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# AN $L^{3/2}$ $SL_2$ KAKEYA MAXIMAL INEQUALITY

JOHN GREEN, TERENCE L. J. HARRIS, AND YUMENG OU

ABSTRACT. We prove a special case of the Kakeya maximal function conjecture in  $\mathbb{R}^3$ , with  $C_\epsilon \delta^{-\epsilon}$  loss, when the centre lines of the tubes are  $SL_2$  lines and the tubes satisfy a 2-dimensional ball condition (implied by the Wolff axioms). We show that the exponent  $p = 3/2$  is sharp and that some loss (such as  $C_\epsilon \delta^{-\epsilon}$ ) is necessary, even in the  $SL_2$  case where the  $\delta$ -tubes have  $\delta$ -separated directions and the cardinality of the tube family is maximal ( $\sim \delta^{-2}$ ).

The  $SL_2$  Kakeya maximal inequality is deduced from an  $L^{3/2}$  inequality for restricted families of projections onto planes. A related  $L^{3/2-\epsilon}$  inequality is also derived for restricted projections onto lines, and an application is given to generic intersections of sets in  $\mathbb{R}^3$  with “light rays” and “light planes”.

## 1. INTRODUCTION

The aim of this work is to prove the following special case of the Kakeya maximal function conjecture:

**Theorem 1.1.** *Let  $\mathbb{T}$  be a collection of  $SL_2$  tubes in  $\mathbb{R}^3$  of length 1 and radius  $\delta$ , satisfying the condition*

$$(1.1) \quad |\{T \in \mathbb{T} : T \subseteq S\}| \leq (r/\delta)^2,$$

for any tube  $S$  of length 2 and radius  $r$ , and any  $r \in [\delta, 1]$ . Then for any  $\epsilon > 0$ ,

$$(1.2) \quad \left\| \sum_{T \in \mathbb{T}} \chi_T \right\|_{L^{3/2}(B_3(0,1))}^{3/2} \leq C_\epsilon \delta^{-\epsilon} |\mathbb{T}| \delta^2.$$

In Theorem 1.1, a tube is an  $SL_2$  tube if its centre line is an  $SL_2$  line. A line  $\ell \subseteq \mathbb{R}^3$  is called an  $SL_2$  line if there exist  $a, b, c, d \in \mathbb{R}$  with  $ad - bc = 1$  such that

$$\ell = \{(a, b, 0) + \lambda(c, d, 1) : \lambda \in \mathbb{R}\},$$

or if there exists  $t_0 \in \mathbb{R}$  and  $(c, d) \in \mathbb{R}^2 \setminus \{0\}$  such that

$$\ell = \{(0, 0, t_0) + \lambda(c, d, 0) : \lambda \in \mathbb{R}\}.$$

The Wolff axioms (see [19, Definition 1.1]) imply the condition (1.1) (and according to [18, Remark 2] they are equivalent in the  $SL_2$  case), which is why Theorem 1.1 is a special case of the Kakeya maximal function conjecture in  $\mathbb{R}^3$  for families of tubes satisfying the Wolff axioms. We note that the Wolff axioms (and (1.1)) hold whenever the tubes in  $\mathbb{T}$  have  $\delta$ -separated directions.

Theorem 1.1 will be deduced from an  $L^{3/2}$  inequality for restricted families of projections onto planes stated below, and the point-line duality principle from [20, 6].

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**Theorem 1.2.** *If  $\alpha > 2$ , then for any Borel measure  $\mu$  on  $B_3(0, 1)$  with  $c_\alpha(\mu) < \infty$ ,*

$$\int_0^{2\pi} \|\pi_{\theta\#}\mu\|_{L^{3/2}}^{3/2} d\theta \leq C_\alpha \mu(\mathbb{R}^3) c_\alpha(\mu)^{1/2}.$$

Here  $\pi_\theta$  is the orthogonal projection onto the plane  $\frac{1}{\sqrt{2}}(\cos \theta, \sin \theta, 1)^\perp$ , and

$$c_\alpha(\mu) = \sup_{r>0, x \in \mathbb{R}^3} \frac{\mu(B(x, r))}{r^\alpha}.$$

As a counterpart to Theorem 1.2, the following version for projections  $\rho_\theta$  onto the line defined as the span of  $\frac{1}{\sqrt{2}}(\cos \theta, \sin \theta, 1)$  will be proved with a similar method, but using a maximal estimate from [28] as a crucial input.

**Theorem 1.3.** *If  $\alpha > 1$  and  $1 \leq p < 3/2$ , then for any Borel measure  $\mu$  on  $B_3(0, 1)$  with  $c_\alpha(\mu) < \infty$ ,*

$$\int_0^{2\pi} \|\rho_{\theta\#}\mu\|_{L^p}^p d\theta \leq C_\alpha \mu(\mathbb{R}^3) c_\alpha(\mu)^{p-1}.$$

The proofs of Theorem 1.2 and Theorem 1.3 here can easily be generalised to cover  $C^2$  curves  $\gamma : [0, 1] \rightarrow S^2$  with  $\det(\gamma, \gamma', \gamma'')$  nonvanishing, but for simplicity they are only written here for the model curve  $\gamma(\theta) = \frac{1}{\sqrt{2}}(\cos \theta, \sin \theta, 1)$ .

In Section 2 it is shown via an explicit construction that there exist closed  $SL_2$  Besicovitch sets of measure zero, using the same point-line duality as that used in [20]. This is then used to show the corollary that a uniform  $L^p$  bound (i.e. without some loss such as  $|\log \delta|$ ) is not possible in the  $SL_2$  Kakeya maximal inequality (for nontrivial  $p$ ). In Section 3, we show that the exponent  $3/2$  in Theorem 1.1, Theorem 1.2, and Theorem 1.3 is sharp. This yields a counterexample to [18, Remark 1], where it was suggested that  $p = 2$  might be possible in the  $SL_2$  case of Theorem 1.1. The version of the  $SL_2$  Kakeya maximal inequality in [18] appears slightly different to that used here, but the equivalence of the various forms is discussed in Subsection 1.2.

By a recent result of Mattila [25, Theorem 1.2], a consequence of Theorem 1.2 and Theorem 1.3 is the following result about generic intersections of sets in  $\mathbb{R}^3$  with “light rays” and “light planes”.

**Theorem 1.4.** *Let  $A \subseteq \mathbb{R}^3$  be  $\mathcal{H}^s$ -measurable with  $0 < \mathcal{H}^s(A) < \infty$ . If  $s > 1$  then for  $(\mathcal{H}^s \times \mathcal{H}^1)$ -a.e.  $(x, \theta) \in A \times [0, 2\pi)$ ,*

$$\dim(A \cap \rho_\theta^{-1}(\rho_\theta(x))) = s - 1,$$

and for a.e.  $\theta \in [0, 2\pi)$ ,

$$\mathcal{H}^1 \{w \in \text{span}(\gamma(\theta)) : \dim(A \cap \rho_\theta^{-1}(w)) = s - 1\} > 0.$$

If  $s > 2$  then for  $(\mathcal{H}^s \times \mathcal{H}^1)$ -a.e.  $(x, \theta) \in A \times [0, 2\pi)$ ,

$$\dim(A \cap \pi_\theta^{-1}(\pi_\theta(x))) = s - 2,$$

and for a.e.  $\theta \in [0, 2\pi)$ ,

$$\mathcal{H}^2 \{w \in \gamma(\theta)^\perp : \dim(A \cap \pi_\theta^{-1}(w)) = s - 2\} > 0.$$

Here and throughout the paper,  $\dim$  denotes the Hausdorff dimension, and  $\mathcal{H}^s$  refers to the  $s$ -dimensional Hausdorff measure in Euclidean space. Theorem 1.4 is a refinement of the Marstrand-Mattila slicing theorem in  $\mathbb{R}^3$ . (see Theorem 10.10

and Theorem 10.11 in [21]), which has the same statement as the above, but with lines and planes varying over all lines and planes through a point, rather than just “light rays” and “light planes”. By Fubini’s theorem and a simple scaling argument, Theorem 1.4 implies the standard Marstrand-Mattila slicing theorem in  $\mathbb{R}^3$ . Although much work has been done on restricted projection families since they were introduced in [15, 4], the only “restricted intersection” type result in Euclidean space of which we are aware is from [24], which is different from Theorem 1.4. There also exist results on the dimension of exceptional sets in intersection theorems (see e.g. [24] for a survey), but Theorem 1.4 seems to be the first refinement of Marstrand’s intersection theorem for (curved) families of subspaces of dimension less than the full Grassmannian.

Theorem 1.4 also implies that if  $A \subseteq \mathbb{R}^3$  is Borel with  $\dim A > 1$ , then  $\mathcal{H}^1(\rho_\theta(A)) > 0$  for a.e.  $\theta \in [0, 2\pi)$ , which was first proved in [13]. This implication follows by applying a theorem of Davies [1] to reduce to the case where  $A$  has positive finite  $s$ -dimensional  $\mathcal{H}^s$ -measure for some  $s > 1$ , and then observing that

$$\mathcal{H}^1(\rho_\theta(A)) = \mathcal{H}^1 \{w \in \text{span}(\gamma(\theta)) : A \cap \rho_\theta^{-1}(w) \neq \emptyset\},$$

and that clearly the nonempty condition above is weaker than the dimension  $s - 1$  condition in Theorem 1.4. Similarly, Theorem 1.4 implies that if  $A \subseteq \mathbb{R}^3$  is Borel with  $\dim A > 2$ , then  $\mathcal{H}^2(\pi_\theta(A)) > 0$  for a.e.  $\theta \in [0, 2\pi)$ , which was first proved in [8]. These length and area statements in turn imply the Hausdorff dimension versions  $\dim(\rho_\theta(A)) = \min\{1, \dim A\}$  and  $\dim(\pi_\theta(A)) = \min\{2, \dim A\}$  for a.e.  $\theta \in [0, 2\pi)$ ; Fässler and Orponen proved that for general families of Euclidean projections, the dimension exponents in projection theorems can always be lowered by randomly adding points to lower dimensional sets to obtain higher dimensional sets (see e.g. [6] for this type of argument), though this method probably loses information about the exceptional set. The a.e. equality  $\dim(\rho_\theta(A)) = \min\{1, \dim A\}$  was first obtained by Käenmäki, Orponen and Venieri [16], and then for  $C^2$  curves by Pramanik, Yang, and Zahl [28] and Gan, Guth, and Maldague [9] (and all of [16, 28, 9] contain better information about the exceptional set than what follows from Theorem 1.4). The Hausdorff dimension theorem for the projections  $\pi_\theta$  was obtained in [8] (and by a different method in [18]). The a.e. equalities  $\dim(\rho_\theta(A)) = \min\{1, \dim A\}$  and  $\dim(\pi_\theta(A)) = \min\{2, \dim A\}$  can also be obtained as corollaries of Theorem 1.3 and Theorem 1.2 directly (by Hölder’s inequality and the definition of Hausdorff dimension).

The  $SL_2$  example related to the Kakeya problem first appeared in Katz and Zahl’s  $5/2 + \epsilon$  lower bound for the Hausdorff dimension of Besicovitch sets in  $\mathbb{R}^3$  [19], where the proof needed to negotiate a hypothetical  $SL_2$  “almost-counterexample”. Later, Wang and Zahl conjectured that  $SL_2$  Besicovitch sets in  $\mathbb{R}^3$  have Hausdorff dimension 3 (as a special case of the Kakeya conjecture in  $\mathbb{R}^3$ ) [29]. This was proved by Fässler and Orponen [5], and also by Katz, Wu, and Zahl [18]. By a standard argument (see e.g. [31, Proposition 10.2]), Theorem 1.1 also implies that  $SL_2$  Besicovitch sets have Hausdorff dimension 3. A type of  $SL_2$  Kakeya maximal inequality first appeared in [18]. The inequality there implies that  $SL_2$  Besicovitch sets have Hausdorff dimension 3, but does not imply the  $SL_2$  maximal inequality in (1.2) for any  $p > 1$  (in place of  $3/2$ ).

Finally, see [23, Sections 6.7 and 6.8] for a general discussion of connections between restricted projection problems and Kakeya-type problems. We also mention

that a related maximal function to the  $SL_2$  Kakeya maximal function was considered in [7], where the optimal  $L^{3/2}$  inequality was obtained. This maximal function was a function of the angle  $\theta \in [0, \pi)$  of the tube, rather than the direction. The tubes considered were also Heisenberg tubes rather than Euclidean tubes, and the problem was related to the projections  $\rho_\theta$  rather than  $\pi_\theta$ .

**1.1. History and novelty of the method.** This subsection outlines a brief history of the “good-bad” decomposition of a measure in the context of this problem, as this may give some motivation for the proofs of Theorems 1.2 and Theorem 1.3. The decomposition of a measure into “good” and “bad” parts was first developed and applied to the planar distance set problem in [10]. This decomposition was modified in [11] to work for restricted projections onto planes in  $\mathbb{R}^3$ . The argument from [11] contained significant loss in the refined decoupling step, arising from defining the bad part of the measure with tubes rather than planks, which are the natural objects to consider in wave packet decompositions adapted to the cone. The reason for using tubes in [11] was partly that the argument used a lemma from Orponen and Venieri’s previous work on the restricted projection problem [27] as a black box to handle the “bad” part, and this lemma was formulated in terms of tubes. In [12], a bootstrapping argument for handling the “bad” part was developed, which removed the dependence on the lemma from [27], but it was not yet realised in [12] that “tubes” could be replaced with “planks”. This improvement was made in the second part of [8], and the possibility of the improvement was suggested by the first part of [8], which developed a different and independent approach to this problem via the “high-low method”. In the second part of [8], it was shown that pushforwards under  $\pi_\theta$  of measures of dimension greater than 2 are almost surely in  $L^1$ , and the method used to control the  $L^1$  norm of the “bad” part of the measure was an incidence bound proved via the “high-low method”. In this work, for projections onto planes we use an entirely recursive approach to bound the  $L^{3/2}$  norm of the “bad” part, which allows us to make efficient use of  $L^2$  bounds on the “good” part. The proof is mostly self-contained (in particular, the proof does not use the “high-low method”), and the only nontrivial result from [8] that we use is the refined decoupling theorem. Note however that we need only  $L^4$  decoupling for the cone to prove Theorem 1.2, which has a much shorter proof than  $L^6$  decoupling.

The novelty in the proof of Theorem 1.2 is a “self-improving” argument for the exponents, where in practice we need to prove Theorem 1.2 for an increasing sequence of exponents  $\{p_n\}_{n=0}^\infty$ , where  $p_0 = 1$ ,  $p_1 = 4/3$  and  $p_\infty = 3/2$ . For each fixed exponent, we use a separate bootstrapping argument to prove a weaker version of Theorem 1.2 over a “small” union of rectangles in each plane, where the meaning of “small” is gradually weakened until we integrate over the whole planes. At each exponent, a crucial input is a bound on the measure of the “small” unions of rectangles, which we get from applying the projection inequality for the previous exponent  $p_{n-1}$ . Because this projection inequality is better for each  $n$  than the previous  $n$ , the best exponent we can obtain in Theorem 1.2 improves at each step and we calculate that it converges to  $3/2$ .

The “good-bad” decomposition was modified to work for projections onto lines in [13], where it was shown that the projections of measures of dimensions greater than 1 are almost surely in  $L^1$ . By modifying the recursive argument from [13] to get an  $L^p$  bound as in the planar case, the best possible exponent via a purely recursive

argument seems to be  $4/3$ , which is not the sharp exponent in Theorem 1.3. To get the exponent  $3/2 - \epsilon$ , we need to use an  $L^{3/2}$  Keakeya bound for “well-spaced curves” of Pramanik, Yang, and Zahl [28] as a black box, and therefore our only substantial contribution to Theorem 1.3 is an “ $\epsilon$ -removal” type argument, which uses  $L^6$  decoupling to leverage the  $\alpha > 1$  assumption in Theorem 1.3 to convert their result into a uniform bound as in Theorem 1.3.

We remark that this work supersedes the results from [14]. For completeness, and since the work from [14] will likely not be submitted for publication, some of the work from [14] is reproduced here (especially in Section 2 and Section 3). We also want to point out that, even though the  $L^p$  bounds in analogues of Theorem 1.2 and Theorem 1.3 in [14] are weaker (at only  $p = \frac{4}{3}$  and  $p = \frac{6}{5}$  respectively), they are sufficient to derive the same restricted intersection results (with light rays and light planes) as in Theorem 1.4.

**1.2. Different forms of the  $SL_2$  Keakeya maximal inequality.** Given a Borel function  $f$  on  $\mathbb{R}^3$ , let  $f_{\delta, SL_2}^* : S^2 \setminus \{(0, 0, \pm 1)\}$  be the  $SL_2$  Keakeya maximal function at scale  $\delta$ , which is defined in the same way as for the standard Keakeya maximal function except that the centre lines  $\ell(T)$  of the  $\delta$ -tubes  $T$  are required to be  $SL_2$  lines:

$$f_{\delta, SL_2}^*(e) = \sup_{\substack{\text{dir}(T)=e \\ T=\delta \times 1\text{-tube} \\ \ell(T) \in SL_2}} \frac{1}{m(T)} \int_T |f|,$$

where  $m(T)$  is the Lebesgue measure of  $T$ . The terms “ $SL_2$ ” and “horizontal” will be used interchangeably, though strictly speaking they are slightly different.

If  $p^* \in (1, \infty)$  is fixed, then the inequality

$$(1.3) \quad \|f_{\delta, SL_2}^*\|_{L^{p^*}(S^2)} \leq C_\epsilon \delta^{-\epsilon} \|f\|_{p^*}$$

is equivalent (by the standard duality argument; see e.g. [31, Lemma 10.4]) to the inequality

$$(1.4) \quad \left\| \sum_{T \in \mathbb{T}} \chi_T \right\|_{p^*}^{p^*} \leq C_\epsilon \delta^{-\epsilon} |\mathbb{T}| \delta^2$$

whenever  $\mathbb{T}$  is a family of  $SL_2$   $\delta$ -tubes of length 1 with  $\delta$ -separated directions. It is known that the inequality (1.4) is equivalent to

$$(1.5) \quad \mathcal{H}^3 \left( \bigcup_{T \in \mathbb{T}} Y(T) \right) \geq C_\epsilon^{-1} \delta^\epsilon \lambda^{p^*} \delta^2 |\mathbb{T}|,$$

whenever  $\mathbb{T}$  is a family of  $SL_2$   $\delta$ -tubes of length 1 with  $\delta$ -separated directions,  $\delta \leq \lambda \leq 1$ , and  $Y$  is a  $\lambda$ -shading of  $\mathbb{T}$  (i.e. a union of  $\delta$ -cubes such that at least a fraction  $\lambda$  of each tube in  $\mathbb{T}$  is covered by the cubes in  $Y$ , meaning that  $\mathcal{H}^3(Y(T)) \geq \lambda \mathcal{H}^3(T)$  for any  $T \in \mathbb{T}$ , where  $Y(T) = Y \cap T$ ). The inequality from [18] states that (1.4) holds with  $p^* = M$  depending on  $\epsilon$  (and possibly tending to  $\infty$  as  $\epsilon \rightarrow 0$ ). The proof of the equivalence of (1.4) and (1.5) is similar to that for the standard Keakeya maximal function, but will be summarised here for readability. That (1.4) implies (1.5) follows from Hölder’s inequality. The reverse implication follows by pigeonholing a set  $Y$  of  $\delta$ -cubes on which the value of  $\sum_{T \in \mathbb{T}} \chi_T$  is approximately constant in order to reverse Hölder’s inequality, then by pigeonholing again to a subset  $\mathbb{T}' \subseteq \mathbb{T}$  (of

potentially smaller cardinality) of tubes containing an approximately equal fraction  $\lambda$  of the cubes from  $Y$  in each tube, and then applying (1.5) to this subset.

An a priori weaker inequality than (1.4) is

$$(1.6) \quad \left\| \sum_{T \in \mathbb{T}} \chi_T \right\|_{L^p(B_3(0,1))}^p \leq C_\epsilon \delta^{-\epsilon},$$

whenever  $\mathbb{T}$  is a family of  $SL_2$   $\delta$ -tubes of length 1 with  $\delta$ -separated directions. Clearly (1.4) implies (1.6), and the implication can be reversed for the standard Kakeya maximal function by rotational averaging (see e.g. [22, Proposition 22.7]), but rotations in  $O(3)$  do not necessarily preserve  $SL_2$  lines, so it does not seem obvious that (1.6) implies (1.4) in the  $SL_2$  case. In Section 3 it is shown that (1.6) cannot hold in the  $SL_2$  case for all  $\epsilon > 0$  unless  $p \leq 3/2$  (and therefore neither can (1.5), (1.4) or (1.3)).

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#### 2. $SL_2$ BESICOVITCH SETS OF MEASURE ZERO

Given  $(x_1, y_1, t_1), (x_2, y_2, t_2) \in \mathbb{R}^3$ , let

$$(x_1, y_1, t_1) * (x_2, y_2, t_2) = \left( x_1 + x_2, y_1 + y_2, t_1 + t_2 + \frac{1}{2}(x_1 y_2 - x_2 y_1) \right),$$

be the Heisenberg product. A line  $\ell \subseteq \mathbb{R}^3$  is called horizontal if there exists  $\theta \in [0, \pi)$  and  $p \in \mathbb{R}^3$  such that

$$\ell = p * \mathbb{V}_\theta,$$

where  $\mathbb{V}_\theta = \{(\lambda e^{i\theta}, 0) : \lambda \in \mathbb{R}\}$ . Horizontal line segments and  $SL_2$  line segments are line segments which are subsets of horizontal and  $SL_2$  lines, respectively. A set  $K \subseteq \mathbb{R}^3$  will be called a horizontal Besicovitch set if  $K$  contains a horizontal unit line segment in every direction in  $S^2 \setminus \{(0, 0, \pm 1)\}$ . Similarly,  $K$  will be called an  $SL_2$  Besicovitch set if  $K$  contains an  $SL_2$  unit line segment in every direction in  $S^2 \setminus \{(0, 0, \pm 1)\}$ . If  $F(x, y, z) = (x, y, 2z)$ , then  $F$  sends horizontal lines to  $SL_2$  lines, and  $\ell \mapsto F(\ell)$  is a bijection between horizontal and  $SL_2$  lines with inverse  $\ell \mapsto F^{-1}(\ell)$ , where  $F^{-1}(x, y, z) = (x, y, z/2)$ . For any horizontal line segment  $I$ , the length of  $F(I)$  is comparable to the length of  $I$ , so  $SL_2$  lines and horizontal lines are more or less the same.

