belongs to a compact set. We denote the limit with $(x_a^\infty, x_b^\infty, \mu^\infty, \mathsf{R}^\infty) \in \mathsf{K} \times [0,1] \times \mathrm{O}(p)$. Also, $\sigma(t_N) \to \sigma$. The set $A_k \cup A'_k$ is closed, hence $(x_a^\infty, x_b^\infty, \mu^\infty \mathsf{R}^\infty \cdot \sigma) \in A_k \cup A'_k$, a contradiction. ■

We now combine Proposition 19 and Lemma 20 to prove the density of good inputs if we avoid the diagonal.

Lemma 21. *Let Assumptions 2.i and 2.ii hold. Let $\mathsf{K} \subset (\mathbb{R}^n \times \mathbb{R}^n) \setminus \mathsf{D}$ be a compact set. The set $Z'(\bar{T}, \mathsf{K})$ is dense in $C^\infty([0, T], \mathbb{R}^p)$.*

Proof. We use the fundamental system of neighborhoods for the topology of $C^\infty([0, \bar{T}], \mathbb{R}^p)$ recalled in Remark 13. Let $u \in C^\infty([0, \bar{T}], \mathbb{R}^p)$, $k \in \mathbb{N}$, and $\varepsilon_0, \ldots, \varepsilon_k > 0$. Let us show that $Z'(\bar{T}, \mathsf{K}) \cap V(u, \varepsilon_0, \ldots, \varepsilon_k) \neq \varnothing$.

We complete the sequence $(\varepsilon_j)_{j \in \mathbb{N}}$ into an arbitrary sequence (such as $\varepsilon_j = 1$ for all $j > k$ for instance). For $j \geqslant 0$, we set B_j to be the open ball of center 0 and radius $|u^{(j)}(0)| + \varepsilon_j$. Since Assumptions 2.i, 2.ii hold, we apply Proposition 19 to find $k' > 0$ and W an open and dense subset of $B_0 \times B_1 \times \cdots \times B_{k'-1}$, such that if $\sigma \in W$, for any $\mu \in [0,1]$, $\mathsf{R} \in \mathrm{O}(p)$, $(x_a, x_b, \mu \mathsf{R} \cdot \sigma) \notin A_k \cup A'_k$. We can assume that $k' \geqslant k+1$, up to replacing W with $W \times B_{k'+1} \times \cdots \times B_k$.

Write $(\sigma_0^u, \ldots, \sigma_{k'-1}^u) = u^{[k'-1]}(0)$, $(\sigma_0, \ldots, \sigma_{k'-1}) = \sigma$, and define the input

$$v^\sigma(t) = u(t) + \sum_{j=0}^{k'-1} \frac{\sigma_j - \sigma_j^u}{j!} t^j.$$

It is clear that $\omega : \sigma \mapsto v^\sigma$ is continuous and allows to define $Q = \omega^{-1}(V(u, \varepsilon_0, \ldots, \varepsilon_k))$, an open neighborhood of σ^u. Furthermore, $Q \subset B_0 \times B_1 \times \cdots \times B_k \times (\mathbb{R}^p)^{k'-k-1}$. Since W is an open and dense subset of $B_0 \times B_1 \times \cdots \times B_{k'-1}$, there exists $\sigma \in W \cap Q$, which by construction satisfies $v^\sigma \in V(u, \varepsilon_0, \ldots, \varepsilon_k)$. By Lemma 20, we then also have that $v^\sigma \in Z'(\bar{T}, \mathsf{K})$, hence the density of $Z'(\bar{T}, \mathsf{K})$. ■

We are finally in position to prove Theorem 6. Theorem 6-*(i)* will follow from Lemmas 14, 21 and 22. The argument we use to prove Theorem 6-*(ii)* is similar to the proof of [26, Theorem 2.1]. The cited theorem only proves existence of analytic universal inputs but a few elements from Theorem 6-*(i)* extend Sussmann's argument to also provides density.

Proof of Theorem 6. Recall that $C^\infty([0,\bar{T}],\mathbb{R}^p)$ endowed with the compact-open topology is a Baire space, i.e., a topological space in which every countable intersection of open and dense sets is dense.

