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Furstenberg entropy spectra of stationary actions of semisimple Lie groups

J eremie Brioussel and Tianyi Zheng

Abstract

We determine Furstenberg entropy spectra of ergodic stationary actions of $SL(d, \mathbb{R})$ and its lattices. The constraints on entropy spectra are derived from a refinement of the Nevo-Zimmer projective factor theorem. The realisation part is achieved by means of building Poisson bundles over stationary random subgroups.

1 Introduction

A compact metrizable space X acted upon continuously by a locally compact group G equipped with a probability measure μ admits a stationary probability measure η , which means a fixed point of the convolution $\mu * \eta = \eta$. The system $G \curvearrowright (X, \eta)$ is referred to as a stationary action of G . The study of stationary actions of semisimple Lie groups was initiated by Furstenberg [Fur63b, Fur63a]. He showed in particular that stationary measures on X are in bijection with measures invariant under a minimal parabolic subgroup P . He also introduced a numerical invariant $h_\mu(X, \eta)$ nowadays referred to as the Furstenberg entropy:

$$h_\mu(X, \eta) = - \int_G \int_X \log \frac{dg^{-1}\eta}{d\eta}(x) d\eta(x) d\mu(g). \quad (1)$$

It is 0 if and only if η is a G -invariant measure.

A systematic study of ergodic stationary actions of semisimple Lie groups was developed by Nevo and Zimmer in a series of articles. They established that any such action admits a maximal projective factor $(G/Q, \nu_Q)$, where Q is a parabolic subgroup of G ; and when G is a higher rank simple Lie group, this factor is trivial if and only if the stationary measure η is actually invariant [NZ02a]. Under a further mixing assumption (called P -mixing), they proved that the stationary system is a relative measure-preserving extension of the maximal projective factor. [NZ99]. This implies in particular that $h_\mu(X, \eta) = h_\mu(G/Q, \nu_Q)$ and it follows that the Furstenberg entropy of P -mixing stationary (G, μ) -spaces can take on only finitely many values [NZ00]. These results no longer hold without higher rank hypothesis, as $PSL(2, \mathbb{R})$ admits infinitely many P -mixing stationary systems (in fact can be taken to be smooth manifolds) with distinct entropy. Nor without the P -mixing hypothesis, as groups with a parabolic subgroup mapping onto $PSL(2, \mathbb{R})$ may have infinite entropy spectrum [NZ00].

The purpose of the present article is to give a complete description of the entropy spectrum of $SL(d, \mathbb{R})$ and of its lattices, equipped with appropriate measures, see Theorem 1.8 below.

Definition 1.1. We say a step distribution μ on G has *finite boundary entropy* if the Furstenberg entropy of the Poisson boundary of (G, μ) is finite. For such a measure μ , we refer to the range of possible Furstenberg entropy values over all ergodic μ -stationary systems as the *Furstenberg entropy spectrum* of (G, μ) :

$$\text{EntSp}(G, \mu) := \{h_\mu(X, \nu) : (X, \nu) \text{ is an ergodic } (G, \mu)\text{-stationary system}\}.$$

Note that by [BS06, Corollary 2.7], (X, ν) is an ergodic (G, μ) -stationary system if and only if ν is extremal in the set of μ -stationary measures on X .

1.1 Structure of stationary systems and constraints on entropy values

We will use:

Notation 1.2. Let Δ denote simple roots of a semisimple Lie group G . For $I \subseteq \Delta$, denote by P_I the standard parabolic subgroup that corresponds to I (see Section 2.3).

Denote by $f : 2^\Delta \rightarrow 2^\Delta$ the map where given $I \subseteq \Delta$, $f(I)$ is the largest subset $I' \subseteq I$ such that the Levi subgroup $L_{I'}$ has no \mathbb{R} -rank 1 noncompact simple factors.

From the proof of Nevo-Zimmer projective factor theorem in [NZ02a], we extract the following statement. Plausibility of such a formulation is hinted in the remarks after [NZ02a, Theorem 11.4].

Theorem 1.3 (A refinement of [NZ02a, Theorem 3]). *Let G be a connected semisimple real Lie group with finite center, μ an admissible measure on G . Suppose (X, ν) is an ergodic (G, μ) -system where ν is not G -invariant. Let λ be the corresponding P -invariant measure on X provided by the Furstenberg isomorphism. Suppose $(G/Q, \nu_Q)$ is the maximal standard projective factor of (X, ν) , where $Q = P_I$, then the measure λ is invariant under the parabolic subgroup $P_{f(I)}$.*

A probability measure μ on G is called *admissible* if $\text{supp} \mu$ generates G as a semigroup, and some convolution power μ^{*k} is absolutely continuous with respect to Haar measure on G . The Furstenberg isomorphism between μ -stationary and P -invariant probability measures on X is described in Section 2.2.2.

When the Levi subgroup L_I has no rank one noncompact factors, $f(I) = I$, Theorem 1.3 implies the following corollary. Compared to [NZ02a, Theorem 3 and 9.1]: the λ -ergodicity assumption of S -action is dropped; instead it is assumed that the Levi subgroup of Q has no rank one factors.

Corollary 1.4. *Let G be a connected semisimple real Lie group with finite center, μ an admissible measure on G . Suppose (X, ν) is an ergodic (G, μ) -system where ν is not G -invariant. Let $(G/Q, \nu_Q)$ be the maximal standard projective factor of (X, ν) . If the Levi subgroup of Q has no \mathbb{R} -rank 1 non-compact simple factors, then $(X, \nu) \rightarrow (G/Q, \nu_Q)$ is a relative measure-preserving extension.*

The conclusion of Theorem 1.3 can be formulated in terms of the boundary map. Denote by $\mathcal{P}(X)$ the space of probability measures on X and let $\beta_\nu : G/P \rightarrow \mathcal{P}(X)$ be the *boundary map* associated with the stationary measure ν (its definition is reviewed in Subsection 2.2.1). Then the measure λ is invariant under $P_{f(I)}$ if and only if β_ν factors through the projection $G/P \rightarrow G/P_{f(I)}$. In this formulation, we can derive an analogous statement for lattices equipped with Furstenberg measures, through an induction procedure for stationary actions in [BH21, BBHP22].

Definition 1.5. Let Γ be a lattice in a semi-simple Lie group G . Denote by P a minimal parabolic subgroup of G . We say a non-degenerate measure μ_0 on Γ is a *Furstenberg measure* if the Poisson boundary of (Γ, μ_0) can be identified with $(G/P, \nu_P)$, where ν_P is in the same measure class as \bar{m}_K , the unique K -invariant probability measure on G/P . Such measures on Γ exist by [Fur71].

In this setting, a (Γ, μ_0) -stationary system (X, ν) gives rise to a Γ -boundary map $\beta_\nu : G/P \rightarrow \mathcal{P}(X)$, although G does not necessarily act on X . We derive the following from Theorem 1.3.

Theorem 1.6. *Let Γ be a lattice in a connected semisimple real Lie group G with finite center. Equip Γ with a Furstenberg measure μ_0 . Suppose (X, ν) is an ergodic (Γ, μ_0) -system where ν is not Γ -invariant. Let $(G/Q, \nu_Q)$ be the maximal standard projective Γ -factor of (X, ν) , where $Q = P_I$. Then the Γ -boundary map $\beta_\nu : G/P \rightarrow \mathcal{P}(X)$ factors through $G/P_{\mathfrak{f}(I)}$, that is a.e., $\beta_\nu(gP)$ depends only on the coset $gP_{\mathfrak{f}(I)}$.*

Constraints on the Furstenberg entropy spectrum follow from the structure theorems.

Theorem 1.7. *Let G be a connected semisimple real Lie group with finite center and denote by Δ simple restricted roots of G . Let μ be an admissible step distribution on G with finite boundary entropy. Then the Furstenberg entropy spectrum of (G, μ) satisfies*

$$\text{EntSp}(G, \mu) \subseteq \bigcup_{I \subseteq \Delta} [h_\mu(G/P_I, \nu_I), h_\mu(G/P_{\mathfrak{f}(I)}, \nu_{\mathfrak{f}(I)})],$$

where ν_I is the (unique) μ -stationary measure on G/P_I .

An analogous statement for lattices equipped with Furstenberg discretization measures is stated in Theorem 4.3.

1.2 Realisation of entropy values

For the free group F_k on k -generators and step distribution μ uniform on the generators and their inverses, Bowen shows in [Bow14] that $\text{EntSp}(F_k, \mu)$ is the full interval $[0, \mathbf{h}_\mu]$, where \mathbf{h}_μ is the Furstenberg entropy of the Poisson boundary of (F_k, μ) , which is equal to the random walk asymptotic entropy. The proof is based on a construction of Poisson bundles over invariant random subgroups (IRSs) and analysis of associated random walks on the coset graphs.

For $G = SL(d, \mathbb{R})$ and its lattices, we realise Furstenberg entropy values within the constraints of Theorem 1.7 and 4.3 via construction of Poisson bundles over stationary systems. Recall the map $I \mapsto \mathfrak{f}(I)$ defined in Notation 1.2. We say a measure μ is B_∞ if it is admissible on G , of compact support and bounded density with respect to the Haar measure.

Theorem 1.8. *Let $G = SL(d, \mathbb{R})$ and denote by $\Delta = \{1, \dots, d-1\}$ its simple roots. Suppose*

- μ is in the B_∞ class on G ,
- or μ is a Furstenberg measure on a lattice $\Gamma < G$ of finite Shannon entropy.

Write $S = \langle \text{supp}\mu \rangle$ for the subgroup generated by $\text{supp}\mu$, then

$$\text{EntSp}(S, \mu) = \bigcup_{I \subseteq \{1, \dots, d-1\}} [h_\mu(G/P_I, \nu_I), h_\mu(G/P_{\mathfrak{f}(I)}, \nu_{\mathfrak{f}(I)})],$$

where ν_I is the μ -stationary measure on G/P_I .

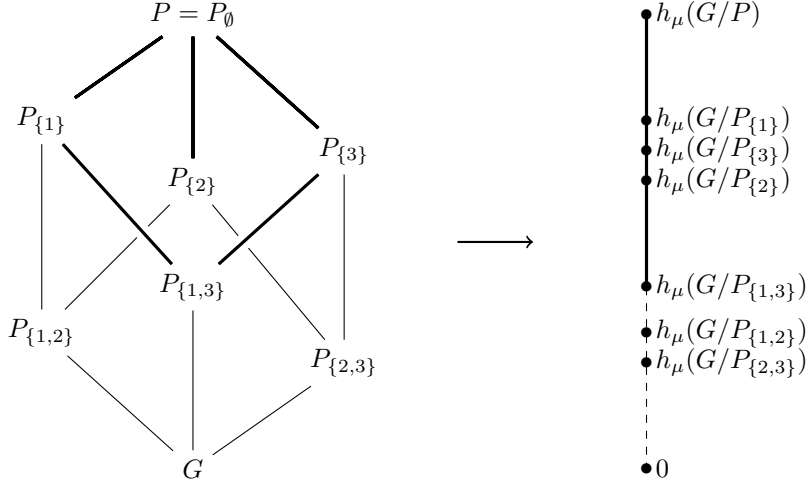


Figure 1: The poset of parabolic subgroups of $SL(4, \mathbb{R})$. Inclusions differing only by a rank 1 factor are drawn in bold. The particular case pictured here occurs when the Lyapunov exponents satisfy $\lambda_2 > 0 > \lambda_3$.

In particular for B_∞ -measures, $SL(2, \mathbb{R})$ has full entropy spectrum, and $SL(3, \mathbb{R})$ has entropy spectrum of the form $\{0\} \cup [h_\mu(G/Q), h_\mu(G/P)]$ for some non-trivial parabolic subgroup $P < Q < G$. A particular instance for $SL(4, \mathbb{R})$ is illustrated in Figure 1.

Remark 1.9. It is natural to ask what are entropy spectra of other simple Lie groups. Note that rank one Lie groups $Sp(n, 1)$, $n \geq 2$, and $F_{4(-20)}$ have Kazhdan's property (T). Therefore by [Nev03], $\{0\}$ is an isolated point in their Furstenberg entropy spectra. For these rank one groups, Theorem 1.7 does not provide sharp constraints on their entropy spectra.

It is classical that for admissible μ considered in Theorem 1.8, the boundary entropy values $h_\mu(G/P_I, \nu_I)$ can be expressed in terms of the Lyapunov spectrum of the μ -random walk, see the Furstenberg formula in Subsection 10.5.

Using Poisson bundles over stationary systems permits to obtain intervals of entropy values. Such an extension beyond the framework over measure-preserving systems is necessary. The bundles over IRS considered in [Bow14] can be described as factors of a system of the form $(X \times B, m \times \nu_B)$, where (X, m) is equipped with an ergodic G -invariant measure m and (B, ν_B) is the Poisson boundary of the μ -random walk on G . For a higher rank connected simple Lie group G , [NZ02b, Theorem 1.2] implies that in this setting, for any G -factor (Z, ν) of $(X \times G/P, m \times \nu_B)$, there is a parabolic subgroup Q of G , such that (Z, ν) is a measure preserving extension of $(G/Q, \bar{\nu}_B)$. Therefore in this case Poisson bundles over measure preserving systems provide at most 2^r Furstenberg entropy values, where r is the \mathbb{R} -rank of G .

For $Q' < Q$, two parabolic subgroups of $G = SL(d, \mathbb{R})$ whose Levi subgroups differ only by a rank 1 factor, we consider stationary G -systems induced from measure-preserving Q -actions. There is a large supply of such systems where Q acts through the quotient $Q \rightarrow PSL(2, \mathbb{R})$. The associated Poisson bundles will be factors of the stationary joinings with the Poisson boundary, rather than direct products. We remark that the random walk models that appear in the construction, based

on stationary joinings, are inherently different from random walks on stationary random graphs considered in [BC12, CPL16].

The proof of realisation of the interval $[h_\mu(G/Q, \nu_Q), h_\mu(G/Q', \nu_{Q'})]$ is based on a continuity argument. As in [Bow14], we construct a family of Poisson bundles (Z_p, λ_p) , parametrised by $p \in [0, 1]$. The key point is to show that the entropy $h_\mu(Z_p, \lambda_p)$ depends continuously on p . Upper and lower semi-continuity are treated separately. Upper semi-continuity follows from standard entropy formulae. More precisely, they provide expressions in terms of infimum of mutual information (or Shannon entropy) of time n random walks. This allows to show that entropy is an infimum of continuous functions. Our approach to lower semi-continuity is based on identification of the Poisson bundles with concrete models. Then the KL-divergence occurring in the definition of entropy can be expressed as the supremum of relative entropies over finite partitions in the model. After verifying certain approximation properties, entropy is expressed as a supremum of continuous functions.

In view of an explicit identification of the Poisson bundles, it is convenient to use measure preserving systems of $SL(2, \mathbb{R})$ induced from IRSs of a lattice F , taken for simplicity to be the Sanov subgroup, free of rank 2. Their respective Poisson boundaries, the boundary circle $\partial\mathbb{H}$ of the hyperbolic plane and the space of ends ∂F , are F -measurably isomorphic, provided F is endowed with a Furstenberg measure, which can be chosen to have finite entropy and finite log-moment. We show that taking Poisson bundles interacts in a compatible way with inducing. These two stages of induction reduce the proof of Theorem 1.8 to the case of free groups, for which a key approximation property is shown in Proposition 9.10. Along the way we show full entropy realisation for a large class of step distributions on F :

Theorem 1.10. *Let F be a free group of finite rank, endowed with a non-degenerate probability measure μ with finite entropy and finite logarithmic moment. Then $\text{EntSp}(F, \mu) = [0, \mathbf{h}_\mu]$.*

This generalizes Bowen’s original result for the case where μ is uniform on the generators and their inverses [Bow14]. In the case of finitely supported μ , full entropy realisation was known by [HY18]. For virtually free groups, full realisation is known for symmetric 4th moment measures by [RGY21], and the existence of a gap is ruled out for first moment measures by [HT15].

Similar arguments can be applied to other groups acting on trees; for which we will investigate elsewhere.

1.3 Organization of the article

Section 2 collects necessary preliminaries. After that the article is divided in three parts, with some additional details provided in two appendices.

1.3.1 Part I: Constraints on entropy spectrum

The first part consists of Section 3 and 4. In Section 3 we apply the line of arguments in [NZ02a], in particular an operation on continuous functions using contracting dynamics and Gauss map considerations, to show a property of maximal projective factor, stated in Theorem 3.2. Consequences of the structure theorem, namely Theorems 1.3, 1.6 and 1.7, are derived in Section 4.

1.3.2 Part II: Poisson bundle over a stationary system

The second part consists of Section 5 to 8, where we develop some general theory on Poisson bundles over stationary systems.

In Section 5, we explain the definition of such Poisson bundles, which starts with stationary joinings. The resulting system (Z, λ) is a G -factor that fits into

$$(X \times B, \eta \vee \nu_B) \rightarrow (Z, \lambda) \rightarrow (X, \eta),$$

where (X, η) is a (G, μ) -stationary system, (B, ν_B) is the Poisson boundary of μ -random walk, and $\eta \vee \nu_B$ denotes the stationary joining of the two. For a stationary system (X, η) which is standard in the sense of Furstenberg-Glasner [FG10], we show that the Poisson bundle over (X, η) can be described as a proximal extension where the fibers are Poisson boundaries of coset Markov chains whose law is given by suitable Doob transforms. A typical example of such (X, η) is induced from a measure-preserving action of Q , where Q is a parabolic group. The fiberwise Markov property is the key ingredient in deriving the entropy formulae in Subsection 5.5. It is standard that the entropy formulae imply upper semi-continuity properties of entropy, as explained in Section 6.

There remains to obtain lower semi-continuity. As a starting point, we formulate in Section 7 entropy criteria for identification of Poisson bundles, which are adapted from the strip and ray criteria originally due to Kaimanovich [Kai00]. Here by identification, we mean explicitly describing a (G, μ) -stationary system $(M, \bar{\lambda})$ and showing it is G -isomorphic to (Z, λ) .

In Section 8 we formulate an approach to prove lower semi-continuity of Furstenberg entropy for some specific systems such as end-compactification bundles. The basic idea is that a symbolic representation provides a natural sequence of finite partitions into cylinder sets, which generate the σ -field on the fiber. We may then write the fiberwise KL-divergence as the supremum of relative entropy on the finite partitions. To ensure lower-semicontinuity, it is sufficient to show that the relative entropy on the chosen finite partitions varies continuously over the base. In Section 8.1 we describe a technical condition, referred to as "locally constant uniform approximation", which implies such continuity.

1.3.3 Part III: entropy realization for free groups, for $SL(d, \mathbb{R})$ and its lattices

The third part consists of Sections 9 and 10. We apply the general framework of Part II to free groups and to $SL(d, \mathbb{R})$.

In Section 9, we revisit Poisson bundles of free groups over IRSs constructed in Bowen [Bow14]. They are supported on subgroups with "tree-like Schreier graphs". We apply the strip criterion Theorem 7.3 to show that these bundles are isomorphic to end compactification bundles over the same base system. The finite partitions allowing approximations are simply the shadows of vertices on a sphere of given radius. We conclude this section with a proof of Theorem 1.10.

Section 10 is devoted to the realization part of Theorem 1.8. Denote by F the free group on two generators. Take a parabolic subgroup $Q < SL(d, \mathbb{R})$ whose Levi subgroup L has a rank-1 factor. In matrix form it means that L has a 2×2 block on the diagonal. Since F is a lattice of $SL(2, \mathbb{R})$, we may induce an IRS of F to a stationary random subgroup (SRS) of $G = SL(d, \mathbb{R})$, see Subsection 10.1. The corresponding SRS is a measure-preserving extension of $(G/Q, \nu_Q)$. Via a discretization argument, we transfer identification results for the free group in Section 9 to identify Poisson bundles over the (co-)induced SRS of G in Subsection 10.2. Roughly speaking, in such a Poisson bundle, a fiber can be described as the space of ends of a tree-like graph equipped with a suitable measure, where the graph and measure depend on the base point. One may view such an identification result as providing a symbolic representation fiberwise for the Poisson bundle. Approximations on the tree-like fibers of the bundle are integrated to obtain lower semi-continuity via Fatou's lemma. Together with Section 6, we conclude the continuity argument.

We mention that the SRS in the construction are supported on non-discrete subgroups of $SL(d, \mathbb{R})$ for $d \geq 3$. This is necessary: by a result of Fraczyk and Gelander [FG23], every discrete SRS of $SL(d, \mathbb{R})$, $d \geq 3$, is an IRS.

1.3.4 Appendices

Appendix A reviews the Nevo-Zimmer operation on continuous functions from [NZ02a] based on contracting dynamics. Operations in the expanding direction were considered earlier: it is first used by Margulis in the proof of the Normal Subgroup Theorem [Mar91, Chapter IV]; and by Nevo-Zimmer in structure theorem under P -mixing assumption [NZ99].

In Appendix B we include proofs of mutual information and entropy formulae for Furstenberg entropy of Poisson bundles, which are stated in Subsection 5.5. These follow from classical arguments being adapted to our setting.

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2 Preliminaries

2.1 Induced actions

We recall basic facts about induced actions, see Zimmer's book [Zim84] for detailed treatment.

Let G be a locally compact group and H a closed subgroup of G . Let m_G be a left Haar measure on G . Let (S, η) be an H -space. Let H act from the right on the product $G \times S$ by $(g, s).h = (gh, h^{-1}.s)$. Denote by $X = G \times_H S$ the space of H -orbits in $G \times S$ and $p : G \times S \rightarrow X$ the natural projection. There is an action of G on the quotient X induced from $G \curvearrowright G \times S$ by $g.(g', s) = (gg', s)$. The space X with the quotient Borel structure and quotient measure from $(G \times S, m_G \times \eta)$ is called the G -space induced from $H \curvearrowright (S, \eta)$. When no ambiguity arises, we write $[g, s]$ for the H -orbit $p(g, s)$.

Another way to describe the induced G -action is through a cocycle. Choose a Borel section $\theta : G/H \rightarrow G$ of the natural projection $G \rightarrow G/H$, such that $\theta([e]) = e$. Let $\alpha : G \times G/H \rightarrow H$ be the cocycle defined as $\alpha(g, [g']) = \theta([gg'])^{-1}g\theta([g'])$. Denote by $(G/H \times_\alpha S, p_*m_G \times \eta)$ the G -space where G acts by $g.([g'], s) = ([gg'], \alpha(g, [g']).s)$. As G -spaces, $G/H \times_\alpha S$ is isomorphic to X via the map

$$([g], s) \rightarrow p(\theta([g]), s),$$

see [Zim78, Proposition 2.2].

2.2 G -spaces and factor maps

We follow the preliminaries in [BS06, Section 2]. Let G be a locally compact second countable group. We say a Lebesgue space (X, η) is a G -space if G acts measurably on X and the probability measure η is quasi-invariant with respect to the G -action. Given a probability measure μ on G , we say (X, η) is a stationary (G, μ) -space if in addition $\mu * \eta = \eta$.

A measure μ on G is admissible if its support generates G as a semi-group and some convolution power is absolutely continuous with respect to Haar measure. It is in the B_∞ class if it is furthermore

absolutely continuous with respect to Haar measure and admits a bounded density with compact support.

By [Var63], we may take a *compact model* for (X, η) , that is, a compact metric space Z on which G acts continuously, equipped with a probability measure η_Z on its Borel σ -algebra, such that (X, η) and (Z, η_Z) are measurably isomorphic G -spaces. The space of probability measures on a compact metric space Z is denoted by $P(Z)$. Equip $P(Z)$ with the weak*-topology.

For a G -map $\pi : (X, \eta) \rightarrow (Y, \nu)$ between two compact G -spaces, there exists a Borel map $\sigma : X \rightarrow Y$ such that $\sigma = \pi$, η -a.e.. We say π is a G -factor map if $Y = \pi(X)$ and $\nu = \pi_*\eta$; the set $\pi^{-1}(\{y\})$ is called the fiber over y . Denote by $D_\pi : Y \rightarrow P(X)$ the disintegration map, which is the unique map with the property that for ν -a.e. y , $D_\pi(y)$ is supported on the fiber $\pi^{-1}(\{y\})$, and $\int_Y D_\pi(y) d\nu(y) = \eta$. We will often write $\eta^y := D_\pi(y)$. A G -factor map $\pi : (X, \eta) \rightarrow (Y, \nu)$ is called a *measure preserving extension* if D_π is G -equivariant, that is, $g.\eta^y = \eta^{g.y}$ for all $g \in G$ and a.e. $y \in Y$.

By Mackey's point realization theorem, G -factors of (X, ν) correspond to G -invariant sub- σ -algebras on X , modulo zero measure subsets.

2.2.1 The boundary map

Denote by (B, ν_B) for the Poisson boundary of (G, μ) and $\text{bnd} : (G^{\mathbb{N}}, \mathbb{P}_\mu) \rightarrow (B, \nu_B)$ the map from the trajectory space to the Poisson boundary.

Let (X, η) be a (G, μ) -stationary system, $\mu * \eta = \eta$. By the martingale convergence theorem, \mathbb{P}_μ -a.s.,

$$\eta_\omega = \lim_{n \rightarrow \infty} \omega_n.\eta \text{ exists.}$$

Since the map $\omega \mapsto \eta_\omega$ is measurable with respect to the invariant σ -field of the random walk, it factorizes through the Poisson boundary of (G, μ) . That is, we have a G -measurable map

$$\begin{aligned} \beta_\eta : B &\rightarrow P(X) \\ \text{bnd}(\omega) &\mapsto \eta_\omega, \end{aligned}$$

where $P(X)$ is the space of probability measures on X . The map β_η is called the boundary map, it is the essentially unique measurable G -map $B \rightarrow P(X)$ which satisfies the barycenter property that

$$\eta = \int_B \beta_\eta(b) d\nu_B(b),$$

see for instance [BS06, Theorem 2.16].

Recall the following terminologies.

- The (G, μ) -space (X, η) is a μ -*boundary* (equivalently μ -*proximal*) if \mathbb{P}_μ -a.s., the measures $\eta_\omega \in \mathcal{M}(X)$ are point masses. In other words, if (X, η) is a G -factor of the Poisson boundary (B, ν_B) .
- A G -factor map $\pi : (X, \eta) \rightarrow (Y, \nu)$ is called a μ -*proximal extension* if \mathbb{P}_μ -a.s., the extension $(X, \eta_\omega) \rightarrow (Y, \nu_\omega)$ is a.s. one-to-one.
- We call a (G, μ) -stationary system (X, η) *standard* if there exists a G -factor map $\pi : (X, \eta) \rightarrow (Y, \nu)$ with (Y, ν) a μ -proximal system and π a measure preserving extension. By [FG10, Proposition 4.2], the structure of a standard system as a measure preserving extension of a proximal system is unique.

2.2.2 Furstenberg isomorphism

Let G be a lsc group equipped with an admissible probability measure μ on G . Assume that the Poisson boundary of the μ -random walk can be identified with a homogenous space G/H . Following [Fur63b, Fur63a], in this situation the boundary map can be interpreted as what is now called the *Furstenberg isomorphism/correspondence*.

Since μ is admissible, then the stationary measure ν is in the quasi-invariant measure class on G/H . Since the action of G on its Poisson boundary is amenable in the sense of Zimmer, we have that the subgroup H is necessarily amenable.