Every horizontal line can be written in the form  $p * \mathbb{V}_\theta$  with  $(p_1, p_2)$  in the span of  $ie^{i\theta}$ , where  $p = (p_1, p_2, p_3)$ . If  $R > 0$  and if such a horizontal line intersects  $B(0, R)$ , then  $|(p_1, p_2)| \leq R$  and the direction of  $p * \mathbb{V}_\theta$  has distance at least  $2/R$  from  $(0, 0, \pm 1)$ . It follows that a bounded set cannot contain a horizontal unit line segment in every direction in  $S^2 \setminus \{0, 0, \pm 1\}$  (and similarly for  $SL_2$  line segments). For this reason, the compact sets of horizontal and  $SL_2$  line segments in Theorem 2.2 below contain only unit line segments in every direction in  $S^2 \setminus \mathcal{N}_\epsilon(\{(0, 0, \pm 1)\})$  for

a given  $\epsilon > 0$ , and to get all directions in  $S^2 \setminus \{(0, 0, \pm 1)\}$  it is necessary to replace “compact” with “closed”.

The following (straightforward) lemma is used to make the proof of Theorem 2.2 below constructive, though it is possible to give a non-constructive proof of Theorem 2.2 below by substituting the application of Lemma 2.1 in the proof of Theorem 2.2 with an averaging/Fubini/scaling argument.

**Lemma 2.1.** *If  $E \subseteq (-\pi/2, \pi/2)$  is Lebesgue measurable with  $\mathcal{H}^1(E) = 0$ , and if  $F$  is the union of great circles defined by*

$$F = \{(\lambda e^{i\theta}, \mu) \in S^2 : (\lambda, \mu) \in S^1, \quad \theta \in E\},$$

*then  $\mathcal{H}^1(F \cap C) = 0$  for any circle  $C \subseteq S^2$  with the property that either  $(0, 0, 1) \notin C$  or  $(0, 0, -1) \notin C$ .*

*Proof.* Let  $E, C$  be given. By stereographic projection on the Riemann sphere (which maps circles to circles or lines) it suffices to show that if  $\tilde{C}$  is either a circle in  $\mathbb{R}^2$  or a line in  $\mathbb{R}^2$  which does not pass through the origin, then  $\mathcal{H}^1(\tilde{F} \cap \tilde{C}) = 0$ , where

$$\tilde{F} = \{\lambda e^{i\theta} : \lambda \in \mathbb{R}, \quad \theta \in E\}.$$

To show this, it suffices to show that for any  $(x_0, y_0) \in \tilde{C} \setminus \{0\}$ , there exists an  $\epsilon > 0$  such that  $\mathcal{H}^1(\tilde{F} \cap \tilde{C} \cap B((x_0, y_0), \epsilon)) = 0$ . By scaling and rotation it may be assumed that  $(x_0, y_0) = (1, 0)$ . If  $\tilde{C}$  is a circle, then let  $(a, b)$  and  $r > 0$  be such that  $\tilde{C}$  is parametrised by  $\gamma(t) = (\gamma_1(t), \gamma_2(t)) = (a + r \cos t, b + r \sin t)$ , where  $\gamma : [0, 2\pi) \rightarrow \tilde{C}$  satisfies  $\gamma(t_0) = (1, 0)$  for some  $t_0 \in [0, 2\pi)$ . Otherwise let  $\gamma(t) = (1, 0) + t(v_1, v_2)$  be the parametrisation of  $\tilde{C}$ , where  $v_2 \neq 0$  and  $t \in \mathbb{R}$ , and define  $t_0 = 0$ . It suffices to show that for  $\delta > 0$  sufficiently small,

$$\mathcal{H}^1\{t \in (t_0 - \delta, t_0 + \delta) : \arctan(\gamma_2(t)/\gamma_1(t)) \in E\} = 0.$$

Equivalently, if  $G : (t_0 - \delta, t_0 + \delta) \rightarrow (-\pi/2, \pi/2)$  is defined by  $G(t) = \arctan(\gamma_2(t)/\gamma_1(t))$ , then  $(G_{\#}\mathcal{H}^1)(E) = 0$  for  $\delta > 0$  sufficiently small. Since  $\mathcal{H}^1(E) = 0$ , it is enough to show that  $G_{\#}\mathcal{H}^1 \ll \mathcal{H}^1$  for  $\delta > 0$  sufficiently small. If  $\tilde{C}$  is a circle and  $(a, b) = 0$ , then  $G'(t) = \frac{d}{dt} \arctan(\gamma_2(t)/\gamma_1(t)) = 1$  for all  $t$  in a neighbourhood of  $t = t_0 = 0$ , and  $G_{\#}\mathcal{H}^1 \ll \mathcal{H}^1$  follows. If  $\tilde{C}$  is a line, then

$$G'(t) = \frac{v_2}{|\gamma(t)|^2} \neq 0,$$

for all  $t$  in a neighbourhood of  $t_0 = 0$ , and again  $G_{\#}\mathcal{H}^1 \ll \mathcal{H}^1$  follows. The remaining case is where  $\tilde{C}$  is a circle and  $(a, b) \neq 0$ . Let  $H(t) = \det(\gamma(t), \gamma'(t))$ . Then

$$G'(t) = \frac{H(t)}{|\gamma(t)|^2},$$

for all  $t$  in a neighbourhood of 0, so if  $H(t)$  is nonvanishing at  $t = t_0$ , it follows that  $G_{\#}\mathcal{H}^1 \ll \mathcal{H}^1$  for  $\delta > 0$  sufficiently small. If  $H(t_0) = 0$ , then  $H(t) \neq 0$  for all  $t \in (t_0 - \delta, t_0 + \delta) \setminus \{t_0\}$  for some  $\delta > 0$  sufficiently small, which follows from the identity  $|H'(t)|^2 + |H''(t)|^2 = r^2(a^2 + b^2)$ . Therefore  $G_{\#}\mathcal{H}^1 \ll \mathcal{H}^1$  for  $\delta > 0$  sufficiently small. This finishes the proof.  $\square$

**Theorem 2.2.**



- (1) For any  $\epsilon > 0$ , there exists a compact set of measure zero containing a horizontal unit line segment in every direction in  $S^2 \setminus \mathcal{N}_\epsilon(\{(0, 0, \pm 1)\})$ .
- (2) There exists a closed set of measure zero containing a horizontal unit line segment in every direction in  $S^2 \setminus \{(0, 0, \pm 1)\}$ .
- (3) There exists a closed set of measure zero containing a horizontal line in every direction in  $S^2 \setminus \{(0, 0, \pm 1)\}$ .

The above claims all hold if “a horizontal” is replaced by “an  $SL_2$ ”.

*Proof.* Let  $A$  be any compact subset of  $\mathbb{R}^3$  such that  $0 < \mathcal{H}^2(A) < \infty$ , such that the projection of  $A$  down to the  $(x, y)$ -plane contains a nonempty open disc, and such that for any non-great circle  $C \subseteq S^2$ ,  $\mathcal{H}^2(P_{v^\perp}(A)) = 0$  for  $\mathcal{H}^1$ -a.e.  $v \in C$ . Such a set can be constructed as follows. Let  $\mathcal{C}$  be the Cantor set in  $\mathbb{R}$  of dimension  $1/2$  obtained by starting with the unit interval  $[0, 1]$  and removing the open middle half interval at each step. It is a standard result that

$$\{2x + y : x, y \in \mathcal{C}\} = [0, 3],$$

as can be seen, e.g., by characterising  $\mathcal{C}$  as those elements of  $[0, 1]$  of the form  $\sum_{j=1}^{\infty} \frac{\varepsilon_j}{4^j}$  with  $\varepsilon_j \in \{0, 3\}$  for all  $j$ . It follows that if  $B = \mathcal{C} \times \mathcal{C} \times [0, 1]$ , then the projection of  $B$  onto the plane spanned by  $\frac{1}{\sqrt{5}}(2, 1, 0)$  and  $(0, 0, 1)$  contains a nonempty open disc. Moreover, if  $v \in S^2$  is of the form  $(\lambda_1 e^{i\theta}, \lambda_2)$ , then

$$\pi_{v^\perp}(B) \subseteq \pi_{(ie^{i\theta}, 0)}(\mathcal{C} \times \mathcal{C} \times \{0\}) + \pi_{(\lambda_2 e^{i\theta}, -\lambda_1)}(\mathbb{R}^3).$$

Hence, by Fubini’s theorem,

$$\{v \in S^2 : \mathcal{H}^2(\pi_{v^\perp}(B)) > 0\} \subseteq F := \{(\lambda_1 e^{i\theta}, \lambda_2) : \theta \in E, (\lambda_1, \lambda_2) \in S^1\},$$

where

$$E = \{\theta \in [0, \pi) : \mathcal{H}^1(\pi_{ie^{i\theta}}(\mathcal{C} \times \mathcal{C})) > 0\}.$$

But  $\mathcal{H}^1(E) = 0$  (see e.g. [22, Theorem 10.1]), and by Lemma 2.1 this implies that  $\mathcal{H}^1(F \cap C) = 0$  for any non-great circle  $C$ . It follows that for any non-great circle  $C$ ,  $\mathcal{H}^2(\pi_{v^\perp}(B)) = 0$  for  $\mathcal{H}^1$ -a.e.  $v \in C$ . Since this property is rotation-invariant, replacing  $B$  by a rotation of  $B$  which sends the plane spanned by  $\frac{1}{\sqrt{5}}(2, 1, 0)$  and  $(0, 0, 1)$  to the  $(x, y)$ -plane in  $\mathbb{R}^3$  yields the required set  $A$ .

For the first claim in the theorem statement, let

$$K = K(A) = \bigcup_{(a,b,c) \in A} \left\{ \left( as + b, s, c + \frac{bs}{2} \right) : |s| \leq \frac{1}{\sqrt{4 + 4a^2 + b^2}} \right\}.$$

Then, by Fubini’s theorem,

$$\mathcal{H}^3(K) \leq \int_{-1}^1 \mathcal{H}^2 \left\{ (x, y) \in \mathbb{R}^2 : (x, y) = \left( as + b, c + \frac{bs}{2} \right) : (a, b, c) \in A \right\} ds.$$

For each  $s \in (-1, 1)$ , the set

$$\left\{ (x, y) \in \mathbb{R}^2 : (x, y) = \left( as + b, c + \frac{bs}{2} \right) : (a, b, c) \in A \right\}$$

can be written as

$$\{(\langle p, (s, 1, 0) \rangle, \langle p, (0, s/2, 1) \rangle) \in \mathbb{R}^2 : p \in A\}.$$

The vectors  $(s, 1, 0)$  and  $(0, s/2, 1)$  are both orthogonal to  $(1, -s, s^2/2)$ . The vectors  $(s, 1, 0)$  and  $(0, s/2, 1)$  are not necessarily orthogonal to each other, but by following the Gram-Schmidt process<sup>1</sup>, the above can be written as

$$A_s \left\{ \left( \langle p, (s, 1, 0) \rangle, \left\langle p, (0, s/2, 1) - \frac{s}{2(1+s^2)}(s, 1, 0) \right\rangle \right) \in \mathbb{R}^2 : p \in A \right\},$$

where  $A_s : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  is the linear map

$$A_s(x, y) = \left( x, y + \frac{sx}{2(1+s^2)} \right).$$

Since  $\det A_s = 1$ ,

$$\begin{aligned} & \mathcal{H}^2 \{ (\langle p, (s, 1, 0) \rangle, \langle p, (0, s/2, 1) \rangle) \in \mathbb{R}^2 : p \in A \} \\ &= \mathcal{H}^2 \left\{ \left( \left\langle p, (s, 1, 0) \right\rangle, \left\langle p, (0, s/2, 1) - \frac{s}{2(1+s^2)}(s, 1, 0) \right\rangle \right) \in \mathbb{R}^2 : p \in A \right\} \\ & \sim \mathcal{H}^2 \left( \pi_{(1, -s, s^2/2)^\perp}(A) \right). \end{aligned}$$

The curve  $(1, -s, s^2/2)$  is a parabola inside the cone  $\eta_2^2 = 2\eta_1\eta_3$ , and this cone is a clockwise rotation of the cone  $\xi_3^2 = \xi_1^2 + \xi_2^2$  by  $\pi/4$  in the  $(\xi_1, \xi_3)$ -plane. It follows that the normalised curve  $\frac{1}{\sqrt{1+s^2+s^4/4}}(1, -s, s^2/2)$  lies in the intersection of the cone  $\eta_2^2 = 2\eta_1\eta_3$  with the sphere  $S^2$ , and this intersection is a non-great circle in  $S^2$ . Hence  $\mathcal{H}^2(\pi_{(1, -s, s^2/2)^\perp}(A)) = 0$  for a.e.  $s \in \mathbb{R}$ , which yields  $\mathcal{H}^3(K) = 0$ . Since the projection of  $A$  down to the  $(x, y)$ -plane contains an open disc  $U \subseteq \mathbb{R}^2$ , for all  $(a, b) \in U$  the set  $K$  contains a horizontal unit line segment with direction  $\frac{1}{a^2+1+(b/2)^2}(a, 1, b/2)$ .

Any translate  $A_{(x_0, y_0)} := A + (x_0, y_0, 0)$  of  $A$  with  $(x_0, y_0) \in \mathbb{R}^2$  has the property that the projection of  $A_{(x_0, y_0)}$  to  $\mathbb{R}^2 \times \{0\}$  contains the open disc  $U_{(x_0, y_0)} := U + (x_0, y_0)$ , and also has the property that for any non-great circle  $C$ , the projection of  $A_{(x_0, y_0)}$  onto  $v^\perp$  has  $\mathcal{H}^2$ -measure zero for  $\mathcal{H}^1$ -a.e.  $v \in C$ . Fix  $(a_0, b_0) \in U$ . For any  $v \in S^2 \setminus \mathbb{R} \times \{0\} \times \mathbb{R}$ , choose  $(a_v, b_v) \in \mathbb{R}^2$  such that  $(a_v + a_0, 1, (b_v + b_0)/2)$  is parallel to  $v$ . Then  $K(A_{(a_v, b_v)})$  contains a horizontal unit line segment with direction  $\frac{1}{a^2+1+(b/2)^2}(a, 1, b/2)$  for all  $(a, b) \in U + (a_v, b_v)$ . This shows that for any  $v \in S^2 \setminus \mathbb{R} \times \{0\} \times \mathbb{R}$ , there is an open neighbourhood  $U_v \subseteq S^2$  of  $v$  and a measure zero compact set  $K(A_{(a_v, b_v)})$  containing a horizontal unit line segment with every direction in  $U_v$ . Since the collection of horizontal lines is invariant under rotations of the  $(x, y)$ -plane (which fix the third coordinate), this implies that for any  $v \in S^2 \setminus \{(0, 0, \pm 1)\}$ , there is an open neighbourhood  $U_v$  of  $v$  and a measure zero compact set containing a horizontal unit line segment with every direction in  $U_v$ . This implies that for any  $\epsilon > 0$ , there exists a measure zero compact set containing a horizontal unit line segment in every direction in  $S^2 \setminus \mathcal{N}_\epsilon(\{(0, 0, \pm 1)\})$ . This verifies the first claim.

For the second claim, take a countable sequence of translates  $A_1, A_2, \dots$  of  $A$  by vectors in  $\mathbb{R}^2 \times \{0\}$  such that  $d(A_j, 0) \rightarrow \infty$ , and such that the projections of the

<sup>1</sup>The same observation was made in [5].

$A_j$ 's onto  $\mathbb{R}^2 \times \{0\}$  cover  $\mathbb{R}^2$ . Let

$$\begin{aligned} K_1 &= (\mathbb{R} \times \{0\} \times \mathbb{R}) \cup \bigcup_{(a,b,c) \in \bigcup_j A_j} K(A_j) \\ &= (\mathbb{R} \times \{0\} \times \mathbb{R}) \cup \bigcup_{(a,b,c) \in \bigcup_j A_j} \left\{ \left( as + b, s, c + \frac{bs}{2} \right) : |s| \leq \frac{1}{\sqrt{4 + 4a^2 + b^2}} \right\}, \end{aligned}$$

and let  $K_2$  be a rotation of  $K_1$  by  $\pi/2$  in the  $(x, y)$ -plane. It has already been shown that  $K = K_1 \cup K_2$  has measure zero and contains a horizontal unit line segment in every direction in  $S^2 \setminus \{(0, 0, \pm 1)\}$ . By the condition  $d(0, A_j) \rightarrow \infty$ , together with the compactness of  $A$  and the definition of  $K$ , it is straightforward to check that  $K$  is closed. This verifies the second claim.

The third claim follows from a similar argument with the same  $A_j$ 's; by taking

$$K_1 = (\mathbb{R} \times \{0\} \times \mathbb{R}) \cup \bigcup_{(a,b,c) \in \bigcup_j A_j} \left\{ \left( as + b, s, c + \frac{bs}{2} \right) : s \in \mathbb{R} \right\}.$$

Then  $K = K_1 \cup K_2$  has measure zero by a similar argument to the above, and is a closed set of measure zero containing a horizontal line in every direction in  $S^2 \setminus \{(0, 0, \pm 1)\}$ .

Finally, the proof for  $SL_2$  lines is similar.  $\square$

**Corollary 2.3.** *If  $p \in (1, \infty]$  and*

$$(2.1) \quad \left\| \sum_{T \in \mathbb{T}} \chi_T \right\|_{L^p(B(0,1))} \leq C_\delta,$$

for any set  $\mathbb{T}$  of  $\delta \times \delta \times 1$   $SL_2$  tubes with  $\delta$ -separated directions in  $S^2$ , then  $\lim_{\delta \rightarrow 0} C_\delta = \infty$ .

Let  $U$  be any Borel subset of  $S^2 \setminus \{(0, 0, \pm 1)\}$  with  $\mathcal{H}^2(U) > 0$ . If, for some fixed  $p \in [1, \infty)$ ,

$$\|f_{\delta, SL_2}^*\|_{L^p(U)} \leq C_{\delta, U} \|f\|_p,$$

for all  $\delta > 0$  and for any non-negative Borel function  $f$  on  $\mathbb{R}^3$ , then  $\lim_{\delta \rightarrow 0} C_{\delta, U} = \infty$ .

*Proof.* For the first part, using Theorem 2.2 let  $B$  be a compact set of measure zero containing a unit line segment in every direction in  $U = S^2 \setminus \mathcal{N}_{0.01}\{(0, 0, \pm 1)\}$ . Let  $\tilde{U} = \{v_1, \dots, v_N\}$  be a maximal  $\delta$ -separated subset of  $U$ , where  $N \sim \delta^{-2}$ . For each  $v_k \in \tilde{U}$ , let  $T_k$  be a  $\delta \times \delta \times 1$   $SL_2$  tube such that  $T \subseteq \mathcal{N}_\delta(B)$ . Then (for a sufficiently large absolute constant  $C$ )

$$1 \lesssim \int_{B(0,C)} \sum_{k=1}^N \chi_{T_k} \leq \mathcal{H}^3(\mathcal{N}_\delta(B))^{1-\frac{1}{p}} \left\| \sum_{k=1}^N \chi_{T_k} \right\|_{L^p(B(0,C))}.$$

Hence, by (2.1),

$$1 \lesssim \mathcal{H}^3(\mathcal{N}_\delta(B))^{1-\frac{1}{p}} C_\delta.$$

Letting  $\delta \rightarrow 0$  proves the first part.

For the second part, let  $V \subseteq U$  be a compact subset of  $U$  such that  $\mathcal{H}^2(V) > 0$ . Let  $B$  be a compact set of measure zero containing an  $SL_2$  unit line segment in every direction in  $V$ , which exists by Theorem 2.2. Then

$$1 \lesssim \left\| (\chi_{\mathcal{N}_\delta(B)})_{\delta, SL_2}^* \right\|_{L^p(V)} \leq C_{\delta, U} \mathcal{H}^3(\mathcal{N}_\delta(B))^{1/p}.$$

Letting  $\delta \rightarrow 0$  proves the second part.  $\square$

### 3. TRAIN TRACKS OF PLANKS

Given a Borel measure  $\mu$  on  $\mathbb{R}^3$ , and  $\alpha \geq 0$ , recall that

$$c_\alpha(\mu) = \sup_{x \in \mathbb{R}^3, r > 0} \frac{\mu(B(x, r))}{r^\alpha}.$$

Let

$$I_\alpha(\mu) = \int \int \frac{1}{|x - y|^\alpha} d\mu(x) d\mu(y).$$

Recall that  $\gamma : [0, 2\pi) \rightarrow S^2$  is defined by  $\gamma(\theta) = \frac{1}{\sqrt{2}}(\cos \theta, \sin \theta, 1)$ , and recall that for each  $\theta \in [0, 2\pi)$ ,  $\pi_\theta$  is the orthogonal projection onto the plane  $\gamma(\theta)^\perp$ . In [26], Oberlin and Oberlin used Erdoğan's  $L^2$  bound for the decay of conical averages of Fourier transforms of fractal measures from [3] to prove a bound on the average  $L^2$  norms of pushforwards of fractal measures under  $\pi_\theta$ . When  $\alpha > 5/2$ , the uncertainty principle suggests that there is no loss in the approach from [26] (up to the endpoint, provided only  $L^2$  norms are considered), and this suggests that the Knapp examples from [3] used to prove sharpness of the  $L^2$  conical decay rates can also be used to prove sharpness of the bound from [26]. The following proposition verifies this intuition.

**Proposition 3.1.** *Let  $\alpha \in [2, 3]$ . If*

$$(3.1) \quad \int_0^{2\pi} \|\pi_{\theta\#}\mu\|_{L^2(\gamma(\theta)^\perp)}^2 d\theta \leq C_\alpha \mu(\mathbb{R}^3) c_\alpha(\mu),$$

for some nonzero finite  $C_\alpha$  depending only on  $\alpha$ , for all Borel measures  $\mu$  on  $B_3(0, 1)$ , then  $\alpha \geq 5/2$ . The same is true if  $\mu(\mathbb{R}^3) c_\alpha(\mu)$  in (3.1) is replaced by  $I_\alpha(\mu)$ .

*Remark 3.2.* Oberlin and Oberlin proved that (3.1) does hold if  $\alpha > 5/2$ , with either  $\mu(\mathbb{R}^3) c_\alpha(\mu)$  or  $I_\alpha(\mu)$  on the right-hand side [26].