To prove item *(i)*, let us consider the non-decreasing sequence $(\mathsf{K}_N)_{N\in\mathbb{N}}$ defined by

$$\mathsf{K}_N = \left\{ (x_a, x_b) \in \mathcal{K}^2 \colon |x_a - x_b| \geqslant \frac{\max_{\mathcal{K}^2} |x_a - x_b|}{N+3} \right\}.$$

According to Lemmas 14 and 21, $\bigcap_{N\in\mathbb{N}} Z'(\bar{T},\mathsf{K}_N)$ is a countable intersection of open and dense subsets of $C^\infty([0,\bar{T}],\mathbb{R}^p)$. It is therefore dense since $C^\infty([0,\bar{T}],\mathbb{R}^p)$ is a Baire space. Let $v \in \bigcap_{N\in\mathbb{N}} Z'(\bar{T},\mathsf{K}_N)$, let us check that $v \in \mathcal{U}_\mathcal{K}$. Let $\bar{x}_a, \bar{x}_b$ be such that $(\bar{x}_a, \bar{x}_b)$ realizes $\max_{\mathcal{K}^2} |x_a - x_b|$. If $x \in \mathcal{K}$ is such that $|x - \bar{x}_a| < |\bar{x}_a - \bar{x}_b|/3$, we automatically get by triangular inequality that $|x - \bar{x}_b| \geqslant 2|\bar{x}_a - \bar{x}_b|/3$. Hence the projected set $\underline{\mathsf{K}}_0$ (see (25)) is equal to $\mathcal{K}$. As a consequence, having $v \in Z'(\bar{T},\mathsf{K}_0)$ implies that there exist $T_0 \in (0,\bar{T}]$ and $q_0 \in \mathbb{N}$ for which $x \mapsto \mathcal{H}_{q_0}(x,t,\mu\mathsf{R}v)$ is an immersion over $\mathcal{K}$ for all $(t,\mu,\mathsf{R}) \in [0,T_0]\times[0,1]\times \mathrm{O}(p)$. Applying Lemma 22, item (ii), thanks of the compactness of $[0,T_0]\times[0,1]\times\mathrm{O}(p)$, there exist $\rho_0,\eta_0 > 0$ such that for all $(t,\mu,\mathsf{R}) \in [0,T_0]\times[0,1]\times\mathrm{O}(p)$ and all $(x_a,x_b)\in\mathcal{K}^2$ such that $|x_a - x_b| < \eta_0$ we have

$$|\mathcal{H}_{q_0}(x_a,t,\mu\mathsf{R}v) - \mathcal{H}_{q_0}(x_b,t,\mu\mathsf{R}v)| \geqslant \rho_0 |x_a - x_b|.$$

Let $N^* \in \mathbb{N}$ be such that $(N^*+3)\eta_0 > \max_{\mathcal{K}^2}|x_a - x_b|$. From $v \in Z'(\bar{T},\mathsf{K}_{N^*})$, there exist $T^* \in (0,\bar{T}]$, $q^* \in \mathbb{N}$ for which $(x_a,x_b) \mapsto \mathcal{H}_{q^*}(x_a,t,\mu\mathsf{R}v) - \mathcal{H}_{q^*}(x_b,t,\mu\mathsf{R}v)$ never vanishes over K_{N^*} (for any $(t,\mu,\mathsf{R}) \in [0,T^*]\times[0,1]\times\mathrm{O}(p)$). Applying Lemma 22, item (i), again from the compactness of $[0,T^*]\times[0,1]\times\mathrm{O}(p)$, there exists $\rho^* > 0$ such that for all $(t,\mu,\mathsf{R}) \in [0,T^*]\times[0,1]\times\mathrm{O}(p)$ and all $(x_a,x_b)\in\mathcal{K}^2$ such that $|x_a - x_b| \geqslant \eta_0$ we have

$$|\mathcal{H}_{q^*}(x_a,t,\mu\mathsf{R}v) - \mathcal{H}_{q^*}(x_b,t,\mu\mathsf{R}v)| \geqslant \rho^* |x_a - x_b|.$$

Taking $T = \min(T_0,T^*) > 0$, $q = \max(q_0,q^*)$ and $\rho = \min(\rho_0,\rho^*) > 0$, we get for all $(t,\mu,\mathsf{R}) \in [0,T]\times[0,1]\times\mathrm{O}(p)$ and all $(x_a,x_b)\in\mathcal{K}^2$:

$$|\mathcal{H}_{q}(x_a,t,\mu\mathsf{R}v) - \mathcal{H}_{q}(x_b,t,\mu\mathsf{R}v)| \geqslant \rho |x_a - x_b|.$$

Since we also have the immersion property on $\mathcal{K}$ (again thanks to $\underline{\mathsf{K}}_0 = \mathcal{K}$), we have proved that $v \in \mathcal{U}_\mathcal{K}$, and point *(i)* stands.