Given a locally compact G -space X , denote by $\mathcal{P}_\mu(X)$ the space of μ -stationary probability measures on X , and $\mathcal{P}_H(X)$ the space of H -invariant probability measures on X . Then by [Fur63a, Lemma 2.1], there is an isomorphism between the affine spaces $\mathcal{P}_\mu(X)$ and $\mathcal{P}_H(X)$, implemented by

$$\begin{aligned} \psi_\mu : \mathcal{P}_\mu(X) &\rightarrow \mathcal{P}_H(X) \\ \eta &\mapsto \lambda = \beta_\eta(H), \end{aligned}$$

where $\beta_\eta : G/H \rightarrow \mathcal{P}(X)$ is the G -boundary map associated with η . Denote by ν_H the μ -harmonic measure on G/H , we have that the barycenter map is implemented by

$$\eta = \int_{G/H} \beta_\eta(w) d\nu_H(w) = \int_{G/H} g.\lambda d\nu_H(gH).$$

Notation 2.1. Let X be a G -space, H a closed subgroup of G . Given a measure ν_0 on G/H and H -invariant measure λ on X , we write $\nu_0 * \lambda := \tilde{\nu}_0 * \lambda = \int_G g.\lambda d\tilde{\nu}_0(g)$, where $\tilde{\nu}_0$ is any lift of ν_0 to $\text{Prob}(G)$. Since λ is H -invariant, it does not depend on the choice of $\tilde{\nu}$. In this setting, denote by X_0 the support of λ . We can then view $G \curvearrowright (X, \nu * \lambda)$ as a factor of the induced system $(G/H \times_\alpha X, \nu \times \lambda)$, see [NZ99, Proposition 2.5].

We will refer to the map $\psi_\mu : \mathcal{P}_\mu(X) \rightarrow \mathcal{P}_H(X)$ as the Furstenberg isomorphism. This isomorphism is continuous with respect to the weak* topology. As a consequence of the isomorphism, we have uniqueness of μ -stationary measure on X is equivalent to uniqueness of H -invariant measure on X .

Recall that the action of G on a compact space X is said to be *strongly proximal* if for any probability measure η on X , there exists a sequence of elements (g_n) in G such that $g_n.\eta$ converges weakly to a δ -mass. When $G \curvearrowright X$ is strongly proximal, by [Fur63a, Theorem 2.3], the G -space $X \times X$ supports a unique stationary measure for μ ; and this measure is concentrated on the diagonal of $X \times X$. It follows in particular that if $G \curvearrowright X$ is strongly proximal, then H -invariant measure on X is unique.

2.2.3 Furstenberg entropy

Recall that given two probability measures P and Q on the same space (Ω, \mathcal{B}) , if P is absolutely continuous with respect to Q , the relative entropy of P with respect to Q , also known as their Kullback-Leibler divergence, is defined as the integral (possibly infinite):

$$D(P||Q) := \int_\Omega \left(\log \frac{dP}{dQ} \right) dP.$$

The Furstenberg entropy, as defined in (1) can be written as $h_\mu(X, \nu) = \int_G D(\nu \| g^{-1}\nu) d\mu(g) = \int_G D(g.\nu \| \nu) d\mu(g)$. We refer to [NZ00, Section 1] for a detailed account on Furstenberg entropy, and only recall here a few well-known properties.

Furstenberg entropy is monotone under factors: when $\pi : (X, \eta) \rightarrow (Y, \nu)$ is a G -map, we have $h_\mu(Y, \nu) \leq h_\mu(X, \eta)$ with equality if and only if π is measure preserving. The Furstenberg entropy of a (G, μ) -space is maximal for the Poisson boundary $0 \leq h_\mu(X, \eta) \leq h_\mu(B, \nu_B)$.

When η' is a probability measure in the measure class of η , then by [Fur63a, Lemma 8.9], we have

$$h_\mu(X, \eta) = - \int_G \int_X \log \frac{dg^{-1}\eta'}{d\eta'}(x) d\eta(x) d\mu(g).$$

This allows to view the Furstenberg entropy as a cohomology invariant. It permits to change the stationary measure η in the Radon-Nikodym derivative in order to compute the entropy.

We also record the following property of the barycenter map.

Lemma 2.2. *Let (C, ν_C) and (X, η) be two nonsingular G -spaces. Suppose there is a G -map $\beta : C \rightarrow \mathcal{P}(X)$ such that η is the barycenter of $\beta_*(\nu_C)$. Then*

$$h_\mu(C, \nu_C) \geq h_\mu(X, \eta).$$

Proof. Consider the product space $C \times X$ on which G acts diagonally, and equip it with the measure λ such that

$$\int_{C \times X} f d\lambda = \int_C \int_X f(c, x) d\beta_c(x) d\nu_C(c).$$

The map $C \times X \rightarrow C$ with $(c, x) \mapsto c$ is a measure-preserving extension as β is equivariant. It follows that $h_\mu(C \times X, \lambda) = h_\mu(C, \nu_C)$. Since $\eta = \text{bar}(\beta_*(\nu_C))$, the coordinate projection $C \times X \rightarrow X$ pushes forward the measure λ to η . Therefore $h_\mu(X, \eta) \leq h_\mu(C \times X, \lambda)$. \square

2.3 Structure of parabolic subgroups

Let G be a semisimple real Lie group, we recall some structure theory, see [Kna02] and also [NZ02a, Section 2]. Denote by \mathfrak{g} the Lie algebra of G . Let $\theta : \mathfrak{g} \rightarrow \mathfrak{g}$ be a Cartan involution on \mathfrak{g} . We have the Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, where \mathfrak{k} (\mathfrak{p} resp.) is the $+1$ (-1 resp.) eigenspace of θ . Let \mathfrak{a} be a maximal commutative subalgebra of \mathfrak{p} . Denote by $\Phi = \Phi(\mathfrak{a}, \mathfrak{g})$ the set of restricted roots. For a fixed ordering on Φ , denote by Δ simple roots. For $\alpha \in \Phi$, denote by \mathfrak{g}_α the restricted root space $\mathfrak{g}_\alpha = \{x \in \mathfrak{g} : [h, x] = \alpha(h)x \text{ for all } h \in \mathfrak{a}\}$, and set $\mathfrak{n} = \bigoplus_{\alpha > 0} \mathfrak{g}_\alpha$. Let $G = KAN$ denote the Iwasawa decomposition, where K, A, N are the analytic subgroups of G corresponding to $\mathfrak{k}, \mathfrak{a}, \mathfrak{n}$.

Conjugacy classes of parabolic subalgebras of \mathfrak{g} are parametrized by subsets of Δ . For $I \subseteq \Delta$, let

$$\mathfrak{a}_I = \bigcap_{\alpha \in I} \ker \alpha.$$

Let $\mathfrak{z}(\mathfrak{h})$ denote the centralizer of the subalgebra \mathfrak{h} in \mathfrak{g} . Let \mathfrak{m}_I be the orthogonal complement of \mathfrak{a}_I in $\mathfrak{z}(\mathfrak{a}_I)$ with respect to the restriction of the Killing form, $\mathfrak{z}(\mathfrak{a}_I) = \mathfrak{m}_I \oplus \mathfrak{a}_I$.

Let $[I]$ denote the set of roots in Φ expressible as integral linear combination of elements in I . Let

$$\mathfrak{n}^I = \bigoplus_{\alpha > 0, \alpha \notin [I]} \mathfrak{g}_\alpha, \quad \mathfrak{n}^{-I} = \bigoplus_{\alpha < 0, \alpha \notin [I]} \mathfrak{g}_\alpha.$$

Then the parabolic subalgebra \mathfrak{p}_I admits the decomposition

$$\mathfrak{p}_I = \mathfrak{m}_I \oplus \mathfrak{a}_I \oplus \mathfrak{n}^I.$$

The parabolic subgroup P_I is the normalizer of the parabolic subalgebra \mathfrak{p}_I . Denote by $A_I = \exp(\mathfrak{a}_I)$, $N_I = \exp(\mathfrak{n}^I)$, $\bar{N}_I = \exp(\mathfrak{n}^{-I})$ and $L_I = Z_G(A_I)$ the centralizer of A_I in G . Then P_I admits the Levi decomposition $P_I = L_I \rtimes N_I$, and the Levi subgroup L_I is reductive : it is the product of M_I with its split component A_I . The decomposition $P_I = M_I A_I N_I$ is called the Langlands decomposition of P_I , see [Kna02, Propositions 7.82 and 7.83].

The minimal parabolic subgroup P corresponds to the empty subset $I = \emptyset$. Write the corresponding decompositions as $\mathfrak{p} = \mathfrak{p}_\emptyset = \mathfrak{m} \oplus \mathfrak{a} \oplus \mathfrak{n}$ and $P = P_\emptyset = MAN$. By [Kna02, Proposition 7.82], we have that $M_I = M_I^0 M$, where M_I^0 is the identity component of M_I . It follows that $P_I = M_I^0 P$.

For $I \subseteq \Delta$, let

$$\begin{aligned} R_I &:= \{\exp(s) : s \in \mathfrak{a}, \alpha_1(s) \leq 0, \alpha_2(s) < 0 \text{ for all } \alpha_1 \in \Delta \text{ and } \alpha_2 \in \Delta \setminus I\}, \\ D_I &:= R_I \cap A_I = \{\exp(s) : s \in \mathfrak{a}_I, \alpha(s) < 0 \text{ for all } \alpha \in \Delta \setminus I\}. \end{aligned} \quad (2)$$

The set D_I is nonempty if and only if $I \neq \Delta$ ([Mar91, Prop I.2.4.2]). For $I \subseteq \Delta$, $s \in R_I$, by [Mar91, Lemma 3.1] the automorphisms $\text{Int}(s)|_{N_I}$ and $\text{Int}(s^{-1})|_{\bar{N}_I}$ are contracting, where $\text{Int}(g).x = gxg^{-1}$ is conjugation by g .

Example 2.3. For $G = SL(5, \mathbb{R})$ and $I = \{1, 3, 4\}$, we have

$$\begin{aligned} L_I &= \begin{pmatrix} * & * & 0 & 0 & 0 \\ * & * & 0 & 0 & 0 \\ 0 & 0 & * & * & * \\ 0 & 0 & * & * & * \\ 0 & 0 & * & * & * \end{pmatrix}, N_I = \begin{pmatrix} 1 & 0 & * & * & * \\ 0 & 1 & * & * & * \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \bar{N}_I = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ * & * & 1 & 0 & 0 \\ * & * & 0 & 1 & 0 \\ * & * & 0 & 0 & 1 \end{pmatrix}, \\ D_I &= \left\{ \begin{pmatrix} e^{-t_1} \mathbf{I}_2 & 0 \\ 0 & e^{-t_2} \mathbf{I}_3 \end{pmatrix}, t_1 > t_2 \right\}, \quad \text{where } \mathbf{I}_k \text{ is the } k \times k \text{ identity matrix.} \end{aligned}$$

3 Properties of the Nevo-Zimmer maximal projective factor

3.1 Statement

Let G be a connected semisimple real Lie group with finite center and P be a minimal parabolic subgroup of G . Suppose ν_P is a probability measure in the G -quasi-invariant measure class on G/P . For a parabolic subgroup $Q > P$, denote by ν_Q the pushforward of ν_P under the projection $G/P \rightarrow G/Q$.

We are given a G -system (X, ν) , where X is taken to be a compact model. Suppose there is a P -invariant measure λ on X such that $\nu = \nu_P * \lambda$, where the convolution is explained in Notation 2.1.

Example 3.1. This is the setting of [NZ02a, Theorem 1]. The measure ν_P is the μ -harmonic measure on G/P for some admissible step distribution μ on G ; (X, ν) is a (G, μ) -stationary system and λ is provided by the Furstenberg isomorphism.

Denote by Δ simple roots of G . Recall that a standard parabolic subgroup P_I , $I \subseteq \Delta$, admits the Levi decomposition $P_I = L_I \rtimes N_I$, where N_I is the unipotent radical of P_I , the Levi subgroup $L_I = Z_G(A_I)$ is reductive. Our goal in this section is to prove the following.

Theorem 3.2. *Let $G, \nu_P, (X, \nu)$ be as above, where $\nu = \nu_P * \lambda$ for a P -invariant, P -ergodic measure λ on X . Suppose $(G/Q, \nu_Q)$ is a maximal projective factor of (X, ν) , where $Q = P_I$ is a parabolic subgroup of G . Then each connected simple non-compact factor with \mathbb{R} -rank ≥ 2 of the Levi subgroup L_I of Q preserves the measure λ .*

The existence of a (unique) maximal projective factor follows from [NZ02a, Lemma 0.1]. Consequences of Theorem 3.2 will be derived in Section 4. Throughout the rest of this section, we assume the setting of Theorem 3.2.

3.2 Q -system and disintegration of the Haar measure m_K

3.2.1 Notations for Borel sections and cocycles

We first set some notations for the cocycles that appear in the induced systems. Fix a choice of Borel sections $\theta : Q/P \rightarrow Q$ and $\tau : G/Q \rightarrow G$ with the property that $\theta([P]) = e$ and $\tau([Q]) = e$. For later convenience, we also require that $\theta(Q/P) \subseteq Q \cap K$ and $\tau(G/Q) \subseteq K$: this is possible because K acts transitively on G/P . Denote by $\beta : G/Q \times G \rightarrow Q$ the cocycle associated with the section τ . Then we have a Borel section

$$\begin{aligned} \vartheta : G/P &\rightarrow G \\ gP &\mapsto \tau(gQ)\theta(\tau(gQ)^{-1}gP). \end{aligned} \tag{3}$$

Denote by α the associated cocycle $G/P \times G \rightarrow G$ with this section. Since $\tau(Q) = e$, we have that ϑ restricted to Q/P agrees with θ . Then α restricted to $Q/P \times Q$ is the cocycle associated with θ . In the notation for a Q -system $Q/P \times_\alpha X_0$, it is understood that the cocycle α is restricted to $Q/P \times Q$.

By the "inducing in stages" property (see e.g., [Zim78, Proposition 2.4]), the two systems $G/Q \times_\beta (Q/P \times_\alpha X_0)$ and $G/P \times_\alpha X_0$ are isomorphic, via a G -isomorphism

$$\begin{aligned} j_P^Q : G/Q \times_\beta (Q/P \times_\alpha X_0) &\rightarrow G/P \times_\alpha X_0 \\ (gQ, qP, x_0) &\mapsto (\tau(gQ)\alpha(qP)P, x_0). \end{aligned} \tag{4}$$

3.2.2 Change to K -invariant measures and decomposition of Haar measure

For our purposes it is convenient to have that the harmonic measure on G/P is K -invariant, such that disintegration of Haar measures can be applied. Recall that ν_0 denotes the μ -harmonic measure on G/P . Denote by λ the P -invariant measure on X given by the Furstenberg isomorphism, $\nu = \nu_P * \lambda$. Equivalently, the boundary map $\beta_\nu : G/P \rightarrow \mathcal{P}(X)$ sends gP to $g.\lambda$.

Denote by $G = KAN$ the Iwasawa decomposition of G and m_K the normalized Haar measure on the compact subgroup K . Since ν_P is assumed to be in the G -quasi-invariant measures on G/P , it follows that $\nu = \nu_P * \lambda$ and $\nu' = m_K * \lambda$ are in the same measure class on X as well, see [NZ99, Corollary 1.5]. The boundary map associated with ν' sends gP to $g.\lambda$, that is, $\beta_{\nu'} = \beta_\nu$. Denote by \bar{m}_K the pushforward of m_K under the natural projection $G \rightarrow G/Q$. Then $(G/Q, \bar{m}_K)$ is a maximal projective factor of (X, ν') if and only if $(G/Q, \nu_Q)$ is a maximal projective factor of (X, ν) . Therefore to prove Theorem 3.2, we may replace (X, ν) by (X, ν') where $\nu' = m_K * \lambda$.

Consider the decomposition of the Haar measure on K over the closed subgroup $K \cap Q$. Since K is compact, both K and $K \cap Q$ are unimodular. Denote by \bar{m}_K the (unique) K -invariant probability

measure on $K/K \cap Q$ and $m_{K \cap Q}$ the Haar measure on $K \cap Q$ normalized to have total mass 1. Recall that in Subsection 3.2.1 we have chosen a Borel section $\tau : G/Q \rightarrow G$ such that $\tau(Q) = e$ and $\tau(G/Q) \subseteq K$. Then the decomposition of Haar measure (see e.g., [Kna02, Theorem 8.36]) implies the disintegration

$$m_K = \int_{G/Q} \tau(y).m_{K \cap Q} d\bar{m}_K(y). \quad (5)$$

3.2.3 Structure of the Q -system

We start in the same way as the proof of [NZ02a, Theorem 11.4]. The assumptions of Theorem 3.2 imply that (X, ν) fits into the sequence of G -spaces:

$$G/P \times_\alpha X_0 \xrightarrow{\xi} X \xrightarrow{\varphi} G/Q. \quad (6)$$

Here $\varphi : X \rightarrow G/Q$ is the G -map to the maximal projective factor, X_0 is the support of the measure $\lambda = \psi_\mu(\nu) = \beta_\nu(P)$ and the factor map $\xi : G/P \times_\alpha X_0 \rightarrow X$ is given by $\xi(gP, x_0) = \vartheta(gP).x_0$, where ϑ is the section map defined in (3). Also note $\xi_*(p_*m_K \times \lambda) = \nu$ where $p : G \rightarrow G/P$ is the quotient map.

By [Zim78, Theorem 2.5] we have that X is induced from an ergodic action of Q . Next we derive some information on the Q -system that arises this way. The main property we will use is that such a Q -system is induced from the P -system (X_0, λ) as well, see Proposition 3.4 below.

The following lemma is well-known, it is based on the fact that the only P -invariant measure on G/Q is the δ_Q -mass at the identity coset Q .

Lemma 3.3. *Suppose (X, ν) fits into the sequence of G -spaces (6). Then we have for λ -a.e. $x_0 \in X_0$,*

$$\varphi \circ \xi(g, x_0) = gQ \text{ for all } g \in G.$$

Proof. Since G/Q is a strongly proximal boundary, by [Fur63a, Theorem 2.2], there is a unique P -invariant probability measure on G/Q . The point mass at the identity coset Q is invariant under P , thus it is the unique P -invariant measure on G/Q .

The measure λ on X_0 , also viewed as a measure supported on the set $\{(P, x_0) : x_0 \in X_0\}$ in $G/P \times_\alpha X_0$, is P -invariant. Therefore its pushforward under $\varphi \circ \xi$ is a P -invariant measure on G/Q , which must be δ_Q . By G -equivariance, we have then for any $g \in G$ and $x_0 \in X'_0$

$$\varphi \circ \xi(gP, x_0) = \varphi \circ \xi(g.(P, \alpha(g, P)^{-1}.x_0)) = g.(P, \alpha(g, P)^{-1}.x_0) = gQ.$$

□

Next we use the disintegration (5) to specify a Q -system, which will be denoted as (Y, η) . Recall that β is the cocycle associated with the section $\tau : G/Q \rightarrow G$.

Proposition 3.4. *Assume (6). Define Y as the subset of X*

$$Y := \xi(Q/P \times_\alpha X_0),$$

*equipped with the measure $\eta := m_{K \cap Q} * \lambda$. Then the induced system $(G/Q \times_\beta Y, \bar{m}_K \times \eta)$ is G -isomorphic to (X, ν) via the map*

$$\begin{aligned} \phi : G/Q \times_\beta Y &\rightarrow X \\ (gQ, y) &\mapsto \tau(gQ).y. \end{aligned}$$

Proof. Define $\tilde{\xi}$ to be the map

$$\begin{aligned}\tilde{\xi} : G/Q \times_{\beta} (Q/P \times_{\alpha} X_0) &\rightarrow G/Q \times_{\beta} Y \\ (gQ, (qP, x_0)) &\mapsto (gQ, \vartheta(qP).x_0).\end{aligned}$$

Recall the notations that $p : G \rightarrow G/P$ is the natural projection, and \bar{m}_K denotes the pushforward of m_K under the projection $G \rightarrow G/Q$. Write $Z = G/Q \times_{\beta} (Q/P \times_{\alpha} X_0)$. By inducing in stages, we have a G -isomorphism $j_P^Q : Z \rightarrow G/P \times_{\alpha} X_0$ as in (4).

Claim 3.5. *Let $\varphi : X \rightarrow G/Q$ be the G -factor map in (6). Then we have a sequence of G -factors:*

$$(Z, \bar{m}_K \times (p_*(m_{K \cap Q}) \times \lambda)) \xrightarrow{\tilde{\xi}} (G/Q \times_{\beta} Y, \bar{m}_K \times \eta) \xrightarrow{\phi} (X, \nu) \xrightarrow{\varphi} (G/Q, \bar{m}_K).$$

Proof of the claim. The measurability and G -equivariance of the maps $\tilde{\xi}$ and ϕ are clear by their definitions. Also by the definitions of the maps the following diagram commutes:

$$\begin{array}{ccc} Z & \xrightarrow{\tilde{\xi}} & G/Q \times_{\beta} Y \\ j_P^Q \downarrow & & \downarrow \phi \\ G/P \times_{\alpha} X_0 & \xrightarrow{\xi} & X. \end{array}$$

Since j_P^Q is a G -isomorphism and ξ is a G -factor map, we see that $\tilde{\xi}$ and ϕ are G -factor maps as well.

We need to verify that the measures follow the maps. The measure η is defined as $m_{K \cap Q} * \lambda$. Since $k^{-1}\vartheta(kP) \in P$ and λ is P invariant, we have $k.\lambda = \vartheta(kP).\lambda$ and then

$$\xi_*(p_*(m_{K \cap Q}) \times \lambda) = \int_{K \cap Q} \vartheta(kP).\lambda dm_{K \cap Q}(k) = \int_{K \cap Q} k.\lambda dm_{K \cap Q}(k) = m_{K \cap Q} * \lambda = \eta.$$

Therefore for the map $\tilde{\xi}$,

$$\tilde{\xi}_*(\bar{m}_K \times (p_*(m_{K \cap Q}) \times \lambda)) = \bar{m}_K \times \xi_*(p_*(m_{K \cap Q}) \times \lambda) = \bar{m}_K \times \eta.$$

Next we verify that $\phi_*(\bar{m}_K \times \eta) = \nu$:

$$\begin{aligned}\phi_*(\bar{m}_K \times \eta) &= \int_{G/Q} \int_{K \cap Q} \tau(y)k.\lambda dm_{K \cap Q}(k) d\bar{m}_K(y) \\ &= \left(\int_{G/Q} \tau(y).m_{K \cap Q} d\bar{m}_K(y) \right) * \lambda = m_K * \lambda = \nu.\end{aligned}$$

In the second line we plugged in the decomposition formula (5). □

For the sequence of G -factor maps in the Claim, we have $\phi \circ \tilde{\xi} = \xi \circ j_P^Q$ and by Lemma 3.3, $\varphi \circ \phi \circ \tilde{\xi}(gQ, qP, x_0) = \tau(gQ)\vartheta(qP)Q = gQ$. To show that the map ϕ is indeed a measurable G -isomorphism, it remains to verify that ϕ is injective almost everywhere. Since $\varphi \circ \phi(gQ, y) = gQ$, necessarily $\phi^{-1}(\{x\}) \subseteq \{\varphi(x)\} \times Y$. When restricted to this fiber, we have $\phi(\varphi(x), y) = \theta(\varphi(x)).y$ which is injective in $y \in Y$. We conclude that ϕ is an isomorphism. □

As shown in the proof of [NZ02a, Theorem 11.4], suppose the Q -system (Y, η) admits a projective factor $(Q/Q_1, \bar{\eta})$, where $Q_1 < Q$ is a proper closed subgroup of Q , then the G -system (X, ν) admits $(G/Q_1, \nu_{Q_1})$ as a projective factor. To proceed, we will carry out the inductive step which applies the Nevo-Zimmer arguments to the Q -systems

$$(Q/P \times_{\alpha} X_0, \bar{m}_{K \cap Q} \times \lambda) \xrightarrow{\xi} (Y, \eta).$$

The goal is to show that if a higher-rank factor of $L_I = Z_G(S_I)$ does not preserve the measure λ , then we will be able to find a nontrivial projective factor $(Q/Q_1, \bar{\eta})$ of (Y, η) , contradicting the assumption that G/Q is the maximal projective factor. This will be carried out in the next subsections.

3.3 The Nevo-Zimmer operation applied to parabolic subgroups

Throughout this subsection, we assume the setting of Theorem 3.2 and (6). We have a Q -system (Y, η) described in Proposition 3.4, which fits in the setting of Appendix A with $Q = P_I$ the lsc group, its closed subgroup $P = P_{\emptyset}$ and

$$\xi_0 : (Q \times_P X_0, m_{K \cap Q} \times \lambda) \rightarrow (Y, \eta).$$

We use notations for parabolic subgroups as in Subsection 2.3. Take the Langlands decomposition $Q = M_I A_I N_I$. Denote by M_1, \dots, M_{ℓ} the noncompact simple factors of the connected component M_I^0 , and $\mathfrak{m}_1, \dots, \mathfrak{m}_{\ell}$ the corresponding Lie algebras. For each noncompact simple factor M_j , let $\mathfrak{a}_j = \mathfrak{m}_j \cap \mathfrak{a}$. Denote by I_i be the subset of I that consists of $\alpha \in I$ such that α vanishes on all \mathfrak{a}_k , $k \neq i$.

Denote by $p : G \rightarrow G/P$ the natural projection. Define $\bar{U}_I = Q \cap \bar{N}$, where $\bar{N} = \bar{N}_{\emptyset}$. Restricted to Q , we have that p maps \bar{U}_I diffeomorphically onto an open dense conull set in $(Q/P, \bar{m}_{K \cap Q})$.

Suppose M_i is a simple factor of M_I^0 with \mathbb{R} -rank at least 2, which is fixed in what follows. We need to show that M_i preserves the measure λ . Take $\varrho \subset I_i$ to be a nonempty proper subset of I_i . Take $s \in D_{\varrho}$, where D_{ϱ} is defined as in (2):

$$D_{\varrho} = \{s \in \cap_{\alpha \in \varrho} \ker(\alpha) : \alpha(s) < 0 \text{ for all } \alpha \in \Delta - \varrho\}.$$

Then \bar{U}_I admits a semi-direct product decomposition as $\bar{U}_I = \bar{U}_{\varrho} \rtimes \bar{V}_{\varrho, I}$, where $\bar{U}_{\varrho} = \bar{N} \cap P_{\varrho}$ and $\bar{V}_{\varrho, I} = \bar{N}_{\varrho} \cap L_I$. Note that since $\emptyset \neq \varrho \subsetneq I_i$, \bar{U}_{ϱ} is a nontrivial subgroup of M_i .

Example 3.6. For $G = SL(5, \mathbb{R})$, with $I = \{1, 3, 4\}$ and $\varrho = \{3\}$, we have

$$\bar{U}_I = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ * & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & * & 1 & 0 \\ 0 & 0 & * & * & 1 \end{pmatrix}, \bar{U}_{\varrho} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & * & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \bar{V}_{\varrho, I} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ * & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & * & * & 1 \end{pmatrix},$$

$$N_{\varrho} = \begin{pmatrix} 1 & * & * & * & * \\ 0 & 1 & * & * & * \\ 0 & 0 & 1 & 0 & * \\ 0 & 0 & 0 & 1 & * \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, D_{\varrho} = \left\{ \begin{pmatrix} e^{-t_1} & 0 & 0 & 0 & 0 \\ 0 & e^{-t_2} & 0 & 0 & 0 \\ 0 & 0 & e^{-t_3} & 0 & 0 \\ 0 & 0 & 0 & e^{-t_3} & 0 \\ 0 & 0 & 0 & 0 & e^{-t_4} \end{pmatrix}, t_1 > t_2 > t_3 > t_4 \right\}.$$

We have a parametrization

$$\begin{aligned}\xi : \bar{U}_\varrho \times \bar{V}_{\varrho,I} \times X_0 &\rightarrow Y \\ (u, v, x_0) &\mapsto uv.x_0.\end{aligned}$$

By the choice that $s \in D_\varrho$, $(s, \bar{U}_\varrho, \bar{V}_{\varrho,I}, N_\varrho)$ satisfies the conditions in Assumption A.2 for the Nevo-Zimmer operation as reviewed in Appendix A. More precisely, the following properties hold:

(i) the map

$$\begin{aligned}p : \bar{U}_\varrho \times \bar{V}_{\varrho,I} &\rightarrow Q/P \\ (u, v) &\mapsto uvP\end{aligned}$$

takes $\bar{U}_\varrho \times \bar{V}_{\varrho,I}$ homeomorphically to a $\bar{m}_{K \cap Q}$ -conull set in Q/P , and moreover the pushforward $p_* \left(m_{\bar{U}_\varrho} \times m_{\bar{V}_{\varrho,I}} \right)$ is in the same measure class as $\bar{m}_{K \cap Q}$.