*Proof of Proposition 3.1.* Let  $\alpha \in [2, 3]$  and suppose that (3.1) holds. Let  $\delta > 0$  be small. Fix any  $\theta_0 \in [0, 2\pi)$ . Let  $\psi$  be a smooth non-negative bump function supported in  $B(0, 1)$ , with  $\psi \sim 1$  on  $B(0, 1/2)$ , such that  $\int \psi = 1$ . Let  $\mu$  be the measure with Radon-Nikodym derivative equal to

$$\mu(x) = \delta^{-3/2} \psi \left( \langle x, \gamma(\theta_0) \rangle, \left\langle \delta^{-1/2} x, \sqrt{2} \gamma'(\theta_0) \right\rangle, \left\langle \delta^{-1} x, \sqrt{2} (\gamma \times \gamma')(\theta_0) \right\rangle \right),$$

i.e.,  $\mu$  is a Schwartz function of  $L^1$  norm 1 supported on a plank of dimensions  $\sim 1 \times \delta^{1/2} \times \delta$  centred at the origin, such that the longest direction is parallel to  $\gamma(\theta_0)$ , the medium direction is parallel to  $\gamma'(\theta_0)$  and the shortest direction is parallel to  $(\gamma \times \gamma')(\theta_0)$ . If  $|y_1| \leq \delta^{1/2}/100$ ,  $|y_2| \leq \delta/100$ , and  $|y_3| \leq 1/100$ , then

$$\mu(y_1 \gamma'(\theta_0) + y_2 (\gamma \times \gamma')(\theta_0) + y_3 \gamma(\theta_0)) \gtrsim \delta^{-3/2}.$$

By second-order Taylor approximation, it follows that if  $|\theta - \theta_0| < c\delta^{1/2}$  for  $c > 0$  a sufficiently small absolute constant, then for all  $|x_1| \leq 10^{-3}\delta^{1/2}$  and  $|x_2| \leq 10^{-3}\delta$ ,

$$\begin{aligned} & (\pi_{\theta\#\mu})(x_1\gamma'(\theta) + x_2(\gamma \times \gamma')(\theta)) \\ & \geq \int_{-10^{-3}}^{10^{-3}} \mu(x_1\gamma'(\theta) + x_2(\gamma \times \gamma')(\theta) + t\gamma(\theta)) dt \gtrsim \delta^{-3/2}. \end{aligned}$$

Therefore

$$\begin{aligned} & \int_0^{2\pi} \|\pi_{\theta\#\mu}\|_{L^2(\gamma(\theta)^\perp)}^2 d\theta \\ & \geq \int_{|\theta-\theta_0| \leq c\delta^{1/2}} \int_{[-10^{-3}\delta^{1/2}, 10^{-3}\delta^{1/2}]} \int_{[-10^{-3}\delta, 10^{-3}\delta]} \\ & \quad |(\pi_{\theta\#\mu})(x_1\gamma'(\theta) + x_2(\gamma \times \gamma')(\theta))|^2 dx_1 dx_2 d\theta \\ & \gtrsim \delta^{-1}. \end{aligned}$$

But  $\mu(\mathbb{R}^3) = 1$ , and it is straightforward to check that

$$c_\alpha(\mu) \sim \delta^{\frac{3}{2}-\alpha}.$$

Similarly  $I_\alpha(\mu) \sim \delta^{\frac{3}{2}-\alpha}$  (e.g. via the Plancherel formula for the energy). Hence, by the assumed inequality (3.1),

$$\delta^{-1} \lesssim \delta^{\frac{3}{2}-\alpha}.$$

Letting  $\delta \rightarrow 0$  gives  $-1 \geq \frac{3}{2} - \alpha$  or  $\alpha \geq 5/2$ .  $\square$

In the previous example,  $c_\alpha(\mu)$  is much larger than  $\mu(\mathbb{R}^3)$ . But if  $\nu$  is the sum of  $N$  copies of  $\mu$  translated in the  $\gamma'(\theta_0)$  and  $(\gamma \times \gamma')(\theta_0)$  directions, such that the projections of these translated copies under  $\pi_\theta$  are pairwise disjoint for  $|\theta - \theta_0| \lesssim \delta^{1/2}$ , then the lower bound for the left-hand side will be multiplied by  $N$ . If the translations are chosen sufficiently sparse, to ensure that  $c_\alpha(\nu)$  is not larger than  $c_\alpha(\mu)$ , then the right-hand side will also be scaled by  $N$ , since  $\mu(\mathbb{R}^3)$  will be scaled by  $N$ . If  $N$  could be chosen large enough to make  $\nu(\mathbb{R}^3) \sim \delta^{3/2-\alpha} \sim c_\alpha(\nu) \sim c_\alpha(\mu)$ , then this would show that no inequality is possible even with the larger right-hand side of  $c_\alpha(\nu)^2$  instead of  $\nu(\mathbb{R}^3)c_\alpha(\nu)$  in (3.1). The proposition below shows that this is possible using “parallel train tracks of planks”. The spacing is based on the example of “parallel train tracks” from [10, Proposition 6.1], where the long spaces are a  $\delta^{-1/2}$ -multiple of the short spaces (the  $R$  in [10] corresponds to  $\delta^{-1}$  here).

**Proposition 3.3.** *Let  $\alpha \in [2, 3]$ . If*

$$(3.2) \quad \int_0^{2\pi} \|\pi_{\theta\#\nu}\|_{L^2(\gamma(\theta)^\perp)}^2 d\theta \leq C_\alpha c_\alpha(\nu)^2,$$

for some nonzero finite  $C_\alpha$  depending only on  $\alpha$ , for all Borel measures  $\nu$  on  $B_3(0, 1)$ , then  $\alpha \geq 5/2$ .

*Proof.* Let  $\alpha \in [2, 3]$  and suppose that (3.2) holds. Let  $\delta > 0$  be small, and fix any  $\theta_0 \in [0, 2\pi)$ . Let  $\mu = \mu_{\delta, \theta_0}$  be the measure from the proof of Proposition 3.1, given by

$$\mu(x) = \delta^{-3/2} \psi \left( \langle x, \gamma(\theta_0) \rangle, \left\langle \delta^{-1/2}x, \sqrt{2}\gamma'(\theta_0) \right\rangle, \left\langle \delta^{-1}x, \sqrt{2}(\gamma \times \gamma')(\theta_0) \right\rangle \right),$$

where  $\psi$  is a smooth non-negative bump function supported in  $B(0, 1)$ , with  $\psi \sim 1$  on  $B(0, 1/2)$ , such that  $\int \psi = 1$ . Let  $\nu$  be the sum of translated copies of  $\mu$ ; spacing  $\delta^{\frac{\alpha}{2}-\frac{1}{2}}$  in the short direction  $(\gamma \times \gamma')(\theta_0)$ , and spacing  $\delta^{\frac{\alpha}{2}-1}$  in the medium direction  $\gamma'(\theta_0)$ , given by

$$\nu(x) = \sum_{|m| \leq 10^{-3}\delta^{\frac{1}{2}-\frac{\alpha}{2}}} \sum_{|n| \leq 10^{-3}\delta^{1-\frac{\alpha}{2}}} \mu\left(x - \delta^{\frac{\alpha}{2}-\frac{1}{2}}m(\gamma \times \gamma')(\theta_0) - n\delta^{\frac{\alpha}{2}-1}\gamma'(\theta_0)\right).$$

There are  $\sim \delta^{\frac{3}{2}-\alpha}$  such copies, and the supports of the projections of these translated copies under  $\pi_\theta$  are pairwise disjoint for  $|\theta - \theta_0| \leq c\delta^{1/2}$ , for a sufficiently small absolute constant  $c$  (using  $\alpha \leq 3$  and second-order Taylor approximation), so the lower bound of  $\delta^{-1}$  from the proof of Proposition 3.1 is multiplied by  $\delta^{\frac{3}{2}-\alpha}$  to get

$$\int_0^{2\pi} \|\pi_{\theta\#}\nu\|_{L^2(\gamma(\theta)^\perp)}^2 d\theta \gtrsim \delta^{\frac{1}{2}-\alpha}.$$

Moreover,  $\nu(\mathbb{R}^3) \sim \delta^{\frac{3}{2}-\alpha}$ , and

$$(3.3) \quad c_\alpha(\nu) \sim \delta^{\frac{3}{2}-\alpha}.$$

The lower bound  $c_\alpha(\nu) \gtrsim \delta^{\frac{3}{2}-\alpha}$  in (3.3) follows from  $c_\alpha(\nu) \gtrsim \nu(\mathbb{R}^3)$ . The upper bound can be shown by considering different ranges of  $r$  separately, as follows. The maximum of  $\nu(B(x, r))/r^\alpha$  over the range  $0 < r \leq \delta$  occurs near  $r = \delta$  and is  $\lesssim \delta^{\frac{3}{2}-\alpha}$ . The maximum for the range  $\delta \leq r \leq \delta^{\frac{\alpha}{2}-\frac{1}{2}}$  also occurs near  $r = \delta$  (since the ball cannot intersect multiple “slats”), and is  $\lesssim \delta^{\frac{3}{2}-\alpha}$ . The maximum in the range  $\delta^{\frac{\alpha}{2}-\frac{1}{2}} \leq r \leq \delta^{1/2}$  occurs near  $r = \delta^{1/2}$ , and is  $\lesssim \delta^{\frac{3}{2}-\alpha}$ . The maximum for the range  $\delta^{1/2} \leq r \leq \delta^{\frac{\alpha}{2}-1}$  also occurs near  $r = \delta^{1/2}$  (since the ball cannot intersect multiple “tracks”), and is  $\lesssim \delta^{\frac{3}{2}-\alpha}$ . The maximum for the range  $\delta^{\frac{\alpha}{2}-1} \leq r \leq 1$  occurs near  $r = 1$ , and is  $\lesssim \delta^{\frac{3}{2}-\alpha}$ . Applying the assumed inequality (3.2) gives

$$\delta^{\frac{1}{2}-\alpha} \lesssim \delta^{3-2\alpha}.$$

Letting  $\delta \rightarrow 0$  gives  $\frac{1}{2} - \alpha \geq 3 - 2\alpha$  or  $\alpha \geq 5/2$ .  $\square$

It may be possible to weaken the requirement  $\alpha \geq 5/2$  by replacing the  $L^2$  norm on the left in the previous examples by an  $L^p$  norm with  $1 < p < 2$ . Proposition 3.4 below shows that at the critical exponent  $\alpha = 2$ , the average  $L^p$  norms of  $\delta$ -discretised measures cannot be bounded by  $C_\epsilon \delta^{-\epsilon} c_2(\mu)^p$  unless  $p \leq 3/2$ . The example is similar to the “train tracks of planks” used in Proposition 3.3, except that as  $\alpha$  approaches 2, the spaces between the parallel “train tracks” tends to a distance  $\sim 1$ , and the example reduces to a single “train track of planks”, which is a plank version of the original “train track” example from [30, p. 563] and [17, p. 151]. When  $\alpha = 2$ , a uniform bound (without the  $C_\epsilon \delta^{-\epsilon}$  factor) is known to not be possible; by considering a purely 2-unrectifiable set in  $\mathbb{R}^3$  and applying the Besicovitch-Federer projection theorem. Finally, the counterexample for projections onto lines below is based on the Knapp example from [3] used to prove sharpness of the  $L^2$  conical decay rates of 1-dimensional fractal measures.

**Proposition 3.4.** *Let  $p \in [1, \infty)$  and let  $\epsilon > 0$ .*

(1) *If, for any  $\delta > 0$ ,*

$$(3.4) \quad \int_0^{2\pi} \|\pi_{\theta\#}\mu\|_{L^p(\gamma(\theta)^\perp)}^p d\theta \leq C_\epsilon \delta^{-\epsilon} c_2(\mu)^p,$$

for all Borel measures  $\mu$  on  $B_3(0, 1)$  of the form

$$\mu = \frac{1}{\delta^3 \mathcal{H}^3(B(0, 1))} \sum_{B \in \mathcal{B}} a_B \chi_B,$$

where  $a_B > 0$  for all  $B \in \mathcal{B}$ , and where  $\mathcal{B}$  is a disjoint family of  $\delta$ -balls, then  $p \leq 3/2 + \epsilon$ .

(2) If, for any  $\delta > 0$ ,

$$(3.5) \quad \int_0^{2\pi} \|\rho_{\theta\#}\mu\|_{L^p(\text{span}(\gamma(\theta)))}^p d\theta \leq C_\epsilon \delta^{-\epsilon} c_1(\mu)^p,$$

for all Borel measures  $\mu$  on  $B_3(0, 1)$  of the form

$$\mu = \frac{1}{\delta^3 \mathcal{H}^3(B(0, 1))} \sum_{B \in \mathcal{B}} a_B \chi_B,$$

where  $a_B > 0$  for all  $B \in \mathcal{B}$ , and where  $\mathcal{B}$  is a disjoint family of  $\delta$ -balls, then  $p \leq 3/2 + \epsilon$ .

*Proof.* Let  $\delta > 0$  be small, fix  $\theta_0 \in [0, 2\pi)$  and assume that (3.4) holds. Let  $\mu = \mu_{\delta, \theta_0}$  be the measure from the proof of Proposition 3.1, given by

$$\mu(x) = \delta^{-3/2} \psi \left( \langle x, \gamma(\theta_0) \rangle, \left\langle \delta^{-1/2} x, \sqrt{2} \gamma'(\theta_0) \right\rangle, \left\langle \delta^{-1} x, \sqrt{2} (\gamma \times \gamma')(\theta_0) \right\rangle \right),$$

where  $\psi$  is a smooth non-negative bump function supported in  $B(0, 1)$ , with  $\psi \sim 1$  on  $B(0, 1/2)$ , such that  $\int \psi = 1$ . Similarly to the proof of Proposition 3.1,

$$(3.6) \quad \begin{aligned} & \int_0^{2\pi} \|\pi_{\theta\#}\mu\|_{L^p(\gamma(\theta)^\perp)}^p d\theta \\ & \geq \int_{|\theta - \theta_0| \leq c\delta^{1/2}} \int_{-10^{-3}\delta^{1/2}}^{-10^{-3}\delta} \int_{-10^{-3}\delta}^{10^{-3}\delta} \delta^{-3p/2} dx_1 dx_2 d\theta \\ & \gtrsim \delta^{2 - \frac{3p}{2}}. \end{aligned}$$

But  $\mu(\mathbb{R}^3) = 1$ , and

$$c_2(\mu) \sim \delta^{-1/2}.$$

Moreover,  $\mu$  is essentially a sum of indicator functions over a boundedly overlapping family of  $\delta$ -balls. Hence, if (3.4) holds with the smaller  $\mu(\mathbb{R}^3) c_2(\mu)^{p-1}$  on the right-hand side instead of  $c_2(\mu)^p$ , then

$$\delta^{2 - \frac{3p}{2}} \lesssim \delta^{\frac{1}{2} - \frac{p}{2} - \epsilon}.$$

Letting  $\delta \rightarrow 0$  gives  $2 - \frac{3p}{2} \geq \frac{1}{2} - \frac{p}{2} - \epsilon$  or  $p \leq 3/2 + \epsilon$ .

In order to get the same restriction on  $p$  if the larger right-hand side  $c_2(\mu)^p$  is assumed, let  $\nu$  be the sum of translated copies of  $\mu$  spacing  $\delta^{1/2}$  in the short direction  $(\gamma \times \gamma')(\theta_0)$ , given by

$$\nu(x) = \sum_{|m| \leq 10^{-3}\delta^{-1/2}} \mu \left( x - \delta^{1/2} m (\gamma \times \gamma')(\theta_0) \right).$$

There are  $\sim \delta^{-1/2}$  such copies, and the supports of the projections of these translated copies under  $\pi_\theta$  are pairwise disjoint for  $|\theta - \theta_0| \leq c\delta^{1/2}$ , for a sufficiently

small absolute constant  $c$ , so the lower bound of  $\delta^{2-\frac{3p}{2}}$  from (3.6) is multiplied by  $\delta^{-1/2}$  to get

$$(3.7) \quad \int_0^{2\pi} \|\pi_{\theta\#}\nu\|_{L^p(\gamma(\theta)^\perp)}^p d\theta \gtrsim \delta^{\frac{3-3p}{2}}.$$

But  $\nu(\mathbb{R}^3) \sim \delta^{-1/2} \sim c_2(\nu)$  (as shown in the proof of Proposition 3.3), and  $\nu$  is essentially a sum of indicator functions over a boundedly overlapping family of  $\delta$ -balls. Hence, by the assumed (3.4),

$$\delta^{\frac{3-3p}{2}} \lesssim \delta^{\frac{-p}{2}-\epsilon}.$$

Letting  $\delta \rightarrow 0$  gives  $\frac{3-3p}{2} \geq \frac{-p}{2} - \epsilon$  or  $p \leq 3/2 + \epsilon$ . This proves the restriction on  $p$  for projections onto planes.

It remains to consider the restriction on  $p$  for projections onto lines. Let  $\mu = \mu_{\delta, \theta_0}$  be the measure from the proof of Proposition 3.1, given by

$$\mu(x) = \delta^{-3/2} \psi \left( \langle x, \gamma(\theta_0) \rangle, \left\langle \delta^{-1/2} x, \sqrt{2} \gamma'(\theta_0) \right\rangle, \left\langle \delta^{-1} x, \sqrt{2} (\gamma \times \gamma')(\theta_0) \right\rangle \right),$$

where  $\psi$  is a smooth non-negative bump function supported in  $B(0, 1)$ , with  $\psi \sim 1$  on  $B(0, 1/2)$ , such that  $\int \psi = 1$ . Then  $\mu(\mathbb{R}^3) \sim c_1(\mu) \sim 1$ , and

$$\int_{[\theta_0 - \delta^{1/2}, \theta_0 + \delta^{1/2}]} |\rho_{\theta\#}\mu|^p d\theta \gtrsim \delta^{\frac{3}{2}-p}.$$

Hence, by the assumed (3.5),

$$\delta^{\frac{3}{2}-p} \lesssim \delta^{-\epsilon}.$$

Letting  $\delta \rightarrow 0$  gives  $p \leq 3/2 + \epsilon$ . This proves the restriction on  $p$  for projections onto lines.  $\square$

For  $(x, y, t) \in \mathbb{R}^3$ , define  $\ell^*(x, y, t)$  to be the line

$$\ell^*(x, y, t) = \left( 0, x, t - \frac{xy}{2} \right) + L_y,$$

where  $L_y$  is the ‘‘light ray’’ in the light cone

$$\tilde{\Gamma} := \{ \eta \in \mathbb{R}^3 : \eta_2^2 = 2\eta_1\eta_3 \},$$

given by

$$L_y = \left\{ \lambda \left( 1, -y, \frac{y^2}{2} \right) : \lambda \in \mathbb{R} \right\}.$$

The cone  $\tilde{\Gamma}$  is the image of the light cone

$$\Gamma := \{ \xi \in \mathbb{R}^3 : \xi_3^2 = \xi_1^2 + \xi_2^2 \},$$

in  $\mathbb{R}^3$ , under the orthogonal transformation  $U(\xi_1, \xi_2, \xi_3) = (\eta_1, \eta_2, \eta_3)$  given by

$$\eta_1 = \frac{\xi_1 + \xi_3}{\sqrt{2}}, \quad \eta_3 = \frac{-\xi_1 + \xi_3}{\sqrt{2}}, \quad \eta_2 = \xi_2.$$

The restriction of  $\ell^*$  to  $|y| \leq \sqrt{2}$ , followed by  $U^*$  (an anti-clockwise rotation by  $\pi/4$  in the  $(\xi_1, \xi_3)$ -plane), parametrises the family of light rays parallel to some  $\gamma(\theta)$  with  $|\theta| \leq \pi/2$ .

Given  $(a, b, c) \in \mathbb{R}^3$ , define  $\ell(a, b, c)$  to be the horizontal line

$$(3.8) \quad \ell(a, b, c) = \{ (b, 0, c) + s(a, 1, b/2) : s \in \mathbb{R} \}.$$

The following is the point-line duality principle from [6].



**Lemma 3.5** ([6, Lemma 4.11]). *Let  $p \in \mathbb{R}^3$  and  $p^* \in \mathbb{H}$ . Then*

$$p \in \ell^*(p^*) \quad \text{if and only if} \quad p^* \in \ell(p).$$

The symmetry of the lemma above means that much of the ideas from [6] for vertical projections in the Heisenberg group can be reversed as in the lemma below.

**Lemma 3.6.** *The measure  $\mathbf{m}$  given by*

$$\mathbf{m}(F) = \int_{-\pi/2}^{\pi/2} \mathcal{H}^2 \{y \in \gamma(\theta)^\perp : \pi_\theta^{-1}(y) \in U^*F\} d\theta,$$

for a Borel set  $F$  of light rays parallel to lines in  $\tilde{\Gamma}$ , is comparable to the pushforward of the Lebesgue measure on  $\{(x_1, x_2, x_3) \in \mathbb{R}^3 : |x_2| \leq \sqrt{2}\}$  under the map  $\ell^*$ , meaning that

$$(3.9) \quad \left( \ell^*_{\#} \mathcal{H}^3 \chi_{|x_2| \leq \sqrt{2}} \right) (F) \sim \mathbf{m}(F),$$

for any Borel set  $F$  of light rays parallel to lines in  $\tilde{\Gamma}$ . As a consequence, for any non-negative Borel function  $f$  on the set of light rays parallel to lines in  $\tilde{\Gamma}$ ,

$$(3.10) \quad \int_{-\pi/2}^{\pi/2} \int_{\gamma(\theta)^\perp} f(U\pi_\theta^{-1}(y)) d\mathcal{H}^2(y) d\theta \sim \int f d \left( \ell^*_{\#} \mathcal{H}^3 \chi_{|x_2| \leq \sqrt{2}} \right).$$

*Proof.* Let  $F$  be given. Let  $m$  be the Lebesgue measure on  $\mathbb{R}^3 \cap \{|y| \leq \sqrt{2}\}$ . Then

$$\begin{aligned} (\ell^*_{\#} m)(F) &= m \left\{ (x, y, t) \in \mathbb{R}^3 : |y| \leq \sqrt{2}, \left( 0, x, t - \frac{xy}{2} \right) + L_y \in F \right\} \\ &= \int_{-\sqrt{2}}^{\sqrt{2}} \mathcal{H}^2 \left\{ (x, t) \in \mathbb{R}^2 : \left( 0, x, t - \frac{xy}{2} \right) + \text{span}(1, -y, y^2/2) \in F \right\} dy \\ &\sim \int_{-\sqrt{2}}^{\sqrt{2}} \mathcal{H}^2 \left\{ x \in (1, -y, y^2/2)^\perp : \pi_{(1, -y, y^2/2)^\perp}^{-1}(x) \in F \right\} dy \\ &= \int_{-\sqrt{2}}^{\sqrt{2}} \mathcal{H}^2 \left\{ x \in (U^*(1, -y, y^2/2))^\perp : \pi_{(U^*(1, -y, y^2/2))^\perp}^{-1}(x) \in U^*F \right\} dy \\ &\sim \int_{-\pi/2}^{\pi/2} \mathcal{H}^2 \left\{ x \in (\cos \theta, \sin \theta, 1)^\perp : \pi_{(\cos \theta, \sin \theta, 1)^\perp}^{-1}(x) \in U^*F \right\} d\theta \\ &= \mathbf{m}(F). \end{aligned}$$

This proves (3.9), and yields (3.10) whenever  $f = \chi_F$  for a Borel set  $F$  of light rays. The equivalence (3.10) for general Borel functions follows by approximating  $f$  with simple functions and applying the monotone convergence theorem.  $\square$

**Proposition 3.7.** *If, for any  $\epsilon > 0$ ,*

$$(3.11) \quad \left\| \sum_{T \in \mathbb{T}} \chi_T \right\|_{L^p(B(0,1))}^p \leq C_\epsilon \delta^{-\epsilon},$$

for any family  $\mathbb{T}$  of  $SL_2$   $\delta$ -tubes of length 1 with  $\delta$ -separated directions, then  $p \leq 3/2$ .