We now prove point *(ii)*. Let $u \in C^\infty([0,\bar{T}],\mathbb{R}^p)$, $k\in\mathbb{N}$, and $\varepsilon_0,\dots,\varepsilon_k > 0$. Let us prove that $\mathcal{U}_\mathcal{K} \cap C^\omega([0,\bar{T}],\mathbb{R}^p) \cap V(u,\varepsilon_0,\dots,\varepsilon_k) \neq \varnothing$.

We first apply Weierstrass density theorem to find a polynomial P belonging to $V(u,\varepsilon_0/2,\dots,\varepsilon_k/2)$. Then we define $(B'_i)_{i\in\mathbb{N}}$ a sequence of open balls centered at $P^{(i)}(0)$ and of radii $r = \min(\varepsilon_0,\dots,\varepsilon_k)\,\mathrm{e}^{-T}/2$. That way for any input $w \in C^\omega([0,T],\mathbb{R}^p)$ such that $w^{(i)}(0) \in B'_i$, we have

$$P(t) - w(t) = \sum_{i=0}^{\infty} \frac{P^{(i)}(0) - w^{(i)}(0)}{i!} t^i,$$

which implies $w \in V(P,\varepsilon_0/2,\dots,\varepsilon_k/2) \subset V(u,\varepsilon_0,\dots,\varepsilon_k)$.

Let $(B_i)_{i\in\mathbb{N}}$ be a sequence of open balls centered at 0 and of radii $\left|P^{(i)}(0)\right| + r$. For every $N\in\mathbb{N}$, we apply Proposition 19 on the compact set K_N to find q_N, an open and dense $W_N \subset B_0 \times \cdots \times B_{q_N-1}$ such that for all $(x_a,x_b)\in\mathsf{K}_N$, $\sigma\in W_N$, $\mu\in[0,1]$, $\mathsf{R}\in\mathrm{O}(p)$, $(x_a,x_b,\mu\mathsf{R}\cdot\sigma) \notin A_{q_N} \cup A'_{q_N}$. Naturally, this implies that if $w^{[q_N-1]}(0) \in W_N$, then $w \in Z'(\bar{T},\mathsf{K}_N)$ from Lemma 20. We complete each W_N into $W'_N = W_N \times \prod_{i=q_N}^\infty B_i$. Each W'_N is an open and dense subset of $\prod_{i=0}^\infty B_i$ endowed with the product topology. Then, because $\prod_{i=0}^\infty B_i$ is a Baire space for the product topology, $\bigcap_{N=0}^\infty W'_N$ is dense. Thus, the intersection $\bigcap_{N=0}^\infty W'_N \cap \prod_{i=0}^\infty B'_i$ is nonempty since $\prod_{i=0}^\infty B'_i$ is open in $\prod_{i=0}^\infty B_i$. For any σ in that intersection, set $v(t) = \sum_{i=0}^\infty \sigma_i t^i/i!$. We have $v \in V(u,\varepsilon_0,\dots,\varepsilon_k) \cap \bigcap_{N=0}^\infty Z'(\bar{T},\mathsf{K}_N) \subset V(u,\varepsilon_0,\dots,\varepsilon_k) \cap \mathcal{U}_\mathcal{K}$, which proves the density and concludes the argument. ∎

5. Conclusion and perspective

We introduced control templates to extend sample-and-hold control while maintaining observability. Using this idea, we first established a separation principle under the assumptions of existence of control templates and a locally exponentially stabilizing state feedback. Second, we proved that control templates are generic among smooth inputs. These combine into the first generic semiglobal dynamic output feedback stabilization result for locally exponentially stabilizable nonlinear control systems. That is, without any uniform observability assumption [27, 15], nor any passivity or dissipativity properties [21, 23].

It remains an open question whether a similar result could be achieved by using a two-modes strategy assuming exponential stabilizability and observability of the null-input as in [24] (the potential difficulty lies in passing from practical to exact stabilization), or as in [11] (it lies in passing from local to semiglobal stabilization). The template control method has the advantage of simultaneous estimation and observation without relying on two-mode separation, making it closer to observer-based methods. The observability assumption we make is still stronger than the ones imposed by Coron in [11] that are almost necessary (see [8]). We believe template control methods could be developed further to address other observability issues, including stabilization at an unobservable target.