(ii) $\text{Int}(s)$ acts trivially on $\bar{U}_\varrho = \bar{N} \cap P_\varrho$.

(iii) $\text{Int}(s^{-1})$ acts as a contracting automorphism on $\bar{V}_{\varrho,I} = \bar{N}_\varrho \cap L_I$; $\text{Int}(s)$ acts as a contracting automorphism on N_ϱ .

Denote by $\tilde{L}^\infty(Y)$ the lifts of functions in $L^\infty(Y, \eta)$ to $\bar{U}_\varrho \times \bar{V}_{\varrho,I} \times X_0$ via ξ . The Nevo-Zimmer operation provides a map

$$\begin{aligned}\mathcal{E}_{\varrho,s} : C(Y) &\rightarrow \tilde{L}^\infty(Y), \\ f &\mapsto \mathcal{E}_{\varrho,s}f, \text{ where } (\mathcal{E}_{\varrho,s}f)(\bar{u}, \bar{v}, \cdot) = \mathbb{E}_\lambda \left[\tilde{f}(\bar{u}, \cdot) | \mathcal{F}^s \right],\end{aligned}$$

where \mathcal{F}^s is the s -invariant sub- σ -algebra of $\mathcal{B}(X_0)$, $\tilde{f}(\bar{u}, \cdot) : X_0 \rightarrow \mathbb{R}$ is defined as $\tilde{f}(\bar{u}, x_0) = f(\bar{u}.x_0)$. For more explanation of this operation we refer to [NZ02a, Section 7].

As discussed in Appendix A.3, there are three possible situations for $\mathcal{E}_{\varrho,s}$.

(I) The subgroup \bar{U}_ϱ preserves the measure λ . This is exactly what we are aiming to prove.

(II1) there exists $f \in C(Y)$ and $\bar{u} \in \bar{U}_\varrho$ such that $\int f d\lambda \neq \int f d\bar{u}.\lambda$; and for a.e. $\bar{u}' \in \bar{U}_\varrho$, the function $x_0 \mapsto \mathcal{E}_{\varrho,s}f(\bar{u}', x_0)$ is λ -constant. In this case by [NZ02a, Proposition 9.2], (Y, η) has a nontrivial homogeneous factor of the form $(Q/Q_1, \bar{\eta})$, which can be taken as the Mackey realization of $\tilde{\mathcal{L}}(Y) \cap \tilde{\mathcal{L}}(Q/P)$.

(II2) The negation of (I) \vee (II1), see the next subsection.

3.4 The Gauss map argument and Case (II2)

In this subsection we assume that for some $\emptyset \neq \varrho \subseteq I_i$ and $s \in D_\varrho$, the operation $\mathcal{E}_{\varrho,s}$ is in Case (II2). Following [NZ02a, Section 4], take X'_0 to be the Mackey realization of the N_ϱ -invariant sub- σ -field of $\mathcal{B}(X_0)$, equipped with the measure λ' from the restriction of λ to $\mathcal{B}_{N_\varrho}(X_0)$. We have now a Q -system (Y', η') that is the largest common Q -factor of Y and $Q \times_P X'_0$,

$$\begin{array}{ccc}
Q \times_P X_0 & \xrightarrow{\xi_0} & Y \\
\downarrow & & \downarrow \\
Q \times_P X'_0 & \longrightarrow & Y'.
\end{array}$$

By constructions, we have the following properties.

Lemma 3.7. *Assume that $\mathcal{E}_{\varrho,s}$ is in Case (II2). The Q -system (Y', η') as above satisfies:*

- (a) *The unipotent radical N_I of Q acts trivially on (Y', η') .*
- (b) *The measure λ' , viewed as a measure on Y' , is not preserved by \bar{U}_ϱ .*

Proof. To see (a), note that since $\varrho \subset I$, we have that $N_I < N_\varrho$. By construction N_ϱ acts trivially on X'_0 , thus its subgroup N_I acts trivially on X'_0 . Since N_I is contained in P and it is a normal subgroup of Q , we have that for $v \in N_I$, $g \in Q$, $vgP = g(g^{-1}vg)P = gP$, that is, N_I acts trivially on Q/P . Then in the cocycle $\beta : Q \times Q/P \rightarrow P$, for $v \in N_I$, $\beta(v, gP) = \theta(vgP)^{-1}v\theta(gP) = \theta(gP)^{-1}v\theta(gP) \in N_I$. That is, $\beta(N_I \times Q/P) \subseteq N_I$. We conclude that N_I acts trivially on $Q/P \times_\beta X'_0$, and as a consequence it also acts trivially on the factor (Y', η') .

Part (b) is Lemma A.6. □

We now recall some facts about Gauss maps from [NZ02a, Stu96]. Let G be a real algebraic group. Denote by \mathfrak{g} or $\text{Lie}(G)$ the Lie algebra of G , and G^0 the identity component of G . If V is a k -dimensional subspace of \mathfrak{g} , denote by $[V]$ the corresponding element of $\text{Gr}_k(\mathfrak{g})$, where $\text{Gr}_k(\mathfrak{g})$ is the Grassmannian of k -planes in \mathfrak{g} . Let $\text{Gr}(\mathfrak{g}) = \cup_{k=1}^{\dim \mathfrak{g}} \text{Gr}_k(\mathfrak{g})$. The group G acts on $\text{Gr}(\mathfrak{g})$ by adjoint action, and we write $g.[V] = [\text{Ad}(g).V]$.

Following the reasoning in [NZ02a, Section 3], for the action $Q \curvearrowright (Y', \eta')$, consider the stabilizer map

$$\begin{aligned}
\psi : Y' &\mapsto \text{Sub}(Q), \\
y' &\mapsto \text{Stab}_Q(y'),
\end{aligned}$$

and the associated Gauss map

$$\begin{aligned}
d\psi : Y' &\mapsto \text{Gr}(\mathfrak{q}), \\
y' &\mapsto [\text{Lie}(\text{Stab}_Q(y'))].
\end{aligned}$$

The map $d\psi$ is a Q -equivariant Borel map.

Since the parabolic subgroup Q is a connected real algebraic group, its action on $\text{Gr}(\text{Lie}(Q))$ is algebraic. Therefore every orbit is locally closed ([Mar91, Proposition 2.1.4]) and the measure $(d\psi)_* \eta'$ is supported on a single orbit ([Zim84, Proposition 2.1.10]). Let $y_0 \in Y'$ be a point such that $(d\psi)_* \eta'$ is supported on the orbit of $[\text{Lie}(\text{Stab}_Q(y_0))]$. Since Y' is a Q -factor of $Q \times_P X'_0$, every Q -orbit in Y' meets X'_0 . Replacing y_0 by another point on $Q.y_0 \cap X'_0$ if necessary, we may assume that $y_0 \in X'_0$. By Lemma 3.7 (b), \bar{U}_ϱ is not contained in $\text{Stab}_Q(y_0)$. On the other hand, $\text{Stab}_Q(y_0)$ contains N_ϱ , which is positive dimensional.

Recall that we are given a higher-rank factor M_i of L_I as in the statement of Theorem 3.2.

Lemma 3.8. *Assume that $\mathcal{E}_{\varrho,s}$ is in Case (II2). Write $H = \text{Stab}_Q(y_0)$, where $y_0 \in X'_0$ is as above. Then we have Q -factor maps*

$$(Y', \eta') \rightarrow Q/N_Q(H) \rightarrow M_i/N_{M_i}(H \cap M_i),$$

where Q acts on $M_i/N_{M_i}(H \cap M_i)$ through the factor M_i . The normalizer $N_{M_i}(H \cap M_i)$ is a proper subgroup of M_i .

Proof. Let $L = L_I$ be the Levi subgroup of Q . By Lemma 3.7 (a), we have that $\text{Stab}_Q(y) = \text{Stab}_L(y) \rtimes N_I$. It follows that $N_Q(H) = N_L(H \cap L) \rtimes N_I$. Therefore $Q/N_Q(H)$ and $L/N_L(H \cap L)$ are isomorphic.

Since M_i is a normal subgroup of L , we have that $N_L(H \cap M_i) > N_L(H \cap L)$. The Levi subgroup L is reductive, it can be written as a product of M_i with other almost simple factors and the split component A_I , all the latter components commute with M_i , thus are contained in $N_L(H \cap M_i)$. It follows that M_i acts transitively on $L/N_L(H \cap M_i)$, which is isomorphic to $M_i/N_{M_i}(H \cap M_i)$ by the second isomorphism theorem.

Recall that the closed subgroup $H \cap M_i$ contains $N_{\varrho} \cap M_i$, which is of positive dimension since $\varrho \subsetneq I_i$. Recall also that $\bar{U}_{\varrho} \leq M_i$. Then $H \cap M_i$ does not contain \bar{U}_{ϱ} by Lemma 3.7 (a). Thus the Lie algebra of $H \cap M_i$ cannot be $\{0\}$ or \mathfrak{m}_i . Since M_i is almost simple, it follows that $H \cap M_i$ is not normal in M_i . \square

Lemma 3.8 allows us to apply [NZ02a, Proposition 3.2] to deduce that in Case (II2), (Y', η') admits a nontrivial projective factor.

Proposition 3.9. *In Case (II2), (Y', η') admits a nontrivial projective factor $(Q/Q_1, \bar{\eta})$, where Q_1 is a parabolic subgroup of G and Q_1 is a proper subgroup of Q .*

Proof. Apply [NZ02a, Proposition 3.2] to the simple Lie group M_i and its algebraic subgroup $F = N_{M_i}(H \cap M_i)$, where F is a proper subgroup by Lemma 3.8. We deduce that F is contained in a proper parabolic subgroup of M_i . Conjugate F by an element of M_i if necessary, there is a non-empty subset $J \subseteq I_i$ such that F is contained in the parabolic group of M_i parametrized by $I_i - J$. Let $Q_1 = P_{I-J}$ be the corresponding parabolic subgroup of G . Then $Q_1 \cap M_i = P_{I_i - J}$, Q/Q_1 is isomorphic to $M_i/P_{I_i - J}$, thus a factor of M_i/F . The statement then follows from Lemma 3.8. \square

3.5 Concluding that M_i preserves the measure λ

End of the proof of Theorem 3.2. Suppose $(G/Q, \nu_Q)$ is a maximal projective factor of (X, ν) , where $Q = P_I$ is a proper parabolic subgroup of G . Let $\lambda = \psi_{\mu}(\nu)$. As explained in Subsection 3.2.2, we may replace μ by $\mu' = m_K * \mu$ and ν by $m_K * \lambda$: since ν and $m_K * \lambda$ are in the same measure class, we have that $(G/Q, \bar{m}_K)$ is the maximal projective factor of $(X, m_K * \lambda)$. Then $(X, m_K * \lambda)$ is induced from the Q -system (Y, η) specified in Proposition 3.4.

Suppose M_i is a simple factor of the Levi subgroup L_I with \mathbb{R} -rank ≥ 2 . Denote by I_i the set of simple roots corresponding to M_i . Take $\varrho \subset I_i$ to be a nonempty proper subset of I_i and take $s \in D_{\varrho}$, where D_{ϱ} is specified in (2). Applying the Nevo-Zimmer operation $\mathcal{E}_{\varrho,s}$ with respect to $(s, \bar{U}_{\varrho}, \bar{V}_{\varrho,I}, N_{\varrho})$ as explained in Subsection 3.3. Then in three cases that can arise, we have

- in Case (I), \bar{U}_{ϱ} preserves the P -invariant measure λ ;
- in Case (III1), $Q \curvearrowright (Y, \eta)$ admits a nontrivial projective factor by [NZ02a, Proposition 9.2];

- in Case (II2), $Q \curvearrowright (Y, \eta)$ admits a nontrivial projective factor by Proposition 3.9.

Since G/Q is assumed to be a maximal projective factor of $(X, m_K * \lambda)$, the situations in (III) or (II2) cannot occur for (ϱ, s) . Indeed, otherwise $(X, m_K * \lambda)$ would admit a projective factor $(G/Q_1, \bar{m}_{Q_1})$ with Q_1 a proper subgroup of Q , contradicting the maximality assumption on G/Q , see the proof of [NZ02a, Theorem 11.4]. It follows that for any ϱ such that $\emptyset \neq \varrho \subsetneq I_i$, \bar{U}_ϱ preserves λ . This implies λ is invariant under M_i , see the last paragraph of the proof of [NZ02a, Theorem 1]. Indeed, since the \mathbb{R} -rank of M_i is at least 2, such subgroups \bar{U}_ϱ generate the opposite unipotent subgroup \bar{N}_i of M_i . It follows that \bar{N}_i preserves the measure λ . Since M_i is generated by \bar{N}_i and $M_i \cap P$, and P preserves the measure λ , we conclude that λ is invariant under M_i . \square

4 Consequences of the structure theorem

4.1 Proof of Theorem 1.3 and Corollary 1.4

By the structure of parabolic subgroups, we derive Theorem 1.3 and Corollary 1.4 stated in the Introduction from Theorem 3.2.

Proof of Theorem 1.3. Write $I' = f(I)$ and $Q' = P_{\mathfrak{f}(I)}$. Let $Q' = M_{I'} A_{I'} N_{I'}$ be the Langlands decomposition. The disconnectedness of $M_{I'}$ is controlled by that of P . More precisely, there is a finite subgroup $F < P$ such that $M_{I'} = M_{I'}^0 F$, see [Kna02, Proposition 7.82]. Denote by E the maximal normal compact subgroup of $M_{I'}^0$, and $M_1, \dots, M_{\ell'}$ the noncompact simple factors. Then $M_{I'}^0$ is an almost direct product of $E, M_1, \dots, M_{\ell'}$. By the definition of $f(I)$, each M_k where $k \in \{1, \dots, \ell'\}$, has \mathbb{R} -rank ≥ 2 , and M_k is a factor of $M_I < Q$. Theorem 3.2 then implies that M_k preserves the measure λ .

Next note that from the decomposition of Lie algebras, see [Kna02, (7.77) and Proposition 7.78], the compact factor E of $M_{I'}^0$ is contained in $M = Z_K(\mathfrak{s}) < P$. In the decomposition $Q' = M_{I'}^0 F A_{I'} N_{I'}$, we have $E F A_{I'} N_{I'} < P$ thus preserves the measure λ ; and the noncompact factors $M_1, \dots, M_{\ell'}$ preserves λ by Theorem 3.2. We conclude that Q' preserves the measure λ . \square

In the case that L_I has no rank-1 non-compact factors, we have $f(I) = I$, thus the statement of Corollary 1.4 follows:

Proof of Corollary 1.4. Under the assumptions, $I' = f(I) = I$, thus by Theorem 1.3, Q preserves the P -invariant measure. Recall Proposition 3.4 that (X, ν) fits into the G -factors sequence

$$(G/Q \times_\beta Y, \nu_Q \times \eta) \rightarrow (X, \nu) \rightarrow (G/Q, \nu_Q),$$

and $\eta = \bar{m}_{K \cap Q} * \lambda$. Since λ is invariant under Q , it follows that $\eta = \lambda$. Thus $(G/Q \times_\beta Y, \nu_Q \times \eta) = (G/Q \times_\beta Y, \nu_Q \times \lambda)$ is a measure-preserving extension of $(G/Q, \nu_Q)$. \square

4.2 Proof of Theorem 1.7: constraints on Furstenberg entropy spectrum

Theorem 1.3 implies the following bound on Furstenberg entropy.

Corollary 4.1. *Suppose $(G/P_I, \bar{\nu}_0)$ is the maximal standard projective factor of (X, ν) . Then*

$$h(G/P_I, \bar{\nu}_0) \leq h(X, \nu) \leq h(G/P_{\mathfrak{f}(I)}, \bar{\nu}_0).$$

Proof. Since it is assumed that $(G/P_I, \bar{\nu}_0)$ is a G -factor of (X, ν) , the first inequality $h(G/P_I, \bar{\nu}_0) \leq h(X, \nu)$ follows. For the second inequality, by the Furstenberg isomorphism, we have that $\nu = \nu_0 * \lambda$, where λ is P -invariant. By Theorem 1.3, the parabolic subgroup $P_{\mathfrak{f}(I)}$ preserves the measure λ . Then we may express $\nu = \nu_0 * \lambda$ as

$$\nu = \int_{G/P} g.\lambda d\nu_0(g) = \int_{G/P_{\mathfrak{f}(I)}} g.\lambda d\bar{\nu}_0(g).$$

The proof of [NZ99, Proposition 2.5] applied to $P_{\mathfrak{f}(I)}$ instead of P shows that (X, ν) can be viewed as a system induced from the measure-preserving $P_{\mathfrak{f}(I)}$ -system (X_0, λ) , with a G -factor map

$$(G \times_{P_{\mathfrak{f}(I)}} X_0, \bar{\nu}_0 \times \lambda) \rightarrow (X, \nu).$$

Since $(G \times_{P_{\mathfrak{f}(I)}} X_0, \bar{\nu}_0 \times \lambda)$ is a measure-preserving extension of $(G/P_{\mathfrak{f}(I)}, \bar{\nu}_0)$, we have that

$$h(G/P_{\mathfrak{f}(I)}, \bar{\nu}_0) = h(G \times_{P_{\mathfrak{f}(I)}} X_0, \bar{\nu}_0 \times \lambda) \geq h(X, \nu).$$

□

Theorem 1.7 follows directly from Corollary 4.1.

4.3 Lattices equipped with Furstenberg measures

In this subsection, we apply the induction procedure for stationary systems in [BH21, BBHP22] to deduce statements on a lattice $\Gamma < G$, equipped with a Furstenberg measure μ_0 as described in Definition 1.5. Indeed in what follows, it is sufficient to assume that $(G/P, \nu_P)$ is the Poisson boundary of (Γ, μ_0) , and ν_P is in the G -quasi-invariant measure class.

Consider an ergodic (Γ, μ_0) -space (X, ν) . Since $(G/P, \nu_P)$ is the Poisson boundary of (Γ, μ_0) , we have the Γ -equivariant boundary map $\beta_\nu : G/P \rightarrow \mathcal{P}(X)$ associated with the stationary measure ν on X . Define for $g \in G$, the measure $\phi_g \in \mathcal{P}(X)$ as the barycenter of $(\beta_\nu)_*(g\nu_P)$. Note that when restricted to Γ , $\gamma \mapsto \phi_\gamma$ is equivariant as $\phi_\gamma = \gamma.\nu$.

Fix a measurable section $\tau : G/\Gamma \rightarrow G$ and denote by $c : G \times G/\Gamma \rightarrow \Gamma$ the corresponding cocycle. Denote by $\tilde{X} = G/\Gamma \times_c X$ the induced space. For a function $f \in L^\infty(G/\Gamma \times_c X, m_{G/\Gamma} \times \nu)$, write $f_z(\cdot) = f(z, \cdot)$ for $z \in G/\Gamma$, and regard $f_z \in L^\infty(X, \nu)$. As in [BH21, Theorem 4.3], define a measure $\tilde{\nu}$ on \tilde{X} as follows:

$$\begin{aligned} \int_{\tilde{X}} f d\tilde{\nu} &= \int_{G/\Gamma} \int_X f_z d\phi_{\tau(z)} dm_{G/\Gamma}(z) \\ &= \int_{G/\Gamma} \int_{G/P} \int_X f_z d\beta_\nu(w) d\nu_P(\tau(z).w) dm_{G/\Gamma}(z). \end{aligned} \quad (7)$$

The formula (7) implies that $\tilde{\nu}$ admits a decomposition as in the Furstenberg isomorphism that $\tilde{\nu} = \nu_P * \tilde{\lambda}$, where $\tilde{\lambda}$ is P -invariant. Indeed, by [BBHP22, Lemma 4.6 and Remark 4.7], the map $\tilde{\beta}_\nu : G/P \rightarrow \mathcal{P}(\tilde{X})$, which is related to the Γ -boundary map $\beta_\nu : G/P \rightarrow \mathcal{P}(X)$ by

$$\int_{\tilde{X}} f d\tilde{\beta}_\nu(w) = \int_{G/\Gamma} \int_X f_z d\beta_\nu(\tau(z)^{-1}.w) dm_{G/\Gamma}(z), \text{ where } w \in G/P, f \in L^\infty(\tilde{X}), \quad (8)$$

is G -equivariant; and the measure $\tilde{\nu}$ defined in (7) is the barycenter of $(\tilde{\beta}_{\tilde{\nu}})_{*}(\nu_P)$. Write $\tilde{\lambda} = \tilde{\beta}_{\tilde{\nu}}(P)$, then $\tilde{\lambda}$ is P -invariant and satisfies $\tilde{\nu} = \nu_P * \tilde{\lambda}$.

An important property of the induction procedure, shown in [BH21, Proof of Theorem B] is the following.

Fact 4.2. $(G/Q, \nu_Q)$ is a Γ -factor of (X, ν) if and only if it is a G -factor of $(\tilde{X}, \tilde{\nu})$.

Here we consider G -factors up to measure classes only. As ν_P and \bar{m}_K are in the same measure class, the measure $\tilde{\nu}_1 = \bar{m}_K * \tilde{\lambda}$ is in the same measure class as $\tilde{\nu}$, thus $(\tilde{X}, \tilde{\nu}_1)$ is a μ -stationary G -space for any admissible K -invariant probability measure μ on G .

As $(\tilde{X}, \tilde{\nu}_1)$ admits a maximal projective factor for (G, μ) , by [NZ02a, Lemma 0.1], the fact implies that (X, ν) admits the same maximal projective factor for (Γ, μ_0) .

Proof of Fact 4.2. If $(G/Q, \nu_Q)$ is a Γ -factor of (X, ν) , we obtain by induction that $(\tilde{X}, \tilde{\nu})$ admits as a G -factor the space $G/\Gamma \times_c G/Q$, which is a measure-preserving extension of $(G/Q, \nu_Q)$, where by assumption on μ_0, ν_Q is in the same measure class as the K -invariant probability measure on G/Q .

Conversely, if $(G/Q, \nu_Q)$ is a G -factor of $\tilde{X} = G/\Gamma \times_c X$, there is a family of maps $(p_z : X \rightarrow G/Q)$ for $z \in G/\Gamma$ such that for every $g \in G$ and for $\tilde{\nu}$ almost every (z, x) in $G/\Gamma \times_c X$, one has $p_{gz}(c(g, z)x) = gp_z(x)$. By standard technics (see [BH21, Proof of Theorem B] for details), one can assume that it actually holds for all $z \in G/\Gamma$ and all $x \in X$. Taking $z = \Gamma$ and restricting to $g = \gamma \in \Gamma$, one gets a Γ -factor map $p_\Gamma : X \rightarrow G/Q$, with ν_Q the unique μ_0 -stationary measure. \square

Proof of Theorem 1.6. Suppose $(G/Q, \nu_Q)$ is the maximal standard projective Γ -factor of (X, ν) , where $Q = P_I$ is a parabolic subgroup. By Fact 4.2, $(G/Q, \nu_Q)$ is also the maximal standard projective G -factor of the induced G -system $(\tilde{X}, \tilde{\nu})$. Apply Theorem 1.3 to $(\tilde{X}, \tilde{\nu})$, where $\tilde{\nu} = \nu_P * \tilde{\lambda}$ as above, we have that $Q' = P_{\mathfrak{f}(I)}$ preserves the measure $\tilde{\lambda}$.

To lighten notations, in what follows write $\beta = \beta_\nu$ and $\tilde{\beta} = \tilde{\beta}_\nu$. We claim that the P -invariant measure $\tilde{\lambda}$ is Q' -invariant if and only if the Γ -boundary map $\beta : G/P \rightarrow \mathcal{P}(X)$ satisfies that $\beta(hqP) = \beta(hP)$ for all $q \in Q'$ and m -a.e. $h \in G$. Indeed, for $g \in G$, by (8) and G -invariance of $m_{G/\Gamma}$ we have that

$$\int_{\tilde{X}} g^{-1} \cdot f d\tilde{\lambda} = \int_{G/\Gamma} \int_X f_z d\beta(\tau(z)^{-1}gP) dm_{G/\Gamma}(z).$$

It follows that $g \cdot \tilde{\lambda} = \tilde{\lambda}$ if and only if for $m_{G/\Gamma}$ -a.e. z , $\beta(\tau(z)^{-1}gP) = \beta(\tau(z)^{-1}P)$. By Γ -equivariance of β , this is equivalent to that $\beta(\gamma^{-1}\tau(z)^{-1}gP) = \beta(\gamma^{-1}\tau(z)^{-1}P)$ for all $\gamma \in \Gamma$. Since $\tau : G/\Gamma \rightarrow G$ is a section, $G = \tau(G/\Gamma)\Gamma$, the claim is verified. We conclude that the property $Q' = P_{\mathfrak{f}(I)}$ preserves the measure $\tilde{\lambda}$ implies the Γ -boundary map $\theta : G/P \rightarrow \mathcal{P}(X)$ factors through $\tilde{\theta} : G/Q' \rightarrow \mathcal{P}(X)$, where $\tilde{\theta}(gQ') = \theta(gP)$ is well-defined almost everywhere. \square

Next we derive constraints on the Furstenberg entropy spectrum of (Γ, μ) in the same manner as in Subsection 4.2.

Theorem 4.3. Let μ_0 be a Furstenberg measure of finite Shannon entropy on a lattice $\Gamma < G$, where G is a connected semisimple real Lie group with finite center. Denote by Δ simple restricted roots of G . The Furstenberg entropy spectrum of (Γ, μ_0) satisfies

$$\text{EntSp}(\Gamma, \mu_0) \subseteq \bigcup_{I \subseteq \Delta} [h_{\mu_0}(G/P_I, \nu_I), h_{\mu_0}(G/P_{\mathfrak{f}(I)}, \nu_{\mathfrak{f}(I)})].$$

Proof. Suppose (X, ν) is an ergodic (Γ, μ_0) -space and $(G/Q, \nu_Q)$ is its maximal projective Γ -factor, $Q = P_I$. Then $h_{\mu_0}(X, \nu) \geq h_{\mu_0}(G/Q, \nu_Q)$. Let $Q' = P_{\Gamma(I)}$. Then by Theorem 1.6, the Γ -boundary map $\theta : G/P \rightarrow \mathcal{P}(X)$ factors through the projection $G/P \rightarrow G/Q'$, that is, $\theta(gP) = \theta(gqP)$ for $q \in Q'$. It follows that ν is the barycenter of $\theta_*(\nu_{Q'})$. Apply Lemma 2.2 to the (Γ, μ_0) -spaces $(G/Q', \nu_{Q'})$ and (X, ν) , we conclude that $h_{\mu_0}(X, \nu) \leq h_{\mu_0}(G/Q', \nu_{Q'})$. We have shown that in this case

$$h_{\mu_0}(G/Q, \nu_Q) \leq h_{\mu_0}(X, \nu) \leq h_{\mu_0}(G/Q', \nu_{Q'}).$$

The statement follows. \square

It is worth emphasizing that in the statements above, it is crucial that the μ_0 -harmonic measure on G/P is in the quasi-invariant measure class. For a general step distribution μ on Γ , the μ -harmonic measure may be singular with respect to \bar{m}_K . In such a case one can not derive constraints on $\text{EntSp}(\Gamma, \mu)$ via the inducing procedure as above.

5 Poisson bundle over a stationary system

In this section we define the μ -Poisson bundle over a stationary system (X, ν) and study its basic properties. Throughout, we assume that μ is nondegenerate in the sense that $\text{supp}\mu$ generates G as a semigroup.

5.1 Stationary joining

For a more detailed reference on stationary joining, see [FG10]. Suppose we are in the setting of Subsection 2.2.1. Denote by $(G^{\mathbb{N}}, \mathbb{P}_\mu)$ the random walk trajectory space and (B, ν_B) the Poisson boundary of (G, μ) . For a (G, μ) -stationary system (X, η) , denote by $\omega \mapsto \eta_\omega$ the almost sure limit of $\omega_n \cdot \eta$ provided by the martingale convergence theorem.