*Proof.* It is first shown that if, for any  $\epsilon > 0$ , (3.11) holds for any family  $\mathbb{T}$  of  $SL_2$   $\delta$ -tubes of length 1 satisfying the weaker property that

$$|\{T \in \mathbb{T} : T \subseteq S\}| \leq (r/\delta)^2,$$

for any tube  $S$  of length 2 and radius  $r$ , and any  $r \in [\delta, 1]$ , then  $p \leq 3/2$ .

Let  $\delta > 0$  be small and choose  $\theta_0 = \pi/2$ . Let  $\mu = \mu_{\delta, \theta_0}$  be a  $\delta^{1/2}$ -multiple of the (slightly modified) measure from the proof of Proposition 3.1, given by

$$\mu(x) = \delta^{-1} \psi \left( \langle x, \gamma(\theta_0) \rangle, \left\langle \delta^{-1/2} x, \sqrt{2} \gamma'(\theta_0) \right\rangle, \left\langle \delta^{-1} x, \sqrt{2} (\gamma \times \gamma')(\theta_0) \right\rangle \right),$$

where  $\psi$  is a smooth non-negative bump function supported in  $B(0, c)$ , with  $\psi \sim 1$  on  $B(0, c/2)$ , such that  $\int \psi = 1$  (here  $c$  is a small absolute constant to be chosen; this is the only difference from the proof of Proposition 3.1 where  $c = 1$ ). As in the proof of Proposition 3.4, let  $\nu$  be the sum of translated copies of  $\mu$  spacing  $\delta^{1/2}$  in the short direction  $(\gamma \times \gamma')(\theta_0)$ , given by

$$\nu(x) = \sum_{|m| \leq c\delta^{-1/2}} \mu \left( x - \delta^{1/2} m (\gamma \times \gamma')(\theta_0) \right).$$

Then  $\nu$  is supported in a ball around the origin of radius  $\sim c$ , and  $\nu$  is very similar to a  $\delta^{1/2}$ -multiple of the measure from the proof of Proposition 3.4, so by a similar argument to (3.7),

$$\delta^{\frac{3}{2}-p} \lesssim \int_{-\pi/2}^{\pi/2} \|\pi_{\theta\#}\nu\|_{L^p(\gamma(\theta)^\perp)}^p d\theta.$$

Let  $\mathcal{B}$  be a family of  $\delta$ -balls, such that the centres of the balls in  $\mathcal{B}$  form a maximal  $\delta$ -separated subset of the support of  $\nu$ . Then

$$\nu \leq \sum_{B \in \mathcal{B}} \delta^{-1} \chi_B.$$

Hence, by Lemma 3.6,

$$\begin{aligned} \int_{-\pi/2}^{\pi/2} \|\pi_{\theta\#}\nu\|_{L^p(\gamma(\theta)^\perp)}^p d\theta &= \int_{-\pi/2}^{\pi/2} \int_{\gamma(\theta)^\perp} \left| \int_{\pi_\theta^{-1}(y)} \nu d\mathcal{H}^1 \right|^p d\mathcal{H}^2(y) d\theta \\ &= \int_{-\pi/2}^{\pi/2} \int_{\gamma(\theta)^\perp} \left| \int_{U_{\pi_\theta^{-1}(y)}} (U_{\#}\nu) d\mathcal{H}^1 \right|^p d\mathcal{H}^2(y) d\theta \\ &\sim \int \left| \int_L U_{\#}\nu d\mathcal{H}^1 \right|^p d \left( \ell_{\#}^* \mathcal{H}^3 \chi_{|x_2| \leq \sqrt{2}} \right) (L) \\ &\lesssim \int |\{B \in \mathcal{B} : UB \cap L \neq \emptyset\}|^p d \left( \ell_{\#}^* \mathcal{H}^3 \chi_{|x_2| \leq \sqrt{2}} \right) (L) \\ (3.12) \quad &= \int_{|p_2^*| \leq \sqrt{2}} |\{B \in \mathcal{B} : UB \cap \ell^*(p^*) \neq \emptyset\}|^p d\mathcal{H}^3(p^*). \end{aligned}$$

If  $|p_2^*| \leq \sqrt{2}$  and  $\ell^*(p^*)$  intersects  $B(0, c)$ , then  $|p^*| \leq 2$  (provided  $c$  is now chosen sufficiently small; this follows easily from the definition of  $\ell^*$ ). Hence

$$(3.13) \quad (3.12) \leq \int_{|p^*| \leq 2} |\{B \in \mathcal{B} : UB \cap \ell^*(p^*) \neq \emptyset\}|^p d\mathcal{H}^3(p^*).$$

By Lemma 3.5 (point-line duality),

$$(3.13) \leq \int_{|p^*| \leq 2} \left| \sum_{B \in \mathcal{B}} \chi_{\ell(UB)}(p^*) \right|^p d\mathcal{H}^3(p^*).$$

The measure  $\nu$  satisfies  $c_2(\nu) \lesssim 1$ , and similarly so does the measure  $\sum_{B \in \mathcal{B}} \delta^{-1} \chi_B$  (which is roughly the same as  $\nu$ ), which means that for any  $\delta \leq r \leq 1$ , the number

of  $\delta$ -balls from  $\mathcal{B}$  intersecting any  $r$ -ball is  $\lesssim (r/\delta)^2$ . By the formula for  $\ell$ , this implies that for any  $r \in [\delta, 1]$  and any tube  $r$ -tube  $T$  of length 2,

$$|\{\ell(UB) : B \in \mathcal{B}, \ell(UB) \subseteq T\}| \lesssim (r/\delta)^2.$$

Therefore, if (3.11) holds for any family of  $SL_2$  tubes satisfying the 2-dimensional ball condition above, then

$$\delta^{\frac{3}{2}-p} \lesssim \int_{|p^*| \leq 2} \left| \sum_{B \in \mathcal{B}} \chi_{\ell(UB)}(p^*) \right|^p d\mathcal{H}^3(p^*) \leq C_\epsilon \delta^{-\epsilon}.$$

This yields  $p \leq 3/2 + \epsilon$ .

To get the same restriction on  $p$  when (3.11) holds only for those sets  $\mathbb{T}$  of  $SL_2$   $\delta$ -tubes of length 1 with  $\delta$ -separated directions, it suffices to show that the collection of tubes  $\ell(UB)$  have  $\sim \delta$ -separated directions, for any subfamily of balls  $B$  in  $\mathcal{B}$  separated by  $100\delta$ . Let  $B_1, B_2 \in \mathcal{B}$  be  $100\delta$ -separated balls in  $\mathcal{B}$ . Let  $(a_j, b_j, c_j)$  be the centre of  $UB_j$ , where  $j \in \{1, 2\}$ . The direction of  $\ell(UB_j)$  is parallel to  $(a_j, 1, b_j/2)$ , so it is required to show that

$$|a_1 - a_2| + |b_1 - b_2| \gtrsim \delta.$$

Write

$$B_2 = B_1 + \lambda_1 \gamma(\pi/2) + \lambda_2 \gamma'(\pi/2) + \lambda_3 (\gamma \times \gamma')(\pi/2),$$

where  $|\lambda_3| \leq 1$ ,  $|\lambda_2| \leq c\delta^{1/2}$  and  $|\lambda_1| \leq 1$ , and  $|(\lambda_1, \lambda_2, \lambda_3)| \geq 100\delta$ . This can be written as

$$B_2 = B_1 + \frac{\lambda_1}{\sqrt{2}}(0, 1, 1) + \frac{\lambda_2}{\sqrt{2}}(-1, 0, 0) + \frac{\lambda_3}{2}(0, -1, 1).$$

Hence

$$UB_2 = UB_1 + \frac{\lambda_1}{\sqrt{2}} \left( \frac{1}{\sqrt{2}}, 1, \frac{1}{\sqrt{2}} \right) + \frac{\lambda_2}{\sqrt{2}} \left( \frac{-1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}} \right) + \frac{\lambda_3}{2} \left( \frac{1}{\sqrt{2}}, -1, \frac{1}{\sqrt{2}} \right).$$

It follows that

$$(a_2, b_2) = (a_1, b_1) + \left( \frac{\lambda_1}{2} - \frac{\lambda_2}{2} + \frac{\lambda_3}{2\sqrt{2}}, \frac{\lambda_1}{\sqrt{2}} - \frac{\lambda_3}{2} \right).$$

By the triangle inequality,

$$\left| \frac{\lambda_1}{2} - \frac{\lambda_2}{2} + \frac{\lambda_3}{2\sqrt{2}} \right| \geq \left| \frac{\lambda_3}{\sqrt{2}} - \frac{\lambda_2}{2} \right| - \frac{1}{\sqrt{2}} \left| \frac{\lambda_1}{\sqrt{2}} - \frac{\lambda_3}{2} \right|.$$

Hence

$$|a_1 - a_2| + \frac{|b_1 - b_2|}{\sqrt{2}} \geq \left| \frac{\lambda_3}{\sqrt{2}} - \frac{\lambda_2}{2} \right|.$$

If  $B_1$  and  $B_2$  lie in different planks, then  $|\lambda_3| \geq \delta^{1/2}$ , and since  $|\lambda_2| \leq c\delta^{1/2}$ , this yields

$$|a_1 - a_2| + |b_1 - b_2| \gtrsim \delta^{1/2}.$$

It remains to consider the case where  $B_1$  and  $B_2$  lie in the same plank. In this case,  $|\lambda_3| \leq \delta$ . If  $|\lambda_1| \geq 10\delta$ , then  $|b_1 - b_2| \sim \left| \frac{\lambda_1}{\sqrt{2}} - \frac{\lambda_3}{2} \right| \gtrsim \delta$ . Otherwise  $|\lambda_1| < 10\delta$  and  $|\lambda_3| \leq \delta$  imply that  $|\lambda_2| \geq 50\delta$ , and this gives  $|a_1 - a_2| = \left| \frac{\lambda_1}{2} - \frac{\lambda_2}{2} + \frac{\lambda_3}{2\sqrt{2}} \right| \gtrsim \delta$ . This shows that the directions of  $100\delta$ -separated balls in  $\mathcal{B}$  are  $\sim \delta$ -separated. It follows

that if (3.11) holds for any family  $\mathbb{T}$  of  $SL_2$   $\delta$ -tubes of length 1 with  $\delta$ -separated directions, then it can be applied to the family  $\{\ell(UB) : B \in \mathcal{B}\}$ , and therefore

$$\delta^{\frac{3}{2}-p} \lesssim \int_{|p^*| \leq 2} \left| \sum_{B \in \mathcal{B}} \chi_{\ell(UB)}(p^*) \right|^p d\mathcal{H}^3(p^*) \leq C_\epsilon \delta^{-\epsilon}.$$

This yields  $p \leq 3/2 + \epsilon$ . Letting  $\epsilon \rightarrow 0$  gives  $p \leq 3/2$ .  $\square$

#### 4. AN $L^{3/2}$ INEQUALITY FOR PROJECTIONS ONTO PLANES

Consider the light cone  $\{\xi \in \mathbb{R}^3 : |(\xi_1, \xi_2)| = |\xi_3|\}$ . The set  $\{\xi \in \mathbb{R}^3 : 2^j \gtrsim |(\xi_1, \xi_2)| \geq |\xi_3| \sim 2^j\}$  will be broken into dyadic conical shells according to the distance to the cone. If this set is rescaled by  $2^{-j}$ , then this is equivalent to dividing the  $\sim 1$  outer neighbourhood of the truncated light cone  $|\xi_3| = |(\xi_1, \xi_2)|$  in  $B(0, 100) \setminus B(0, 1/100)$  into dyadic conical shells according to the distance to the cone. The purpose of this decomposition is that on each dyadic conical shell, the change of variables used in what follows will have constant Jacobian on that shell. The part with distance  $\sim 2^{j-k}$  from the cone (before scaling), where  $0 \leq k < j$ , is roughly a scaling of the part of  $\sim 2^{-k}$  outer neighbourhood of the truncated light cone separated by a distance  $\sim 2^{-k}$  from the cone, by  $2^j$ . The scaled down neighbourhood can be covered by a finitely overlapping collection of boxes similar to the standard cover of the  $2^{-k}$ -neighbourhood of the truncated light cone; of dimensions  $\sim 1 \times 2^{-k/2} \times 2^{-k}$ , and this covering can then be scaled up. If  $k = j$  then the shell with distance  $\lesssim 1$  from the cone (inside  $B(0, 100 \cdot 2^j) \setminus B(0, 2^j/100)$ ) is a rescaling of the  $\sim 2^{-j}$ -neighbourhood of the truncated light cone. This neighbourhood can be covered by the standard  $\lesssim 1$  overlapping cover of the  $2^{-j}$ -neighbourhood of the truncated light cone by boxes of dimensions  $1 \times 2^{-j/2} \times 2^{-j}$ , and this covering can then be scaled up.

The decomposition will be made a bit more precise. For each  $\theta \in [0, 2\pi)$  and  $0 \leq k < j$ , let

$$\begin{aligned} \tau(\theta, j, k, +) &= \left\{ \lambda_1(\gamma \times \gamma')(\theta) + \lambda_2 \gamma'(\theta) + \lambda_3 \gamma(\theta) : \right. \\ &\quad \left. |\lambda_1| \sim 2^j, |\lambda_2| \lesssim 2^{j-k/2}, |\lambda_3| \sim 2^{j-k}, \lambda_1 > 0, \lambda_3 < 0 \right\}, \end{aligned}$$

and for  $k = j$  let

$$\begin{aligned} \tau(\theta, j, j, +) &= \left\{ \lambda_1(\gamma \times \gamma')(\theta) + \lambda_2 \gamma'(\theta) + \lambda_3 \gamma(\theta) : \right. \\ &\quad \left. |\lambda_1| \sim 2^j, |\lambda_2| \lesssim 2^{j/2}, |\lambda_3| \lesssim 1, \lambda_1 > 0 \right\}, \end{aligned}$$

and similarly with signs reversed to get boxes near the backward light cone. For each such  $\tau$ , we write  $\theta = \theta_\tau$ . For each  $k$ , choose a maximal  $\sim 2^{-k/2}$ -separated set  $\Theta_k$  of  $[0, 2\pi)$ , let  $\Lambda_{j,k} = \{\tau(\theta, j, k, \pm) : \theta \in \Theta_k\}$ , such that the set  $\Lambda = \bigcup_{j \geq 1} \bigcup_{0 \leq k \leq j} \Lambda_{j,k}$  forms a boundedly overlapping cover of the set  $\{\xi \in \mathbb{R}^3 : 1 \lesssim |\xi_3| \leq |(\xi_1, \xi_2)| \lesssim |\xi_3|\}$ . Let  $\{\psi_\tau\}_{\tau \in \Lambda}$  be a smooth partition of unity such that each  $\psi_\tau$  is supported in  $\tau$  and such that the  $\psi_\tau$ 's sum to 1 on the set  $\{\xi \in \mathbb{R}^3 : 1 \lesssim |\xi_3| \leq |(\xi_1, \xi_2)| \lesssim |\xi_3|\}$ . It may be assumed that the derivatives of  $\psi_\tau$  satisfy natural bounds; meaning that the size of the  $m$ -th derivative of  $\psi_\tau$  in the direction  $(\gamma \times \gamma')(\theta_\tau)$  is  $\lesssim_m 2^{-jm}$ , the size of the  $m$ -th derivative of  $\psi_\tau$  in the direction  $\gamma'(\theta_\tau)$  is  $\lesssim_m 2^{-(j-k/2)m}$ , and the

size of the  $m$ -th derivative of  $\psi_\tau$  in the direction  $\gamma(\theta_\tau)$  is  $\lesssim_m 2^{-(j-k)m}$  (see [12, Lemma 3.3] for a more precise statement).

Given  $\delta > 0$  (which may be thought of as vanishingly small), for each  $\tau$  cover  $\mathbb{R}^3$  by a boundedly overlapping collection of (rescaled) planks  $T \in \mathbb{T}_\tau$  of dimensions  $2^{k\delta-(j-k)} \times 2^{k(-1/2+\delta)-(j-k)} \times 2^{k(-1+\delta)-(j-k)}$  dual to  $\tau$  (meaning that the long direction of  $T$  is the short direction of  $\tau$ , and vice-versa). Given a measure  $\mu$  on  $B_3(0, 1)$  and a plank  $T$ , define

$$M_T \mu = \eta_T \left( \mu * \widetilde{\psi}_\tau \right),$$

where, for each  $T$ ,  $\eta_T$  is a smooth function essentially supported on  $T$  such that  $\text{supp} \widehat{\eta_T} \subset \tau$  and  $\sum_{T \in \mathbb{T}_\tau} \eta_T(x) = 1$ .

Lemma 4.1 below states that the projections of measures of dimension greater than 2 have small  $L^p$  norms on small unions of rectangles, on average, for exponents  $1 \leq p \leq 3/2$ . The main use of Lemma 4.1 will be in proving that pushforwards of measures of dimension greater than 2 are almost surely in  $L^{3/2}$ ; by splitting  $\mu$  into “good” and “bad” parts, and bounding the  $L^{3/2}$  norm of the projections of the “bad” part of a measure by the  $L^{3/2}$  norm of the projections of the original measure on small unions of rectangles.

Let  $\phi$  be a smooth bump function equal to 1 on  $B_3(0, 1)$  and vanishing outside  $B_3(0, 2)$ . For each  $R > 0$  let  $\phi_R(x) = R^3 \phi(Rx)$ .

**Lemma 4.1.** *Let  $\alpha > 2$  and  $1 \leq p \leq \frac{3}{2}$ . Then for any  $\beta > 0$ , there exists a  $\kappa > 0$  (which may depend on  $\alpha$ ,  $\beta$ , and  $p$ ) such that*

$$(4.1) \quad \int_0^{2\pi} \int_{\bigcup_{D \in \mathbb{D}_\theta} D} |\pi_{\theta\#}(\mu * \phi_R)|^p d\mathcal{H}^2 d\theta \leq C_{\alpha, \beta, p} R^{-\kappa} \mu(\mathbb{R}^3) c_\alpha(\mu)^{p-1},$$

for all  $\beta$ -admissible datum  $(\mu, R, \{\mathbb{D}_\theta\})$ . Here,  $(\mu, R, \{\mathbb{D}_\theta\})$  is called  $\beta$ -admissible if  $R > 1$ ,  $\mu$  is a Borel measure on  $B_3(0, 1)$  with  $c_\alpha(\mu) < \infty$ , and  $\{\mathbb{D}_\theta\}$  is a family of sets, where each  $\mathbb{D}_\theta$  is a set of rectangles of dimensions  $R^{-1/2} \times R^{-1}$  in  $\gamma(\theta)^\perp$  with long direction parallel to  $\gamma'(\theta)$  and short direction parallel to  $(\gamma \times \gamma')(\theta)$ , with  $(\pi_{\theta\#}(\mu * \phi_R))(D) \geq c_\alpha(\mu) R^{\beta-3/2}$  for each  $D \in \mathbb{D}_\theta$ , such that the set  $\{(\theta, x) \in [0, 2\pi) \times \mathbb{R}^3 : x \in \bigcup_{D \in \mathbb{D}_\theta} D\}$  is Borel measurable.

*Remark 4.2.* Having  $\mu * \phi_R$  instead of  $\mu$  in (4.4) is similar to requiring that  $\mu$  is a linear combination of characteristic functions over a boundedly overlapping collection of  $R^{-1}$ -balls.

*Proof.* Apparently, the result follows from the endpoint cases  $p = 1$  and  $p = \frac{3}{2}$  and interpolation. Instead of proving the two endpoint cases directly, we will in fact show that the claimed result holds true for a sequence of increasing exponents  $\{p_n\}$ . Briefly speaking, the proof is based on a self-improving argument: if (4.1) holds for the exponent  $p_n$ , then it holds for  $p_{n+1}$ .

Here are the details. Define the sequence  $\{p_n\}_{n \in \mathbb{N} \cup \{0, \infty\}}$  by

$$(4.2) \quad \frac{1}{p_n} = \frac{2}{3} + \frac{1}{3 \cdot 4^n}, \quad n \in \mathbb{N} \cup \{0, \infty\}.$$

We claim that the following holds true: if either

- (1)  $n \in \{0, 1\}$ , or if

(2)  $2 \leq n \leq \infty$ , and for all  $m \in \mathbb{N} \setminus \{0\}$  with  $m < n$ , for any  $\beta > 0$ , there exists a  $\kappa > 0$  (depending on  $\alpha, \beta, m$ ) such that

$$(4.3) \quad \int_0^{2\pi} \int_{\bigcup_{D \in \mathbb{D}_\theta} D} |\pi_{\theta\#}(\mu * \phi_R)|^{p_m} d\mathcal{H}^2 d\theta \leq C_{\alpha, \beta, m} R^{-\kappa} \mu(\mathbb{R}^3) c_\alpha(\mu)^{p_m-1},$$

for all  $\beta$ -admissible datum  $(\mu, R, \{\mathbb{D}_\theta\})$ ,

then for any  $\beta > 0$ , there exists a  $\kappa > 0$  (which may depend on  $\alpha, \beta$ , and  $n$ ) such that

$$(4.4) \quad \int_0^{2\pi} \int_{\bigcup_{D \in \mathbb{D}_\theta} D} |\pi_{\theta\#}(\mu * \phi_R)|^{p_n} d\mathcal{H}^2 d\theta \leq C_{\alpha, \beta, n} R^{-\kappa} \mu(\mathbb{R}^3) c_\alpha(\mu)^{p_n-1},$$

for all  $\beta$ -admissible datum  $(\mu, R, \{\mathbb{D}_\theta\})$ .

It is easy to see that the claim would imply the desired result. Indeed, the endpoint case  $p = 1$  follows directly from setting  $n = 0$  (for which (4.4) holds unconditionally). Induction then shows that (4.4) holds for all finite  $n$ . Lastly, applying the claim again to  $n = \infty$  proves the endpoint case  $p = \frac{3}{2}$ .