Appendix. Additional technical lemmas

Lemma 22. *Let n and m be two positive integers, K be a compact subset of $\mathbb{R}^n \times \mathbb{R}^n$, Π be a topological space, and $F : \mathbb{R}^n \times \Pi \to \mathbb{R}^m$ be a continuous map. Define $F_\pi := x \mapsto F(x, \pi)$, for all $\pi \in \Pi$.*

Let D_η and $\underline{\mathsf{K}}$ be as in (24) and (25), respectively.

(i) There exists a continuous function $\rho_1 : \Pi \times \mathbb{R}_+ \to \mathbb{R}_+$ such that for all $(x_a, x_b, \pi) \in (\mathsf{K} \setminus \mathring{\mathsf{D}}_\eta) \times \Pi$,

$$|F_\pi(x_a) - F_\pi(x_b)| \geqslant \rho_1(\pi, \eta)|x_a - x_b|,$$

and if $(x_a, x_b) \mapsto F_\pi(x_a) - F_\pi(x_b)$ never vanishes over $\mathsf{K} \setminus \mathring{\mathsf{D}}_\eta$ for any $\pi \in \Pi$, then ρ_1 is positive.

(ii) Assume F_π is C^1 over $\underline{\mathsf{K}}$ and that $(x, \pi) \mapsto \frac{\partial F_\pi}{\partial x}(x)$ is continuous. There exist $\eta_2, \rho_2 : \Pi \to \mathbb{R}_+$ continuous such that for all $(\pi, x_a, x_b) \in \Pi \times \mathsf{K} \cap \mathsf{D}_{\eta_2(\pi)}$,

$$|F_\pi(x_a) - F_\pi(x_b)| \geqslant \rho_2(\pi)|x_a - x_b|,$$

and if $\frac{\partial F_\pi}{\partial x}(x)$ is injective for all $(x, \pi) \in \underline{\mathsf{K}} \times \Pi$, then η_2 and ρ_2 are positive.

(iii) If Π is compact and F_π is an injective immersion over a compact $\mathcal{K} \subset \mathbb{R}^n$ for all $\pi \in \Pi$, then there exists $\rho > 0$ such that

$$|F_\pi(x_a) - F_\pi(x_b)| \geqslant \rho|x_a - x_b|, \tag{28}$$

for all $(x_a, x_b, \pi) \in \mathcal{K}^2 \times \Pi$, and there exists a continuous map $F^{\mathrm{inv}} : \mathbb{R}^m \times \Pi \to \mathbb{R}^n$ that is globally Lipschitz with respect to its first variable, and Lipschitz constant uniform with respect to the second, and such that $F^{\mathrm{inv}}(F(x, \pi), \pi) = x$ for all $(x, \pi) \in \mathcal{K} \times \Pi$.

Proof. This lemma is an adaptation of [2, Lemma 3.2], where we investigate the dependency of ρ_1 and ρ_2 with respect to the parameter π. The last item is a direct consequence of [5, Lemma A.12], but we give a proof for the sake of completeness. Suppose $\mathcal{K}$ and K contain at least two elements, for otherwise the result is obvious.

Proof of *(i)*. For all $\pi \in \Pi$ and all $\eta \in [0, d]$ with $d := \max_{(x_a, x_b) \in \mathsf{K}} |x_a - x_b|$, define

$$\rho_1(\pi, \eta) = \min_{(x_a, x_b) \in \mathsf{K} \setminus \mathring{\mathsf{D}}_\eta} \frac{|F_\pi(x_a) - F_\pi(x_b)|}{|x_a - x_b|}.$$

Note that ρ_1 is well-defined and continuous since F is continuous and $\mathsf{K} \setminus \mathring{\mathsf{D}}_\eta$ is a nonempty compact set since $\eta \leqslant d$. Moreover, ρ_1 is positive if $(x_a, x_b, \pi) \mapsto F_\pi(x_a) - F_\pi(x_b)$ does not vanish on $(\mathsf{K} \setminus \mathring{\mathsf{D}}_\eta) \times \Pi$. If $\eta > d$, define $\rho_1(\pi, \eta) = \rho_1(\pi, d)$.