Let (X, η) and (Y, λ) be two (G, μ) -stationary systems. Let the group G act on the product space $X \times Y$ diagonally. The **stationary joining** of the two, denoted by $(X \times Y, \eta \curlywedge \lambda)$ is the system with measure

$$\eta \curlywedge \lambda = \int_{G^{\mathbb{N}}} \eta_\omega \times \lambda_\omega d\mathbb{P}_\mu(\omega) = \int_B \beta_\eta(b) \times \beta_\lambda(b) d\nu_B(b).$$

In our notation \mathbb{P}_μ denotes the law of μ -random walk trajectories from the identity e . Then $g \cdot \mathbb{P}_\mu = \mathbb{P}_\mu^g$ is the law of the trajectories starting from g . We use the same notation as stationary joining for the measure on $X \times G^{\mathbb{N}}$ given by

$$\eta \curlywedge \mathbb{P}_\mu = \int_{G^{\mathbb{N}}} \eta_\omega \times \delta_\omega d\mathbb{P}_\mu(\omega).$$

When η is G -invariant, $\eta \curlywedge \mathbb{P}_\mu = \eta \times \mathbb{P}_\mu$.

On the space $X \times G^{\mathbb{N}}$ we have a skew transformation

$$T : (x, (\omega_1, \omega_2, \dots)) \mapsto (\omega_1^{-1} \cdot x, (\omega_1^{-1} \omega_2, \omega_1^{-1} \omega_3, \dots)).$$

The arguments of [Fur02, Theorem 3.1] (see also [Kif86, Theorem I.2.1]) immediately imply that:

Fact 5.1. *The transformation T preserves the measure $\eta \curlywedge \mathbb{P}_\mu$ on $X \times G^{\mathbb{N}}$. If $G \curvearrowright (X, \eta)$ is ergodic, then T is an ergodic transformation on $(X \times G^{\mathbb{N}}, \eta \curlywedge \mathbb{P}_\mu)$.*

5.2 Definition of the Poisson bundle

Denote by $\text{Sub}(G)$ the space of closed subgroup of G , equipped with the Chabauty topology. We assume that our stationary system (X, η) comes together with a G -equivariant measurable map $L : X \rightarrow \text{Sub}(G)$, denoted by $x \mapsto L_x$. For example, $L_x = \text{Stab}_G(x)$. The pushforward of η under this map is a μ -stationary measure on $\text{Sub}(G)$, often referred to as a *stationary random subgroup* (in short SRS) of G . Denote by

$$W_\Omega := \{(x, (L_x\omega_1, L_x\omega_2, \dots)) : x \in X, (\omega_1, \omega_2, \dots) \in G^\mathbb{N}\} = \bigsqcup_{x \in X} \{x\} \times (L_x \backslash G)^\mathbb{N}$$

the space of trajectories in coset spaces. The group G acts on W_Ω by $g \cdot (x, (L_x\omega_n)) = (g.x, (L_{g.x}g\omega_n))$. Consider the map

$$\begin{aligned} \vartheta : X \times G^\mathbb{N} &\rightarrow W_\Omega \\ (x, \omega) &\mapsto (x, (L_x\omega_1, L_x\omega_2, \dots)), \end{aligned}$$

which is a G -equivariant. Write

$$\overline{\mathbb{P}}_\mu := \vartheta_* (\eta \curlywedge \mathbb{P}_\mu)$$

for the pushforward of the measure $\eta \curlywedge \mathbb{P}_\mu$.

On the space W_Ω we have a time shift operator \mathcal{S} defined as

$$\mathcal{S}(x, (L_x\omega_1, L_x\omega_2, \dots)) = (x, (L_x\omega_2, L_x\omega_3, \dots)),$$

which commutes with the G action on W_Ω . Consider the invariant σ -field \mathcal{I} under \mathcal{S} , that is,

$$\mathcal{I} := \{A \in \mathcal{B}(W_\Omega) : \mathcal{S}^{-1}(A) = A\}.$$

Definition 5.2. Let (Z, λ) be the Mackey point realisation of (the completion of) the invariant σ -field \mathcal{I} equipped with the measure $\overline{\mathbb{P}}_\mu$. We call (Z, λ) a (G, μ) -Poisson bundle over the stationary system (X, η) .

Remark 5.3. The Poisson bundle (Z, λ) is G -ergodic if (X, η) is G -ergodic. Indeed, (Z, λ) is a G -factor of the stationary joining of (X, η) and the Poisson boundary of (G, μ) . The ergodicity of the latter follows from Fact 5.1.

Denote by $\theta : (W_\Omega, \overline{\mathbb{P}}_\mu) \rightarrow (Z, \lambda)$ the factor map which induces an isomorphism of measured G -spaces between $(W_\Omega, \mathcal{I}, \overline{\mathbb{P}}_\mu|_{\mathcal{I}})$ and $(Z, \mathcal{B}(Z), \lambda)$, where $\overline{\mathbb{P}}_\mu|_{\mathcal{I}}$ denotes the restriction of the probability measure $\overline{\mathbb{P}}_\mu$ to the invariant σ -field \mathcal{I} .

In each fiber $(L_x \backslash G)^\mathbb{N}$ we obtain an invariant σ -field \mathcal{I}_x , which almost surely coincides with the invariant σ -field of the shift operator restricted to this fiber. Therefore the fiber over x in Z is (almost surely) identified with the Poisson boundary $\rho^{-1}(x) = B_{L_x \backslash G}$, i.e. the space of ergodic components of the shift operator. Up to measure zero, $Z = \bigsqcup_{x \in X} \{x\} \times B_{L_x \backslash G}$. Fiberwise, $\theta(x, \cdot) : (L_x \backslash G)^\mathbb{N} \rightarrow B_{L_x \backslash G}$ can be viewed as a map which sends a trajectory on the coset space to its image in the boundary $B_{L_x \backslash G}$. It plays a role analogous to the map $\text{bnd} : G^\mathbb{N} \rightarrow B$.

Remark 5.4. When η is a G -invariant measure, we have $\eta \curlywedge \mathbb{P}_\mu = \eta \times \mathbb{P}_\mu$. Thus over a measure preserving system (X, η) , the bundle (Z, λ) is the same Poisson bundle as considered in [Bow14].

Denote by $\overline{\mathbb{P}}_\mu = \int_X \overline{\mathbb{P}}_{\mu, x} d\eta(x)$ the disintegration of the measure $\overline{\mathbb{P}}_\mu$ over the factor map $W_\Omega \rightarrow X$, that is, for $x \in X$, the distribution of $(L_x\omega_n)_{n=1}^\infty$ is $\overline{\mathbb{P}}_{\mu, x}$. When η is not an invariant measure, in general the fiberwise process $(L_x\omega_n)_{n=1}^\infty$ is not a Markov chain. To ensure fiberwise Markov property, we will consider the special case of standard systems in the next subsection.

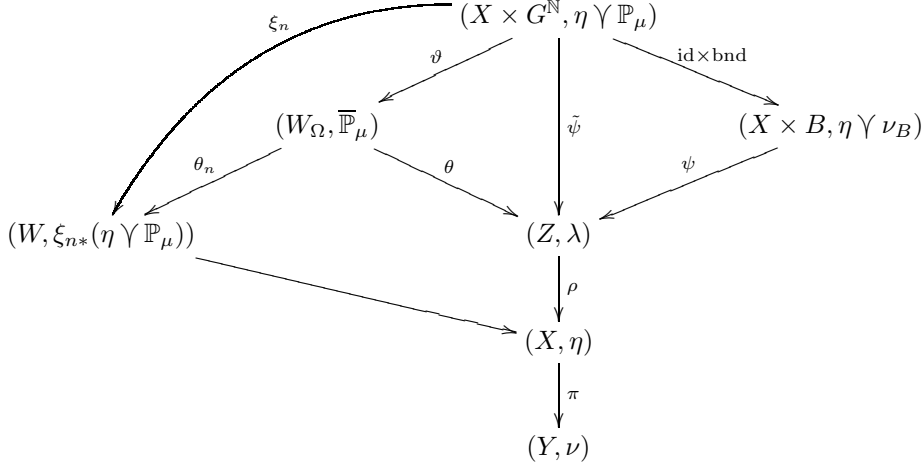


Figure 2: Commutative diagram of G -spaces.

5.3 The special case of standard systems

Let (X, η) be a standard system in the sense of [FG10], that is, $\pi : (X, \eta) \rightarrow (Y, \nu)$ a measure preserving extension and (Y, ν) a μ -boundary. Then ν_ω is a point mass \mathbb{P}_μ -a.s. In this case write $\nu_\omega = \delta_{\beta_Y(\omega)}$ where $\beta_Y : G^{\mathbb{N}} \rightarrow Y$ factors through the Poisson boundary of (G, μ) . In the notation of the boundary map we have $\beta_Y(\text{bnd}(\omega)) = \delta_{\beta_Y(\omega)}$. We use the same symbol β here, understanding that for a μ -boundary (Y, ν) , β_Y and β_ν are consistent when identifying points with δ -masses.

Denote by $y \mapsto \eta^y$ the disintegration of η over $\pi : (X, \eta) \rightarrow (Y, \nu)$. A useful property is that for a standard system, disintegration measures coincide almost surely with conditional measures:

Proposition 5.5 ([BS06, Prop 2.19]). *Let (X, η) be a standard system with the structure $\pi : (X, \eta) \rightarrow (Y, \nu)$. Then*

$$\eta_\omega = \eta^{\beta_Y(\omega)} \text{ for } \mathbb{P}_\mu\text{-a.e. } \omega.$$

Consider disintegration $\mathbb{P}_\mu = \int_Y \mathbb{P}_\mu^y d\nu(y)$ over the map $\beta_Y : (G^{\mathbb{N}}, \mathbb{P}_\mu) \rightarrow (Y, \nu)$. Since the measurable spaces we consider are all Borel spaces, regular conditional distributions exist. By uniqueness of disintegration, we have that \mathbb{P}_μ^y is the conditional distribution of ω given $\{\beta_Y(\omega) = y\}$. It is known that this conditional measure is the law of the Doob transformed random walk determined by the Radon-Nikodym derivative $\varphi_g(y) = \frac{dg\nu}{d\nu}(y)$, see [Kai00, Section 3]. Explicitly, a trajectory $(\omega_1, \omega_2, \dots)$ with law \mathbb{P}_μ^y is a Markov chain with transition kernel

$$\mathbb{P}_\mu^y(g, A) = \int_G \mathbf{1}_A(gs) \frac{\varphi_{gs}(y)}{\varphi_g(y)} d\mu(s). \quad (9)$$

Recall that $\vartheta_*(\eta \curlywedge \mathbb{P}_\mu) = \bar{\mathbb{P}}_\mu = \int_X \bar{\mathbb{P}}_{\mu,x} d\eta(x)$ denotes the disintegration over $W_\Omega \rightarrow X$.

Lemma 5.6. *When (X, η) is standard with the structure $\pi : (X, \eta) \rightarrow (Y, \nu)$, we have*

$$\eta \curlywedge \mathbb{P}_\mu = \int_Y \eta^y \times \mathbb{P}_\mu^y d\nu(y) = \int_X \eta^{\pi(x)} \times \mathbb{P}_\mu^{\pi(x)} d\eta(x).$$

The coset process with law $\bar{\mathbb{P}}_{\mu,x}$ can be sampled as follows. First sample $x \in X$ according to the stationary measure η ; then take the Doob transformed random walk ω on G conditioned on $\{\beta_Y(\omega) = \pi(x)\}$. The projected trajectory $(x, (L_x\omega_1, L_x\omega_2, \dots))$ on the coset space $L_x \backslash G$ has distribution $\bar{\mathbb{P}}_{\mu,x} = \vartheta_* \mathbb{P}_\mu^{\pi(x)}$.

Proof. Take a product set $A \times B \subseteq X \times G^{\mathbb{N}}$. By Proposition 5.5 and as $\mathbb{P}_\mu^y(\beta_Y(\omega) = y) = 1$, we have

$$\begin{aligned} \eta \curlywedge \mathbb{P}_\mu(A \times B) &= \int_{G^{\mathbb{N}}} \eta_\omega(A) \delta_\omega(B) d\mathbb{P}_\mu(\omega) = \int_Y \int_{G^{\mathbb{N}}} \eta^{\beta_Y(\omega)}(A) \mathbf{1}_B(\omega) d\mathbb{P}_\mu^y(\omega) d\nu(y) \\ &= \int_Y \eta^y(A) \int_{G^{\mathbb{N}}} \mathbf{1}_B(\omega) d\mathbb{P}_\mu^y(\omega) d\nu(y) = \int_Y \eta^y(A) \mathbb{P}_\mu^y(B) d\nu(y). \end{aligned}$$

It follows that $\bar{\mathbb{P}}_\mu = \vartheta_*(\eta \curlywedge \mathbb{P}_\mu)$ can be described as in the statement. \square

Note that the Doob transformed random walk on G is a Markov chain. In order to retain the Markov property when projected to $L_x \backslash G$, we impose the following assumption on the map $x \mapsto L_x$.

(S) Stabilizer assumption to ensure fiberwise Markov property. Suppose (X, η) is a standard (G, μ) -system with the structure $\pi : (X, \eta) \rightarrow (Y, \nu)$ a measure preserving extension and (Y, ν) a μ -proximal system. We assume that L is a G -equivariant map $X \rightarrow \text{Sub}(G)$ such that for every $x \in X$,

$$L_x < \text{Stab}_G(\pi(x)), \quad (10)$$

that is, L_x is contained in the G -stabilizer of the point $\pi(x) \in Y$.

This assumption is satisfied for instance when $L_x = \text{Stab}_G(x)$: since Y is a G -factor of X , we have $L_x < \text{Stab}_G(\pi(x))$. In particular, assumption **(S)** is always satisfied when (X, η) is G -invariant.

Proposition 5.7. *Under **(S)**, for any $x \in X$, the coset trajectory $(L_x\omega_1, L_x\omega_2, \dots) \in (L_x \backslash G)^{\mathbb{N}}$ of law $\bar{\mathbb{P}}_{\mu,x}$ follows a Markov chain whose transition kernel is given by*

$$\begin{aligned} P_{\mu,x} : (L_x \backslash G) \times \mathcal{B}(L_x \backslash G) &\rightarrow [0, 1] \\ P_{\mu,x}(L_x g, A) &= \int_G \mathbf{1}_A(L_x g s) \frac{\varphi_{gs}(\pi(x))}{\varphi_g(\pi(x))} d\mu(s), \end{aligned} \quad (11)$$

where $\varphi_g(y) = \frac{dg\nu}{d\nu}(y)$ is the Radon-Nikodym derivative.

Proof. The Doob transformed trajectory $(\omega_1, \omega_2, \dots) \in G^{\mathbb{N}}$ with law $\mathbb{P}_\mu^{\pi(x)}$ is a Markov chain with transition kernel given by (9) with $y = \pi(x)$. The containment condition (10) implies that the function

$$g \mapsto \varphi_g(\pi(x)) = \frac{dg\nu}{d\nu}(\pi(x))$$

is constant on the coset $L_x g$, for any $g \in G$. Therefore the Markov chain of the group trajectory induces the claimed Markov chain of the coset trajectory. The proposition follows as Lemma 5.6 gives

$$\int_X \bar{\mathbb{P}}_{\mu,x} d\eta(x) = \bar{\mathbb{P}}_\mu = \vartheta_*(\eta \curlywedge \mathbb{P}_\mu) = \int_X \vartheta_* \left(\eta^{\pi(x)} \times \mathbb{P}_\mu^{\pi(x)} \right) d\eta(x).$$

\square

On the Poisson boundary (B, ν_B) of the μ -random walk, since $g.\nu_B = \text{bnd}_*(g.\mathbb{P}_\mu)$ for $g \in G$, the measure $g.\nu_B$ can be regarded as the harmonic measure of the random walk starting from g . A similar property holds in the current setting. Denote by $\overline{\mathbb{P}}_{\mu,x,g}$ the law of the Markov chain on $L_x \backslash G$ with transition kernel $P_{\mu,x}$ as in (11), starting from the coset $L_x g$. Recall the fiberwise boundary maps $\theta(x, \cdot) : (L_x \backslash G)^\mathbb{N} \rightarrow B_{L_x \backslash G}$ from the definition of the bundle (Z, λ) .

Lemma 5.8. *Under (S), we have for disintegration over $W_\Omega \rightarrow X$,*

$$(g.\overline{\mathbb{P}}_\mu)_x = \overline{\mathbb{P}}_{\mu,x,g},$$

and for disintegration over $Z \rightarrow X$,

$$(g.\lambda)_x = \theta(x, \cdot)_* (\overline{\mathbb{P}}_{\mu,x,g}).$$

Proof. We first verify that the transition probabilities satisfy that for $A \subseteq \{x\} \times L_x \backslash G \subseteq W$,

$$P_{\mu,x}^n(L_x g^{-1}, A) = P_{\mu,g.x}^n(L_{g.x}, g.A). \quad (12)$$

By Proposition 5.7,

$$P_{\mu,x}^n(L_x g^{-1}, A) = \int_G \mathbf{1}_A(x, L_x g^{-1}s) \frac{\varphi_{g^{-1}s}(\pi(x))}{\varphi_{g^{-1}}(\pi(x))} d\mu^{(n)}(s), \quad (13)$$

where φ_g is the Radon-Nikodym derivative $\frac{dg\nu}{d\nu}$ on (Y, ν) . If $(x, L_x g^{-1}s) \in A$ then $g.(x, L_x g^{-1}s) \in g.A$, where $g.(x, L_x g^{-1}s) = (g.x, gL_x g^{-1}s) = (g.x, L_{g.x}s)$. Therefore $\mathbf{1}_A(x, L_x g^{-1}s) = \mathbf{1}_{g.A}(g.x, L_{g.x}s)$. Recall the general formula that $\frac{dg_1 g_2 \nu}{d\nu}(y) = \frac{dg_2 \nu}{dg_1^{-1} \nu}(g_1^{-1}.y)$. We have then

$$\frac{\varphi_{g^{-1}s}(\pi(x))}{\varphi_{g^{-1}}(\pi(x))} = \frac{dg^{-1}s\nu}{dg^{-1}\nu}(\pi(x)) = \frac{ds\nu}{d\nu}(g.\pi(x)) = \frac{ds\nu}{d\nu}(\pi(g.x)).$$

Plugging back in (13), we have that

$$P_{\mu,x}^n(L_x g^{-1}, A) = \int_G \mathbf{1}_{g.A}(g.x, L_{g.x}s) \frac{ds\nu}{d\nu}(\pi(g.x)) d\mu^{(n)}(s) = P_{\mu,g.x}^n(L_{g.x}, g.A).$$

It follows then from the Markov property that for $A \subseteq \{x\} \times (L_x \backslash G)^\mathbb{N} \subseteq W_\Omega$, we have

$$\overline{\mathbb{P}}_{\mu,x,g^{-1}}(A) = \overline{\mathbb{P}}_{\mu,g.x,id}(g.A). \quad (14)$$

Next we verify the first identity. Take a subset $C \subseteq W_\Omega$, we have

$$\begin{aligned} (g.\vartheta_*(\eta \curlywedge \mathbb{P}_\mu))(C) &= \vartheta_*(\eta \curlywedge \mathbb{P}_\mu)(g^{-1}.C) \\ &= \int_X \delta_x \otimes \overline{\mathbb{P}}_{\mu,x,id}(g^{-1}.C) d\eta(x) \\ &= \int_X \delta_{g.x} \otimes \overline{\mathbb{P}}_{\mu,g.x,g}(C) d\eta(x) \text{ (by (14))} \\ &= \int_X \delta_x \otimes \overline{\mathbb{P}}_{\mu,x,g}(C) d\eta(g^{-1}.x) \text{ (by change of variable } x \rightarrow g.x). \end{aligned}$$

By uniqueness of disintegration, we conclude that $(g.\vartheta_*(\eta \curlywedge \mathbb{P}_\mu))_x = \overline{\mathbb{P}}_{\mu,x,g}$. The second identity follows from equivariance of the map $\theta : W_\Omega \rightarrow Z$. \square

We now explain another description of the Poisson bundle over a standard system (X, η) . Under **(S)**, consider the Doob-transformed random walk on G with law $\mathbb{P}_\mu^{\pi(x)}$. Recall that (Y, ν) is assumed to be a quotient of the Poisson boundary (B, ν_B) of (G, μ) and we denote by $\beta_Y : (B, \nu_B) \rightarrow (Y, \nu)$ the factor map. Let $\nu_B = \int_Y \nu_B^y d\nu(y)$ be the disintegration. The shift operator \mathcal{S} maps $(\omega_1, \omega_2, \dots)$ to $(\omega_2, \omega_3, \dots)$. Take the \mathcal{S} -invariant sub- σ -field \mathcal{I}^y in $(G^\mathbb{N}, \mathbb{P}_\mu^y)$. By Proposition 5.5, we have that the fiber $(\beta_Y^{-1}(\{y\}), \nu_B^y)$ from the disintegration is a model for \mathcal{I}^y equipped with conditional measure \mathbb{P}_μ^y . Note that (10) implies that L_x preserves $\beta_Y^{-1}(\{\pi(x)\})$.

Proposition 5.9. *In the notation introduced above, under **(S)**, in the Poisson bundle $(Z, \lambda) \rightarrow (X, \eta)$ the fiber over $x \in X$ can be described as the ergodic components of $L_x \curvearrowright (\beta_Y^{-1}(\{y\}), \nu_B^y)$, where $y = \pi(x)$.*

Proof. The proof is the same as the statement for Poisson boundary of random walks on Schreier graphs, see [Can14, Prop 4.1.2.], also the commutative diagram on [Bow14, P. 489]. \square

5.4 Tail σ -field

In preparation for the random walk entropy formulae, we now consider the relation between the invariant and tail σ -fields. We denote the bundle of coset spaces by

$$W := \{(x, L_x \omega) : x \in X, \omega \in G\} = \bigsqcup_{x \in X} \{x\} \times L_x \backslash G.$$

On the space $(X \times G^\mathbb{N}, \eta \vee \mathbb{P}_\mu)$, we have a sequence of random variables

$$\begin{aligned} \xi_n &: X \times G^\mathbb{N} \rightarrow W \\ (x, \omega) &\mapsto (x, L_x \omega_n). \end{aligned}$$

In other terms, $\xi_n = \theta_n \circ \vartheta$, where θ_n is taking time n position of a coset trajectory in a fiber of $W_\Omega \rightarrow X$. See commutative diagram in Figure 2.

Let \mathcal{T} be the tail σ -field of $(\xi_n)_{n=1}^\infty$:

$$\mathcal{T} := \bigcap_{n=1}^\infty \sigma(\xi_n, \xi_{n+1}, \xi_{n+2}, \dots).$$

It is clear from the definitions that $\mathcal{I} \subseteq \mathcal{T}$.

The restriction of the map ξ_n defined above to the fiber over a given $x \in X$ is the map

$$\xi_n(x, \cdot) : G^\mathbb{N} \rightarrow L_x \backslash G.$$

We write ξ_n^x for the random variable taking values in $L_x \backslash G$ with law $\xi_n(x, \cdot)_* (\eta \vee \mathbb{P}_\mu) = \xi_{n*} \mathbb{P}_\mu^{\pi(x)}$ by Lemma 5.6. (With a slight abuse, we identify $L_x \backslash G$ with $\{x\} \times L_x \backslash G$.) By Proposition 5.7, $(\xi_n^x)_{n=0}^\infty$ is the Markov chain with transition probabilities $P_{\mu, x}$ starting at the identity coset. Let \mathcal{T}_x be the tail σ -field of this Markov chain. It is clear that \mathcal{T}_x almost surely coincides with the restriction of the tail σ -field \mathcal{T} to the fiber over x .

In order to relate the Furstenberg entropy of the Poisson bundle to the entropy of the random walk, we need to identify the invariant and tail σ -fields, up to null sets. For general Markov chains, the tail σ -field and the invariant σ -field do not necessarily agree modulo null sets, see [Kai92]. To ensure that fiberwise $\mathcal{I}_x \equiv \mathcal{T}_x$ modulo null sets, we assume

(**T**) Tail assumption. The random walk step distribution μ on G satisfies that there exists $n \in \mathbb{N}$ and $\epsilon > 0$ such that

$$d_{\text{TV}} \left(\mu^{(n)}, \mu^{(n+1)} \right) < 1 - \epsilon.$$

This assumption is satisfied for example by admissible measures on a locally compact group, and for μ on a countable group with $\mu(e) > 0$. As for any (G, μ) -space we have $h_{\mu^{(n)}}(X, \eta) = nh_{\mu}(X, \eta)$, we will assume without further mention that this tail assumption is satisfied when we are concerned with Furstenberg entropy realization problem.

By [Kai92, Theorem 2.7], assumption (**T**) is sufficient to guarantee $\mathcal{I}_x \equiv \mathcal{T}_x$ modulo null sets:

Corollary 5.10. *Suppose the step distribution μ on G satisfies assumption (T). Under (**S**), for any $x \in X$, the fiberwise invariant and tail σ -fields are equal up to null sets : $\mathcal{I}_x \equiv \mathcal{T}_x$. A fortiori, the invariant and tail σ -fields agree $\mathcal{I} \equiv \mathcal{T}$ modulo null sets.*

Proof. By Proposition 5.7, the fiberwise process (ξ_n^x) is a Markov chain. Moreover, the formula for the Markov kernel implies that

$$d_{\text{TV}} \left(P_{\mu, x}^n (L_x g, \cdot), P_{\mu, x}^{n+1} (L_x g, \cdot) \right) \leq d_{\text{TV}} \left(\mu^{(n)}, \mu^{(n+1)} \right).$$

Apply [Kai92, Theorem 2.7] to (ξ_n^x) , we conclude that under assumption (T), $\mathcal{I}_x \equiv \mathcal{T}_x$ modulo null sets. \square

5.5 Formulae for Furstenberg entropy

Consider a Poisson bundle (Z, λ) over a standard system (X, η) satisfying assumption (**S**), together with the fiberwise Markov chain (ξ_n^x) and the tail σ -field \mathcal{T}_x , as defined above.

We denote the conditional mutual information of ξ_n^x and \mathcal{T}_x by

$$I(\xi_1, \mathcal{T} | X, \eta) := \int_X I(\xi_1^x, \mathcal{T}_x) d\eta(x).$$

Similarly, when G is countable, we denote the conditional Shannon entropy by

$$H(\xi_n | X, \eta) = \int_X H(\xi_n^x) d\eta(x).$$

We refer to Appendix B for definitions and basic properties of mutual information and entropy.

The next proposition shows that for a Poisson bundle (Z, λ) over a standard stationary system (X, η) , its Furstenberg entropy can be expressed as the sum of the Furstenberg entropy of the base (X, η) and mutual information from fiberwise Markov chains. It will be useful for showing upper-semi continuity properties.

Proposition 5.11. *Let (Z, λ) be the Poisson bundle over a standard system (X, η) satisfying (**S**). Then*

$$h_{\mu}(Z, \lambda) = h_{\mu}(Y, \nu) + I(\xi_1, \mathcal{T} | X, \eta).$$

The proof of this proposition, which follows classical arguments of Derriennic [Der85] is given for completeness at the end of Appendix B.2.

In the case where $(X, \eta) = (Y, \nu)$ is a μ -boundary together with $L : Y \rightarrow \text{Sub}(G)$ the trivial map that $L_y = \{id\}$ for all y , then the Poisson bundle over (Y, ν) is the Poisson boundary (B, ν_B) of the μ -random walk. Proposition 5.11 recovers the known formula:

$$h(B, \nu_B) = h(Y, \nu) + \int_Y \mathbb{I}(\mathbb{P}_{\mu,1}^y, \mathcal{T}_y) d\nu(y) = h(Y, \nu) + \int_Y \inf_{n \in \mathbb{N}} \mathbb{I}(\mathbb{P}_{\mu,1}^y, \mathbb{P}_{\mu,n}^y) d\nu(y).$$

which follows from combining [Kai00, Theorem 4.5] and [Der85].