We now prove the claim. For a given  $n$ , the proof is a bootstrapping argument which gradually lowers the value of  $\beta$ , which for simplicity will be written as a proof by contradiction. Define  $p_m = p_0 = 1$  for  $m < 0$ . Let  $n \in \mathbb{N} \cup \{0, \infty\}$  be given and let  $\beta_{\text{inf}}$  be the infimum over all positive  $\beta$  for which the conclusion of the lemma is true. Suppose for a contradiction that  $\beta_{\text{inf}} > 0$ . Clearly  $\beta_{\text{inf}} \leq 3/2$  (in particular it is finite); this follows from the inequality  $\mu(\mathbb{R}^3) \leq c_\alpha(\mu)$ . Choose a constant  $c \in (0, 1)$  sufficiently close to 1 to ensure that  $1 - c < \frac{\alpha-2}{1000}$ , then let  $\beta_0 \in (\beta_{\text{inf}}, 2\beta_{\text{inf}})$  be sufficiently close to  $\beta_{\text{inf}}$ . To obtain a contradiction, it will suffice to show that the lemma holds for all  $\beta > c\beta_0$ . Fix such a  $\beta > c\beta_0$ . Let  $\delta > 0$  be such that  $\delta \ll \kappa(\beta_0)$  and  $\delta \ll \beta$ . The measure  $\mu * \phi_R$  will be re-labelled as  $\mu$  to simplify the notation (so it may be assumed that  $\hat{\mu}$  is negligible outside  $B(0, R^{1+\delta})$ ).

We first derive an estimate of the measure of the integration domain. Let

$$F = \left\{ (\theta, x) \in [0, 2\pi) \times \mathbb{R}^3 : x \in \bigcup_{D \in \mathbb{D}_\theta} D \right\}.$$

By the assumption that  $(\pi_{\theta\#}\mu)(D) \geq R^{\beta-3/2} c_\alpha(\mu)$  and the Vitali covering lemma (to handle the possibly non-disjoint case),

$$(\mathcal{H}^1 \times \mathcal{H}^2)(F) \lesssim c_\alpha(\mu)^{-1} R^{-\beta} \int_0^{2\pi} \int_{\bigcup_{D \in \mathbb{D}_\theta} D} |\pi_{\theta\#}\mu| d\mathcal{H}^2 d\theta.$$

If  $n < \infty$ , then by Hölder's inequality applied to  $\chi_F$  and  $|\pi_{\theta\#}\mu|$ ,

$$(\mathcal{H}^1 \times \mathcal{H}^2)(F)^{1/p_{n-1}} \lesssim c_\alpha(\mu)^{-1} R^{-\beta} \left( \int_0^{2\pi} \int_{\bigcup_{D \in \mathbb{D}_\theta} D} |\pi_{\theta\#}\mu|^{p_{n-1}} d\mathcal{H}^2 d\theta \right)^{1/p_{n-1}}.$$

Therefore

$$(\mathcal{H}^1 \times \mathcal{H}^2)(F) \lesssim c_\alpha(\mu)^{-p_{n-1}} R^{-\beta p_{n-1}} \int_0^{2\pi} \int_{\bigcup_{D \in \mathbb{D}_\theta} D} |\pi_{\theta\#}\mu|^{p_{n-1}} d\mathcal{H}^2 d\theta.$$

By the assumed (4.3) (which also holds trivially without the  $R^{-\kappa}$  decay if  $m \in \{-1, 0\}$ ), this gives

$$(\mathcal{H}^1 \times \mathcal{H}^2)(F) \lesssim \mu(\mathbb{R}^3) c_\alpha(\mu)^{-1} R^{-\beta p_{n-1}}.$$

To include the case  $n = \infty$ , one can replace the above by a slightly weaker estimate

$$(4.5) \quad (\mathcal{H}^1 \times \mathcal{H}^2)(F) \lesssim \mu(\mathbb{R}^3) c_\alpha(\mu)^{-1} R^{\delta - \beta p_{n-1}},$$

which, when  $n = \infty$ , follows by re-running the above argument with  $p_{n-1}$  replaced by  $p_m$  for some sufficiently large finite  $m$  (more precisely, taking  $m < \infty$  such that  $p_m > \frac{3}{2} - \frac{\delta}{\beta}$ ). Here,  $n - 1 := \infty$  if  $n = \infty$ . Since the measure  $\mathcal{H}^2$  on  $\mathbb{R}^3$  is not  $\sigma$ -finite, to avoid issues of applying Fubini's theorem above and below we identify the measure  $\mathcal{H}^1 \times \mathcal{H}^2$  on  $[0, 2\pi) \times \mathbb{R}^3$  with that obtained by integrating first in  $x \in \gamma(\theta)^\perp$  then in  $\theta \in [0, 2\pi)$  (rather than in the other order; we could alternatively rewrite the measure as a measure on  $[0, 2\pi) \times \mathbb{R}^2$  by using coordinates).

Note that the above is the only place in the proof that the assumption (4.3) is used. From now on, estimate (4.5) of the integration domain is all we need.

Let  $J$  be such that  $2^J \sim R^\epsilon$ , where  $\epsilon > 0$  is chosen so that  $\delta \ll \epsilon \ll \min\{\beta, \alpha - 2\}$ . Given  $(j, k)$  and  $\tau \in \Lambda_{j,k}$ , let

$$(4.6) \quad \mathbb{T}_{\tau,b} = \left\{ T \in \mathbb{T}_\tau : \mu(4T) \geq 2^{-k(\frac{3}{2} - \beta_0)} 2^{-(j-k)\alpha + 1000j\epsilon} c_\alpha(\mu) \right\}, \quad \mathbb{T}_{\tau,g} = \mathbb{T}_\tau \setminus \mathbb{T}_{\tau,b},$$

and define the bad and good part of the measure as

$$(4.7) \quad \mu_b = \sum_{j \geq J} \sum_{k \in [j\epsilon, j]} \sum_{\tau \in \Lambda_{j,k}} \sum_{T \in \mathbb{T}_{\tau,b}} M_T \mu, \quad \mu_g = \mu - \mu_b.$$

Note that by the trivial upper bound on each  $\mu(4T)$  implied by the Frostman condition of  $\mu$ ,  $\mathbb{T}_{\tau,b}$  is empty whenever  $\tau \in \Lambda_{j,k}$  and  $0 \leq k \leq j\epsilon$ , so the range of  $k$  in the definition of  $\mu_b$  in (4.7) could be replaced by  $k \in [0, j]$ . This will be used later.

By the triangle inequality, followed by Hölder's inequality (using (4.5)),

$$(4.8) \quad \int_0^{2\pi} \int_{\bigcup_{D \in \mathbb{D}_\theta} D} |\pi_{\theta\#} \mu|^{p_n} d\mathcal{H}^2 d\theta \lesssim \int_0^{2\pi} \int |\pi_{\theta\#} \mu_b|^{p_n} d\mathcal{H}^2 d\theta \\ + \left( \frac{\mu(\mathbb{R}^3)}{c_\alpha(\mu)} \right)^{1 - \frac{p_n}{2}} R^{\delta - \beta p_{n-1} (1 - \frac{p_n}{2})} \left( \int_0^{2\pi} \int |\pi_{\theta\#} \mu_g|^2 d\mathcal{H}^2 d\theta \right)^{p_n/2}.$$

In what follows, negligible tail terms of the form  $R^{-N} \mu(\mathbb{R}^3)$  (for arbitrarily large  $N$ ) will be left out, since whenever such terms dominate, the desired inequalities follow trivially. In particular, this means that it suffices to consider scales  $J \leq j \lesssim \log R$ .

We first estimate the bad part, which doesn't rely on (4.5). By the triangle inequality, followed by Hölder's inequality again (noting that  $j \lesssim \log R$ ),

$$(4.9) \quad \int_0^{2\pi} \int |\pi_{\theta\#} \mu_b|^{p_n} d\mathcal{H}^2 d\theta \\ \lesssim (\log R)^{O(1)} \sum_{j \geq J} \sum_{k \in [j\epsilon, j]} \int_0^{2\pi} \int \left| \sum_{\tau \in \Lambda_{j,k}} \sum_{T \in \mathbb{T}_{\tau,b}} \pi_{\theta\#} M_T \mu \right|^{p_n} d\mathcal{H}^2 d\theta.$$

Fix a pair  $(j, k)$  occurring in the above sum. We will prove the following:

**Claim.**

$$(4.10) \quad \int_0^{2\pi} \int \left| \sum_{\tau \in \Lambda_{j,k}} \sum_{T \in \mathbb{T}_{\tau,b}} \pi_{\theta\#} M_T \mu \right|^{p_n} d\mathcal{H}^2 d\theta \\ \lesssim 2^{kO(\delta)} \int_0^{2\pi} \sum_{\substack{\tau \in \Lambda_{j,k}: \\ |\theta_\tau - \theta| \leq 2^{k(\delta-1/2)}}} \sum_{T \in \mathbb{T}_{\tau,b}} \mu(2T)^{p_n} (m(T)2^{j-k})^{-(p_n-1)} d\theta,$$

where  $\theta_\tau$  is defined by  $\tau = \tau(\theta_\tau, j, k)$ , and  $m(T)$  is the Lebesgue measure of  $T$ .

*Proof of the Claim.* In order to prove (4.10), we will sometimes need to make use of the intermediate inequality

$$(4.11) \quad \int_0^{2\pi} \sum_{\substack{\tau \in \Lambda_{j,k}: \\ |\theta_\tau - \theta| \leq 2^{k(\delta-1/2)}}} \sum_{T \in \mathbb{T}_{\tau,b}} \|\pi_{\theta\#} M_T \mu\|_{L^{p_n}(\mathcal{H}^2)}^{p_n} d\theta \\ \lesssim 2^{kO(\delta)} \int_0^{2\pi} \sum_{\substack{\tau \in \Lambda_{j,k}: \\ |\theta_\tau - \theta| \leq 2^{k(\delta-1/2)}}} \sum_{T \in \mathbb{T}_{\tau,b}} \mu(2T)^{p_n} (m(T)2^{j-k})^{-(p_n-1)} d\theta.$$

The inequality (4.11) holds up to a rapidly decaying tail, and is a consequence of the uncertainty principle (more precisely, since each  $M_T \mu$  is a wave packet that is concentrated on  $T$  and locally constant, up to a rapidly decaying tail, one has  $\|\pi_{\theta\#} M_T \mu\|_{L^{p_n}}^{p_n} \sim \|\pi_{\theta\#} M_T \mu\|_{L^1}^{p_n} m(\pi_\theta(T))^{1-p_n}$ . The estimate then follows from Lemma 4 of [8]).

We now prove (4.10). First, note that the left-hand side of (4.10) is

$$\lesssim \int_0^{2\pi} \int \left| \sum_{\substack{\tau \in \Lambda_{j,k}: \\ |\theta_\tau - \theta| \leq 2^{k(\delta-1/2)}}} \sum_{T \in \mathbb{T}_{\tau,b}} \pi_{\theta\#} M_T \mu \right|^{p_n} d\mathcal{H}^2 d\theta \\ + \int_0^{2\pi} \int \left| \sum_{\substack{\tau \in \Lambda_{j,k}: \\ |\theta_\tau - \theta| > 2^{k(\delta-1/2)}}} \sum_{T \in \mathbb{T}_{\tau,b}} \pi_{\theta\#} M_T \mu \right|^{p_n} d\mathcal{H}^2 d\theta.$$

The second term is negligible by the trivial  $L^\infty$  bound on each  $\pi_{\theta\#} M_T \mu$ , followed by the same stationary phase bound used in the  $L^1$  case (see Lemma 5 of [8]). So it suffices to only consider  $\tau$  such that  $\theta_\tau$  is close to  $\theta$ .



Assume first that  $k = j$ . By the  $\lesssim 2^{kO(\delta)}$ -overlapping property of the  $\pi_\theta(T)$ 's (this property uses  $j = k$ ), one gets the bound

$$\begin{aligned} \int_0^{2\pi} \int \left| \sum_{\substack{\tau \in \Lambda_{j,k}: \\ |\theta_\tau - \theta| \leq 2^{k(\delta-1/2)}}} \sum_{T \in \mathbb{T}_{\tau,b}} \pi_{\theta\#} M_T \mu \right|^{p_n} d\mathcal{H}^2 d\theta \\ \lesssim 2^{kO(\delta)} \int_0^{2\pi} \sum_{\substack{\tau \in \Lambda_{j,k}: \\ |\theta_\tau - \theta| \leq 2^{k(\delta-1/2)}}} \sum_{T \in \mathbb{T}_{\tau,b}} \|\pi_{\theta\#} M_T \mu\|_{L^{p_n}(\mathcal{H}^2)}^{p_n} d\theta, \end{aligned}$$

and then (4.10) follows from (4.11).

Now consider the case  $k < j$ . If  $p$  were equal to 2 then the inequality

$$\begin{aligned} \int_0^{2\pi} \int \left| \sum_{\substack{\tau \in \Lambda_{j,k}: \\ |\theta_\tau - \theta| \leq 2^{k(\delta-1/2)}}} \sum_{T \in \mathbb{T}_{\tau,b}} \pi_{\theta\#} M_T \mu \right|^p d\mathcal{H}^2 d\theta \\ \lesssim 2^{kO(\delta)} \int_0^{2\pi} \sum_{\substack{\tau \in \Lambda_{j,k}: \\ |\theta_\tau - \theta| \leq 2^{k(\delta-1/2)}}} \sum_{T \in \mathbb{T}_{\tau,b}} \mu(2T)^p (m(T)2^{j-k})^{-(p-1)} d\theta \end{aligned}$$

would follow by orthogonality. More precisely, by applying Plancherel in two dimensions, and then changing variables to an integral over  $\mathbb{R}^3$  (fixing  $(j, k)$  ensures that the Jacobian of the change of variables is the constant  $2^{-j+k/2}$ ; we use (4.15) below and the straightforward observation that if  $\eta_1 \gamma'(\theta) + \eta_2 (\gamma \times \gamma')(\theta)$  intersects  $\tau \in \Lambda_{j,k}$ , then  $|\eta_1| \sim 2^{j-k/2}$ , whose justification can be found in [8, Eq. 106]),

$$\begin{aligned} (4.12) \quad \int_0^{2\pi} \int \left| \sum_{\substack{\tau \in \Lambda_{j,k}: \\ |\theta_\tau - \theta| \leq 2^{k(\delta-1/2)}}} \sum_{T \in \mathbb{T}_{\tau,b}} \pi_{\theta\#} M_T \mu \right|^2 d\mathcal{H}^2 d\theta \\ \lesssim 2^{-j+k/2} \int_{\mathbb{R}^3} \left| \sum_{\substack{\tau \in \Lambda_{j,k}: \\ |\theta_\tau - \theta| \leq 2^{k(\delta-1/2)}}} \sum_{T \in \mathbb{T}_{\tau,b}} M_T \mu(x) \right|^2 dx. \end{aligned}$$

We then apply Plancherel's theorem in  $\mathbb{R}^3$  to decouple the  $L^2$  norms (as in [8, Eq. 113]), to bound the above by

$$2^{-j+k/2} \sum_{\substack{\tau \in \Lambda_{j,k}: \\ |\theta_\tau - \theta| \leq 2^{k(\delta-1/2)}}} \sum_{T \in \mathbb{T}_{\tau,b}} \int_{\mathbb{R}^3} |M_T \mu(x)|^2 dx.$$

Each  $M_T \mu$  is just a wave packet, so it is easy to bound the above by

$$2^{-j+k/2+kO(\delta)} \sum_{\substack{\tau \in \Lambda_{j,k}: \\ |\theta_\tau - \theta| \leq 2^{k(\delta-1/2)}}} \sum_{T \in \mathbb{T}_{\tau,b}} \mu(2T)^2 m(T)^{-1},$$

and by Fubini, the above is comparable to the right-hand side of (4.12) when  $p = 2$ .

If  $p$  is equal to 1, then (4.12) follows from the triangle inequality and the stationary phase bound from Lemma 5 of [8]. This proves (4.12) if  $p = 1$ .

The arguments above for (4.12) in the case  $p \in \{1, 2\}$  work equally well if the collections of  $T$ 's in the left-hand side is replaced by any subcollection, so (4.12) holds more generally for any subcollection of  $T$ 's in the left-hand side, if  $p \in \{1, 2\}$ .

The inequalities (4.12) and (4.10) for the exponent  $p = p_n$  will be shown via the following application of the interpolation argument from [2, Exercise 9.21]. By pigeonholing, there exists a subcollection  $\mathbb{T}$  of  $T$ 's in the left-hand side of (4.10), such that  $\mu(2T)$  is constant up to a factor of 2, and such that

$$\int_0^{2\pi} \int \left| \sum_{\tau \in \Lambda_{j,k}} \sum_{T \in \mathbb{T}_{\tau,b}} \pi_{\theta\#} M_T \mu \right|^{p_n} d\mathcal{H}^2 d\theta \lesssim 2^{kO(\delta)} \int_0^{2\pi} \int \left| \sum_{T \in \mathbb{T}} \pi_{\theta\#} M_T \mu \right|^{p_n} d\mathcal{H}^2 d\theta.$$

If we write  $p_n = (1 - \phi) \cdot 1 + \phi \cdot 2$  and apply Hölder's inequality with  $q = 1/(1 - \phi)$  and  $q' = 1/\phi$ , this bounds the above by

$$2^{kO(\delta)} \left( \int_0^{2\pi} \int \left| \sum_{T \in \mathbb{T}} \pi_{\theta\#} M_T \mu \right| d\mathcal{H}^2 d\theta \right)^{1-\phi} \left( \int_0^{2\pi} \int \left| \sum_{T \in \mathbb{T}} \pi_{\theta\#} M_T \mu \right|^2 d\mathcal{H}^2 d\theta \right)^{\phi}.$$

If we apply (4.12) in the cases  $p = 1$  and  $p = 2$  (with the sum in left-hand side replaced by a sum over  $\mathbb{T}$ ), the above is bounded by

$$2^{kO(\delta)} \left( \int_0^{2\pi} \sum_{\substack{\tau \in \Lambda_{j,k}: \\ |\theta_\tau - \theta| \leq 2^{k(\delta-1/2)}}} \sum_{T \in \mathbb{T}_{\tau,b} \cap \mathbb{T}} \mu(2T) d\theta \right)^{1-\phi} \times \left( \int_0^{2\pi} \sum_{\substack{\tau \in \Lambda_{j,k}: \\ |\theta_\tau - \theta| \leq 2^{k(\delta-1/2)}}} \sum_{T \in \mathbb{T}_{\tau,b} \cap \mathbb{T}} \mu(2T)^2 (m(T)2^{j-k})^{-1} d\theta \right)^{\phi}.$$

By Fubini's theorem and the essentially constant property of  $\mu(2T)$ , the above is bounded by

$$2^{kO(\delta)} \left( |\mathbb{T}| \mu(2T) 2^{-k/2} \right)^{1-\phi} \left( |\mathbb{T}| \mu(2T)^2 (m(T)2^{j-k})^{-1} 2^{-k/2} \right)^{\phi}.$$

This simplifies to

$$2^{kO(\delta)} |\mathbb{T}| \mu(2T)^{p_n} 2^{-k/2} (m(T)2^{j-k})^{-(p_n-1)},$$

which, by using Fubini's theorem and the essentially constant property of  $\mu(2T)$  again, is bounded by

$$2^{kO(\delta)} \int_0^{2\pi} \sum_{\substack{\tau \in \Lambda_{j,k}: \\ |\theta_\tau - \theta| \leq 2^{k(\delta-1/2)}}} \sum_{T \in \mathbb{T}_{\tau,b}} \mu(2T)^{p_n} (m(T)2^{j-k})^{-(p_n-1)} d\theta.$$

This matches the right-hand side of (4.10), so this finishes the proof of (4.10).  $\square$

The right-hand side of (4.10) is bounded by

$$2^{kO(\delta)} \int_0^{2\pi} \sum_m \sum_{\substack{\tau \in \Lambda_{j,k}: \\ |\theta_\tau - \theta| \leq 2^{k(\delta-1/2)}}} \sum_{\substack{T \in \mathbb{T}_{\tau,b}: \\ T \cap B_m \neq \emptyset}} \mu(2T)^{p_n} (m(T)2^{j-k})^{-(p_n-1)} d\theta,$$

where, for each pair  $(j, k)$ ,  $\{B_m\}_m$  is a boundedly overlapping cover of  $\mathbb{R}^3$  by balls of radius  $2^{-(j-k)+k\delta}$ . The size of the balls  $B_m$  is chosen so that, for any fixed  $B_m$ , after a rescaling later, the badness of planks  $T$  (intersecting  $B_m$ ) implies the  $\beta_0$ -admissibility, i.e. the rescaled version of the measure  $\mu$  restricted on  $B_m$  and the rescaled planks satisfy the hypotheses for (4.4).