Proof of *(ii)*. Since $(x, \pi) \mapsto \frac{\partial F_\pi}{\partial x}(x)$ is continuous, there exists a continuous $\mathcal{E} : (\mathbb{R}^n)^2 \times \Pi \to \mathbb{R}^n$ such that $\mathcal{E}(0, 0, \pi) = 0$ for all $\pi \in \Pi$, and

$$F_\pi(x_a) - F_\pi(x_b) = \frac{\partial F_\pi}{\partial x}(x_b)(x_a - x_b) + |x_a - x_b|\mathcal{E}(x_a, x_b, \pi),$$

for all $(x_a, x_b, \pi) \in (\mathbb{R}^n)^2 \times \Pi$. Define

$$\rho_2(\pi) = \min_{(x,\xi) \in \underline{\mathsf{K}} \times \mathbb{S}^{n-1}} \frac{1}{2} \left| \frac{\partial F_\pi}{\partial x}(x)\xi \right|.$$

Note that ρ_2 is well-defined and continuous since $(x, \pi) \mapsto \frac{\partial F_\pi}{\partial x}(x)$ is continuous and $\underline{\mathsf{K}}$ is a nonempty compact set. If F_π is an immersion for all $(x, \pi) \in \underline{\mathsf{K}} \times \Pi$, then ρ_2 is positive. Define

$$\eta_2(\pi) = \min_{\substack{(x_a,x_b) \in \mathsf{K} \\ |\mathcal{E}(x_a,x_b,\pi)| \geqslant \rho_2(\pi)}} |x_a - x_b|.$$

Note that η_2 is well-defined and continuous since $\mathcal{E}$ is continuous, $\mathcal{E}(0, 0, \pi) = 0$, and K is a nonempty compact set. Moreover, $\eta_2(\pi) > 0$ if $\rho_2(\pi) > 0$. If $(x_a, x_b) \in \mathsf{K} \cap \mathsf{D}_{\eta_2(\pi)}$, then $|\mathcal{E}(x_a, x_b, \pi)| \leqslant \rho_2(v)$, which yields

$$|F_\pi(x_a) - F_\pi(x_b)| \geqslant \rho_2(v)|x_a - x_b|.$$

Proof of *(iii)*. Apply (i) and (ii) with $\mathsf{K} = \mathcal{K}^2$, and define $\rho = \min_\Pi\{\rho_1(\cdot, \min_\Pi \eta_2), \rho_2\}$, which is positive since Π is compact. Then (28) holds for all $(x_a, x_b, \pi) \in \mathcal{K}^2 \times \Pi$. For each $\pi \in \Pi$, the injective map F_π admits a left inverse $G_\pi : F_\pi(\mathcal{K}) \to \mathcal{K}$ which is, because of (28), $(1/\rho)$-Lipschitz continuous. Moreover, according to [18, Theorem 1], the mapping $F^{\mathrm{inv}} : \mathbb{R}^m \times \Pi \to \mathbb{R}^n$ given by

$$F^{\mathrm{inv}}(z, \pi) = \left(\min_{\tilde{z} \in F_\pi(\mathcal{K})} \{G_{\pi,j}(\tilde{z}) + |\tilde{z} - z|/\rho\} \right)_{1 \leqslant j \leqslant n},$$

$G_{\pi,j}$ being the jth-coordinate of G_π, is such that for each $\pi \in \Pi$, $F^{\mathrm{inv}}(\cdot, \pi)$ is a $(\sqrt{n}/\rho)$-Lipschitz extention of G_π. Defined as such, F^{inv} is also continuous. ■

Lemma 23. *For any $u, \tilde{u} \in \mathbb{R}^p$ and any $\mathsf{R} \in \mathcal{R}(u)$, there exists $\widetilde{\mathsf{R}} \in \mathcal{R}(\tilde{u})$ such that $\left\| |u|\mathsf{R} - |\tilde{u}|\widetilde{\mathsf{R}} \right\| \leqslant |u - \tilde{u}|$.*