For countable groups, we have the following formulae for Furstenberg entropy of the Poisson bundle in terms of Shannon entropy of the fiberwise random walks. This can be viewed as a generalization of the formula for Poisson bundle over an IRS in [Bow14].

Theorem 5.12 (Random walk entropy formula). *Assume G is a countable group endowed with a probability measure μ of finite Shannon entropy. Let (X, η) be a standard system over (Y, ν) , with $x \mapsto L_x$ satisfying (S). Then the Poisson bundle (Z, λ) over (X, η) satisfies*

$$\begin{aligned} h_\mu(Z, \lambda) &= h_\mu(Y, \nu) + \lim_{n \rightarrow \infty} \int_X (H(\xi_n^x) - H(\xi_{n-1}^x)) d\eta(x) \\ &= h_\mu(Y, \nu) + \lim_{n \rightarrow \infty} \frac{1}{n} H(\xi_n | X, \eta). \end{aligned}$$

In both lines \lim can be replaced by $\inf_{n \in \mathbb{N}}$.

A proof of Theorem 5.12 is provided in Appendix B.3.

6 Upper semi-continuity of Furstenberg entropy

In this section, we consider a family of standard stationary (G, μ) -systems (X, η_p) , for $p \in [0, 1]$, over the same μ -boundary (Y, ν) , independent of p . As in Section 5, we denote by (Z_p, λ_p) the associated Poisson bundles, where $\lambda_p = \vartheta(\eta_p \vee \mathbb{P}_\mu)$ are given by the diagram in Subsection 5.2. Let $L : X \rightarrow \text{Sub}(G)$ be a G -equivariant map satisfying the assumption (S).

Our goal is to obtain upper-semi-continuity of the map $p \mapsto h_\mu(Z_p, \lambda_p)$. We prove it under two further assumptions on the family of measures (η_p) .

6.1 Two assumptions on a path of systems

We equip $\text{Sub}(G)$ with the Chabauty topology and $\text{Prob}(\text{Sub}(G))$ with the weak* topology.

(C) Fiberwise continuity assumption. Let $\eta_p = \int_Y \eta_p^y d\nu(y)$ denote the disintegration over (Y, ν) .

We assume that for ν -a.e. $y \in Y$, the union of supports $\cup_{p \in [0,1]} \text{supp}(L_* \eta_p^y)$ is included in a closed subset $S_y \subset \text{Sub}(G)$ and the map $p \mapsto L_* \eta_p^y$ is continuous with respect to the weak* topology on $\text{Prob}(S_y)$.

For Poisson bundles over an IRS, the space Y is a point. In this case, the fiberwise continuity assumption (C) is simply continuity of the map $p \mapsto L_* \eta_p$.

Our second assumption, slightly technical, is designed to obtain continuity of the maps $p \mapsto \mathbb{I}(\xi_1, \xi_n | X, \eta_p)$. It is possible that this map is necessarily continuous under (C) (this is the case for G discrete), which would permit to avoid Lemma 6.2 below.

Definition 6.1. A subset $S \subset \text{Sub}(G)$ has the property of *local coincidence in Chabauty topology* if a sequence $(H_k)_{k \in \mathbb{N}}$ of subgroups in S converges to $\bar{H} \in S$ in Chabauty topology if and only if for any exhaustion $(K_n)_{n=1}^\infty$ of G by compact subsets, we have

$$\sup \{n : H_k \cap K_n = \bar{H} \cap K_n\} \xrightarrow[k \rightarrow \infty]{} \infty.$$

This property means that two subgroups H_1, H_2 in S are close in Chabauty topology if and only if they coincide on large subsets $H_1 \cap K_n = H_2 \cap K_n$.

Clearly $\text{Sub}(G)$ has this property when G is discrete. It also holds when S is the collection of subgroups of a fixed discrete subgroup of G . Some non-discrete examples appear in Subsection 10.1. The space of one-dimensional subgroups of \mathbb{R}^2 does not have local coincidence property.

(L) Local coincidence assumption. Under **(C)**, we assume that for ν -a.e. $y \in Y$, the set S_y has the local coincidence property.

Assumption **(L)** is empty when the group G is discrete.

Lemma 6.2. *Let G be a locally compact group with a probability measure μ of compact support and finite boundary entropy. Consider a path of systems (X, η_p) as above satisfying **(S)**, **(C)** and **(L)**. Then the map $p \mapsto \mathbb{I}(\xi_1, \xi_n | X, \eta_p)$ is continuous.*

Proof. Let us denote ξ_n^L the image in $L \setminus G$ of the time n position of the Doob transformed random walk of law \mathbb{P}_μ^y . Let K denote the support of the measure μ , then time n position belongs to K^n . By the local coincidence assumption **(L)**, if two subgroups $L, L' \in S_y$ are close enough in Chabauty topology, then the two random variables (ξ_1^L, ξ_n^L) and $(\xi_1^{L'}, \xi_n^{L'})$ have the same law, so $\mathbb{I}(\xi_1^L, \xi_n^L) = \mathbb{I}(\xi_1^{L'}, \xi_n^{L'})$. It implies that the map $L \mapsto \mathbb{I}(\xi_1^L, \xi_n^L)$ is continuous on S_y . By compactness of S_y , the map $P(S_y) \rightarrow \mathbb{R}$ given by $\kappa \mapsto \int_{S_y} \mathbb{I}(\xi_1^L, \xi_n^L) d\kappa(L)$ is weak* continuous. Composing with the map $p \mapsto L_* \eta_p^y$, the fiberwise continuity assumption **(C)** gives continuity of $p \mapsto \int_{S_y} \mathbb{I}(\xi_1^x, \xi_n^x) d\eta_p^y(x)$ for ν -a.e. y . Now by disintegration

$$\mathbb{I}(\xi_1, \xi_n | X, \eta_p) = \int_X \mathbb{I}(\xi_1^x, \xi_n^x) d\eta_p(x) = \int_Y \int_{S_y} \mathbb{I}(\xi_1^x, \xi_n^x) d\eta_p^y(x) d\nu(y)$$

so the result follows as the convergence is dominated by $\mathbb{I}(\mathbb{P}_{\mu,1}^y, \mathbb{P}_{\mu,n}^y)$ which belongs to $L^1(Y, \nu)$ by Lemma B.4 in Appendix B.2. \square

6.2 For locally compact groups

We assume here that G is a locally compact group, endowed with a probability measure μ of compact support and finite boundary entropy $h_\mu(B, \nu_B) < \infty$. The following restrictive setting will be sufficient for our construction later.

Corollary 6.3. *Let G be a locally compact group with a probability measure μ of compact support and finite boundary entropy. Consider a path $(X, \eta_p)_{p \in [0,1]}$ of standard stationary (G, μ) -systems over (Y, ν) , satisfying **(S)**, **(C)** and **(L)**. Then the map $p \mapsto h_\mu(Z_p, \lambda_p)$ is upper semi-continuous.*

Proof. We use Proposition 5.11. By Lemma B.4, mutual information satisfies

$$\mathbb{I}(\xi_1, \mathcal{T} | X, \eta_p) = \inf_n \int_X \mathbb{I}(\xi_1^x, \xi_n^x) d\eta_p(x) = \inf_n \mathbb{I}(\xi_1, \xi_n | X, \eta_p).$$

As an infimum of continuous functions is upper semi-continuous, Lemma 6.2 gives the corollary. \square

6.3 For countable groups

In the countable setting, we show upper semi-continuity of $p \mapsto h_\mu(Z_p, \lambda_p)$ for the more general class of step distributions with finite Shannon entropy.

Corollary 6.4. *Assume G is a countable discrete group endowed with a probability measure μ of finite Shannon entropy. Let (X, η_p) be a path of standard systems satisfying **(S)** and **(C)**. Then the map $p \mapsto h_\mu(Z_p, \lambda_p)$ is upper semi-continuous.*

When μ has finite support, the statement is also covered by Corollary 6.3. We first record the following lemma.

Lemma 6.5. *For each $n \geq 1$, the map $p \mapsto H(\xi_n|X, \eta_p)$ is continuous.*

The proof is in two steps. First approximate by measures with finite support, then show continuity in this case. The second step is similar to Lemma 6.2.

Proof. Let $\mathbb{P}_{\mu,n}^y$ denote the law of step n of the Doob transformed random walk started at identity. Then $\mu^{(n)} = \int_Y \mathbb{P}_{\mu,n}^y d\nu(y)$. By concavity of entropy $\int_Y H(\mathbb{P}_{\mu,n}^y) d\nu(y) \leq H(\mu^{(n)})$, so $\mathbb{P}_{\mu,n}^y$ has finite entropy for ν -a.e. y .

Given a subset $K \subset G$, an arbitrary probability measure $\zeta \in P(G)$ can be decomposed as $\zeta = \zeta(K)\zeta|_K + \zeta(K^c)\zeta|_{K^c}$ where K^c denotes the complement of K in G , and $H(\zeta|_{K_j}) \rightarrow H(\zeta)$ for any exhaustion (K_j) of G . Moreover for $L \in \text{Sub}(G)$, we have

$$|H(\theta_{L*}\zeta|_K) - H(\theta_{L*}\zeta)| \leq |H(\zeta|_K) - H(\zeta)|$$

where $\theta_L : G \rightarrow L \backslash G$ denotes the quotient map. It follows that for a given $\varepsilon > 0$, we can find a large enough finite set K and $Y_1 \subset Y$ such that

$$\forall y \in Y_1, \quad \left| H(\mathbb{P}_{\mu,n|K}^y) - H(\mathbb{P}_{\mu,n}^y) \right| \leq \varepsilon H(\mathbb{P}_{\mu,n}^y) \quad \text{and} \quad \int_{Y \setminus Y_1} H(\mathbb{P}_{\mu,n}^y) d\nu(y) \leq \varepsilon H(\mu^{(n)})$$

The above inequalities show that the map

$$p \mapsto H(\xi_n|X, \eta_p) = \int_X H(\xi_n^x) d\eta_p(x) = \int_Y \int_{\pi^{-1}(y)} H(\theta_{L_x*}\mathbb{P}_{\mu,n}^y) d\eta_p^y(x) d\nu(y)$$

is the uniform limit of a sequence of maps of the form

$$p \mapsto H_{n,K}(p) := \int_Y \int_{\pi^{-1}(y)} H(\theta_{L_x*}\mathbb{P}_{\mu,n|K}^y) d\eta_p^y(x) d\nu(y).$$

There remains to show that the maps $H_{n,K}(p)$ are continuous. Observe that the map $L \mapsto H(\theta_{L*}\mathbb{P}_{\mu,n|K}^y)$ is continuous. Indeed, if $L, L' \in \text{Sub}(G)$ are close enough in Chabauty topology, their coset partitions have the same intersection with K , and so $\theta_{L*}\mathbb{P}_{\mu,n|K}^y = \theta_{L'*}\mathbb{P}_{\mu,n|K}^y$. Then for any $y \in Y$, the map $P(G) \rightarrow \mathbb{R}$ given by $\kappa \mapsto \int_X H(\theta_{L_x*}\mathbb{P}_{\mu,n|K}^y) d\kappa(L)$ is continuous in weak* topology. We compose with $p \mapsto L_*\eta_p^y$, which is continuous for ν -a.e. Y by **(C)**, and get continuity of $p \mapsto \int_{\pi^{-1}(y)} H(\theta_{L_x*}\mathbb{P}_{\mu,n|K}^y) d\eta_p^y(x)$. As these maps are dominated by $H(\mathbb{P}_{\mu,n}^y)$ in $L^1(Y, \nu)$, we conclude that $H_{n,K}(p)$ is continuous for each K . \square

Proof of Corollary 6.4. By Theorem 5.12, we have $h_\mu(Z_p, \lambda_p) = h_\mu(X, \eta_p) + \inf_n \frac{1}{n} H(\xi_n|X, \eta_p)$. By Lemma 6.5, $\frac{1}{n} H(\xi_n|X, \eta_p)$ is a continuous function of p . We conclude as an infimum of continuous functions is upper semi-continuous. \square

7 Tools for identification of Poisson bundles

Throughout this section assume that G is a discrete countable group and we are under assumption **(S)** as in Subsection 5.3. The goal is to show that the entropy criteria for identification of Poisson boundaries, originally due to Kaimanovich [Kai00], can be adapted to the current setting. Identification of Poisson bundles is the starting point of the lower semicontinuity argument for Furstenberg entropy in later sections.

Suppose we have a system, denoted by $(M, \bar{\lambda})$, which is a G -factor of the Poisson bundle (Z, λ) that fits into the sequence of G -factors

$$(X \times B, \eta \vee \nu_B) \rightarrow (Z, \lambda) \rightarrow (M, \bar{\lambda}) \rightarrow (X, \eta), \quad (15)$$

where the composition $(X \times B, \eta \vee \nu_B) \rightarrow (X, \eta)$ is the coordinate projection $X \times B \rightarrow X$. Since the Poisson bundle (Z, λ) is a proximal extension of (X, η) , it is a proximal extension of $(M, \bar{\lambda})$ as well, by [FG10, Prop. 4.1]. By [NZ00, Prop. 1.9], it follows that (Z, λ) is G -measurable isomorphic to $(M, \bar{\lambda})$ if and only if $h_\mu(Z, \lambda) = h_\mu(M, \bar{\lambda})$.

Under **(S)**, over $x \in X$, we have that the coset random walk $(L_x \omega_n)$ is the projection of the Doob transformed random walk $\mathbb{P}_\mu^{\pi(y)}$ to the coset space $L_x \backslash G$. Since $(M, \bar{\lambda})$ fits into (15), the fiber of M over a point $x \in X$ is covered by the Poisson boundary of the coset random walk $(L_x \omega_n)$. Denote by $\theta_M : W_\Omega \rightarrow M$ the lift of the map $Z \rightarrow M$, where W_Ω is the space of coset trajectories defined in subsection 5.2. In this setting we have that in the disintegration of $\bar{\mathbb{P}}_\mu$ over θ_M , the fiber measure $(\bar{\mathbb{P}}_\mu)_{(x, \zeta)}$, considered as a distribution on $G^\mathbb{N}$, is the law of a Markov chain $(L_x \omega_n)$ conditioned on $\theta_M(x, L_x \omega) = (x, \zeta)$. To summarize, we have:

Fact 7.1. *In the setting above, the Doob transform of the coset Markov chain $(L_x \omega_n)$ conditioned on $\theta_M(x, L_x \omega) = (x, \zeta)$ has transition kernel*

$$P_{\mu, x}^\zeta(L_x g, A) = \sum_{s \in G} \mathbf{1}_A(L_x g s) \frac{dgs \cdot \bar{\lambda}}{dg \cdot \lambda}(x, \zeta) d\mu(s).$$

Applying Shannon's theorem, see Proposition B.6, to the extension $(Z, \lambda) \rightarrow (M, \bar{\lambda})$, we have that

Corollary 7.2. *The difference between Furstenberg entropy of (Z, λ) and $(M, \bar{\lambda})$ is the $\eta \vee \mathbb{P}_\mu$ -a.s. limit*

$$h(Z, \lambda) - h(M, \bar{\lambda}) = \lim_{n \rightarrow \infty} -\frac{1}{n} \log P_{\mu, x}^{\zeta, n}(L_x, L_x \omega_n). \quad (16)$$

In particular, $h(Z, \lambda) = h(M, \bar{\lambda})$ if and only if for $\bar{\lambda}$ -a.e. (x, ζ) , the Doob transformed coset Markov chain $(L_x \omega_n)$ conditioned on $\theta_M(x, L_x \omega) = (x, \zeta)$ has 0 asymptotic entropy.

7.1 Strip approximation for bundles over IRS

The strip approximation criterion, due to Kaimanovich [Kai00, Section 6], is a powerful tool for identification of Poisson boundary in the presence of some form of hyperbolicity.

Considers bilateral paths in $G^\mathbb{Z}$. Given a step distribution μ on G , denote by $\check{\mu}$ the reflected measure $\check{\mu}(g) = \mu(g^{-1})$. Take the product space $(G^\mathbb{N}, \mathbb{P}_\mu) \times (G^\mathbb{N}, \mathbb{P}_{\check{\mu}})$ and the map $(x, \check{x}) \mapsto \omega \in G^\mathbb{Z}$, where $\omega_0 = id$, $\omega_n = x_n$, $\omega_{-n} = \check{x}_n$ for $n \in \mathbb{N}$. We write $\tilde{\mathbb{P}}_\mu$ for the pushforward of $\mathbb{P}_\mu \times \mathbb{P}_{\check{\mu}}$ under this

map and call $(G^{\mathbb{Z}}, \tilde{\mathbb{P}}_\mu)$ a bilateral path space. Denote by (B_+, ν_+) the Poisson boundary $((B_-, \nu_-)$ resp.) of the μ -random walk ($\check{\mu}$ -random walk resp.) on G and $\text{bnd}_+ : (G^{\mathbb{N}}, \mathbb{P}_\mu) \rightarrow (B_+, \nu_+)$ the associated boundary map (bnd_- resp.). Then we have a map from the bilateral paths to the product of the Poisson boundaries

$$\begin{aligned} \text{bnd}_+ \times \text{bnd}_- : (G^{\mathbb{Z}}, \tilde{\mathbb{P}}_\mu) &\rightarrow (B_+, \nu_+) \times (B_-, \nu_-) \\ \omega &\mapsto (\text{bnd}_+((\omega_n)_{n \in \mathbb{N}}), \text{bnd}_-((\omega_{-n})_{n \in \mathbb{N}})). \end{aligned}$$

Bilateral path space does not fit into the general stationary joining framework considered in Section 5. However when the measure η in the base space (X, η) is G -invariant, we may take the product space $(X \times G^{\mathbb{Z}}, \eta \times \tilde{\mathbb{P}}_\mu)$. As in Subsection 5.1, it admits skew transform

$$\tilde{T}(x, \omega) = (\omega_1^{-1} \cdot x, (\omega_1^{-1} \omega_{n+1})_{n \in \mathbb{Z}}).$$

In the same way as Fact 5.1, one can verify that if (X, η) is an ergodic p.m.p. G -system, then $\tilde{T} \curvearrowright (X \times G^{\mathbb{Z}}, \eta \times \tilde{\mathbb{P}}_\mu)$ is a p.m.p. ergodic transformation.

In this setting, the following version of strip approximation holds. Suppose in both positive and negative time directions, we have candidates for the Poisson bundle that fit into

$$(X \times B_\pm, \eta \times \nu_\pm) \rightarrow (Z, \lambda_\pm) \rightarrow (M_\pm, \bar{\lambda}_\pm) \rightarrow (X, \eta)$$

respectively. Denote by

$$\begin{aligned} (X \times G^{\mathbb{Z}}, \eta \times \tilde{\mathbb{P}}_\mu) &\rightarrow (M_\pm, \bar{\lambda}_\pm) \\ (x, \omega) &\mapsto (x, \zeta_\pm(x, \omega)) \end{aligned}$$

the maps factorising the above. Further assume that G is equipped with a distance d . Denote by $|g| = d(\text{id}_G, g)$ and $B_{L \setminus G}(R)$ the ball of radius R centred at L in the coset space $L \setminus G$ with induced distance. For example, when G is finitely generated, these can be word distances in the group and Schreier graphs.

Theorem 7.3. *Let G be a countable group with μ of finite entropy. Suppose (X, η) is G -invariant. Assume that we have a measurable assignment of strips*

$$S(x, \omega) = S(x, \zeta_+(x, \omega), \zeta_-(x, \omega)) \subseteq L_x \setminus G$$

that satisfies

- (i) *compatibility:* $L_x \omega_1 \in S(x, \omega)$ if and only if $\omega_1^{-1} L_x \omega_1 \in S(\tilde{T}(x, \omega))$,
- (ii) *positive probability of containing the root:* $(\eta \times \tilde{\mathbb{P}}_\mu)(L_x \in S(x, \omega)) > 0$,
- (iii) *subexponential size:* for any $\epsilon > 0$ and $\eta \times \tilde{\mathbb{P}}_\mu$ -a.e. (x, ω) ,

$$\limsup \frac{1}{n} \log |S(x, \omega) \cap B_{L_x \setminus G}(|L_x \omega_n|)| \leq \epsilon.$$

Then $(M_+, \bar{\lambda}_+)$ is G -isomorphic to the Poisson bundle (Z_+, λ_+) ; and $(M_-, \bar{\lambda}_-)$ is G -isomorphic to the Poisson bundle (Z_-, λ_-) .

Remark 7.4. In Theorem 7.3, we require the compatibility condition (i) and positive probability of the event that the root is on the strip (ii) as a replacement for G -equivariance of strips in the original Kaimanovich strip criterion. An illustration can be found in Figure 3.

Proof of Theorem 7.3. The proof relies on the Birkhoff ergodic theorem applied to the skew transformation \tilde{T} and Shannon's Theorem as in Proposition B.6.

Denote by A the subset $\{(x, \omega) : L_x \in S(x, \omega)\}$ of $X \times G^{\mathbb{Z}}$. By (ii), A has positive probability under $\eta \times \tilde{\mathbb{P}}_\mu$. Apply the Birkhoff ergodic theorem to $\tilde{T} \curvearrowright (X \times G^{\mathbb{Z}}, \eta \times \tilde{\mathbb{P}}_\mu)$, we have that for a.e. (x, ω) , the set of times n such that $\tilde{T}^n(x, \omega) \in A$ has positive limiting frequency. Note that

$$\begin{aligned} \{\tilde{T}^n(x, \omega) \in A\} &= \left\{L_{\omega_n^{-1} \cdot x} \in S\left(\omega_n^{-1} \cdot x, (\omega_n^{-1} \omega_{m+n})_{m \in \mathbb{Z}}\right)\right\} \\ &= \{L_x \omega_n \in S(x, \omega)\} \text{ by compatibility (i).} \end{aligned}$$

It follows that for a.e. (x, ω) , the set $\{n \in \mathbb{N} : L_x \omega_n \in S(x, \omega)\}$ has positive density.

Assume by contradiction that $h(Z_+, \lambda_+) - h(M_+, \bar{\lambda}_+) = \delta > 0$ and take $\epsilon = \frac{\delta}{3}$. By Corollary 7.2, for any $p > 0$ there is a subset $\tilde{V} \subset X \times G^{\mathbb{Z}}$ with $(\eta \times \tilde{\mathbb{P}}_\mu)(\tilde{V}) \geq 1 - p$ and there is $N \in \mathbb{N}$ such that for $(x, \omega) \in \tilde{V}$ and $n \geq N$

$$P_{\mu, x}^{\zeta_+(x, \omega), n}(L_x, L_x \omega_n) \leq e^{-2n\epsilon}. \quad (17)$$

Recall that as in previous sections we have the disintegration of measure over M_+ that $\eta \times \mathbb{P}_\mu = \int_{M_+} (\eta \times \mathbb{P}_\mu)_{(x, \zeta_+)} d\bar{\lambda}_+$, and moreover, fiberwise $P_{\mu, x}^{\zeta_+}$ is the transition kernel of the Doob transformed random walk conditioned on $\zeta_+(x, \omega) = (x, \zeta_+)$. We have then

$$\begin{aligned} &\eta \times \tilde{\mathbb{P}}_\mu \left(L_x \omega_n \in S(x, \omega), (x, \omega) \in \tilde{V} \right) \\ &= \eta \times \tilde{\mathbb{P}}_\mu (L_x \omega_n \in S(x, \omega) \cap B_{L_x \setminus G}(|L_x \omega_n|), (x, \omega) \in \tilde{V}) \\ &= \int_{M_+} \left(\eta \times \tilde{\mathbb{P}}_\mu \right)_{x, \zeta_+} \left(L_x \omega_n \in S(x, \omega) \cap B_{L_x \setminus G}(|L_x \omega_n|), (x, \omega) \in \tilde{V} \right) d\bar{\lambda}_+(x, \zeta_+) \\ &\leq \int_{M_+} \left(\sup_{(x, \omega) \in \tilde{V}} P_{\mu, x}^{\zeta_+(x, \omega), n}(L_x, L_x \omega_n) \right) |S(x, \omega) \cap B_{L_x \setminus G}(|L_x \omega_n|)| d\bar{\lambda}_+(x, \zeta_+) \\ &\leq e^{-n\epsilon}, \end{aligned}$$

where the last line uses the bound (17) and the subexponential size assumption (iii). By the Borel-Cantelli lemma, and as p is arbitrary, we deduce that

$$\eta \times \tilde{\mathbb{P}}_\mu (\{L_x \omega_n \in S(x, \omega) \text{ for infinitely many } n \in \mathbb{N}\}) = 0.$$

contradicting the positive limiting frequency of times spent on the strip. The statement for (M_-, λ_-) follows from applying the same argument to negative indices. \square

Remark 7.5. In the setting of general locally compact groups, one can not apply the subadditive ergodic theorem to derive an analogue of Proposition B.6. The recent work of Forghani and Tiozzo [FT22] shows a version of Shannon’s theorem for random walks on locally compact groups; and the techniques there could be adapted to our setting. We will consider the Poisson bundle identification problem for free groups, then inducing to $SL(d, \mathbb{R})$. For this reason we do not pursue the direction to formulate results for locally compact groups in this section.

7.2 Ray approximation criteria for Poisson bundles over standard systems

For future reference, we state a version of the ray approximation criterion for Poisson bundles over standard systems, which is more generally applicable than the strip criterion. Such a criterion is originally due to Kaimanovich [Kai00]. The version stated here is adapted from the enhanced criterion of Lyons and Peres [LP21].

Theorem 7.6 (Ray approximation [Kai00, LP21]). *Let G be a countable group endowed with μ of finite entropy. Let $(M, \bar{\lambda})$ be a G -system that fits in (15), denote by $\theta_M : W_\Omega \rightarrow M$ the factor map. Suppose for any $\epsilon > 0$, there is a subset $U \subseteq M$ with positive measure $\bar{\lambda}(U) > 0$ such that there is a sequence of measurable maps*

$$(x, \zeta) \mapsto A_n^\epsilon(x, \zeta),$$

where $(x, \zeta) \in U$ and $A_n^\epsilon(x, \zeta) \subseteq L_x \backslash G$ satisfying that

- (i) $\limsup_{n \rightarrow \infty} \bar{\mathbb{P}}_\mu(\exists m \geq n : L_x \omega_m \in A_n^\epsilon(\theta_M(x, L_x \omega)) | \theta_M(x, L_x \omega) \in U) > 0$,
- (ii) $\limsup_{n \rightarrow \infty} \frac{1}{n} \log |A_n^\epsilon(x, \zeta)| \leq \epsilon$ for all $(x, \zeta) \in U$.

Then $(M, \bar{\lambda})$ is G -measurable isomorphic to the Poisson bundle (Z, λ) over (X, η) .

A proof of Theorem 7.6 is provided in Subsection B.4.

8 A setting for lower semi-continuity argument

We now return to the general setting of bundles over stationary systems. Suppose X and Y are locally compact metrizable spaces and $\pi : X \rightarrow Y$ is a Borel G -factor map where G is a lsc group. For the remainder of this section, let $\pi : X \rightarrow Y$ and (Y, ν) be a fixed μ -stationary system. In this section we do not need to assume that (Y, ν) is a μ -boundary.

Suppose we have a (topological) bundle M over X where the fiber over a point $x \in X$ is a topological space M_x . We assume:

- (M) Fiberwise measures. The space M_x is equipped with a family of probability measures $\{\alpha_{x,g}\}_{g \in G}$ in the same measure class. Moreover, for every $g \in G$, the Radon-Nikodym derivative $d\alpha_{x,g}/d\alpha_{x,e} \in L^\infty(M_x, \alpha_{x,e})$. Let $C_{x,g} < \infty$ be an upper bound for the L^∞ -norm of $d\alpha_{x,g}/d\alpha_{x,e}$.

Consider a path of measures on X , $p \mapsto \eta_p$ such that $\pi_*(\eta_p) = \nu$ and (X, η_p) is a relative measure-preserving extension of (Y, ν) for all $p \in [0, 1]$. As before, let $\eta = \int_Y \eta^y d\nu(y)$ be the disintegration of η over Y . Similar to (C), suppose

(C') Fiberwise continuity. The map $p \mapsto \eta_p^y$ is continuous for ν -a.e. y , and for each $y \in Y$, there is a compact subset $S_y \subseteq X$, such that $S_y \supseteq \cup_{p \in [0,1]} \text{supp} \eta_p^y$. Equip S_y with the subspace topology and $\text{Prob}(S_y)$ the weak*-topology.