For fixed  $(j, k)$ ,  $m$ ,  $\tau$ , and a  $T$  occurring in the sum above, recalling that the shortest sidelength of the plank  $T$  is  $2^{k\delta-j}$ , so

$$\begin{aligned} & \mu(2T)^{p_n} (m(T)2^{j-k})^{-(p_n-1)} \\ & \lesssim \left( \int_{\pi_\theta(2T)} |\pi_{\theta\#}(\mu_{m,j,k} * \phi_{2^{j-k\delta}})| d\mathcal{H}^2 \right)^{p_n} (m(T)2^{j-k})^{-(p_n-1)}, \end{aligned}$$

where  $\mu_{m,j,k}$  is the restriction of  $\mu$  to  $100B_m$ . By Hölder's inequality applied to the above,

$$\mu(2T)^{p_n} (m(T)2^{j-k})^{-(p_n-1)} \lesssim 2^{kO(\delta)} \int_{\pi_\theta(2T)} |\pi_{\theta\#}(\mu_{m,j,k} * \phi_{2^{j-k\delta}})|^{p_n} d\mathcal{H}^2.$$

If we substitute this inequality into (4.10), and then sum (4.10) over  $j$  and  $k$  in (4.9), we get

$$\begin{aligned} & \int_0^{2\pi} \int |\pi_{\theta\#}\mu_b|^{p_n} d\mathcal{H}^2 d\theta \lesssim (\log R)^{O(1)} \sum_{j \geq J} \sum_{k \in [j\epsilon, j]} 2^{kO(\delta)} \sum_m \\ & \int_0^{2\pi} \sum_{\substack{\tau \in \Lambda_{j,k}: \\ |\theta_\tau - \theta| \leq 2^{k(\delta-1/2)}}} \sum_{\substack{T \in \mathbb{T}_{\tau,b}: \\ T \cap B_m \neq \emptyset}} \int_{\pi_\theta(2T)} |\pi_{\theta\#}(\mu_{m,j,k} * \phi_{2^{j-k\delta}})|^{p_n} d\mathcal{H}^2 d\theta. \end{aligned}$$

The rectangles  $\pi_\theta(2T)$  in the sum above are  $\lesssim 2^{kO(\delta)}$ -overlapping over the two innermost sums, so the above implies that

$$\begin{aligned} & \int_0^{2\pi} \int |\pi_{\theta\#}\mu_b|^{p_n} d\mathcal{H}^2 d\theta \lesssim (\log R)^{O(1)} \sum_{j \geq J} \sum_{k \in [j\epsilon, j]} 2^{kO(\delta)} \sum_m \\ & \int_0^{2\pi} \int \bigcup_{\substack{\tau \in \Lambda_{j,k}: \\ |\theta_\tau - \theta| \leq 2^{k(\delta-1/2)}}} \bigcup_{\substack{T \in \mathbb{T}_{\tau,b}: \\ T \cap B_m \neq \emptyset}} \pi_\theta(2T) |\pi_{\theta\#}(\mu_{m,j,k} * \phi_{2^{j-k\delta}})|^{p_n} d\mathcal{H}^2 d\theta. \end{aligned}$$

We will abbreviate  $\nu_{m,j,k} = \mu_{m,j,k} * \phi_{2^{j-k\delta}}$  to simplify the notation. By rescaling each ball  $B_m$  to a ball of radius 1, using that orthogonal projections commute with

rescalings,

$$\begin{aligned} \int_0^{2\pi} \int |\pi_{\theta\#\mu_b}|^{p_n} d\mathcal{H}^2 d\theta &\lesssim (\log R)^{O(1)} \sum_{j \geq J} \sum_{k \in [j\epsilon, j]} 2^{kO(\delta)} \sum_m 2^{2(j-k)(p_n-1)} \\ &\int_0^{2\pi} \int \bigcup_{\substack{\tau \in \Lambda_{j,k}: \\ |\theta_\tau - \theta| \leq 2^{k(\delta-1/2)}} \cup_{\substack{T \in \mathbb{T}_{\tau,b}: \\ T \cap B_m \neq \emptyset}} \pi_{\theta\#} 2^{j-k-k\delta} \nu_{m,j,k} \Big|^{p_n} d\mathcal{H}^2 d\theta. \end{aligned}$$

Each rescaled measure  $2^{j-k-k\delta} \nu_{m,j,k}$  and collection of rescaled planks  $2^{j-k-k\delta} 2T$  satisfy the hypotheses for (4.4); by the definition of  $\beta_0$  and of the ‘‘bad’’ planks. Moreover,  $c_\alpha \left( 2^{j-k-k\delta} \nu_{m,j,k} \right) \leq 2^{-\alpha(j-k)+kO(\delta)} c_\alpha(\mu)$ . Hence, by the conclusion of (4.4) for  $\beta_0$  and the assumption that  $\alpha > 2$ ,

$$\begin{aligned} \int_0^{2\pi} \int |\pi_{\theta\#\mu_b}|^{p_n} d\mathcal{H}^2 d\theta &\lesssim (\log R)^{O(1)} \sum_{j \geq J} \sum_{k \in [j\epsilon, j]} \sum_m \mu(100B_m) 2^{kO(\delta)} c_\alpha(\mu)^{p_n-1} 2^{-k\kappa} \\ &\lesssim 2^{-J\kappa/2} \mu(\mathbb{R}^3) c_\alpha(\mu)^{p_n-1} \lesssim R^{-\epsilon\kappa/100} \mu(\mathbb{R}^3) c_\alpha(\mu)^{p_n-1}. \end{aligned}$$

It remains to bound the ‘‘good’’ part. The  $L^2$  method we use is similar to the second part of [8], but we include all details for completeness. By (4.8), it suffices to show that

$$(4.13) \quad \int_0^{2\pi} \int |\pi_{\theta\#\mu_g}|^2 d\mathcal{H}^2 d\theta \lesssim \mu(\mathbb{R}^3) c_\alpha(\mu) R^{-100\delta+p_{n-1}\beta(\frac{2}{p_n}-1)}.$$

By Plancherel’s theorem in 2 dimensions,

$$\int_0^{2\pi} \int |\pi_{\theta\#\mu_g}|^2 d\mathcal{H}^2 d\theta = \int_0^{2\pi} \int |\widehat{\mu}_g(\eta_1 \gamma'(\theta) + \eta_2(\gamma \times \gamma')(\theta))|^2 d\eta_1 d\eta_2 d\theta.$$

Therefore, to prove (4.13), it suffices to show that

$$\begin{aligned} \int_0^{2\pi} \int_{\{|\eta| \geq 2^{2J}\}} |\widehat{\mu}_g(\eta_1 \gamma'(\theta) + \eta_2(\gamma \times \gamma')(\theta))|^2 d\eta_1 d\eta_2 d\theta \\ \lesssim \mu(\mathbb{R}^3) c_\alpha(\mu) R^{-100\delta+p_{n-1}\beta(\frac{2}{p_n}-1)}, \end{aligned}$$

since the part near the origin can be handled by a trivial bound on  $\|\widehat{\mu}_g\|_{L^\infty(B(0,2^{2J}))}$  by  $\|\widehat{\mu}\|_{L^\infty} + \|\widehat{\mu}_b\|_{L^\infty(B(0,2^{2J}))} \lesssim 2^{O(1)J} \lesssim R^{O(1)\epsilon}$  (this is where we use that  $\epsilon \ll \beta$ ). We break the left-hand side above into two integrals:

$$\begin{aligned} \int_0^{2\pi} \int_{\{|\eta_1| \geq |\eta|^{1-\epsilon}\} \cap \{|\eta| \geq 2^{2J}\}} |\widehat{\mu}_g(\eta_1 \gamma'(\theta) + \eta_2(\gamma \times \gamma')(\theta))|^2 d\eta_1 d\eta_2 d\theta \\ + \int_0^{2\pi} \int_{\{|\eta_1| \leq |\eta|^{1-\epsilon}\} \cap \{|\eta| \geq 2^{2J}\}} |\widehat{\mu}_g(\eta_1 \gamma'(\theta) + \eta_2(\gamma \times \gamma')(\theta))|^2 d\eta_1 d\eta_2 d\theta. \end{aligned}$$

For the first integral,  $\widehat{\mu}_b$  is negligible on the domain of integration, since  $\mathbb{T}_{\tau,b}$  is empty whenever  $\tau \in \Lambda_{j,k}$  and  $k \leq 10j\epsilon$  (as mentioned previously, this follows from

the definition of  $\mathbb{T}_{\tau,b}$  and the fractal property of  $\mu$ ). Therefore, to bound the first integral, it suffices to prove that

$$(4.14) \quad \int_0^{2\pi} \int_{\{|\eta_1| \geq |\eta|^{1-\epsilon}\} \cap \{|\eta| \geq 2^{2^J}\}} |\widehat{\mu}(\eta_1 \gamma'(\theta) + \eta_2(\gamma \times \gamma')(\theta))|^2 d\eta_1 d\eta_2 d\theta \\ \lesssim \mu(\mathbb{R}^3) c_\alpha(\mu).$$

The change of variables

$$(\xi_1, \xi_2, \xi_3) = \eta_1 \gamma'(\theta) + \eta_2(\gamma \times \gamma')(\theta)$$

has Jacobian

$$(4.15) \quad \left| \frac{\partial(\eta_1, \eta_2, \theta)}{\partial(\xi_1, \xi_2, \xi_3)} \right| \sim |\eta_1|^{-1},$$

so the integral in (4.14) is bounded by

$$\int |\xi|^{\epsilon-1} |\widehat{\mu}(\xi)|^2 d\xi.$$

By the Fourier formula for the energy (see e.g. [22, Theorem 3.10]), the above is comparable to  $I_{2+\epsilon}(\mu)$ , which is bounded by  $\mu(\mathbb{R}^3) c_\alpha(\mu)$  since  $\alpha > 2 + \epsilon$  (see for instance the inequality at the end of p. 19 in [22]).

Therefore, it remains to prove that

$$(4.16) \quad \int_0^{2\pi} \int_{\{|\eta_1| \leq |\eta|^{1-\epsilon}\} \cap \{|\eta| \geq 2^{2^J}\}} |\widehat{\mu}_g(\eta_1 \gamma'(\theta) + \eta_2(\gamma \times \gamma')(\theta))|^2 d\eta_1 d\eta_2 d\theta \\ \lesssim \mu(\mathbb{R}^3) c_\alpha(\mu) R^{-100\delta + p_{n-1}\beta(\frac{2}{p_n} - 1)}.$$

As remarked above, that  $\mathbb{T}_{\tau,b}$  is empty whenever  $k \leq 10j\epsilon$  implies that the definition of  $\mu_b$  in (4.7) can be written as

$$\mu_b = \sum_{j \geq J} \sum_{k \in [0, j]} \sum_{\tau \in \Lambda_{j,k}} \sum_{T \in \mathbb{T}_{\tau,b}} M_T \mu, \quad \mu_g = \mu - \mu_b.$$

It follows that on the domain of integration in (4.16),  $\mu_g$  can be replaced by the sum over the good planks (ignoring rapidly decaying error terms), and it suffices to prove that

$$\int_0^{2\pi} \int_{\{|\eta_1| \leq |\eta|^{1-\epsilon}\} \cap \{|\eta| \geq 2^{2^J}\}} \left| \sum_{j \geq J} \sum_{k \in [j\epsilon/1000, j]} \sum_{\tau \in \Lambda_{j,k}} \sum_{T \in \mathbb{T}_{\tau,g}} \widehat{M}_T \mu(\eta_1 \gamma'(\theta) + \eta_2(\gamma \times \gamma')(\theta)) \right|^2 d\eta_1 d\eta_2 d\theta \\ \lesssim \mu(\mathbb{R}^3) c_\alpha(\mu) R^{-100\delta + p_{n-1}\beta(\frac{2}{p_n} - 1)}.$$

By disjointness, it is enough to show that

$$\begin{aligned} & \sum_{j \geq J} \sum_{k \in [j\epsilon/1000, j]} \sum_{\tau \in \Lambda_{j,k}} \int_0^{2\pi} \int_{\{|\eta_1| \leq |\eta|^{1-\epsilon}\} \cap \{|\eta| \geq 2^{2j}\}} \\ & \left| \sum_{T \in \mathbb{T}_{\tau,g}} \widehat{M_T \mu}(\eta_1 \gamma'(\theta) + \eta_2(\gamma \times \gamma')(\theta)) \right|^2 d\eta_1 d\eta_2 d\theta \\ & \lesssim \mu(\mathbb{R}^3) c_\alpha(\mu) R^{-100\delta + p_n - 1\beta(\frac{2}{p_n} - 1)}. \end{aligned}$$

If  $k < j$  and  $\tau \in \Lambda_{j,k}$ , then by a change of variables and the formula (4.15),

$$\begin{aligned} & \int_0^{2\pi} \int_{\{|\eta_1| \leq |\eta|^{1-\epsilon}\} \cap \{|\eta| \geq 2^{2j}\}} \left| \sum_{T \in \mathbb{T}_{\tau,g}} \widehat{M_T \mu}(\eta_1 \gamma'(\theta) + \eta_2(\gamma \times \gamma')(\theta)) \right|^2 d\eta_1 d\eta_2 d\theta \\ & \lesssim 2^{-j+k/2} \int_{\mathbb{R}^3} \left| \sum_{T \in \mathbb{T}_{\tau,g}} \widehat{M_T \mu}(\xi) \right|^2 d\xi, \end{aligned}$$

where we used that if  $\eta_1 \gamma'(\theta) + \eta_2(\gamma \times \gamma')(\theta)$  intersects  $\tau \in \Lambda_{j,k}$ , then  $|\eta_1| \sim 2^{j-k/2}$ . We then use essential orthogonality of wave packets to get

$$(4.17) \quad \begin{aligned} & \int_0^{2\pi} \int_{\{|\eta_1| \leq |\eta|^{1-\epsilon}\} \cap \{|\eta| \geq 2^{2j}\}} \left| \sum_{T \in \mathbb{T}_{\tau,g}} \widehat{M_T \mu}(\eta_1 \gamma'(\theta) + \eta_2(\gamma \times \gamma')(\theta)) \right|^2 d\eta_1 d\eta_2 d\theta \\ & \lesssim 2^{-j+k/2+kO(\delta)} \sum_{T \in \mathbb{T}_{\tau,g}} \int_{\mathbb{R}^3} \left| \widehat{M_T \mu}(\xi) \right|^2 d\xi. \end{aligned}$$

The above inequality also holds if  $k = j$  by the following argument. Use Plancherel's theorem in 2 dimensions to change the integral in the left-hand side of (4.17) back to an integral of  $\sum_{T \in \mathbb{T}_{\tau,g}} \pi_{\theta\#} M_T \mu$ . The only significant  $\theta$  in the integration over  $\theta$  satisfies  $|\theta - \theta_\tau| \lesssim 2^{-k/2}$ , so the sets  $\pi_\theta(T)$  are  $\lesssim 2^{kO(\delta)}$ -overlapping. This means that the  $L^2$  norms can be decoupled, and then (4.17) follows by a calculation using that each  $M_T \mu$  is a wave packet.

Therefore, it suffices to prove that

$$(4.18) \quad \begin{aligned} & \sum_{j \geq J} \sum_{k \in [j\epsilon/1000, j]} 2^{-j+k/2+kO(\delta)} \sum_{\tau \in \Lambda_{j,k}} \sum_{T \in \mathbb{T}_{\tau,g}} \|M_T \mu\|_2^2 \\ & \lesssim \mu(\mathbb{R}^3) c_\alpha(\mu) R^{-100\delta + p_n - 1\beta(\frac{2}{p_n} - 1)}. \end{aligned}$$

Fix a pair  $(j, k)$  and  $m$ , where  $\{B_m\}$  is a boundedly overlapping covering of  $\mathbb{R}^3$  by balls of radius  $2^{-(j-k)+k\delta}$ . We abbreviate  $\mathbb{T}_{\tau,g,m} = \{T \in \mathbb{T}_{\tau,g} : T \cap B_m \neq \emptyset\}$ . By writing out the  $L^2$  norm as an inner product and expanding out the definition of one of the  $M_T \mu$ 's, the innermost double sum can be written as

$$(4.19) \quad \sum_{\tau \in \Lambda_{j,k}} \sum_{T \in \mathbb{T}_{\tau,g,m}} \|M_T \mu\|_2^2 = \int \sum_{\tau \in \Lambda_{j,k}} \sum_{T \in \mathbb{T}_{\tau,g,m}} (\eta_T M_T \mu) * \check{\psi}_\tau d\mu.$$

By the Cauchy-Schwarz inequality with respect to the measure  $\mu$ ,

$$(4.20) \quad (4.19) \leq \mu(100B_m)^{1/2} \left( \int \left| \sum_{\tau \in \Lambda_{j,k}} \sum_{T \in \mathbb{T}_{\tau,g,m}} (\eta_T M_T \mu) * \widetilde{\psi}_\tau \right|^2 d\mu \right)^{1/2}.$$

Let  $p_{\text{dec}} \in (2, 6]$  be an exponent to be chosen. We abbreviate  $f_T = (\eta_T M_T \mu) * \widetilde{\psi}_\tau$ . By dyadic pigeonholing, we can find a subset  $\mathbb{W} \subseteq \bigcup_{\tau \in \Lambda_{j,k}} \mathbb{T}_{\tau,g,m}$  such that  $\|f_T\|_{p_{\text{dec}}}$  is constant over  $T \in \mathbb{W}$  up to a factor of 2, such that

$$(4.21) \quad (4.20) \leq O(j) \mu(100B_m)^{1/2} \left( \int \left| \sum_{T \in \mathbb{W}} f_T \right|^2 d\mu \right)^{1/2}.$$

By partitioning the support of  $\mu$  into boundedly overlapping balls of radius  $2^{-j}$ , we can find a dyadic number  $M$  and a union  $Y$  of balls of radius  $2^{-j}$  such that each ball  $B \subseteq Y$  intersects  $\sim M$  planks  $2T$  with  $T \in \mathbb{W}$ , and such that

$$(4.22) \quad (4.21) \lesssim O(j) \mu(100B_m)^{1/2} \left( \int_Y \left| \sum_{T \in \mathbb{W}} f_T \right|^2 d\mu \right)^{1/2}.$$

By Hölder's inequality with respect to Lebesgue measure,

$$(4.23) \quad \left( \int_Y \left| \sum_{T \in \mathbb{W}} f_T \right|^2 d\mu \right)^{1/2} \leq \left( \int_Y \left| \sum_{T \in \mathbb{W}} f_T \right|^{p_{\text{dec}}} d\mu \right)^{\frac{1}{p_{\text{dec}}}} \left( \int_Y \mu^{\frac{p_{\text{dec}}}{p_{\text{dec}}-2}} d\mu \right)^{\frac{1}{2} - \frac{1}{p_{\text{dec}}}},$$

By a rescaling, followed by an application of the refined decoupling inequality (Theorem A.1), the first factor in the right-hand side of (4.23) satisfies

$$(4.24) \quad \left( \int_Y \left| \sum_{T \in \mathbb{W}} f_T \right|^{p_{\text{dec}}} d\mu \right)^{\frac{1}{p_{\text{dec}}}} \lesssim 2^{k\epsilon} \left( \frac{M}{|\mathbb{W}|} \right)^{\frac{1}{2} - \frac{1}{p_{\text{dec}}}} \left( \sum_{T \in \mathbb{W}} \|f_T\|_{p_{\text{dec}}}^2 \right)^{1/2}.$$

By the definition of  $M$ , and the definition of “bad” planks in (4.6), the second factor in the right-hand side of (4.23) satisfies

$$(4.25) \quad \begin{aligned} & \left( \int_Y \mu^{\frac{p_{\text{dec}}}{p_{\text{dec}}-2}} d\mu \right)^{\frac{1}{2} - \frac{1}{p_{\text{dec}}}} \\ & \lesssim \left( \int_Y d\mu \right)^{\frac{1}{2} - \frac{1}{p_{\text{dec}}}} 2^{\frac{j(3-\alpha)}{p_{\text{dec}}}} c_\alpha(\mu)^{\frac{1}{p_{\text{dec}}}} \\ & \lesssim \left( \frac{1}{M} \sum_{T \in \mathbb{W}} \int_{Y \cap T} d\mu \right)^{\frac{1}{2} - \frac{1}{p_{\text{dec}}}} 2^{\frac{j(3-\alpha)}{p_{\text{dec}}}} c_\alpha(\mu)^{\frac{1}{p_{\text{dec}}}} \\ & \lesssim \left( \frac{|\mathbb{W}|}{M} 2^{-\alpha(j-k) - k(3/2 - \beta_0) + 1000j\epsilon} c_\alpha(\mu) \right)^{\frac{1}{2} - \frac{1}{p_{\text{dec}}}} 2^{\frac{j(3-\alpha)}{p_{\text{dec}}}} c_\alpha(\mu)^{\frac{1}{p_{\text{dec}}}} \\ & = \left( \frac{|\mathbb{W}|}{M} \right)^{\frac{1}{2} - \frac{1}{p_{\text{dec}}}} 2^{\left(\frac{1}{2} - \frac{1}{p_{\text{dec}}}\right)(-\alpha(j-k) - k(3/2 - \beta_0) + 1000j\epsilon) + \frac{j(3-\alpha)}{p_{\text{dec}}}} c_\alpha(\mu)^{\frac{1}{2}}. \end{aligned}$$

Substituting (4.24) and (4.25) into (4.23) yields

$$(4.26) \quad \left( \int_Y \left| \sum_{T \in \mathbb{W}} f_T \right|^2 d\mu \right)^{1/2} \\ \lesssim 2^{k\epsilon + \left(\frac{1}{2} - \frac{1}{p_{\text{dec}}}\right)(-\alpha(j-k) - k(3/2 - \beta_0) + 1000j\epsilon) + \frac{j(3-\alpha)}{p_{\text{dec}}} c_\alpha(\mu)^{\frac{1}{2}} \left( \sum_{T \in \mathbb{W}} \|f_T\|_{p_{\text{dec}}}^2 \right)^{1/2}.$$

By recalling that  $f_T = (\eta_T M_T \mu) * \widetilde{\psi}_\tau$  and applying the Hausdorff-Young inequality, followed by Hölder's inequality,

$$\|f_T\|_{p_{\text{dec}}} \lesssim \|M_T \mu\|_2 2^{(3j - \frac{3k}{2})\left(\frac{1}{2} - \frac{1}{p_{\text{dec}}}\right)}.$$

Therefore, (4.26) becomes

$$(4.27) \quad \left( \int_Y \left| \sum_{T \in \mathbb{W}} f_T \right|^2 d\mu \right)^{1/2} \\ \lesssim 2^{k\epsilon + \left(\frac{1}{2} - \frac{1}{p_{\text{dec}}}\right)(-\alpha(j-k) - k(3/2 - \beta_0) + 1000j\epsilon) + \frac{j(3-\alpha)}{p_{\text{dec}}} + (3j - \frac{3k}{2})\left(\frac{1}{2} - \frac{1}{p_{\text{dec}}}\right)} \\ \times c_\alpha(\mu)^{\frac{1}{2}} \left( \sum_{\tau \in \Lambda_{j,k}} \sum_{T \in \mathbb{T}_{\tau,g,m}} \|M_T \mu\|_2^2 \right)^{1/2},$$

where we now sum over all “good” planks in  $\mathbb{T}_{\tau,g,m}$  rather than just  $\mathbb{W}$ , since the property of  $\mathbb{W}$  will no longer be needed. By substituting (4.27) into (4.22), then into (4.21), then into (4.20), and then finally into (4.19), we get

$$\sum_{\tau \in \Lambda_{j,k}} \sum_{T \in \mathbb{T}_{\tau,g,m}} \|M_T \mu\|_2^2 \\ \lesssim 2^{jO(\epsilon) + \left(\frac{1}{2} - \frac{1}{p_{\text{dec}}}\right)(-\alpha(j-k) - k(3/2 - \beta_0) + 1000j\epsilon) + \frac{j(3-\alpha)}{p_{\text{dec}}} + (3j - \frac{3k}{2})\left(\frac{1}{2} - \frac{1}{p_{\text{dec}}}\right)} \\ \times \mu(100B_m)^{\frac{1}{2}} c_\alpha(\mu)^{\frac{1}{2}} \left( \sum_{\tau \in \Lambda_{j,k}} \sum_{T \in \mathbb{T}_{\tau,g,m}} \|M_T \mu\|_2^2 \right)^{1/2}.$$

By cancelling the common factor in both sides, this simplifies to

$$\sum_{\tau \in \Lambda_{j,k}} \sum_{T \in \mathbb{T}_{\tau,g,m}} \|M_T \mu\|_2^2 \lesssim 2^{jO(\epsilon) + j(3-\alpha) - 2k\left(\frac{1}{2} - \frac{1}{p_{\text{dec}}}\right)(3-\alpha-\beta_0)} \mu(100B_m) c_\alpha(\mu).$$

Summing over  $m$  gives

$$\sum_{\tau \in \Lambda_{j,k}} \sum_{T \in \mathbb{T}_{\tau,g}} \|M_T \mu\|_2^2 \lesssim 2^{jO(\epsilon) + j(3-\alpha) - 2k\left(\frac{1}{2} - \frac{1}{p_{\text{dec}}}\right)(3-\alpha-\beta_0)} \mu(\mathbb{R}^3) c_\alpha(\mu).$$



If we substitute this bound into the left-hand side of (4.18), we get

$$(4.28) \quad \sum_{j \geq J} \sum_{k \in [\epsilon j/1000, j]} \sum_{\tau \in \Lambda_{j,k}} \sum_{T \in \mathbb{T}_{\tau,g}} 2^{-j+k/2+kO(\delta)} \|M_T \mu\|_2^2 \lesssim \mu(\mathbb{R}^3) c_\alpha(\mu) \times \\ \sum_{J \leq j \leq (1+\delta) \log_2(R)} \sum_{k \in [\epsilon j/1000, j]} 2^{-j+k/2+jO(\epsilon)} 2^{j(3-\alpha)-2k} \left(\frac{1}{2} - \frac{1}{p_{\text{dec}}}\right)^{(3-\alpha-\beta_0)}.$$

By taking  $p_{\text{dec}} = 4$  and summing the two geometric series, (4.28) simplifies to

$$(4.29) \quad (4.28) \lesssim \mu(\mathbb{R}^3) c_\alpha(\mu) R^{O(\epsilon)} \max \left\{ 1, R^{\frac{\beta_0}{2} - \frac{(\alpha-2)}{2}} \right\}.$$

Using that  $\delta \ll \epsilon \ll \min\{\beta, \alpha - 2\}$ , to verify that (4.29) implies (4.18), it suffices to check that

$$\frac{\beta_0}{2} \leq p_{n-1} \beta \left( \frac{2}{p_n} - 1 \right) + (\alpha - 2)/100.$$

Since  $\beta > c\beta_0$ , where  $1 - c \ll \alpha - 2$ , it suffices to check that

$$(4.30) \quad \frac{1}{2} \leq p_{n-1} \left( \frac{2}{p_n} - 1 \right).$$

But by (4.2), equality<sup>2</sup> holds in (4.30) for any integer  $n \in [1, \infty]$ , and (4.30) also holds if  $n = 0$  (using that  $p_{-1} = 1$ ). We have verified (4.13), so this finishes the proof.  $\square$

We are now ready to prove the following main result of the section, which includes Theorem 1.2 as a special case.