Proof. If $u = 0$ or $\tilde{u} = 0$, the result is obvious. Otherwise, pick $\mathsf{S} \in \mathrm{O}(p)$ such that $\mathsf{S}u/|u| = \tilde{u}/|\tilde{u}|$ and $\mathsf{S}|_{\mathrm{span}\{u,\tilde{u}\}^\perp} = \mathrm{Id}$. Thus, $\mathsf{SR} \in \mathcal{R}(\tilde{u})$ and $\langle v, \mathsf{S}v \rangle = |v|^2 \cos(u, \tilde{u})$ for any $v \in \mathrm{span}\{u, \tilde{u}\}$. Now, let $w \in \mathbb{S}^{p-1}$ be such that $\left\| |u|\mathsf{R} - |\tilde{u}|\mathsf{SR} \right\| = \left| |u|w - |\tilde{u}|\mathsf{S}w \right|$ and decompose $w = v + v^\perp$, with $v \in \mathrm{span}\{u, \tilde{u}\}$ and $v^\perp \in \mathrm{span}\{u, \tilde{u}\}^\perp$. Developing yields

$$\begin{aligned}
\left\| |u|\mathsf{R} - |\tilde{u}|\mathsf{SR} \right\|^2 &= \left| |u|w - |\tilde{u}|\mathsf{S}w \right|^2 \\
&= \left| |u|v - |\tilde{u}|\mathsf{S}v \right|^2 + \left| |u|v^\perp - |\tilde{u}|\mathsf{S}v^\perp \right|^2 \\
&= \left(|u|^2 - 2|u||\tilde{u}| \cos(u, \tilde{u}) + |\tilde{u}|^2 \right) |v|^2 + \left| |u|v^\perp - |\tilde{u}|v^\perp \right|^2 \\
&= \left| u - \tilde{u} \right|^2 \left| v \right|^2 + \left| |u| - |\tilde{u}| \right|^2 \left| v^\perp \right|^2 \\
&\leqslant \left| u - \tilde{u} \right|^2,
\end{aligned}$$

which proves the result for $\widetilde{\mathsf{R}} = \mathsf{SR}$. ■

Proof of Lemma 2. For all $i \in \mathbb{N}$, define $E_i := \{(x_a, x_b) \in \mathbb{R}^n \times \mathbb{R}^n \colon \mathcal{H}_i(x_a, 0, u) = \mathcal{H}_i(x_b, 0, u)\}$. Then $(E_i)_{i\in\mathbb{N}}$ is a non-increasing sequence of analytic sets. According to [19, Chapter 5, Corollary 1], the sequence $(\widetilde{E}_i)_{i\in\mathbb{N}}$ given by $\widetilde{E}_i = E_i \cap (\mathcal{K} \times \mathcal{K})$ is stationary since $\mathcal{K}$ is compact. Hence, $\bigcap_{i\in\mathbb{N}} \widetilde{E}_i = \widetilde{E}_k$ for some $k \in \mathbb{N}$. Under the analyticity assumptions, if, for $x_a, x_b \in \mathcal{K}$, $h \circ X(x_a, \cdot, u) \neq h \circ X(x_b, \cdot, u)$ then the derivatives at $t = 0$ of these two functions differ at some rank, i.e., $\mathcal{H}_i(x_a, 0, u) \neq \mathcal{H}_i(x_b, 0, u)$ for some i. Since u is analytic, the observability assumption yields $\bigcap_{i\in\mathbb{N}} \widetilde{E}_i = \{(x, x), x \in \mathcal{K}\}$. Thus $\widetilde{E}_k = \{(x, x), x \in \mathcal{K}\}$, i.e. (1) is differentially observable of order k over $\mathcal{K}$. ■

References

[1] A. Ajami, J.-P. Gauthier, and L. Sacchelli. Dynamic output stabilization of control systems: An unobservable kinematic drone model. *Automatica J. IFAC*, 125:109383, 2021.

[2] V. Andrieu. Convergence speed of nonlinear Luenberger observers. *SIAM Journal on Control and Optimization*, 52(5):2831–2856, 2014.

[3] V. Andrieu and L. Praly. A unifying point of view on output feedback designs for global asymptotic stabilization. *Automatica J. IFAC*, 45(8):1789–1798, 2009.

[4] V. I. Arnold, S. M. Gusein-Zade, and A. N. Varchenko. *Singularities of differentiable maps. Volume 1.* Modern Birkhäuser Classics. Birkhäuser/Springer, New York, 2012. Classification of critical points, caustics and wave fronts, Translated from the Russian by Ian Porteous based on a previous translation by Mark Reynolds, Reprint of the 1985 edition.

[5] P. Bernard. *Observer design for nonlinear systems*, volume 479. Springer, 2019.

[6] P. Bernard and L. Praly. Estimation of position and resistance of a sensorless PMSM: A nonlinear Luenberger approach for a nonobservable system. *IEEE Transactions on Automatic Control*, 66(2):481–496, 2021.