Under (M), we equip the bundle M with a family of measures $\alpha_p = (\alpha_{g,p})_{g \in G}$, where $\alpha_{g,p}$ is defined by its disintegration $\int_X \alpha_{x,g} d\eta_p(x)$ over the map $M \rightarrow X$ with measure η_p on X . We define the entropy of α_p as

$$\begin{aligned} h_\mu(M, \alpha_p) &:= h_\mu(Y, \nu) + \int_G \int_X D(\alpha_{x,g} \parallel \alpha_{x,e}) \varphi_g(\pi(x)) d\eta_p(x) d\mu(g) \\ &= h_\mu(Y, \nu) + \int_G \int_Y \int_{S_y} D(\alpha_{x,g} \parallel \alpha_{x,e}) d\eta_p^y d\nu(y) d\mu(g) \end{aligned} \quad (18)$$

where $\varphi_g(y) = \frac{d\nu}{d\nu}(y)$ is the Radon-Nikodym derivative in (Y, ν) . We refer to Appendix B.2 for definition and basic properties of the KL-divergence $D(\alpha \parallel \beta)$. This definition of entropy is consistent:

Lemma 8.1. *Assume the composition of factor maps $X \times B \xrightarrow{\zeta} M \rightarrow X$ is the projection on the first factor and let $\alpha_{x,g} := (\zeta_*(g\nu_B))_x$, then $h_\mu(M, \alpha)$ is the Furstenberg entropy of $(M, \zeta_*\nu_B)$.*

Proof. This follows from Proposition 5.11 and Lemma B.4 (iv), with $X \times B \xrightarrow{\zeta} M$ in place of $X \times B \xrightarrow{\psi} Z$. Recall that by Lemma 5.8, $(g\lambda)_x = (\psi_*g\nu_B)_x$. \square

For the rest of this section we will focus on fiberwise approximations to the KL-divergence $D(\alpha_{x,g} \parallel \alpha_{x,e})$. This will be sufficient for our purposes:

Lemma 8.2. *Under (M), (C'), if for ν -a.e. $y \in Y$, μ -a.e. $g \in G$, the map $x \mapsto D(\alpha_{x,g} \parallel \alpha_{x,e})$ is lower semi-continuous on S_y , then $p \mapsto h_\mu(M, \alpha_p)$ is lower semi-continuous.*

Proof. The lower semi-continuity assumption on $D(\alpha_{x,g} \parallel \alpha_{x,e})$ implies that it can be written as an increasing limit of non-negative continuous functions f_n on S_y . Let $p_m \rightarrow p$, then fiberwise continuity (C') implies

$$\int_{S_y} f_n(y) d\eta_p^y = \lim_{m \rightarrow \infty} \int_{S_y} f_n(y) d\eta_{p_m}^y \leq \liminf_{m \rightarrow \infty} \int_{S_y} D(\alpha_{x,g} \parallel \alpha_{x,e}) d\eta_{p_m}^y.$$

Monotone convergence theorem implies $\int_{S_y} D(\alpha_{x,g} \parallel \alpha_{x,e}) d\eta_p^y = \lim_{n \rightarrow \infty} \int_{S_y} f_n(y) d\eta_p^y$. Thus $p \mapsto \int_{S_y} D(\alpha_{x,g} \parallel \alpha_{x,e}) d\eta_p^y$ is lower semi-continuous. By the integral formula (18), the statement follows from Fatou's lemma. \square

8.1 The case of uniform fiberwise approximation

In this subsection we consider approximations of measures on M_x . Assume:

(P) Generating finite partitions. For each $x \in X$, there is a refining sequence of finite measurable partitions $\mathcal{P}_{x,n}$ of M_x , $n \in \mathbb{N}$, such that the union $\cup_{n \in \mathbb{N}} \mathcal{P}_{x,n}$ generates the Borel σ -field \mathcal{B}_x of M_x .

The KL-divergence of two Borel probability measures $\beta_{x,1}$ and $\beta_{x,2}$ on M_x is then given by

$$D(\beta_{x,1} \parallel \beta_{x,2}) = \sup_n H_{\beta_{x,1} \parallel \beta_{x,2}}(\mathcal{P}_{x,n}), \quad \text{where} \quad H_{\beta_{x,1} \parallel \beta_{x,2}}(\mathcal{P}_{x,n}) = \sum_{A \in \mathcal{P}_{x,n}} \beta_{x,1}(A) \log \frac{\beta_{x,1}(A)}{\beta_{x,2}(A)}.$$

See Appendix B.2. We show continuity of the maps $x \mapsto H_{\beta_{x,1} \parallel \beta_{x,2}}(\mathcal{P}_{x,n})$ under assumptions of approximations.

Let S be a subset of X , equipped with subspace topology. We say a collection of probability spaces $(\mathcal{P}_{x,n}, q_x)$, where q_x is a probability measure on the partition $\mathcal{P}_{x,n}$, is *locally constant* on S if for every $x \in X'$, there is an open neighborhood $O(x)$ of it in X' such that for any $x' \in O(x)$, the spaces $(\mathcal{P}_{x,n}, q_x)$ and $(\mathcal{P}_{x',n}, q_{x'})$ are isomorphic.

Assume **(P)**. Let $(\beta_x)_{x \in X'}$ be a collection of probability measures with each β_x supported on M_x . We say that

- this collection admits *approximations* on X' if for $x \in X'$ and $n, t \in \mathbb{N}$, there is a positive measure $q_{x,n}^t$ on $\mathcal{P}_{x,n}$ such that

$$\max_{A \in \mathcal{P}_{x,n}} \left| 1 - \frac{\beta_x(A)}{q_{x,n}^t(A)} \right| \leq \varepsilon_{x,n}(t) \quad \text{with} \quad \lim_{t \rightarrow \infty} \varepsilon_{x,n}(t) = 0.$$

- Such approximations are *uniform* on X' if in addition,

$$\lim_{t \rightarrow \infty} \sup_{x \in X'} \varepsilon_{x,n}(t) = 0.$$

- Such approximations are *locally constant* if for $n, t \in \mathbb{N}$ the collection $(\mathcal{P}_{x,n}, q_{x,n}^t)$ is locally constant.

Proposition 8.3. *Let $(\beta_{x,1})_{x \in X'}$ and $(\beta_{x,2})_{x \in X'}$ be two collections of fiber probability measures, where $\beta_{x,i}$ is supported on M_x . Suppose each $(\beta_{x,i})_{x \in X'}$ admit locally constant uniform approximations on S , $i \in \{1, 2\}$; and there is a constant $C > 0$ such that $1/C \leq \|d\beta_{x,1}/d\beta_{x,2}\|_\infty \leq C$ for all $x \in X'$. Then the following map is continuous:*

$$\begin{aligned} S &\rightarrow \mathbb{R}_{\geq 0} \\ x &\mapsto H_{\beta_{x,1} \parallel \beta_{x,2}}(\mathcal{P}_{x,n}). \end{aligned}$$

It follows that $x \mapsto D(\beta_{x,1} \parallel \beta_{x,2})$ is lower semi-continuous on S .

Remark 8.4. For our applications, the subset S will be totally disconnected and satisfy the local coincidence property **(L)**. The locally constant approximation condition is natural in that context. See Proposition 8.6 for a formulation with weaker assumptions.

Proof of Proposition 8.3. Let $q_{x,n,i}^t$ be the approximation measures of $\beta_{x,i}$ on the finite partition $\mathcal{P}_{x,n}$ with the corresponding error bound $\varepsilon_{x,n,i}(t)$. Note that

$$\begin{aligned} \max_{A \in \mathcal{P}_{x,n}} \frac{q_{x,n,2}^t(A)}{q_{x,n,1}^t(A)} &\leq \frac{1}{(1 - \varepsilon_{x,n,1}(t))(1 - \varepsilon_{x,n,2}(t))} \max_{A \in \mathcal{P}_{x,n}} \frac{\beta_{x,2}(A)}{\beta_{x,1}(A)} \\ &\leq \frac{C}{(1 - \varepsilon_{x,n,1}(t))(1 - \varepsilon_{x,n,2}(t))} =: C_{n,t}. \end{aligned}$$

Lemma B.3 implies that

$$\begin{aligned} H_{q_{x,n,2}^t \| q_{x,n,1}^t}(\mathcal{P}_{x,n}) - H_{\beta_{x,2} \| \beta_{x,1}}(\mathcal{P}_{x,n}) &\leq 2C_{n,t}^{1/2} \max_{A \in \mathcal{P}_{x,n}} \left| 1 - \frac{\beta_{x,2}(A)}{q_{x,n,2}^t(A)} \right| + \log \left(\max_{A \in \mathcal{P}_{x,n}} \frac{\beta_{x,1}(A)}{q_{x,n,1}^t(A)} \right) \\ &\leq 2C_{n,t}^{1/2} \varepsilon_{x,n,2}(t) + \varepsilon_{x,n,1}(t); \end{aligned} \quad (19)$$

and

$$\begin{aligned} H_{\beta_{x,2} \| \beta_{x,1}}(\mathcal{P}_{x,n}) - H_{q_{x,n,2}^t \| q_{x,n,1}^t}(\mathcal{P}_{x,n}) &\leq 2C^{1/2} \max_{A \in \mathcal{P}_{x,n}} \left| 1 - \frac{q_{x,n,2}^t(A)}{\beta_{x,2}(A)} \right| + \log \left(\max_{A \in \mathcal{P}_{x,n}} \frac{q_{x,n,1}^t(A)}{\beta_{x,1}(A)} \right) \\ &\leq 2C^{1/2} \frac{\varepsilon_{x,n,2}(t)}{1 - \varepsilon_{x,n,2}(t)} + \frac{\varepsilon_{x,n,1}(t)}{1 - \varepsilon_{x,n,1}(t)}. \end{aligned} \quad (20)$$

Write $\epsilon_n(t) = \sup_{x \in X'}(\varepsilon_{x,n,2}(t) + \varepsilon_{x,n,1}(t))$, then the uniform assumption states that $\epsilon_n(t) \xrightarrow{t \rightarrow \infty} 0$. Therefore (19) and (20) show that the sequence of continuous (actually locally constant) functions $x \mapsto H_{q_{x,n,2}^t \| q_{x,n,1}^t}(\mathcal{P}_{x,n})$ converges uniformly to the function $x \mapsto H_{\beta_{x,2} \| \beta_{x,1}}(\mathcal{P}_{x,n})$ as $t \rightarrow \infty$. Thus by the uniform convergence theorem, the limit function is continuous as well. By (P), the partitions $\mathcal{P}_{x,n}$ generate the Borel σ -field \mathcal{B}_x of M_x , we have that $H_{\beta_{x,2} \| \beta_{x,1}}(\mathcal{P}_{x,n}) \nearrow D(\beta_{x,1} \| \beta_{x,2})$ when $n \rightarrow \infty$. It follows that $x \mapsto D(\beta_{x,1} \| \beta_{x,2})$ is lower semi-continuous on S . \square

Corollary 8.5. *In the setting of (C'), (M) and (P), suppose for ν -a.e. $y \in Y$, the family of measures $(\alpha_{x,g})_{x \in S_y}$ admits locally constant uniform approximations on S_y , then the map $p \mapsto h_\mu(M, \alpha_p)$ is lower semi-continuous.*

Proof. This follows from Lemma 8.2 and Proposition 8.3 for $\beta_{x,1} = \alpha_{x,g}$ and $\beta_{x,2} = \alpha_{x,e}$. \square

8.2 A more general criterion with integral bounds

For completeness, we record in this subsection a relaxed version of Corollary 8.5, which ensures lower semi-continuity of the map $p \mapsto h_\mu(M, \alpha_p)$.

Under (P), we assume that for each $g \in G$ and $n \in \mathbb{N}$, there is a probability $q_{x,g}^n$ defined on the partition $\mathcal{P}_{x,n}$ that approximates $\alpha_{x,g}$ in the sense that there is a constant $\varepsilon_x(g, n) > 0$ such that

$$\max_{A \in \mathcal{P}_{x,n}} \left| 1 - \frac{\alpha_{x,g}(A)}{q_{x,g}^n(A)} \right| \quad \text{and} \quad \lim_{n \rightarrow \infty} \varepsilon_x(g, n) = 0.$$

Recall that in (M), $C_{x,g}$ is an upper bound for the L^∞ -norm of the Radon-Nikodym derivative $d\alpha_{x,g}/d\alpha_{x,e}$. Similar to the bound in (19), define

$$\Delta_{x,g}(n) := 2 \left(\frac{C_{x,g}}{(1 - \varepsilon_x(e, n))(1 - \varepsilon_x(g, n))} \right)^{1/2} \varepsilon_x(g, n) + \varepsilon_x(e, n). \quad (21)$$

Proposition 8.6. *In the setting of (C'), (M) and (P), suppose in addition that for each $y \in Y$,*

- for all $n \in \mathbb{N}$, the map $x \mapsto H_{q_{x,e}^n \| q_{x,g}^n}(\mathcal{P}_{x,n})$ is continuous on S_y ,
- for $\mu \times \nu$ -a.e. (g, y) , the error terms $\Delta_{x,g}(n)$ defined in (21) is dominated by some function $\psi_g(x)$ which is integrable with respect to every $\eta \in \mathfrak{F}_1$.

- there is a sequence $(\delta_n(g, y))_{n \in \mathbb{N}}$ that converges to 0 and

$$\int_{S_y} \Delta_{x,g}(n) d\eta_p^y(x) \leq \delta_n(g, y) \text{ for all } p \in [0, 1].$$

Then the map $p \mapsto h_\mu(M, \alpha_p)$ is lower semi-continuous.

Proof. As in Lemma 8.2, it suffices to show that $p \mapsto \int_{S_y} D(\alpha_{x,g} \parallel \alpha_{x,e}) d\eta_p^y(x)$ is lower semi-continuous. As in the proof of Proposition 8.3, Lemma B.3 implies

$$D(\alpha_{x,e} \parallel \alpha_{x,g}) = \sup_{n \in \mathbb{N}} \left\{ H_{q_{x,e}^n \parallel q_{x,g}^n}(\mathcal{P}_{x,n}) - \Delta_{x,g}(n) \right\}. \quad (22)$$

Write $\delta_n = \delta_n(g, y)$, we have

$$\begin{aligned} \int_{S_y} D(\alpha_{x,g} \parallel \alpha_{x,e}) d\eta^y(x) &= \int_{S_y} \sup_n \left\{ H_{q_{x,g}^n \parallel q_{x,e}^n}(\mathcal{P}_{x,n}) - \Delta_{x,g}(n) \right\} d\eta^y(x) \\ &\geq \sup_n \left\{ \int_{S_y} H_{q_{x,g}^n \parallel q_{x,e}^n}(\mathcal{P}_{x,n}) d\eta_p^y - \int_{S_y} \Delta_{x,g}(n) d\eta^y(x) \right\} \\ &\geq \sup_n \left\{ \int_{S_y} H_{q_{x,g}^n \parallel q_{x,e}^n}(\mathcal{P}_{x,n}) d\eta^y - \delta_n \right\}. \end{aligned}$$

In the other direction, (22) and the dominated convergence theorem implies that

$$\int_{S_y} D(\alpha_{x,g} \parallel \alpha_{x,e}) d\eta^y(x) = \lim_n \int_{S_y} H_{q_{x,g}^n \parallel q_{x,e}^n}(\mathcal{P}_{x,n}) d\eta^y(x).$$

Since $\delta_n \rightarrow 0$ as $n \rightarrow \infty$, we have then

$$\int_{S_y} D(\alpha_{x,g} \parallel \alpha_{x,e}) d\eta^y(x) = \sup_n \left\{ \int_{S_y} H_{q_{x,g}^n \parallel q_{x,e}^n}(\mathcal{P}_{x,n}) d\eta^y - \delta_n \right\}. \quad (23)$$

By the continuity assumptions (C1) and (C2), the function $\eta \mapsto \int_{S_y} H_{q_{x,e}^n \parallel q_{x,g}^n}(\mathcal{P}_{x,n}) d\eta^y - \delta_n$ is continuous on \mathfrak{P}_1 . Then (23) implies the statement. \square

9 Bowen-Poisson bundle for free groups

In this section, we apply the tools of the previous sections to the Poisson bundles over IRSs of the free group considered in [Bow14].

Let F be the free group \mathbf{F}_k on $k \geq 2$ generators. Denote its standard generating set as $S = \{a_1, \dots, a_k\}$. The Schreier graph of a subgroup H of F has vertex set $H \backslash F$ (the space of cosets) and edge set $\{(Hg, Hgs) : g \in F, s \in S\}$.

Following a terminology of Bowen, we call a subgroup H of F , or rather its Schreier graph $H \backslash F$, *tree-like* if the only simple loops are self-loops, i.e. have length 1 – see [Bow14, Section 4]. This precisely means that between any two vertices of the Schreier graph, there is a unique path without backtrack nor self-loop from one to the other. Algebraically, a subgroup H is tree-like if and only

if it is generated by elements of the form $gs g^{-1}$ for g in F and s in the generating set (and we can assume there is a bijection between such pairs (g, s) and the loops of the Schreier graph). We denote by $\partial(H \setminus F)$ the space of ends of $H \setminus F$. Denote by Tree_F the subset of $\text{Sub}(F)$ which consists of H with tree-like Schreier graphs. It is a conjugacy invariant closed subset of $\text{Sub}(F)$.

9.1 Quasi-transitive tree-like Schreier graphs

We first consider the situation where $H_0 \in \text{Tree}_F$ has a normalizer $N_F(H_0)$ of finite index in F . The normalizer $N_F(H_0)$ acts from the left on the Schreier graph of $H_0 \setminus F$ by automorphisms; it extends to a continuous action on the space of ends $\partial(H_0 \setminus F)$. We also assume that the tree-like Schreier graph $H_0 \setminus F$ has infinitely many ends.

Example 9.1. *Following [Bow14, Section 4.3], given an integer $\ell \geq 2$, take the subgroup K_ℓ of $\mathbf{F}_2 = \langle a, b \rangle$ which is generated by all elements of the form ghg^{-1} , where $g \in \langle a^\ell, b^\ell \rangle$ and $h \in \{a^k b a^{-k}, b^k a b^{-k} : k = 1, 2, \dots, \ell - 1\}$. The coset Schreier graph $K_\ell \setminus F$ is tree-like, and the normalizer $N_F(K_\ell)$ is of finite index in F .*

It is known by [CS89] that for a locally finite infinite tree \mathbb{T} , for any random walk step distribution μ on $\text{Aut}(\mathbb{T})$ such that $\text{supp}\mu$ is not contained in an amenable subgroup of $\text{Aut}(\mathbb{T})$, the sequence $\omega_n.v$, where (ω_n) is a μ -random walk, converges to an end with probability 1. This convergence result uses a martingale argument originally due to Furstenberg. Along this line of reasoning, we have:

Lemma 9.2. *Let μ be a non-degenerate step distribution on F and $(\omega_n)_{n=0}^\infty$ be a μ -random walk. Suppose $H_0 \in \text{Tree}_F$ has a normalizer $N_F(H_0)$ of finite index in F and that the Schreier graph $H_0 \setminus F$ has infinitely many ends. Then the coset random walk $(H_0 \omega_n)_{n=0}^\infty$ converges to an end in $\partial(H_0 \setminus F)$ with probability 1.*

Proof. Let τ_n be the n -th return time of the random walk $(\omega_n)_{n=0}^\infty$ to the finite index subgroup $N = N_F(H_0)$. Denote by μ_τ the distribution of ω_{τ_1} . Note that $\text{supp}\mu_\tau$ generates N . Denote by o the identity coset in $H_0 \setminus F$, we have that $\omega_{\tau_n}.o = H_0 \omega_{\tau_n}$. Apply the convergence theorem [CS89] to the μ_τ -random walk $(\omega_{\tau_n})_{n=0}^\infty$ on $N < \text{Aut}(H_0 \setminus F)$, we have that with probability 1, $\omega_{\tau_n}.o$ converges to an end. On this full measure set of $\omega \in F^\mathbb{N}$, denote by $\lambda_\omega^{(\tau)}$ the end where $H_0 \omega_{\tau_n}$ converges to.

Take an infinite reduced word $\xi = x_1 x_2 \dots \in \partial F$ such that for any $\gamma \in F$, in the Schreier graph $H_0 \setminus F$, the sequence $(H_0 \gamma x_1 \dots x_n)$ converges to an end in $\partial(H_0 \setminus F)$. Denote the end as $H_0 \gamma \xi$. Such infinite words exist: since simple random walk on $H_0 \setminus F$ is transient and converges to an end starting from any vertex, we have that ν_0 -a.e. $\xi \in \partial F$ has the property required, where ν_0 is the harmonic measure on ∂F of simple random walk on F . Here transience follows from the assumption that the quasi-transitive graph $H_0 \setminus F$ is not quasi-isometric to \mathbb{Z} . Fix a choice of such ξ . Let $\nu_\gamma(n)$ be the distribution of $H_0 \gamma \omega_n \xi$ on $\partial(H_0 \setminus F)$. By compactness and a standard diagonal argument, there is a subsequence (n_i) such that $\nu_\gamma(n_i)$ converges in the weak* topology for all $\gamma \in F$. Denote by ν_γ the limit of $\nu_\gamma(n_i)$. The limits satisfy the harmonicity condition $\sum_{s \in F} \nu_\gamma s \mu(s) = \nu_\gamma$, $\gamma \in F$. By the martingale convergence theorem, along the μ -random walk trajectory, ν_{ω_n} converges to a limit measure ν_ω in the weak* topology for \mathbb{P}_μ -a.e. ω .

Next we show that for \mathbb{P}_μ -a.e. ω , the limit ν_ω is the point mass at the end $\lambda_\omega^{(\tau)}$. It suffices to show the subsequence $(\nu_{\omega_{\tau_n}})$ converges weakly to $\delta_{\lambda_\omega^{(\tau)}}$. Since N normalizes H_0 , we have for $g \in N$,

$$\nu_g = \lim_{i \rightarrow \infty} \sum_{s \in F} \delta_{\{H_0 g s \xi\}} \mu^{(n_i)}(s) = \lim_{i \rightarrow \infty} \sum_{s \in F} \delta_{\{g H_0 s \xi\}} \mu^{(n_i)}(s) = g.\nu_e. \quad (24)$$

The μ -harmonicity condition satisfied by $\{\nu_g\}$ then implies ν_e is μ_τ stationary. Then the measures ν_g , $g \in N$, are non-atomic, see e.g., the first paragraph in the proof of [CS89, Theorem]. Recall that for \mathbb{P}_μ -a.e. ω , along the subsequence (ω_{τ_n}) , $\omega_{\tau_n}.o$ converges to the end $\lambda_\omega^{(\tau)}$. By [CS89, Lemma 2.2], $\omega_{\tau_n}.x$ converges to $\lambda_\omega^{(\tau)}$ for all $x \in H_0 \setminus F \cup \partial(H_0 \setminus F)$ except possibly one point. Therefore $\omega_{\tau_n}.\nu_e$ converges to the point mass at $\lambda_\omega^{(\tau)}$. We conclude that $\nu_\omega = \delta_{\lambda_\omega^{(\tau)}}$.

Finally, suppose that the sequence $(H_0\omega_n)_{n=0}^\infty$ has other accumulation points than $\lambda_\omega^{(\tau)}$: there is a subsequence $(H_0\omega_{m_j})$ that converges to a different end λ' . Since N is finite index in F , passing to a further subsequence if necessary, we may assume that there is a $\gamma \in F$ such that $\omega_{m_j} \in N\gamma$ for all j . Then $\omega_{m_j}\gamma^{-1}.o = H_0\omega_{m_j}\gamma^{-1}$ converges to the end λ' as well. The same calculation as in (24) shows that $\nu_{\omega_{m_j}} = (\omega_{m_j}\gamma^{-1}).\nu_\gamma$. Again by [CS89, Lemma 2.2], $(\nu_{\omega_{m_j}})$ converges weakly to $\delta_{\lambda'}$. However we have shown that ν_{ω_n} converges weakly to $\delta_{\lambda_\omega^{(\tau)}}$. Therefore $\lambda' = \lambda_\omega^{(\tau)}$. We conclude that there is no other accumulation points and $(H_0\omega_n)_{n=0}^\infty$ converges to $\lambda_\omega^{(\tau)}$ for \mathbb{P}_μ -a.e. ω . \square

Under the finite entropy and finite log-moment assumption on μ , we note the following strengthening of the convergence statement in Lemma 9.2. This property will be useful in the lifting argument in the next subsection. Given a point $x \in H_0 \setminus F$ and an element $g \in F$ (viewed as a reduced word), for $0 \leq \ell \leq |g|$, denote by g_ℓ the length ℓ prefix of g , and $[x; g]$ the set $\{x, xg_1, xg_2, \dots, xg\}$.

Lemma 9.3. *In the setting of Lemma 9.2, assume also μ has finite entropy and finite log-moment. Then we have for any finite set B in $H_0 \setminus F$,*

$$\mathbb{P}_\mu(B \cap [H_0\omega_n; \omega_n^{-1}\omega_{n+1}] \neq \emptyset \text{ infinitely often}) = 0.$$

Proof. It suffices to prove it for the case where B consists of a single point, $B = \{x_0\}$. Let h be the entropy of the Poisson bundle over conjugates of H_0 . By the convergence lemma 9.2, we know that the tail σ -field of $(H_0\omega_n)_{n=0}^\infty$ is nontrivial, and thus the Furstenberg entropy of the Poisson bundle is positive: $h = h_\mu(Z, \lambda) > 0$. Take any $0 < \epsilon < h/3$. Consider the subset of vertices

$$A_n = \left\{ x \in H_0 \setminus F : -\frac{1}{n} \log P_{\mu, H_0}^n(o, x) \geq h - \epsilon \right\},$$

and the event

$$C_n = \{ \omega : \log |\omega_n^{-1}\omega_{n+1}| \leq n\epsilon, H_0\omega_n \in A_n, x_0 \in [H_0\omega_n; \omega_n^{-1}\omega_{n+1}] \}.$$

Given an element $g \in F$, for $\ell \leq |g|$, denote by g_ℓ the length ℓ prefix of g . Then we have:

$$\begin{aligned} \mathbb{P}_\mu(C_n) &\leq \sum_{g: |g| \leq e^{n\epsilon}} \mu(g) \sum_{\ell=0}^{|g|} \mathbb{P}_\mu(H_0\omega_n \in A_n, H_0\omega_n g_\ell = x_0) \\ &= \sum_{g: |g| \leq e^{n\epsilon}} \mu(g) \sum_{\ell=0}^{|g|} \mathbb{P}_\mu(H_0\omega_n = x_0 g_\ell^{-1}) \mathbf{1}_{A_n}(x_0 g_\ell^{-1}) \\ &\leq \sum_{g: |g| \leq e^{n\epsilon}} \mu(g) \sum_{\ell=0}^{|g|} e^{-n(h-\epsilon)} \text{ (by definition of the set } A_n) \\ &\leq ce^{n\epsilon} e^{-n(h-\epsilon)} \leq ce^{-nh/3}. \end{aligned}$$

By the Borel-Cantelli lemma, we have that $\mathbb{P}_\mu(\omega \in C_n \text{ i.o.}) = 0$. By the Shannon theorem B.6, $\mathbb{P}_\mu(H_0\omega_n \notin A_n \text{ i.o.}) = 0$, and recall that finite log-moment implies that $\mathbb{P}_\mu(\log|\omega_n^{-1}\omega_{n+1}| > n\epsilon \text{ i.o.}) = 0$. The statement follows from taking a union of these three events. \square

9.2 Identification over the covering construction

We describe the end-compactification bundle and identify it with the Poisson bundle.