**Theorem 4.3.** *Let  $\alpha > 2$  and  $1 \leq p \leq \frac{3}{2}$ . Then*

$$(4.31) \quad \int_0^{2\pi} \int |\pi_{\theta\#}\mu|^p d\mathcal{H}^2 d\theta \leq C_{\alpha,p} \mu(\mathbb{R}^3) c_\alpha(\mu)^{p-1},$$

for any Borel measure  $\mu$  on  $B_3(0, 1)$  with  $c_\alpha(\mu) < \infty$ .

*Proof.* Since (4.31) is trivial if the projections are a.e. absolutely continuous, it suffices to prove (4.31) for  $p > 1$ . By interpolation, it suffices to prove the endpoint case  $p = \frac{3}{2}$ . Since  $p = 3/2 > 1$ , by a limiting argument, it may be assumed that  $\mu$  is a smooth function.

Take  $\beta > 0$  such that  $\beta \ll \alpha - 2$ , and let  $\delta, \epsilon > 0$  be such that  $\delta \ll \epsilon \ll \beta$  and  $\delta \ll \kappa = \kappa(\beta)$ , the parameter in Lemma 4.1. Write  $\mu = \mu_b + \mu_g$ , where the “bad” part is defined as in (4.7), except that we take  $J = 1$  and the parameter  $\beta$  is used instead of  $\beta_0$  in (4.6). By the triangle inequality, and then by following the same

<sup>2</sup>This step could only be improved by using  $p_{\text{dec}} > 4$  if we could assume that  $\beta$  is small, but for  $\beta$  close to 1 there is no gain from taking  $p_{\text{dec}} > 4$ . From the train-track example in Proposition 3.4, we expect the  $\beta = 1$  case to be critical. Equality at  $p_{n-1} = p_n = 3/2$  explains why the exponent  $3/2$  is important and cannot be increased.

argument as in the proof of Lemma 4.1,

$$\begin{aligned} \left( \int_0^{2\pi} \int |\pi_{\theta\#}\mu_b|^p d\mathcal{H}^2 d\theta \right)^{1/p} &\lesssim \sum_{j \geq 1} \sum_{k \in [j\epsilon, j]} \\ &\left( O(j) 2^{2(j-k)(p-1)+kO(\delta)} \sum_m \int_0^{2\pi} \int_{\substack{\tau \in \Lambda_{j,k}: \\ |\theta_\tau - \theta| \leq 2^{k(\delta-1/2)}} \cup \substack{T \in \mathbb{T}_{\tau,b}: \\ T \cap B_m \neq \emptyset}} \pi_{\theta}(2^{j-k-k\delta} 2T) \right. \\ &\quad \left. \left| \pi_{\theta\#} 2^{j-k-k\delta} (\mu_{m,j,k} * \phi_{2^{j-k\delta}}) \right|^p d\mathcal{H}^2 d\theta \right)^{1/p}. \end{aligned}$$

By the  $p = \frac{3}{2}$  case of Lemma 4.1, (4.32)

$$\int_0^{2\pi} \int |\pi_{\theta\#}\mu_b|^p d\mathcal{H}^2 d\theta \lesssim c_\alpha(\mu)^{p-1} \mu(\mathbb{R}^3) \left( \sum_{j \geq 1} 2^{-j\epsilon\kappa/1000} \right)^p \lesssim c_\alpha(\mu)^{p-1} \mu(\mathbb{R}^3).$$

This bounds the bad part, so it remains to bound the good part.

If we follow the same argument as in Lemma 4.1 to bound the good part, the only modification that we need to make is that at (4.20), we need to use the Fourier support of the integrand being contained inside a ball of radius  $\sim 2^j$  to replace  $\mu$  by its smoothed out version  $\mu_j$  at scale  $2^{-j}$ . The rest of the proof works without substantial changes, and yields

$$\begin{aligned} \int_0^{2\pi} \int |\pi_{\theta\#}\mu_g|^2 d\mathcal{H}^2 d\theta &\lesssim I_{2+100\epsilon}(\mu) \\ &+ \mu(\mathbb{R}^3) c_\alpha(\mu) \sum_{j \geq 1} \sum_{k \in [\epsilon j/1000, j]} 2^{-j+k/2+jO(\epsilon)} 2^{j(3-\alpha)-2k\left(\frac{1}{2}-\frac{1}{p_{\text{dec}}}\right)(3-\alpha-\beta)}. \end{aligned}$$

By taking  $p_{\text{dec}} = 4$  and using  $\alpha > 2$  and  $\epsilon \ll \beta \ll \alpha - 2$ , the second term is a decaying geometric series, which yields

$$\int_0^{2\pi} \int |\pi_{\theta\#}\mu_g|^2 d\mathcal{H}^2 d\theta \lesssim \mu(\mathbb{R}^3) c_\alpha(\mu).$$

Note that the argument above for the bad part in (4.32) also works if one sets  $p = 1$  and applies the corresponding conclusion in Lemma 4.1. Hence, the above shows that  $\pi_{\theta\#}\mu_b \in L^1$  for a.e.  $\theta \in [0, 2\pi)$ , and that

$$\int_0^{2\pi} \int |\pi_{\theta\#}\mu_b| d\mathcal{H}^2 d\theta \lesssim \mu(\mathbb{R}^3).$$

By the trivial  $L^1$  estimate of  $\pi_{\theta\#}\mu$  (using that  $\mu$  is a smooth function) and the triangle inequality, this gives

$$\int_0^{2\pi} \int |\pi_{\theta\#}\mu_g| d\mathcal{H}^2 d\theta \lesssim \mu(\mathbb{R}^3).$$

By writing  $3/2 = p = (1 - \phi)1 + \phi \cdot 2$  where  $\phi = p - 1$ , and then applying Hölder's inequality to the function

$$|\pi_{\theta\#}\mu_g|^p = |\pi_{\theta\#}\mu_g|^{(1-\phi) \cdot 1} \cdot |\pi_{\theta\#}\mu_g|^{\phi \cdot 2},$$

with  $q = \frac{1}{1-\phi}$  and  $q' = \frac{1}{\phi}$ , this yields

$$(4.33) \quad \int_0^{2\pi} \int |\pi_{\theta\#}\mu_g|^p d\mathcal{H}^2 d\theta \lesssim c_\alpha(\mu)^{p-1} \mu(\mathbb{R}^3).$$

Combining (4.32) and (4.33) gives, by the triangle inequality

$$\int_0^{2\pi} \int |\pi_{\theta\#}\mu|^p d\mathcal{H}^2 d\theta \lesssim c_\alpha(\mu)^{p-1} \mu(\mathbb{R}^3). \quad \square$$

*Proof of Theorem 1.1.* The class of horizontal (or  $SL_2$ ) lines is invariant under rotations of the plane  $\mathbb{R}^2 \times \{0\}$ , so by the triangle inequality it may be assumed that all tubes  $T \in \mathbb{T}$  have  $\ell(T) = p * \mathbb{V}_\theta$  with  $|\theta| \geq \pi/4$ . Recall that  $U(\xi_1, \xi_2, \xi_3) = (\eta_1, \eta_2, \eta_3)$  is the orthogonal transformation given by

$$\eta_1 = \frac{\xi_1 + \xi_3}{\sqrt{2}}, \quad \eta_3 = \frac{-\xi_1 + \xi_3}{\sqrt{2}}, \quad \eta_2 = \xi_2.$$

For each  $T \in \mathbb{T}$ , let  $w_T$  be such that  $\ell(Uw_T)$  is the centre line of  $T$ . Let  $\mu = \delta^{-1} \sum_{T \in \mathbb{T}} \chi_{B(w_T, C\delta)}$ , where  $C$  is a large constant to be chosen. If we run the point-line duality argument from (3.12) in reverse, we get

$$(4.34) \quad \begin{aligned} & \int_{-\pi/2}^{\pi/2} \|\pi_{\theta\#}\mu\|_{L^{3/2}(\gamma(\theta)^\perp)}^{3/2} d\theta \\ &= \int_{-\pi/2}^{\pi/2} \int_{\gamma(\theta)^\perp} \left| \int_{\pi_\theta^{-1}(y)} \mu d\mathcal{H}^1 \right|^{3/2} d\mathcal{H}^2(y) d\theta \\ &= \int_{-\pi/2}^{\pi/2} \int_{\gamma(\theta)^\perp} \left| \int_{U\pi_\theta^{-1}(y)} (U\#\mu) d\mathcal{H}^1 \right|^{3/2} d\mathcal{H}^2(y) d\theta \\ &\sim \int \left| \int_L U\#\mu d\mathcal{H}^1 \right|^{3/2} d\left(\ell_\#^* \mathcal{H}^3 \chi_{|x_2| \leq \sqrt{2}}\right)(L) \quad (\text{by Lemma 3.6}) \end{aligned}$$

$$(4.35) \quad \begin{aligned} & \gtrsim \int |\{T \in \mathbb{T} : UB(w_T, 100\delta) \cap L \neq \emptyset\}|^{3/2} d\left(\ell_\#^* \mathcal{H}^3 \chi_{|x_2| \leq \sqrt{2}}\right)(L) \\ &= \int_{|p_2^*| \leq \sqrt{2}} |\{T \in \mathbb{T} : UB(w_T, 100\delta) \cap \ell^*(p^*) \neq \emptyset\}|^{3/2} d\mathcal{H}^3(p^*) \\ &\geq \int_{|p^*| \leq 1} |\{T \in \mathbb{T} : UB(w_T, 100\delta) \cap \ell^*(p^*) \neq \emptyset\}|^{3/2} d\mathcal{H}^3(p^*) \end{aligned}$$

$$(4.36) \quad = \int_{|p^*| \leq 1} |\{T \in \mathbb{T} : p^* \in \ell(UB(w_T, 100\delta))\}|^{3/2} d\mathcal{H}^3(p^*)$$

$$= \int_{|p^*| \leq 1} \left| \sum_{T \in \mathbb{T}} \chi_{\ell(UB(w_T, 100\delta))}(p^*) \right|^{3/2} d\mathcal{H}^3(p^*)$$

$$(4.37) \quad \geq \int_{B_3(0,1)} \left| \sum_{T \in \mathbb{T}} \chi_T \right|^{3/2},$$

where we now choose  $C$  large enough to ensure that (4.35) holds. In more detail, if  $C$  is large enough then for any  $SL_2$  line  $L$ ,

$$\int_L U_{\#}\mu \, d\mathcal{H}^1 \gtrsim |\{T \in \mathbb{T} : UB(w_T, 100\delta) \cap L \neq \emptyset\}|.$$

By the formula defining  $\mu$ , to verify the above it suffices to check that for any  $T \in \mathbb{T}$  such that  $UB(w_T, 100\delta) \cap L \neq \emptyset$ ,

$$\mathcal{H}^1(L \cap UB(w_T, C\delta)) \gtrsim \delta,$$

which is clearly true if  $C > 1000$ . The inequality (4.36) used Lemma 3.5. The last line (4.37) used

$$\ell(UB(w_T, 100\delta)) \cap B_3(0, 1) \supseteq \mathcal{N}_\delta(\ell(Uw_T)) \cap B_3(0, 1) \supseteq T \cap B_3(0, 1),$$

where the first part follows from the definition of  $\ell$  in (3.8) (it is locally Lipschitz), and the second part follows from the assumption that  $\ell(Uw_T)$  is the centre line of  $T$ .

If we apply Theorem 1.2 to (4.34) with  $\alpha = 2 + \epsilon$ , we get

$$(4.38) \quad \left\| \sum_{T \in \mathbb{T}} \chi_T \right\|_{L^{3/2}(B(0,1))}^{3/2} \lesssim \int \|\pi_{\theta\#}\mu\|_{3/2}^{3/2} \, d\theta \leq C_\epsilon \delta^{-\epsilon/2} \mu(\mathbb{R}^3) c_2(\mu)^{1/2}.$$

The above inequality used that  $c_{2+\epsilon}(\mu) \lesssim \delta^{-\epsilon} c_2(\mu)$  whenever  $\mu$  is a linear combination of indicator functions over a boundedly overlapping collection of  $\delta$ -balls. But  $\mu(\mathbb{R}^3) \lesssim \delta^2 |\mathbb{T}|$ , and  $c_2(\mu) \lesssim 1$  (this is where we use the 2-dimensional ball condition (1.1) on the family of tubes  $\mathbb{T}$ ), so (4.38) implies (1.2) and proves Theorem 1.1.  $\square$

## 5. AN $L^p$ INEQUALITY FOR PROJECTIONS ONTO LINES

**Lemma 5.1.** *Let  $p$  be such that  $1 \leq p < 3/2$ . For any  $\beta > 0$ , there exists an  $\kappa > 0$  (which may depend on  $\beta$  and  $p$ ), such that*

$$(5.1) \quad \int_0^{2\pi} \int_{\bigcup_{D \in \mathbb{D}_\theta} D} |\rho_{\theta\#}(\mu * \phi_R)|^p \, d\mathcal{H}^1 \, d\theta \leq C_{\beta,p} R^{-\kappa} \mu(\mathbb{R}^3) c_1(\mu)^{p-1},$$

for any  $R > 1$ , for any Borel measure  $\mu$  on  $B_3(0, 1)$  with  $c_1(\mu) < \infty$ , and for any family of sets  $\{\mathbb{D}_\theta\}$ , where each  $\mathbb{D}_\theta$  is a set of intervals of radius  $R^{-1}$  in  $\text{span}(\gamma(\theta))$ , with  $(\rho_{\theta\#}(\mu * \phi_R))(D) \geq c_1(\mu) R^{\beta-1}$  for each  $D \in \mathbb{D}_\theta$ , and such that the set  $\{(\theta, x) \in [0, 2\pi) \times \mathbb{R}^3 : x \in \bigcup_{D \in \mathbb{D}_\theta} D\}$  is Borel measurable.

*Proof.* To prove (5.1), by pigeonholing we can replace  $\mu$  by a sum of indicator functions over a disjoint family of  $R^{-1}$ -balls, i.e.,

$$(5.2) \quad \mu = \frac{R^3 \mu(\mathbb{R}^3)}{|\mathcal{B}|(4\pi/3)} \sum_{B \in \mathcal{B}} \chi_B.$$

By scaling we can additionally assume that  $c_1(\mu) = 1$ . Then (5.1) is equivalent to

$$\left( \frac{R^3 \mu(\mathbb{R}^3)}{|\mathcal{B}|} \right)^p \int_0^{2\pi} \int_{\bigcup_{D \in \mathbb{D}_\theta} D} \left| \rho_{\theta\#} \left( \sum_{B \in \mathcal{B}} \chi_B \right) \right|^p \, d\mathcal{H}^1 \, d\theta \leq C_{\beta,p} R^{-\kappa} \mu(\mathbb{R}^3).$$

By integrating out the projections  $\rho_\theta$ , it is enough to show that

$$(5.3) \quad \left( \frac{R \mu(\mathbb{R}^3)}{|\mathcal{B}|} \right)^p \int_0^{2\pi} \int_{\bigcup_{D \in \mathbb{D}_\theta} D} \left| \sum_{B \in \mathcal{B}} \chi_{\rho_\theta(B)} \right|^p \, d\mathcal{H}^1 \, d\theta \leq C_{\beta,p} R^{-\kappa} \mu(\mathbb{R}^3).$$

For each  $z \in \mathbb{R}^3$ , let  $\Gamma_z := \{(\theta, \langle z, \gamma(\theta) \rangle) : \theta \in [0, 2\pi)\}$ , which is a planar curve. Let  $\delta = R^{-1}$ , and let  $\Gamma_z^\delta$  be the  $\delta$ -neighbourhood of the planar curve  $\Gamma_z$ . Let  $Z'_\delta$  be the set of centres of balls in  $\mathcal{B}$ . It is straightforward to verify that

$$\sum_{B \in \mathcal{B}} \chi_{\rho_\theta(B)}(y) \leq \sum_{z \in Z'_\delta} \chi_{\Gamma_z^\delta}(\theta, \langle y, \gamma(\theta) \rangle),$$

which was observed in [28, Eq. 2.12], and also in [16]. To show (5.3) it is therefore enough to check that

$$\left( \frac{R\mu(\mathbb{R}^3)}{|\mathcal{B}|} \right)^p \int_0^{2\pi} \int_{\bigcup_{D \in \mathbb{D}_\theta} D} \left| \sum_{z \in Z'_\delta} \chi_{\Gamma_z^\delta}(\theta, \langle y, \gamma(\theta) \rangle) \right|^p d\mathcal{H}^1(y) d\theta \leq C_{\beta,p} R^{-\kappa} \mu(\mathbb{R}^3).$$

By Hölder's inequality applied to the above, it suffices to check that, for any  $\epsilon > 0$  (where  $\epsilon$  is much smaller than  $\kappa$ ),

$$(5.4) \quad \int_0^{2\pi} \int \left| \sum_{z \in Z'_\delta} \chi_{\Gamma_z^\delta}(\theta, \langle y, \gamma(\theta) \rangle) \right|^{3/2} d\mathcal{H}^1(y) d\theta \leq C_\epsilon R^\epsilon \mu(\mathbb{R}^3)^{-1/2} R^{-3/2} |\mathcal{B}|^{3/2},$$

where we used the Vitali covering lemma to estimate the measure of

$$\left\{ (\theta, x) \in [0, 2\pi) \times \mathbb{R}^3 : x \in \bigcup_{D \in \mathbb{D}_\theta} D \right\}.$$

By the Frostman condition on the measure  $\mu$ , this follows from the general  $L^{3/2}$  maximal function estimate in [28] (more precisely see Remark 5.10 on p. 32 of [28]) which states (as a special case of [28, Eq. 5.63]) with  $\alpha = \zeta = 1$ )

$$(5.5) \quad \int_0^{2\pi} \int \left| \sum_{z \in Z'_\delta} \chi_{\Gamma_z^\delta}(\theta, \langle y, \gamma(\theta) \rangle) \right|^{3/2} d\mathcal{H}^1(y) d\theta \leq C_\epsilon \delta^{-\epsilon} A^{1/2} \delta |Z'_\delta|,$$

whenever  $Z'_\delta$  is a collection of points in  $B_3(0, 1)$  such that

$$(5.6) \quad |\{z \in Z'_\delta \cap B(z_0, r)\}| \leq A(r/\delta), \quad r \in [\delta, 1].$$

The formula (5.2) and the assumption that  $c_1(\mu) \leq 1$  yields that (5.6) holds with  $A \sim \delta |Z'_\delta| \mu(\mathbb{R}^3)^{-1}$ , so (5.5) implies (5.4), and this finishes the proof.  $\square$

*Proof of Theorem 1.3.* The theorem for  $p = 1$  follows from the theorem for any  $p > 1$ , so it may be assumed that  $p > 1$ . Therefore, by a limiting argument, it may be assumed that  $\mu$  is a smooth function. Take  $\beta > 0$  such that  $\beta \ll \alpha - 1$ , and let  $\delta, \epsilon > 0$  be such that  $\delta \ll \epsilon \ll \beta$ , and  $\delta \ll \kappa(\beta)$ , the parameter in Lemma 5.1. Write  $\mu = \mu_b + \mu_g$ , where

$$\mu_b := \sum_{j \geq 1} \sum_{\tau \in \Lambda_j} \sum_{T \in \mathbb{T}_{\tau,b}} M_T \mu, \quad \mu_g := \mu - \mu_b,$$

$$M_T \mu = \eta_T \left( \mu * \tilde{\psi}_\tau \right),$$

and where  $\{\psi_\tau\}_{\tau \in \Lambda}$  is a smooth partition of unity on the  $\sim 1$  neighbourhood of the light cone minus the set  $B_3(0, 1)$ ,  $\Lambda = \bigcup_{j \geq 1} \Lambda_j$ , and each  $\Lambda_j$  is a boundedly overlapping cover of the  $\sim 1$  neighbourhood of the light cone intersected with  $B(0, 2^j) \setminus B(0, 2^{j-1})$ , by boxes of dimensions  $\sim 1 \times 2^{j/2} \times 2^j$ . For each  $\tau$ , the set  $\mathbb{T}_\tau$

is a finitely overlapping cover of  $\mathbb{R}^3$  by planks  $T$  of dimensions  $2^{j\delta} \times 2^{-j(1/2-\delta)} \times 2^{-j(1-\delta)}$  dual to  $\tau$ , and

$$(5.7) \quad \mathbb{T}_{\tau,b} = \left\{ T \in \mathbb{T}_\tau : \mu(4T) \geq 2^{-j(1-\beta)} c_\alpha(\mu) \right\}, \quad \mathbb{T}_{\tau,g} = \mathbb{T}_\tau \setminus \mathbb{T}_{\tau,b}.$$

Fix  $j \geq 1$ . It will be shown that

$$(5.8) \quad \int_0^{2\pi} \int \left| \sum_{\tau \in \Lambda_j} \sum_{T \in \mathbb{T}_{\tau,b}} \rho_{\theta\#} M_T \mu \right|^p d\mathcal{H}^1 d\theta \\ \lesssim 2^{jO(\delta)} \int_0^{2\pi} \sum_{\substack{\tau \in \Lambda_j: \\ |\theta_\tau - \theta| \leq 2^{j(\delta-1/2)}}} \sum_{T \in \mathbb{T}_{\tau,b}} \mu(2T)^p 2^{j(p-1)} d\theta,$$

where  $\theta_\tau$  is such that  $\gamma(\theta_\tau)$  is parallel to the long axis of  $\tau$  (the medium axis of  $\tau$  is parallel to  $\gamma'(\theta_\tau)$  and the short axis of  $\tau$  is parallel to  $(\gamma \times \gamma')(\theta_\tau)$ ). If  $p$  were equal to 2 then (5.8) would be a consequence of the following argument. For each  $\theta$ , apply Plancherel's theorem in one dimension to change the left-hand side of (5.8) to a surface integral over the light cone at distance  $\sim 2^j$  from the origin. The uncertainty principle, using that  $\mu$  is supported in the unit ball, changes this integral to an integral over  $\mathbb{R}^3$  (essentially supported in the  $\sim 1$  neighbourhood of the light cone in  $\mathbb{R}^3$  at distance  $\sim 2^j$  from the origin). Plancherel's theorem in  $\mathbb{R}^3$  can then be used to decouple the  $L^2$  norms (this is similar to the argument from Eq. 2.16 to Eq. 2.18 in [13]), and then (5.8) follows from the uncertainty principle (and (5.8) continues to hold with the set of planks in the left-hand side of (5.8) replaced by any subcollection). If  $p$  were equal to 1 then (5.8) would follow from the triangle inequality and stationary phase (see e.g. Lemma 2.2 of [13]). This implies (5.8) if either  $p = 1$  or  $p = 2$ . The case  $p \in (1, 2)$  of (5.8) then holds by interpolation; by pigeonholing it may be assumed that the sum over the  $T$ 's in the left-hand side of (5.8) is over a subset of the  $T$ 's such that  $\mu(2T)$  is constant up to a factor of 2, and then (5.8) follows by writing  $p = (1 - \phi) \cdot 1 + \phi \cdot 2$  and applying Hölder's inequality with  $q = 1/(1 - \phi)$  and  $q' = 1/\phi$ . This proves (5.8).