[7] L. Brivadis, J.-P. Gauthier, and L. Sacchelli. Output feedback stabilization of non-uniformly observable systems. *Proceedings of the Steklov Institute of Mathematics*, 321(1):69–83, 2023.

[8] L. Brivadis, J.-P. Gauthier, L. Sacchelli, and U. Serres. New perspectives on output feedback stabilization at an unobservable target. *ESAIM: Control, Optimisation and Calculus of Variations*, 27:102, 2021.

[9] L. Brivadis and L. Sacchelli. A switching technique for output feedback stabilization at an unobservable target. In *2021 60th IEEE Conference on Decision and Control (CDC)*, pages 3942–3947, 2021.

[10] L. Brivadis and L. Sacchelli. Output feedback stabilization of non-uniformly observable systems by means of a switched Kalman-like observer. *IFAC-PapersOnLine*, 56(2):9517–9522, 2023. 22nd IFAC World Congress.

[11] J.-M. Coron. On the stabilization of controllable and observable systems by an output feedback law. *Mathematics of Control, Signals and Systems*, 7:187–216, 1994.

[12] R. Engelking. General topology. *Sigma series in pure mathematics*, 6, 1989.

[13] J.-P. Gauthier and I. Kupka. *Deterministic observation theory and applications*. Cambridge University Press, Cambridge, 2001.

[14] R. Goedel, R. G. Sanfelice, and A. R. Teel. Hybrid dynamical systems: modeling stability, and robustness. *Princeton, NJ, USA*, 2012.

[15] P. Jouan and J.-P. Gauthier. Finite singularities of nonlinear systems. Output stabilization, observability, and observers. *Journal of Dynamical and Control Systems*, 2(2):255–288, 1996.

[16] W. Lin. When is a nonlinear system semiglobally asymptotically stabilizable by digital feedback? *IEEE Transactions on Automatic Control*, 65(11):4584–4599, 2020.

[17] M. Maghenem, W. Pasillas-Lépine, A. Loría, and M. Aguado-Rojas. Hybrid observer-based asymptotic stabilization of non-uniformly observable systems: a case study. *IEEE Transactions on Automatic Control*, pages 1–8, 2023.

[18] E. J. McShane. Extension of range of functions. *Bulletin of the American Mathematical Society*, 40(12):837 – 842, 1934.

[19] R. Narasimhan. *Introduction to the theory of analytic spaces*. Lecture Notes in Mathematics, No. 25. Springer-Verlag, Berlin-New York, 1966.

[20] L. Praly and D. Bresch-Pietri. *Fonctions de Lyapunov : stabilité*. Spartacus-Idh, 2022.

[21] C. Prieur and L. Praly. A tentative direct Lyapunov design of output feedbacks. *IFAC Proceedings Volumes*, 37(13):847–852, 2004. 6th IFAC Symposium on Nonlinear Control Systems 2004 (NOLCOS 2004), Stuttgart, Germany, 1-3 September, 2004.

[22] A. Rapaport and D. Dochain. A robust asymptotic observer for systems that converge to unobservable states—a batch reactor case study. *IEEE Transactions on Automatic Control*, 65(6):2693–2699, 2020.

[23] L. Sacchelli, L. Brivadis, V. Andrieu, U. Serres, and J.-P. Gauthier. Dynamic output feedback stabilization of non-uniformly observable dissipative systems. *IFAC-PapersOnLine*, 53(2):4923–4928, 2020. 21st IFAC World Congress.

[24] H. Shim and A. R. Teel. Asymptotic controllability and observability imply semiglobal practical asymptotic stabilizability by sampled-data output feedback. *Automatica*, 39(3):441–454, 2003.

[25] D. Surroop, P. Combes, P. Martin, and P. Rouchon. Adding virtual measurements by PWM-induced signal injection. In *2020 American Control Conference (ACC)*, pages 2692–2698, 2020.

[26] H. J. Sussmann. Single-input observability of continuous-time systems. *Math. Systems Theory*, 12(4):371–393, 1979.

[27] A. Teel and L. Praly. Global stabilizability and observability imply semi-global stabilizability by output feedback. *Systems & Control Letters*, 22(5):313–325, 1994.

[28] A. Teel and L. Praly. Tools for semiglobal stabilization by partial state and output feedback. *SIAM Journal on Control and Optimization*, 33(5):1443–1488, 1995.