9.2.1 A bundle of end-compactifications

Denote by M the end compactification bundle over Tree_F : the fiber over $H \in \text{Tree}_F$ is the space of ends $\partial(H \setminus F)$. The group F acts on M as follows. For $(H, \zeta) \in M$, let $\xi \in \partial F$ be such that the sequence $H\xi_n$ converges to ζ on $H \setminus F$, where ξ_n is the length n prefix of ξ . Then F acts on M by $\gamma \cdot (H, \zeta) = (H^\gamma, \zeta')$, where ζ' is the end in $\partial(H^\gamma \setminus F)$ that $(H^\gamma \gamma \xi_n)_{n=1}^\infty$ converges to.

Fact 9.4. *The F -action on M described above is well-defined.*

Proof. Suppose ξ, ξ' are two infinite reduced words such that $H\xi_n$ and $H\xi'_n$ converge to the same end $\zeta \in \partial(H \setminus F)$. Then on the tree-like Schreier graph $H \setminus F$, the Gromov product $(H\xi_n | H\xi'_n)_H \xrightarrow{n \rightarrow \infty} \infty$. On the graph $H^\gamma \setminus F$, which is related to $H \setminus F$ by rerooting, we have

$$(H^\gamma \gamma \xi_n | H^\gamma \gamma \xi'_n)_{H^\gamma} \geq (H\xi_n | H\xi'_n)_H - |\gamma|_F.$$

Thus $H^\gamma \gamma \xi_n$ and $H^\gamma \gamma \xi'_n$ converges to the same end in $\partial(H^\gamma \setminus F)$. \square

Throughout the rest of this subsection, let μ be a nondegenerate step distribution on F of finite entropy and finite log-moment. Let (ω_n) be a μ -random walk on F . Denote by ν the hitting distribution on ∂F of the μ -random walk.

Let $H_0 \in \text{Tree}_F$ be as in Lemma 9.2 that the Schreier graph $H_0 \setminus F$ is a quasi-transitive tree-like graph with infinitely many ends. Denote by

$$\text{Tree}_F^{H_0} = \{H \in \text{Tree}_F : H < H_0^\gamma \text{ for some } \gamma \in F\}, \quad (25)$$

that is, subgroups H such that, up to rerooting, the Schreier graph $H \setminus F$ is a tree-like graph that covers $H_0 \setminus F$.

The property stated in Lemma 9.3 naturally lifts to covering graphs. Thus we have the following convergence to ends lemma.

Lemma 9.5. *Let H_0 be as in Lemma 9.2. For any $H \in \text{Tree}_F^{H_0}$ and any finite set K in $H \setminus F$, \mathbb{P}_μ -almost surely $K \cap [H\omega_n; \omega_n^{-1}\omega_{n+1}] \neq \emptyset$ for only finitely many n . In particular, $H\omega_n$ converges to an end in $\partial(H \setminus F)$ when $n \rightarrow \infty$.*

Lemma 9.5 implies that there is a measurable F -invariant ν -conull subset $A \subseteq \partial F$ such that the map $\zeta_H : A \rightarrow \partial(H \setminus F)$ is defined for all $H \in \text{Tree}_F^{H_0}$: if ω_n converges to $\xi \in A$, then $L\omega_n$ converges to $\zeta_H(\xi)$.

Suppose ρ is an F -invariant measure supported on $\text{Tree}_F^{H_0}$. We equip the bundle $M \rightarrow \text{Tree}_F$ with a measure $\bar{\lambda}$ such that the disintegration of $\bar{\lambda}$ is

$$\bar{\lambda} = \int_{\text{Tree}_F^{H_0}} (\zeta_H)_* \nu d\rho(H).$$

That is, in M the fiber $\partial(H \setminus F)$ over H is equipped with the measure $(\zeta_H)_* \nu$, which is the hitting distribution of the random walk $(H\omega_n)_{n=0}^\infty$ on the ends space $\partial(H \setminus F)$.

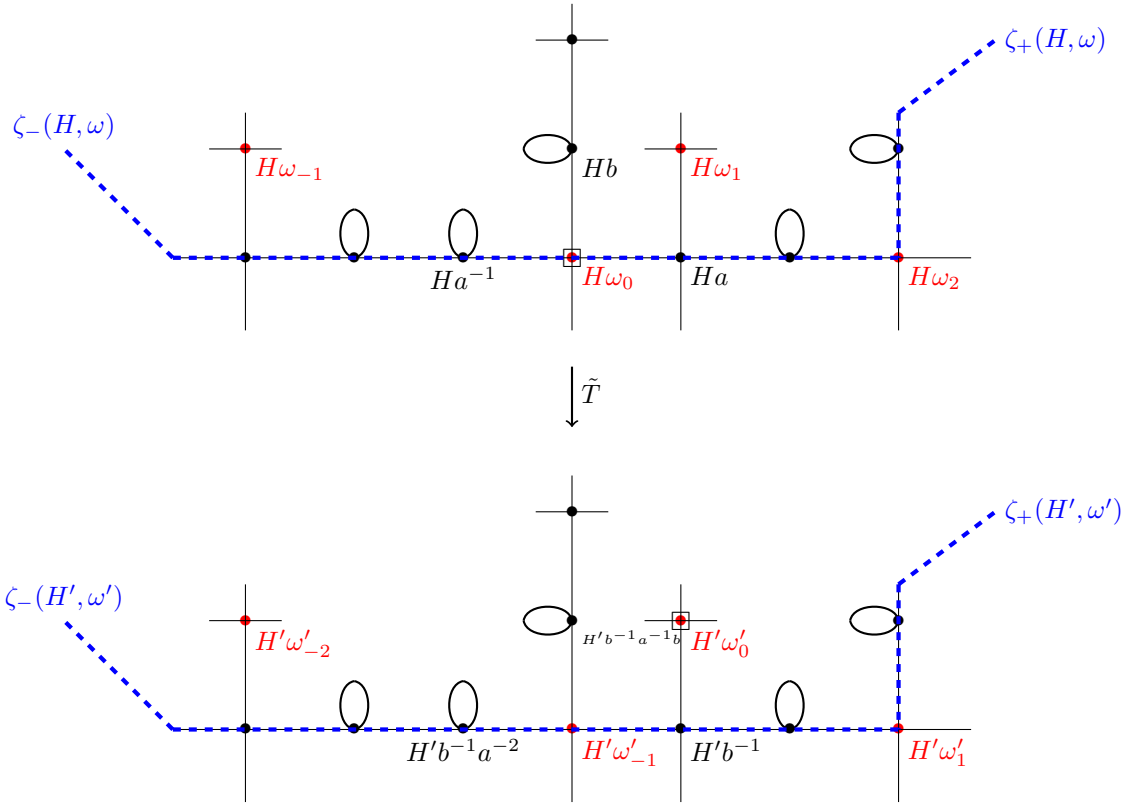


Figure 3: A trajectory $H\omega$ in $H \setminus F$, and its image under the skew transformation $(H', \omega') := \tilde{T}(H, \omega) = (H\omega_1^{-1}, (\omega_1^{-1}\omega_{n+1})_n)$. The root of each Schreier graph is marked by a square. The compatibility condition (i) of Theorem 7.3 is satisfied as $H\omega_1$ is on the strip of (H, ω) if and only if the root $H' = H\omega_1^{-1}$ is on the strip of (H', ω') .

9.2.2 Identification of bundles

Notation 9.6 (Shadows). Let $H \in \text{Tree}_F$, choose the identity coset H as the base point o in $H \setminus F$. For a vertex $v \in H \setminus F$, denote by $\text{Shd}(v)$ the set of geodesic rays (finite or infinite) based at o that pass through v . We view $\text{Shd}(v)$ as a subset of $(H \setminus F) \cup \partial(H \setminus F)$. Denote by

$$\mathcal{U}_H(v) := \text{Shd}(v) \cap \partial(H \setminus F).$$

We note the following lower bound on the hitting probabilities of shadows, which will be used to apply the strip criterion in Proposition 9.8.

Lemma 9.7. *For each $n \in \mathbb{N}$, there exists a constant $c = c(H_0, n, \mu) > 0$ such that for any $H \in \text{Tree}_F^{H_0}$, the hitting distribution satisfies*

$$(\zeta_H)_* \nu(\mathcal{U}_H(u)) \geq c$$

for any $u \in H \setminus F$ within distance n to the root $o = H$.

Proof. First consider the Schreier graph $H_0 \setminus F$. Let B be a connected finite subset of $H_0 \setminus F$, containing the root $o = H_0$. We claim that for any $v \in H_0 \setminus F$ such that $v \notin B$,

$$p_v(B) := \mathbb{P}_\mu(B \cap [v\omega_t; \omega_t^{-1}\omega_{t+1}] = \emptyset \text{ for all } t \in \mathbb{N}) > 0.$$

Suppose the claim is not true for some v . The connected component of v in $H_0 \setminus F - B$ is $\text{Shd}(v_0)$ of some vertex v_0 at distance 1 to B . Then it follows from non-degeneracy of μ that for all $x \in \text{Shd}(v_0)$, with probability 1, $[x\omega_t; \omega_t^{-1}\omega_{t+1}]$ intersects B for some $t \in \mathbb{N}$. Since the hitting distribution $(\zeta_{H_0})_* \nu$ charges the cylinder set $\mathcal{U}(v_0)$ with positive probability, this contradicts with Lemma 9.3.

Let $H \in \text{Tree}_F^{H_0}$. Recall (25) that $H < H_0^\gamma$ for some word γ in F representative of one of the finitely many cosets of $F/N_H(F)$. Assume $|\gamma| \leq n$. On the Schreier graph $H \setminus F$, let u be a vertex within distance n to the root $o = H$, and choose $g \in F$ a representative such that $u = Hg$ and $|g| \leq n$. Then choose an element $g' \in F$ with $|g'| \leq 2n$ such that on $H \setminus F$, $Hgg' \in \text{Shd}(u)$; and on $H_0^\gamma \setminus F$, $|H_0^\gamma gg'| > n$. By non-degeneracy of μ , there is $m_0 \in \mathbb{N}$ such that $\mu^{(m_0)}$ charges every element in the ball of radius $3n$ around identity in F . Consider the event that in m_0 steps, the μ -random walk on F is at gg' , and after time m_0 , the induced trajectory on $H_0^\gamma \setminus F$ never sweeps cross the ball of radius n around H_0^γ . The covering property implies that the corresponding trajectory $(H\omega_t)$ never sweeps cross the ball of radius n around H after time m_0 , in particular, it stays in the shadow of u . As the n -ball centred at H_0^γ in $H_0^\gamma \setminus F$ is isometric the n -ball centred at $H_0\gamma^{-1}$ in $H_0 \setminus F$, it follows that

$$(\zeta_H)_* \nu(\mathcal{U}(u)) \geq \min_{B(e_F, 3n)} \mu^{(m_0)} \cdot \min \{p_v(B(H_0\gamma^{-1}, n)) : v \notin B(H_0\gamma^{-1}, n), |v| \leq 3n\}.$$

□

Since ρ is F -invariant, we are in the setting of Subsection 7.1, with $X = \text{Tree}_F^{H_0}$ and $x \mapsto L_x$ the identity map. We apply the strip criterion to identify the Poisson bundle with the end compactification bundle. Recall that as a measurable F -space, $(M, \bar{\lambda})$ fits into the sequence of F -factors

$$\left(\text{Tree}_F^{H_0} \times \partial F, \rho \times \nu \right) \xrightarrow{\zeta} (M, \bar{\lambda}) \rightarrow \left(\text{Tree}_F^{H_0}, \rho \right),$$

where the first map ζ sends (H, ξ) to $(H, \zeta_H(\xi))$; and the second map is the coordinate projection $(H, \zeta_H(\xi)) \mapsto H$.

Proposition 9.8. *Let H_0 be as in Lemma 9.2 and ρ an ergodic F -invariant probability measure on $\text{Tree}_F^{H_0}$. The Bowen-Poisson bundle over $(\text{Tree}_F^{H_0}, \rho)$ is F -measurably isomorphic to the end compactification bundle $(M, \bar{\lambda})$ defined above.*

Proof. We apply the strip criterion in Theorem 7.3. Consider the bilateral path space $(F^{\mathbb{Z}}, \tilde{\mathbb{P}}_\mu)$. Denote by $\zeta_+(H, \omega)$ the composition $\zeta_H \circ \text{bnd}_+$, which is the end of $H \setminus F$ that the random walk $H\omega_n$ converges to in the positive time direction $n \rightarrow \infty$. Similarly, denote by $\zeta_-(H, \omega)$ the composition $\zeta_H \circ \text{bnd}_-$ in the negative time direction.

Take the strip $S(H, \omega)$ to be the (unique) geodesic on the tree-like Schreier graph $H \setminus F$ connecting $\zeta_+(H, \omega)$ to $\zeta_-(H, \omega)$. Since the geodesic does not depend on the location of the root, see Figure 3, the choice of strips satisfies the compatibility condition (i) in Theorem 7.3.

We now verify the positivity condition (ii). Since the graph $H \setminus F$ is tree-like, we have that for $b_+ \in \mathcal{U}(Hs)$ and $b_- \in \mathcal{U}(Hs')$ where s, s' are two elements of F such that Hs, Hs' are two distinct vertices distance 1 from H . the geodesic connecting b_+ and b_- passes through the identity coset H . Therefore for $Hs \neq Hs'$, we have

$$\begin{aligned} (\rho \times \tilde{\mathbb{P}}_\mu)(H \in S(H, \omega)) &\geq \tilde{\mathbb{P}}_\mu(\zeta_+(H, \omega) \in \mathcal{U}(Hs), \zeta_-(H, \omega) \in \mathcal{U}(Hs')) \\ &\geq c(1, \ell, \mu)c(1, \ell, \check{\mu}) > 0, \end{aligned}$$

where the positive constants $c(H_0, 1, \mu), c(H_0, 1, \check{\mu})$ are provided by Lemma 9.7. We have verified condition (ii).

Since μ is assumed to have finite log-moment, we have that $\log |\omega_n|/n \rightarrow 0$ when $n \rightarrow \infty$ for \mathbb{P}_μ -a.e. ω . Since the strips are chosen to be geodesics, the intersection of a strip with any ball of radius r is bounded by $2r$. Condition (iii) is verified. The statement then follows from Theorem 7.3. \square

9.2.3 Another interpretation of the hitting distribution

In the ends compactification bundle M , the fiberwise measure $\bar{\lambda}^H = (\zeta_H)_* \nu$ is the hitting distribution of the random walk $(H\omega_n)$ on $\partial(H \setminus F)$. We have the diagram

$$\begin{array}{ccc} (F^{\mathbb{N}}, \mathbb{P}_\mu) & \xrightarrow{\text{bnd}} & (\partial F, \nu) \\ \downarrow & & \downarrow \zeta_H \\ ((H \setminus F)^{\mathbb{N}}, \bar{\mathbb{P}}_\mu) & \longrightarrow & (\partial(H \setminus F), \bar{\lambda}^H). \end{array}$$

In the diagram above, a point $\xi \in \partial F$ is viewed as an end where the random walk (ω_n) converges to. For later use in constructions for $SL(d, \mathbb{R})$, here we consider another way of interpreting the map $\zeta_H : (\partial F, \nu) \rightarrow (\partial(H \setminus F), \bar{\lambda}^H)$. A point in ∂F is represented uniquely as an infinite reducible word in the alphabet $\{a^{\pm 1}, b^{\pm 1}\}$. Denote by ξ_n the length n prefix of the word ξ . We view ξ_n as an element in F . Then the point $\xi \in \partial F$ induces a sequence of points $(H\xi_n)$, which form a nearest neighbor path on the Schreier graph $H \setminus F$.

Proposition 9.9. *In the setting of Lemma 9.5, for ν -a.e. $\xi \in \partial F$, the nearest neighbor sequence $(H\xi_n)$ converges to the end $\zeta_H(\xi) \in \partial(H \setminus F)$.*

Proof. For a μ -random walk trajectory $\omega = (\omega_n)$ on F that converges to a point $\xi \in \partial F$, we claim that for each $k \in \mathbb{N}$, there is a time n_k such that the prefix ξ_k is on the geodesic connecting ω_{n_k} to ω_{n_k+1} . Indeed, since ω converges to ξ , $n_k = \max\{n : \xi_k \text{ is not a prefix of } \omega_n\}$ is finite. Then ξ_k is a prefix of ω_{n_k+1} and the common prefix of ω_{n_k} and ω_{n_k+1} has length $< k$. It follows then ξ_k is on the geodesic path connecting ω_{n_k} and ω_{n_k+1} .

By Lemma 9.5, we have for any finite set K in $H \setminus F$, almost surely $K \cap [H\omega_n; \omega_n^{-1}\omega_{n+1}] \neq \emptyset$ for only finitely many n . Since ξ_k is on the geodesic connecting ω_{n_k} to ω_{n_k+1} , $H\xi_k \in [L_x\omega_{n_k}; \omega_{n_k}^{-1}\omega_{n_k+1}]$. It follows then for ν -a.e. ξ , the set $\{k \in \mathbb{N} : H\xi_k \in K\}$ is finite. Therefore $(H\xi_k)$ converges to the end in $\partial(H \setminus F)$. Moreover, since ξ_k is the length k prefix of ω_{n_k+1} , we have that the Gromov product of $H\xi_k$ and $H\omega_{n_k+1}$ goes to infinity as $k \rightarrow \infty$. Therefore the two sequences $(H\xi_k)$ and $(H\omega_{n_k+1})$ converge to the same end, which is $\zeta_H(\xi)$. \square

9.3 Approximations on end-compactification bundles of F

Let $H_0 \in \text{Tree}_F$ be as in Lemma 9.2 and μ be a non-degenerate step distribution on F with finite entropy and finite log-moment. By Proposition 9.8, the bundle M with fiber $\partial(L_x \setminus F)$ over $x \in \text{Tree}_F^{H_0}$ equipped with hitting distribution of the coset random walk, is the Poisson bundle over the same base $(\text{Tree}_F^{H_0}, \rho)$ with $x \mapsto L_x$ identity map. Next we show how the bundle M fits into the setting of Proposition 8.3, and obtain lower semi-continuity of entropy.

Proposition 9.10. *Let β be a probability measure in the measure class of the μ -harmonic measure ν on ∂F . Moreover suppose $\|d\beta/d\nu\|_\infty, \|d\nu/d\beta\|_\infty < \infty$. Then $\beta_x = (\zeta_x)_*\beta$ admits locally constant uniform approximations on $\text{Tree}_F^{H_0}$.*

Since the Schreier graph of $L_x \setminus F$ is tree-like, we have a natural sequence of partitions of $\partial(L_x \setminus F)$ given by cylinder sets that are shadows of vertices. Consider the sphere $S_x(n) = \{v \in L_x \setminus F : d_{L_x \setminus F}(o, v) = n\}$ of the Schreier graph $L_x \setminus F$. Since the graph is tree-like, we have that

$$\mathcal{P}_{x,n} = \{\mathcal{U}(v)\}_{v \in S_x(n)},$$

where the shadow $\mathcal{U}(v)$ is defined in Notation 9.6, forms a partition of $\partial(L_x \setminus F)$ by clopen subsets. This sequence of partitions satisfy:

- $\mathcal{P}_{x,n+1}$ is a refinement of $\mathcal{P}_{x,n}$,
- the Borel σ -field of $\partial(H \setminus F)$ is generated by the partitions $\bigvee_{n=0}^\infty \mathcal{P}_{x,n}$.

As shown in Lemma 9.5, for $x \in \text{Tree}_F^{H_0}$, we have a map $\zeta_x : \partial F \rightarrow \partial(L_x \setminus F)$ such that $(\zeta_x)_*\nu$ is the hitting distribution of the random walk $L_x\omega_n$ on the Schreier graph $L_x \setminus F$.

For locally constant approximations to $(\zeta_x)_*(g\nu)$ on such partitions, one option is to take the measure of $\mathcal{U}(v)$ to be the probability that the coset random walk $L_x g\omega_t$ is in $\text{Shd}(v)$ and up to time t , the random walk never exited the ball of radius r_t around L_x , for a suitable choice of the radius r_t . One can indeed verify the conditions to apply Proposition 8.3 for such a approximations. Instead of this natural choice, for the convenience of inducing to $SL(d, \mathbb{R})$ in the next subsections, we use the interpretation of the hitting distributions in Subsection 9.2.3, which leads naturally to Proposition 9.10.

Denote by ξ_t the length t prefix of an infinite word $\xi \in \partial F$. Then a point $\xi \in \partial F$ induces a sequence of points $(H\xi_t)_{t \in \mathbb{N}}$ on the Schreier graph $H \setminus F$. Apply Proposition 9.9 to $x \in \text{Tree}_F^{H_0}$, we

have that for ν -a.e. ξ , the sequence $L_x \xi_t$ converges to an end, denoted by $\zeta_x(\xi)$ in $\partial(L_x \setminus F)$, when $t \rightarrow \infty$.

Suppose β is a probability measure on ∂F in the measure class of the μ -harmonic measure ν . Write $\beta_x = (\zeta_x)_* \beta$. Define a measure $\phi_{x,\beta}^t$ on the partition $\mathcal{P}_{x,n}$ by setting

$$\phi_{x,\beta}^t(\mathcal{U}(v)) := \beta(\{\xi \in \partial F : L_x \xi_t \in \text{Shd}(v)\}). \quad (26)$$

For $t > n$, $\phi_{x,\beta}^t$ is a measure on $\mathcal{P}_{x,n}$. Moreover, $\phi_{x,\beta}^t$ depends only on the ball of radius t around L_x in the Schreier graphs $L_x \setminus F$, which by definition of the Chabauty topology implies that $x \mapsto (\mathcal{P}_{x,n}, \phi_{x,\beta}^t)$ is locally constant on $\text{Tree}_F^{H_0}$.

Lemma 9.11 (Verifies uniform approximation). *Suppose there is a constant $C_\beta > 0$ such that $1/C_\beta \leq d\beta/d\nu \leq C_\beta$. Then there is a function $\epsilon(t) \xrightarrow{t \rightarrow \infty} 0$, which only depends on n, H_0, μ , such that for all $x \in \text{Tree}_F^{H_0}$,*

$$\max_{A \in \mathcal{P}_{x,n}} \left| 1 - \frac{\beta_x(A)}{\phi_{x,\beta}^t(A)} \right| \leq C_\beta^2 \epsilon(t).$$

Proof. Let $v \in S_x(n)$. To ease notations, in this proof we write $\phi_{x,\beta}^t(v)$ in place of $\phi_{x,\beta}^t(\mathcal{U}(v))$, similarly for β_x . The tree-like structure of the Schreier graphs guarantees that the total variation distance between β and $\phi_{x,\beta}^t$, both restricted to $\mathcal{P}_{x,n}$, is no more than the measure, under β , that there is some $s \geq t$ such that the path $L_x \xi_s$ returns to the ball of radius n in $L_x \setminus F$. For $x \in \text{Tree}_F^{H_0}$, we have that $L_x < H_0^\gamma$ for some $\gamma \in F$, thus for graph distances, $d(H_0^\gamma, H_0^\gamma g) \leq d(L_x, L_x g)$ for any $g \in F$. Recall also H_0 has only finitely many conjugates in F . Thus by this covering property, we have

$$\begin{aligned} \frac{1}{2} \sum_{v \in S_x(n)} |\beta_x(v) - \phi_{x,\beta}^t(v)| &\leq \beta(\{\xi \in \partial F : \exists s \geq t, L_x \xi_s \in B_{L_x \setminus F}(n)\}) \\ &\leq \left\| \frac{d\beta}{d\nu} \right\|_\infty \nu\left(\left\{\xi \in \partial F : \exists s \geq t, \exists \gamma \in F, H_0^\gamma \xi_s \in B_{H_0^\gamma \setminus F}(n)\right\}\right) \\ &=: \left\| \frac{d\beta}{d\nu} \right\|_\infty \epsilon_0(t) =: \epsilon(t). \end{aligned}$$

The term $\epsilon_0(t)$ does not depend on x , and $\epsilon_0(t) \rightarrow 0$ as $t \rightarrow \infty$ by Proposition 9.9.

The hitting measure $\beta_x(v) = (\zeta_x)_* \nu(\mathcal{U}(v))$ is equal to the probability that the trajectory $(L_x \xi_s)_{s=1}^\infty$, eventually remains in $\text{Shd}(v)$. By Lemma 9.7, we have a lower bound $\beta_x(v) \geq c(n, H_0, \mu)$. It follows that

$$\max_{A \in \mathcal{P}_{x,n}} \left| 1 - \frac{\phi_{x,\beta}^t(A)}{\beta_x(A)} \right| \leq \frac{1}{\min_{v \in S_x(n)} \beta_x(v)} \sum_{v \in S_x(n)} |\beta_x(v) - \phi_{x,\beta}^t(v)| \leq \frac{2}{c(n, H_0, \mu)} \left\| \frac{d\beta}{d\nu} \right\|_\infty \epsilon_0(t).$$

□

Proof of Proposition 9.10. The approximations $\phi_{x,\beta}^t$ defined in (26) are locally constant and uniform by Lemma 9.11. □

9.4 Entropy realization for the free group

In this subsection we conclude the proof of Theorem 1.10. Let μ be an admissible probability measure on F with finite entropy and finite log-moment. Recall that we are in the standard setting of Section 5, with $X = \text{Tree}_F \subset \text{Sub}(F)$ an F -space. Take a path of ergodic IRS $(\rho_{\ell,p})_{p \in [0,1]}$ supported on X as in Bowen [Bow14, p.505], which is briefly described in the next paragraph.

For an integer $\ell \geq 2$, let $H_0 = K_\ell$ be the quasi-transitive subgroup of Example 9.1. Its Schreier graph is tree-like with infinitely many ends and H_0 has a finite index normalizer. Informally, an $\rho_{\ell,p}$ sample is obtained by taking a random covering of the Schreier graph $K_\ell \setminus F$ where each loop is “opened” independently according to a p Bernoulli distribution (or equivalently, the generating pair (g, s) of a loop is removed from the set of generators of K_ℓ). One can check directly from this description that for $p \in (0, 1)$, $F \curvearrowright (\text{Tree}_F, \rho_{\ell,p})$ is a weakly mixing extension of a finite transitive system. The properties we need in what follows are: the map $p \mapsto \rho_{\ell,p}$ is weak* continuous, $\rho_{\ell,0}$ is the uniform measure on conjugates of K_ℓ , and $\rho_{\ell,1}$ is the δ -mass on the trivial group $\{e\}$. These are shown in [Bow14, Section 4.3].

The situation fits into the setting of Section 6, we obtain a family of measured F -bundles $(Z_p, \lambda_{\ell,p})$ over IRS’s $(X, \rho_{\ell,p})$ standard over the same trivial μ -boundary (Y is a point here).

Proposition 9.12. *In the setting above, the map $p \mapsto h_\mu(Z_p, \lambda_{\ell,p})$ is continuous.*

Proof. By [Bow14, p.505], Assumption (C) is satisfied. The assumption (L) is automatically satisfied in the discrete setting. Lemma 6.4 shows that $p \mapsto h_\mu(Z_p, \rho_{\ell,p})$ is upper semi-continuous.

Now by Proposition 9.8, the Poisson bundle $(Z_p, \lambda_{\ell,p})$ is F -isomorphic to the end compactification bundle $(M, \bar{\lambda}_{\ell,p})$ whose fiber over x is the topological space $M_x = \partial(L_x \setminus F)$ with hitting distribution $\bar{\lambda}_p^x = (\zeta_x)_*(\nu_B)$. Then

$$h_\mu(Z_p, \lambda_{\ell,p}) = h_\mu(M, \bar{\lambda}_{\ell,p}).$$

Write $\alpha_{x,g} = (\zeta_x)_*(g\nu_B)$ for the hitting distribution of the coset random walk $(L_x g \omega_n)$ starting at $L_x g$. By stationarity of ν_B , we have that the Radon-Nikodym derivative $dg\nu_B/d\nu_B$ is bounded from above and below. It follows that Assumption (M) is satisfied and this fits into the setting of Section 8. By Lemma 8.1, we have

$$h_\mu(M, \bar{\lambda}_{\ell,p}) = h_\mu(M, \alpha_p).$$

Proposition 9.10 provides locally constant uniform approximations. Corollary 8.5 gives lower semi-continuity of $p \mapsto h_\mu(M, \alpha_p)$. Combine with the upper semi-continuity, the statement follows. \square

Proof of Theorem 1.10. With Proposition 9.12, the proof concludes in the same way as in [Bow14]: by the intermediate value theorem, each Furstenberg entropy value between $h_\mu(Z_0, \lambda_{\ell,0})$ and $h_\mu(Z_1, \lambda_{\ell,1})$ is attained. Now by [Bow14, Lemma 4.7], $\rho_{\ell,1}$ is the trivial subgroup, so $h_\mu(Z_1, \lambda_{\ell,1}) = h_\mu(B, \nu_B)$ is maximal.