By Hölder's inequality, the integrand in the right-hand side of (5.8) satisfies, for any  $\theta \in [0, 2\pi)$ ,

$$(5.9) \quad \sum_{\substack{\tau \in \Lambda_j: \\ |\theta_\tau - \theta| \leq 2^{j(\delta-1/2)}}} \sum_{T \in \mathbb{T}_{\tau,b}} \mu(2T)^p 2^{j(p-1)} \lesssim \\ 2^{jO(\delta)} \int \bigcup_{\substack{\tau \in \Lambda_j: \\ |\theta_\tau - \theta| \leq 2^{j(\delta-1/2)}}} \bigcup_{T \in \mathbb{T}_{\tau,b}} \rho_{\theta\#} (\mu * \phi_{2^{j(1-\delta)}})^p d\mathcal{H}^1,$$

where we used the inequality  $\|x\|_p \leq \|x\|_1$  for sequences to deal with the possibility of multiple planks in a slab<sup>3</sup>.

<sup>3</sup>This step was not needed in the case of projections onto planes, and was overlooked in [14].

By the triangle inequality, followed by (5.8), and then (5.9),

$$\left( \int_0^{2\pi} \int |\rho_{\theta\#}\mu_b|^p d\mathcal{H}^1 d\theta \right)^{1/p} \lesssim \sum_{j \geq 1} \left( 2^{jO(\delta)} \int_0^{2\pi} \int_{\bigcup_{\substack{\tau \in \Lambda_j: \\ |\theta_\tau - \theta| \leq 2^{j(\delta-1/2)}}} \bigcup_{T \in \mathbb{T}_{\tau, b}} \rho_{\theta}(100T)} |\rho_{\theta\#}(\mu * \phi_{2^{j(1-\delta)}})|^p d\mathcal{H}^1 d\theta \right)^{1/p}.$$

By Lemma 5.1, this yields

$$(5.10) \quad \int_0^{2\pi} \int |\rho_{\theta\#}\mu_b|^p d\mathcal{H}^1 d\theta \lesssim c_\alpha(\mu)^{p-1} \mu(\mathbb{R}^3) \left( \sum_{j \geq 1} 2^{-j\kappa/10^{10}} \right)^p \lesssim c_\alpha(\mu)^{p-1} \mu(\mathbb{R}^3).$$

This bounds the “bad” part.

It remains to bound the “good” part. From this point on, the  $L^2$  method we use is similar to [13], but we include the details for completeness. It will be shown that

$$(5.11) \quad \int_0^{2\pi} \int |\rho_{\theta\#}\mu_g|^2 d\mathcal{H}^1 d\theta \lesssim \mu(\mathbb{R}^3) c_\alpha(\mu) \sum_{j \geq 1} 2^{j \left( \frac{5-2\alpha}{p_{\text{dec}}} - \frac{1}{2} + \beta \left( \frac{p_{\text{dec}}-2}{p_{\text{dec}}} \right) \right) + jO(\delta)},$$

where  $2 \leq p_{\text{dec}} \leq 6$ . Before proving (5.11), we will show that it implies the theorem. By taking  $p_{\text{dec}} = 6$  and using  $\delta \ll \beta \ll \alpha - 1$ , (5.11) is a decaying geometric series, which yields

$$\int_0^{2\pi} \int |\rho_{\theta\#}\mu_g|^2 d\mathcal{H}^1 d\theta \lesssim \mu(\mathbb{R}^3) c_\alpha(\mu).$$

By taking  $p = 1$ , the previous argument for the bad part to obtain (5.10) shows that  $\rho_{\theta\#}\mu_b \in L^1$  for a.e.  $\theta \in [0, 2\pi)$ , and that

$$\int_0^{2\pi} \int |\rho_{\theta\#}\mu_b| d\mathcal{H}^1 d\theta \lesssim \mu(\mathbb{R}^3).$$

By the triangle inequality, this implies that

$$\int_0^{2\pi} \int |\rho_{\theta\#}\mu_g| d\mathcal{H}^1 d\theta \lesssim \mu(\mathbb{R}^3).$$

For general  $p$  with  $1 \leq p < 3/2$ , by writing  $p = (1 - \phi)1 + \phi \cdot 2$  where  $\phi = p - 1$ , and by applying Hölder’s inequality to the function

$$|\rho_{\theta\#}\mu_g|^p = |\rho_{\theta\#}\mu_g|^{(1-\phi) \cdot 1} \cdot |\rho_{\theta\#}\mu_g|^{\phi \cdot 2},$$

with  $q = \frac{1}{1-\phi}$  and  $q' = \frac{1}{\phi}$ , this yields

$$(5.12) \quad \int_0^{2\pi} \int |\rho_{\theta\#}\mu_g|^p d\mathcal{H}^1 d\theta \lesssim c_\alpha(\mu)^{p-1} \mu(\mathbb{R}^3).$$

Combining (5.10) and (5.12) gives

$$\int_0^{2\pi} \int |\rho_{\theta\#}\mu|^p d\mathcal{H}^1 d\theta \lesssim c_\alpha(\mu)^{p-1} \mu(\mathbb{R}^3).$$

This proves the theorem, assuming (5.11).

It remains to prove (5.11). Because we consider only the  $L^2$  norm of the good part in (5.11), which was already dealt with in [13], the argument for (5.11) below

is very similar to the one in [13], but we include the details to make the proof self-contained. By the triangle inequality and by symmetry, to prove (5.11) it suffices to show that for any  $j \geq 1$ ,

$$(5.13) \quad \int_0^{2\pi} \int_{2^{j-1}}^{2^j} |\widehat{\mu}_g(t\gamma(\theta))|^2 dt d\theta \lesssim \mu(\mathbb{R}^3) c_\alpha(\mu) 2^j \left( \frac{5-2\alpha}{p_{\text{dec}}} - \frac{1}{2} + \beta \left( \frac{p_{\text{dec}}-2}{p_{\text{dec}}} \right) \right) + jO(\epsilon),$$

for any  $p_{\text{dec}} \in (2, 6]$ . By the definition of  $\mu_g$ , it suffices to show that for any  $j \geq 1$ ,

$$\begin{aligned} \int_0^{2\pi} \int_{2^{j-10}}^{2^{j+10}} \left| \sum_{\tau \in \Lambda_j} \sum_{T \in \mathbb{T}_{\tau,g}} \widehat{M_T \mu}(t\gamma(\theta)) \right|^2 dt d\theta \\ \lesssim \mu(\mathbb{R}^3) c_\alpha(\mu) 2^j \left( \frac{5-2\alpha}{p_{\text{dec}}} - \frac{1}{2} + \beta \left( \frac{p_{\text{dec}}-2}{p_{\text{dec}}} \right) \right) + jO(\epsilon), \end{aligned}$$

where, above and below, we can assume that rapidly decaying tail terms do not dominate (since in this case (5.13) would be trivial). Since  $\mu$  is supported in the unit ball, the uncertainty principle can be used to change the above surface integral to an integral over  $\mathbb{R}^3$ . In more detail, we partition the part of the light cone over which we are integrating into balls of radius  $2^{-10j\delta}$ , and then use the locally constant property of the integrand over each such ball to replace the surface integral  $dt d\theta$  over the ball intersected with the light cone with the 3-dimensional integral  $dx$  over the ball. Because the  $dt d\theta$  measure above of such a ball intersected with the light cone is  $\lesssim 2^{-j}$  (as  $t \sim 2^j$  and is varying on an interval of length  $\sim 1$ , the angle  $\theta$  such that  $t\gamma(\theta)$  intersects the ball must lie in an interval of size  $\sim 2^{-j}$ ), we gain a factor of  $2^{-j}$  when changing  $dt d\theta$  to  $dx$ . We then replace the integral over the  $2^{-10j\delta}$ -neighbourhood of the light cone with an integral over all of  $\mathbb{R}^3$ , and then use Plancherel's theorem in  $\mathbb{R}^3$  to decouple the  $L^2$  norms. Due to the  $2^{-j}$  gain, it is enough to show that

$$(5.14) \quad \sum_{\tau \in \Lambda_j} \sum_{T \in \mathbb{T}_{\tau,g}} \int_{\mathbb{R}^3} |M_T \mu|^2 dx \lesssim \mu(\mathbb{R}^3) c_\alpha(\mu) 2^j \left( \frac{5-2\alpha}{p_{\text{dec}}} + \frac{1}{2} + \beta \left( \frac{p_{\text{dec}}-2}{p_{\text{dec}}} \right) \right) + jO(\epsilon).$$

By writing out the  $L^2$  norm above as an inner product, applying the definition  $M_T \mu = \eta_T (\mu * \widetilde{\psi}_T)$  to one of the  $M_T \mu$ 's, and using Fubini's theorem to interchange the  $\mu$  integral with all the others, the left-hand side of (5.14) is equal to

$$(5.15) \quad \int \sum_{\tau \in \Lambda_j} \sum_{T \in \mathbb{T}_{\tau,g}} (\eta_T M_T \mu) * \widetilde{\psi}_T d\mu.$$

By the Cauchy-Schwarz inequality (with respect to the measure  $\mu$ ),

$$(5.16) \quad (5.15) \lesssim \mu(\mathbb{R}^3)^{1/2} \left( \int \left| \sum_{\tau \in \Lambda_j} \sum_{T \in \mathbb{T}_{\tau,g}} (\eta_T M_T \mu) * \widetilde{\psi}_T \right|^2 d\mu \right)^{1/2}.$$

For each  $T$  above, abbreviate  $f_T = (\eta_T M_T \mu) * \widetilde{\psi}_T$ . By pigeonholing, the sum over  $T$  in the left-hand side above can be restricted to a subset  $\mathbb{W}$  of  $\bigcup_{\tau \in \Lambda_j} \mathbb{T}_{\tau,g}$  on which  $\|f_T\|_{p_{\text{dec}}}$  is constant over  $T \in \mathbb{W}$  up to a factor of 2. Because the integrand on the left-hand side of (5.16) has frequencies of size  $\lesssim 2^j$ , the measure  $\mu$  being integrated against can be replaced by its smoothed out version  $\mu * \phi_{2^j}$  at scale  $2^{-j}$ . We then partition the domain into balls  $B$  of radius  $2^{-j}$ , and pigeonhole these



balls  $B$  according to the number  $M$  of planks  $T \in \mathbb{W}$  such that  $2T$  has nonempty intersection with  $B$ . Let  $Y$  be the union over these balls  $B$ . With this setup, (5.16) will be bounded by

$$(5.17) \quad \log(2^j)^{O(1)} \mu(\mathbb{R}^3)^{1/2} \left( \int_Y \left| \sum_{T \in \mathbb{W}} f_T \right|^2 d(\mu * \phi_j) \right)^{1/2}.$$

By Hölder's inequality (with respect to Lebesgue measure),

$$(5.18) \quad \left( \int_Y \left| \sum_{T \in \mathbb{W}} f_T \right|^2 d(\mu * \phi_j) \right)^{1/2} \leq \left( \int_Y \left| \sum_{T \in \mathbb{W}} f_T \right|^{p_{\text{dec}}} \right)^{\frac{1}{p_{\text{dec}}}} \left( \int_Y (\mu * \phi_j)^{\frac{p_{\text{dec}}}{p_{\text{dec}}-2}} \right)^{\frac{1}{2} - \frac{1}{p_{\text{dec}}}}.$$

By the refined decoupling theorem (see Theorem A.1), the first factor in (5.18) satisfies

$$(5.19) \quad \left( \int_Y \left| \sum_{T \in \mathbb{W}} f_T \right|^{p_{\text{dec}}} \right)^{\frac{1}{p_{\text{dec}}}} \leq C_\epsilon 2^{j\epsilon} \left( \frac{M}{|\mathbb{W}|} \right)^{\frac{1}{2} - \frac{1}{p_{\text{dec}}}} \left( \sum_{T \in \mathbb{W}} \|f_T\|_{p_{\text{dec}}}^2 \right)^{1/2}.$$

By using (5.7), the second factor in (5.18) satisfies

$$(5.20) \quad \begin{aligned} & \left( \int_Y (\mu * \phi_j)^{\frac{p_{\text{dec}}}{p_{\text{dec}}-2}} \right)^{\frac{1}{2} - \frac{1}{p_{\text{dec}}}} \\ & \leq \left( \frac{1}{M} \sum_{T \in \mathbb{W}} \int_{Y \cap 2T} (\mu * \phi_j)^{\frac{p_{\text{dec}}}{p_{\text{dec}}-2}} \right)^{\frac{1}{2} - \frac{1}{p_{\text{dec}}}} \\ & \lesssim \left( \frac{1}{M} \sum_{T \in \mathbb{W}} \mu(4T) \right)^{\frac{1}{2} - \frac{1}{p_{\text{dec}}}} c_\alpha(\mu)^{\frac{1}{p_{\text{dec}}}} 2^{\frac{j(3-\alpha)}{p_{\text{dec}}}} \\ & \lesssim c_\alpha(\mu)^{1/2} \left( \frac{|\mathbb{W}|}{M} \right)^{\frac{1}{2} - \frac{1}{p_{\text{dec}}}} 2^{-j(1-\beta)\left(\frac{1}{2} - \frac{1}{p_{\text{dec}}}\right) + \frac{j(3-\alpha)}{p_{\text{dec}}}}. \end{aligned}$$

Substituting (5.19) and (5.20) into (5.17) and then (5.18) yields

$$(5.21) \quad \left( \int \left| \sum_{\tau \in \Lambda_j} \sum_{T \in \mathbb{T}_{\tau,g}} (\eta_T M_T \mu) * \tilde{\psi}_\tau \right|^2 d\mu \right)^{1/2} \lesssim c_\alpha(\mu)^{1/2} 2^{-j(1-\beta)\left(\frac{1}{2} - \frac{1}{p_{\text{dec}}}\right) + \frac{j(3-\alpha)}{p_{\text{dec}}}} + jO(\epsilon) \left( \sum_{\tau \in \Lambda_j} \sum_{T \in \mathbb{T}_{\tau,b}} \|f_T\|_{p_{\text{dec}}}^2 \right)^{1/2}.$$

By comparing the left-hand side of the above with (5.15) and then (5.14), and then by using  $\|f_T\|_{p_{\text{dec}}} \lesssim 2^{jO(\delta)} \|M_T \mu\|_2 2^{\frac{3j}{2}\left(\frac{1}{2} - \frac{1}{p_{\text{dec}}}\right)}$  (which follows from the definition of  $f_T$  and the Hausdorff-Young inequality, followed by Hölder's inequality), (5.21)

implies that

$$\begin{aligned} \sum_{\tau \in \Lambda_j} \sum_{T \in \mathbb{T}_{\tau,g}} \|M_T \mu\|_2^2 &\lesssim \left( \sum_{\tau \in \Lambda_j} \sum_{T \in \mathbb{T}_{\tau,g}} \|M_T \mu\|_2^2 \right)^{1/2} \\ &\times \mu(\mathbb{R}^3)^{1/2} c_\alpha(\mu)^{1/2} 2^{\left(\frac{3j}{2} - j(1-\beta)\right) \left(\frac{1}{2} - \frac{1}{p_{\text{dec}}}\right) + \frac{j(3-\alpha)}{p_{\text{dec}}} + jO(\epsilon)}. \end{aligned}$$

This simplifies to

$$\sum_{\tau \in \Lambda_j} \sum_{T \in \mathbb{T}_{\tau,g}} \|M_T \mu\|_2^2 \lesssim \mu(\mathbb{R}^3) c_\alpha(\mu) 2^{j(1+2\beta) \left(\frac{1}{2} - \frac{1}{p_{\text{dec}}}\right) + \frac{2j(3-\alpha)}{p_{\text{dec}}} + jO(\epsilon)}.$$

This is the same as (5.14), so this finishes the proof.  $\square$

## 6. SOME OPEN PROBLEMS

- (1) An open problem that lies in-between the stated intersection and projection theorems, is whether there is a “restricted” Besicovitch-Federer projection theorem in  $\mathbb{R}^3$ .
- (2) To replace  $p < 3/2$  in Theorem 1.3 with  $p = 3/2$  may require a version of [28, Eq. 1.9] with  $p(\alpha) > 3/2$  instead of an  $L^{3/2}$  inequality, where  $\alpha > 1$  is the dimension of the discretised set  $Z'_\delta$  and  $p(\alpha) \rightarrow 3/2$  as  $\alpha \rightarrow 1$ . We are not sure if this can be obtained from the methods in [28]. The exponent  $p = 3/2$  is only sharp in Theorem 1.2 and Theorem 1.3 if  $p$  is not allowed to depend on  $\alpha$ . In Theorem 1.2, if  $p = p(\alpha)$  is allowed to depend on  $\alpha$ , then up to the endpoint the sharp exponent is likely the one which naturally interpolates between  $p = 3/2$  at  $\alpha = 2$  and  $p = 2$  at  $\alpha = 5/2$ , and similarly for Theorem 1.3.
- (3) Another open problem is whether  $B(0, 1)$  in Theorem 1.1 can be replaced by  $\mathbb{R}^3$ ; unlike for the standard Keakeya maximal function, the local and global versions do not seem to be obviously equivalent, since horizontal or  $SL_2$  lines are not preserved by Euclidean translations.

## APPENDIX A. REFINED DECOUPLING

Given  $R \geq 1$  and a small constant  $c > 0$ , let  $\Theta_R$  be a  $\sim cR^{-1/2}$ -separated subset of  $[0, 2\pi)$ . For each  $\theta \in \Theta_R$ , let  $\tau(\theta)$  be a plank of dimensions  $1 \times R^{-1/2} \times R^{-1}$  of distance  $\sim 1$  from the origin, tangent to the cone at  $\gamma(\theta)$ . Given  $\delta > 0$ , for each  $\tau$ , let  $\mathbb{T}_\tau$  be a collection of planks of dimensions  $R^\delta \times R^{1/2+\delta} \times R^{1+\delta}$  dual to  $\tau$ , forming a finitely overlapping cover of  $\mathbb{R}^3$ .

**Theorem A.1** ([8, Theorem 9]). *Let  $p_{\text{dec}} \in [2, 6]$ . Then if  $c > 0$  is sufficiently small, then for any  $\epsilon > 0$ , there exists  $\delta_0 > 0$  such that the following holds for all  $0 < \delta < \delta_0$ . Let  $R > 1$ . and suppose that  $f = \sum_{T \in \mathbb{W}} f_T$ , where  $\mathbb{W} \subseteq \bigcup_{\theta \in \Theta_R} \mathbb{T}_{\tau(\theta)}$ , where each  $f_T$  has  $\widehat{f_T}$  supported in  $\tau$ , with  $f_T$  essentially supported on  $T$  in the sense that  $\|f_T\|_{L^\infty(\mathbb{R}^3 \setminus T)} \leq AR^{-10000}$  where  $A$  is a large constant. Suppose that  $\|f_T\|_p$  is constant over  $T \in \mathbb{W}$  up to a factor of 2, and that  $Y$  is a disjoint union of radius 1 balls in  $B(0, R)$ , such that each radius 1 ball  $B \subseteq Y$  intersects at most*

$M$  planks  $2T$  with  $T \in \mathbb{W}$ . Then

$$\left\| \sum_{T \in \mathbb{W}} f_T \right\|_{L^{p_{\text{dec}}}(Y)} \leq C_{\epsilon, \delta, A, c} R^\epsilon \left( \frac{M}{|\mathbb{W}|} \right)^{\frac{1}{2} - \frac{1}{p_{\text{dec}}}} \left( \sum_{T \in \mathbb{W}} \|f_T\|_{p_{\text{dec}}}^2 \right)^{1/2}.$$

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