Finally by [Bow14, Proof of Lemma 4.7], the sequence of measures $(\rho_{\ell,0})_{\ell \geq 2}$ converges in weak* topology towards a measure $\kappa = \sum_{i=1}^{\text{rank}(F)} \delta_{A_i}$, where A_i is the normal closure of the cyclic group $\langle a_i \rangle$. Since $A_i \setminus F$ is isomorphic to \mathbb{Z} , any coset random walk has trivial Poisson boundary. It follows that the Poisson bundle over (X, κ) has zero Furstenberg entropy. We conclude that

$$\limsup_{\ell \rightarrow \infty} h_\mu(Z_0, \lambda_{\ell,0}) = 0$$

by the upper semi-continuity Corollary 6.4. \square

10 Poisson bundles for $SL(d, \mathbb{R})$

In this section we complete the proof of Theorem 1.8. Throughout this section, let $G = SL(d, \mathbb{R})$ and μ be either an admissible measure or a Furstenberg discretization measure supported on a lattice. In both cases, the Poisson boundary of the μ -random walk can be identified as $(G/P, \nu_P)$, where ν_P is in the measure class of the unique K -invariant measure.

10.1 Stationary system induced from IRS of F

The set of simple roots of $G = SL(d, \mathbb{R})$ is $\Delta = \{e_1 - e_2, \dots, e_{d-1} - e_d\}$, which is naturally identified with the set $\{1, \dots, d-1\}$. Let $I \subseteq \Delta$ and list $\Delta - I = \{i_1, \dots, i_{\ell-1}\}$ in increasing order. Then associated with I is the partition $d = d_1 + \dots + d_\ell$, where $d_j = i_j - i_{j-1}$, $i_0 = 0$ and $i_\ell = d$. The minimal parabolic subgroup $P = P(n, \mathbb{R})$ is the subgroup of upper triangular matrices. The parabolic subgroup $Q = P_I$ is stabilizer of the standard flag $V_1 \subset V_2 \subset \dots \subset V_\ell$, where V_j is spanned by the i_j -first standard basis vectors. The Levi subgroup of P_I consists of block diagonal matrices

$$L_I = \left\{ \begin{bmatrix} M_1 & & & \\ & M_2 & & \\ & & \ddots & \\ & & & M_\ell \end{bmatrix}, M_j \in GL(d_j, \mathbb{R}), \det(M_1) \dots \det(M_\ell) = 1 \right\}. \quad (27)$$

Throughout this subsection we consider the situation where $I \subseteq \Delta$ is such that $SL(2, \mathbb{R})$ is a factor of L_I , i.e., one of blocks is 2×2 . Take such a block, that is, $k \in \{1, \dots, \ell\}$ with $d_k = 2$. Regard $SL(2, \mathbb{R})$ as a subgroup of L_I , embedded in M_k and all the other blocks are identities. Denote by

$$p_k : Q \rightarrow G_2 := \{M \in GL(2, \mathbb{R}), |\det(M)| = 1\}$$

the quotient map which is the composition of $Q = L_I \times V_I \rightarrow L_I$ and $L_I \rightarrow G_2$ which sends the 2×2 -block M_k to $\frac{1}{\sqrt{|\det M_k|}} M_k$. Note that G_2 is isomorphic to $SL(2, \mathbb{R}) \rtimes \mathbb{Z}/2\mathbb{Z}$.

By the structure of parabolic subgroups, we have $Q/P = \prod_{j=1}^{\ell} SL(d_j, \mathbb{R})/P(d_j, \mathbb{R})$, where $P(n, \mathbb{R})$ is the minimal parabolic subgroup of $SL(n, \mathbb{R})$. The unique $K \cap Q$ invariant measure on Q/P is $\bar{m}_{K \cap Q} = \prod_{j=1}^{\ell} \bar{m}_{SO(d_j)}$, where $\bar{m}_{SO(n)}$ denotes the unique $SO(n)$ -invariant probability measure on $SL(n, \mathbb{R})/P(n, \mathbb{R})$. Denote by

$$\bar{p}_k : (Q/P, \bar{m}_{K \cap Q}) \rightarrow (SL(2, \mathbb{R})/P(2, \mathbb{R}), \bar{m}_{SO(2)})$$

the projection to the k -th component in the product, induced by the projection p_k .

For clarity of later arguments, it is convenient to fix an embedding of $F = F_2$ as a lattice in $SL(2, \mathbb{R})$. Take

$$A = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, B = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}.$$

The group $\langle A, B \rangle$ is called the Sanov subgroup, it is a free group of rank 2. Denote by \mathbb{H} the upper half plane model of the 2-dimensional hyperbolic space, where $SL(2, \mathbb{R})$ acts by Mobius transforms. Take the ideal rectangle R_0 with vertices $-1, 0, 1, \infty$ on \mathbb{H} . It is the union of two adjacent ideal triangles with vertices $-1, 0, \infty$ and $0, 1, \infty$ in the Farey tessellation by ideal triangles. The orbit

of R_0 under $\langle A, B \rangle$ forms a tessellation of the hyperbolic plane. The dual graph of the tessellation is a tree, it can be identified as the standard Cayley graph of the free group $F = \langle A, B \rangle$.

We follow the classical method to code hyperbolic geodesics with the tessellation. Take the map $\psi : \mathbb{H} \rightarrow F$ by sending a point x in the tile γR_0 to γ . Given a base point $x_0 \in \mathbb{H}$ and an irrational point $z \in \partial\mathbb{H}$, for the geodesic from x_0 to z , record the sequence of tiles that it passes through: $(\gamma_0 R_0, \gamma_1 R_0, \gamma_2 R_0, \dots)$, where each $\gamma_n \in F$. Since the sequence (γ_n) comes from a geodesic, which in particular can not backtrack, we have that γ_n converges to an infinite reduced word $\gamma_\infty \in \partial F$ as $n \rightarrow \infty$. By basic properties of the Farey tessellation, see e.g., [Ser85], we have:

Lemma 10.1. *The map ψ extends continuously to*

$$\begin{aligned} \psi : \partial\mathbb{H} - \mathbb{Q} &\rightarrow \partial F \\ z &\mapsto \gamma_\infty, \end{aligned}$$

which is F -equivariant, injective on $\partial\mathbb{H} - \mathbb{Q}$.

We continue to use the same notations for sections and cocycles as in Subsection 3.2.1.

Take a fundamental domain of F in $SL(2, \mathbb{R})$ so that its image on \mathbb{H} is the ideal rectangle R_0 . Lift it to a fundamental domain Ω_0 of F in G_2 , which is a 2-cover of $SL(2, \mathbb{R})$. Let $\sigma : G_2 \times \Omega_0 \rightarrow F$ be the associated cocycle. Denote by $\beta : G \times G/Q \rightarrow Q$ the cocycle associated with a chosen measurable section $\tau : G/Q \rightarrow G$. Let $F \curvearrowright (X_0, \rho)$ be an ergodic measure preserving action of F . Consider the G -space

$$X = G/Q \times_\beta (\Omega_0 \times_\sigma X_0) \text{ equipped with measure } \eta_\rho^\mu = \nu_Q \times m_{\Omega_0} \times \rho, \quad (28)$$

where the Q action on $\Omega_0 \times_\sigma \text{Sub}(F)$ is through the quotient map $p_k : Q \rightarrow G_2$, ν_Q is the μ -stationary probability measure on G/Q , and m_{Ω_0} is the restriction of the Haar measure on G_2 on Ω_0 , normalized to be a probability measure. Note that (X, η_ρ^μ) is a relative-measure preserving extension of $(G/Q, \nu_Q)$. In the terminology of Subsection 5.3, (X, η_ρ^μ) is a standard system over the μ -boundary $(G/Q, \nu_Q)$.

Take the stabilizer map $X \rightarrow \text{Sub}(G)$, $x \mapsto \text{Stab}_G(x)$, then the pushforward of η_ρ^μ is a μ -stationary random subgroup (SRS). This SRS may be viewed as co-induced from $(\text{Sub}(F), \rho)$ in the specific way described above. Here as customary, we identify ρ and the IRS $\text{Stab}_* \rho$, where $\text{Stab} : X_0 \rightarrow \text{Sub}(F)$ maps a point to its stabilizer. Note that the operation is different from the canonical co-induction of IRSs in the setting of countable groups [KQ19].

Notation 10.2. Given an IRS ρ of F and a step distribution μ on G , denote by (Z, λ_ρ^μ) the Poisson bundle associated with the μ -random walk, over the standard system $(X, \eta_\rho^\mu) \xrightarrow{\pi} (G/Q, \nu_Q)$ as in (28), where $L_x = \text{Stab}_G(x)$.

The bundle depends on Q and the choice of rank one factor M_k , but the dependence is suppressed in the notation.

Since $Y = G/Q$ is a factor of X , the stabilizer assumption **(S)** of Subsection 5.3 is satisfied. By Fact 5.3, $S = \langle \text{supp} \mu \rangle$ acts ergodically on (Z, λ_ρ^μ) if S acts ergodically on (X, η_ρ^μ) .

10.2 Identification of Poisson bundles for $SL(d, \mathbb{R})$

Recall that we assume μ is a step distribution on $G = SL(d, \mathbb{R})$ such that $(G/P, \nu_P)$ is the Poisson boundary of the μ -random walk, and ν_P is in the quasi-invariant measure class of \bar{m}_K .

Denote by $\nu_P = \int_Y \nu_P^y d\nu_Q(y)$ the disintegration of the harmonic measure ν_P over the quotient map $G/P \rightarrow Y = G/Q$. Note that the support of ν_P^y is $\tau(y)Q/P$ and L_x acts on $\tau(y)Q/P$ where $y = \pi(x)$.

As in the setting of Lemma 9.2, let $H_0 \in \text{Tree}_F$ be a subgroup whose associated Schreier graph is a quasi-transitive tree-like graph with infinitely many ends. Let the subspace $\text{Tree}_F^{H_0} \subset \text{Sub}(F)$ be described as in (25). Take ρ an F -invariant measure supported on $\text{Tree}_F^{H_0}$.

Associated with the μ -random walk on G , take the Poisson bundle (Z, λ_ρ^μ) over the standard system (X, η_ρ^μ) defined in Notation 10.2. By Proposition 5.9, in the Poisson bundle (Z, λ_ρ^μ) over (X, η_ρ^μ) , the fiber over $x \in X$ is the space of ergodic components $L_x // (\tau(y)Q/P, \nu_P^y)$, where $y = \pi(x)$.

Our task in this subsection is to identify (Z, λ_ρ^μ) with a concrete model. Recall that ρ is supported on $\text{Tree}_F^{H_0}$. Denote by $M_F \rightarrow \text{Tree}_F^{H_0}$ the end-compactification F -bundle where the fiber over a subgroup $H \in \text{Tree}_F^{H_0}$ is the space of ends $\partial(H \setminus F)$, see Subsection 9.2. Retain the same notation as in (28), induce $F \curvearrowright M_F$ to a G -space

$$M = G/Q \times_\beta (\Omega_0 \times_\sigma M_F). \quad (29)$$

We will show that the space M , equipped with a suitable measure, is G -measurably isomorphic to the Poisson bundle (Z, λ_ρ^μ) , see Proposition 10.5. The identification will play a key role in the lower semi-continuity argument.

10.2.1 The case of K -invariant harmonic measure

We first consider the case where the step distribution μ is such that its Poisson boundary is $(G/P, \bar{m}_K)$, where \bar{m}_K is the K -invariant probability measure on G/P . To emphasize K -invariance, in this case we write (Z, λ_ρ^K) for the associated Poisson bundle over (X, η_ρ^K) , where (X, η_ρ^K) is defined in (28) with $\nu_P = \bar{m}_K$.

Regard F as the Sanov subgroup in $SL(2, \mathbb{R})$. It acts on the boundary $SL(2, \mathbb{R})/P(2, \mathbb{R})$. Given the IRS ρ of F , we take an F -bundle $(E_F, m_\rho^K) \rightarrow (\text{Sub}(F), \rho)$ where the fiber over $H \in \text{Sub}(F)$ is the ergodic decomposition $H // (SL(2, \mathbb{R})/P(2, \mathbb{R}), \bar{m}_{SO(2)})$. We now show that the Poisson bundle (Z, λ_ρ^K) can be seen as induced from the F -system (E_F, m_ρ^K) .

Lemma 10.3. *Let (Z, λ_ρ^K) be the Poisson bundle over (X, η_ρ) . There is a G -measurable isomorphism*

$$\Psi_0 : (Z, \lambda_\rho^K) \rightarrow (G/Q \times_\beta (\Omega_0 \times_\sigma E_F), \bar{m}_{G/Q} \times m_{\Omega_0} \times m_\rho^K).$$

Proof. By (5), we have that in the disintegration of \bar{m}_K over $G/P \rightarrow G/Q$, the fiber measure over a point y is $\tau(y) \cdot \bar{m}_{K \cap Q}$. Let $p_k : Q \rightarrow G_2$ and $\bar{p}_k : Q/P \rightarrow SL(2, \mathbb{R})/P(2, \mathbb{R})$ be the projection maps as in Subsection 10.1, where k is the index of the chosen 2×2 block.

Then in the Poisson bundle $(Z, \lambda_\rho^K) \rightarrow (X, \eta_\rho)$, the fiber over $x = (y, r, H)$ is the ergodic decomposition $L_x // (\tau(y)Q/P, \tau(y) \cdot \bar{m}_{K \cap Q})$. The stabilizer map $x \mapsto L_x$ is given explicitly by

$$\begin{aligned} \text{Stab}_G : G/Q \times_\beta (\Omega_0 \times_\sigma \text{Sub}(F)) &\rightarrow \text{Sub}(G) \\ (y, r, H) &\mapsto \tau(y) p_k^{-1} (r H r^{-1}) \tau(y)^{-1}. \end{aligned} \quad (30)$$

A point in Z can be recorded as (x, A) , where $x = (y, r, H) \in X$ and A is an L_x -invariant measurable subset in the coset $\tau(y)Q/P$. By the description of the subgroup L_x in (30), we have that $\tau(y)^{-1}A$ is a subset of Q/P which is invariant under $L_Q(r, H) := p_k^{-1} (r H r^{-1})$. Since the

subgroup $L_Q(r, H)$ of Q acts transitively on the components $SL(d_j, \mathbb{R})/P(d_j, \mathbb{R})$ for $j \neq k$, we have that $\tau(y)^{-1}A$ is of the form $A_k \times \prod_{j \neq k} SL(d_j, \mathbb{R})/P(d_j, \mathbb{R})$, where A_k is a rHr^{-1} -invariant subset in $SL(2, \mathbb{R})/P(2, \mathbb{R})$. Then $r^{-1}A_k$ is an H -invariant event in $SL(2, \mathbb{R})/P(2, \mathbb{R})$, thus in the fiber of E_F over H . To summarize, we have seen an isomorphism $\Psi_0 : Z \rightarrow G/Q \times_{\beta} (\Omega_0 \times_{\sigma} E_F)$, $((y, r, H), A) \mapsto (y, r, (H, r^{-1}\bar{p}_k(\tau(y)^{-1}A)))$. It follows from K -invariance that $(\Psi_0)_*\lambda_{\rho}^K = \bar{m}_{G/Q} \times m_{\Omega_0} \times m_{\rho}^K$. \square

Lemma 10.3 shows that the identification problem for (Z, λ_{ρ}^K) is reduced to (E_F, m_{ρ}^K) . Relying on Furstenberg discretization, (E_F, m_{ρ}^K) can be identified as a F -Poisson bundle, and further an end compactification bundle as follows.

Recall that we have an F -equivariant map $\psi : \mathbb{H} \cup (\partial\mathbb{H} - \mathbb{Q}) \rightarrow F \cup \partial F$ from the Farey tessellation as in Lemma 10.1. View $SL(2, \mathbb{R})/P(2, \mathbb{R})$ as the ideal boundary of the hyperbolic plane \mathbb{H} , the $SO(2)$ -invariant measure $m_0 = \bar{m}_{SO(2)}$ corresponds to the Lebesgue measure. By [Fur73, Theorem 15.2], see also [Mar91, Prop VI. 4.1], for the lattice $F < SL(2, \mathbb{R})$, there is a nondegenerate step distribution κ_F on F with finite Shannon entropy and finite log-moment (with respect to word distance on F) such that $(\partial\mathbb{H}, m_0)$ is F -measurably isomorphic to the Poisson boundary of (F, κ_F) . We mention that one can also apply [CM07, Theorem 0.3] to the free group F acting on \mathbb{H} to see such a measure κ_F exists. Then by the description of Proposition 5.9, we have that (E_F, m_{ρ}^K) is F -measurably isomorphic to the Poisson bundle associated with κ_F -random walk over the same base $(\text{Sub}(F), \rho)$.

Associated with the κ_F -random walk $(\omega_n)_{n=0}^{\infty}$, as in Subsection 9.2, we have the end-compactification bundle $(M_F, \bar{\lambda}_{\rho}) \rightarrow (\text{Tree}_F^{H_0}, \rho)$, where in the disintegration of $\bar{\lambda}_{\rho}$, the fiber $\partial(H \setminus F)$ over $H \in \text{supp} \rho$ is endowed with the hitting distribution of the coset trajectory $(H\omega_n)_{n=0}^{\infty}$.

Lemma 10.4. *The bundle (E_F, m_{ρ}) is F -measurably isomorphic to the Poisson bundle, and also to the end-compactification bundle $(M_F, \bar{\lambda}_{\rho})$ associated with the κ_F -random walk over the same base $(\text{Sub}(F), \rho)$.*

Proof. Recall that by [Kai00] we have that almost surely a κ_F -random walk trajectory converges to an end in ∂F , and the Poisson boundary can be identified as ∂F equipped with the hitting measure. Given the map ψ associated with the Farey tessellation, we have that the hitting measure on ∂F is the same as the pushforward $\psi_*\bar{m}_{SO(2)}$.

Now for $H \in \text{supp} \rho$, consider H -ergodic components:

$$\begin{array}{ccc} (\partial\mathbb{H}, \bar{m}_{SO(2)}) & \xrightarrow{\psi} & (\partial F, \psi_*\bar{m}_{SO(2)}) \\ \downarrow & & \downarrow \\ H // (\partial\mathbb{H}, \bar{m}_{SO(2)}) & \xrightarrow{\bar{\psi}_H} & H // (\partial F, \psi_*\bar{m}_{SO(2)}), \end{array}$$

since ψ is an F -measurable isomorphism, it follows that $\bar{\psi}_H$ is a measurable isomorphism as well. Since $(\partial F, \psi_*\bar{m}_{SO(2)})$ is the Poisson boundary of (F, κ_F) , we have that the bundle over $(\text{Tree}_F^{H_0}, \rho)$ with fiber $H // (\partial F, \psi_*\bar{m}_{SO(2)})$ over H is the Poisson bundle over the same base associated with the κ_F -random walk. Applying Proposition 9.8 to the random walk κ_F , we have that the κ_F -Poisson bundle over $(\text{Tree}_F^{H_0}, \rho)$ is F -measurably isomorphic to the end-compactification bundle $(M_F, \bar{\lambda}_{\rho})$.

The isomorphism in the statement is implemented fiberwise by $\bar{\psi}_H : H // (\partial\mathbb{H}, m_0) \rightarrow \partial(H \setminus F)$, where for m_0 -a.e. $\xi \in \partial\mathbb{H}$, denote by $(\gamma_n)_{n=0}^\infty$ the sequence of tiles in the Farey tessellation that it passes through, in other words $(\gamma_n)_{n=0}^\infty = \psi(\xi)$; then by Proposition 9.9, $\psi_H(\xi)$ is the end that $(H\gamma_n)_{n=0}^\infty$ converges to. \square

Combine Lemma 10.3 and 10.4, we have that the Poisson bundle (Z, λ_ρ^K) is G -measurably isomorphic to the bundle M over the same base (X, η_ρ^K) , where the fiber over $x = (y, r, H)$ is the space of ends $\partial(H \setminus F)$, endowed with the hitting distribution of the κ_F -coset random walk $(H\omega_n)_{n=0}^\infty$.

10.2.2 The more general case

Next we consider the case where μ is such that $(G/P, \nu_P)$ is the Poisson boundary of (G, μ) and ν_P is in the measure class of \bar{m}_K . Let (X, η_ρ) be defined as in (28). Since ν_P is in the measure class of \bar{m}_K on G/P , it follows that the μ -Poisson bundle (Z, λ_ρ^μ) over (X, η_ρ) can be realized on the same space Z as in the K -invariant case and the measure λ_ρ^μ is in the measure class of λ_ρ^K . It remains to describe the corresponding measure on the end-compactification bundle M .

By Lemma 10.3 and 10.4, we have a G -measurable isomorphism, which is a composition of Ψ_0 and fiberwise maps $\bar{\psi}_H$:

$$\begin{aligned} \Psi : Z \rightarrow M &:= G/Q \times_\beta (\Omega_0 \times_\sigma M_F) \\ ((y, r, H), A) &\mapsto (y, r, (H, \bar{\psi}_H(r^{-1}\bar{p}_k(\tau(y)^{-1}A))) \end{aligned} \quad (31)$$

where \bar{p}_k is the projection $Q/P = \prod_{j=1}^\ell SL(d_j, \mathbb{R})/P(d_j, \mathbb{R}) \rightarrow SL(2, \mathbb{R})/P(2, \mathbb{R})$ to the k -th component, and $\bar{\psi}_H : H // (\partial\mathbb{H}, m_0) \rightarrow \partial(H \setminus F)$ is the map specified in the proof of Lemma 10.4.

Denote by $\nu_P = \int_{G/Q} \nu_P^y d\nu_Q(y)$ the disintegration of ν_P over G/Q . Note that ν_P^y is in the measure class of $\tau(y) \cdot \bar{m}_{K \cap Q}$.

Proposition 10.5. *Suppose μ is such that $(G/P, \nu_P)$ is the Poisson boundary of the μ -random walk and ν_P is in the quasi-invariant measure class of \bar{m}_K . The map Ψ defined as (31) is a G -measurable isomorphism between the μ -Poisson bundle (Z, λ_ρ^μ) over (X, η_ρ^μ) and the bundle (M, α_ρ) . In the disintegration $\alpha_\rho = \int_X \alpha_\rho^x d\eta_\rho(x)$ over $(M, \alpha_\rho) \rightarrow (X, \eta_\rho)$, we have*

$$\alpha_\rho^x = (\bar{\psi}_H)_* (r^{-1}(\bar{p}_k)_* (\tau(y)^{-1} \cdot \nu_P^y)) \text{ where } x = (y, r, H).$$

Proof. Lemma 10.3 and 10.4 imply that $\Psi : Z \rightarrow M$ is a G -measurable isomorphism. The expression for α_ρ^x is obtained from a change of variable given the explicit formula (31). \square

10.3 Lower semi-continuity for Poisson bundles of $SL(d, \mathbb{R})$

In this subsection we prove lower semi-continuity statement via the identification of the Poisson bundle (Z, λ_ρ^μ) and the bundle (M, α_ρ) . Assume μ is a step distribution on $G = SL(d, \mathbb{R})$ satisfying:

- (B) Bounded Radon-Nikodym derivatives. The Poisson boundary of the μ -random walk can be identified as $(G/P, \nu_P)$, where ν_P is in the quasi-invariant measure class on G/P and moreover, the Radon-Nikodym derivatives $d\nu_P/d\bar{m}_K$ and $d\bar{m}_K/d\nu_P$ are in $L^\infty(G/P, \bar{m}_K)$.

Example 10.6. *The main sources of examples of step distributions that satisfy (B) are from the works of Furstenberg [Fur63b, Fur63a]:*

- (i) μ is a left K -invariant admissible measure on G .
- (ii) μ is an admissible B_∞ measure on G .
- (iii) μ is a Furstenberg measure on a lattice $\Gamma < G$.

Under assumption **(B)**, we explain how the system (M, α_ρ) in Proposition 10.5 fits into the setting of Section 8. Write $Y' = G/Q \times_\beta \Omega_0$ and equip it with the measure $\nu = \nu_Q \times m_{\Omega_0}$. Note that (Y', ν) is a measure-preserving extension of $(G/Q, \nu_Q)$; and the purpose of taking this is to apply Proposition 9.10 in the fibers of $M \rightarrow Y'$.

Recall that $M = G/Q \times_\beta (\Omega_0 \times_\sigma M_F)$ is induced from the end-compactification bundle M_F of F over Tree_F . View M as a bundle over $X = G/Q \times_\beta (\Omega_0 \times_\sigma \text{Sub}(F))$, the fiber over $x = (y, r, H)$, is the space $M_x = \partial(H \setminus F)$. By Proposition 10.5, disintegrate the measure $g.\alpha_\rho = \Psi_*(g.\lambda_\rho)$ over $M \rightarrow X$, we have that the fiber measures in $g.\alpha_\rho = \int_X \alpha_{x,g} d(g.\eta_\rho)(x)$ are given by

$$\alpha_{x,g} = (\zeta_H)_* (r^{-1}(\bar{p}_k)_* (\tau(y)^{-1}(g\nu_P)^y)), \text{ where } x = (y, r, H).$$

Suppose $H_0 \in \text{Tree}_F$ is such that its normalizer $N_F(H_0)$ is of finite index in F and that the Schreier graph $H_0 \setminus F$ has infinitely many ends. Let Tree_{F, H_0} be as defined in (25). In the same way as in the free group case, when the Schreier graph $H \setminus F$ is tree-like, we take the sequence of finite partitions of $M_x = \partial(H \setminus F)$ to be $\mathcal{P}_{x,n} = \{\mathcal{U}(v)\}_{v \in S_H(n)}$, where the shadow $\mathcal{U}(v)$ in $\partial(H \setminus F)$ is defined in Notation 9.6. We have that the conditions **(M)**, **(C')** and **(P)** are satisfied by construction. Proposition 9.10 on the free group implies the following.

Proposition 10.7. *In the setting above, under **(B)**, for $y' = (y, r) \in Y' = G/Q \times_\beta \Omega_0$, the map*

$$S_{y'} := \{(y, r)\} \times \text{Tree}_{F, H_0} \rightarrow \mathbb{R}_{\geq 0}$$

$$x \mapsto H_{\alpha_{x,g} \| \alpha_{x,e}}(\mathcal{P}_{x,n})$$

is continuous with respect to the Chabauty topology on Tree_{F, H_0} . It follows that $x \mapsto D(\alpha_{x,g} \| \alpha_{x,e})$ is lower semi-continuous on $S_{y'}$.

Proof. Given $y' = (y, r) \in Y'$, denote by β_g the measure $r^{-1}(p_k)_* (\tau(y)^{-1} \cdot (g\nu_P)^y)$ on $\partial\mathbb{H} = SL(2, \mathbb{R})/P(2, \mathbb{R})$. The Radon-Nikodym derivative $d\beta_g/dm_0$ is bounded by

$$\left\| \frac{d\beta_g}{dm_0} \right\|_\infty \leq \left\| \frac{d\beta_g}{dr^{-1}.m_0} \right\|_\infty \left\| \frac{dr^{-1}.m_0}{dm_0} \right\|_\infty \leq \left\| \frac{d\tau(y)^{-1}(g\nu_P)^y}{d\bar{m}_{K \cap Q}} \right\|_\infty \left\| \frac{dr^{-1}.m_0}{dm_0} \right\|_\infty,$$

where $\tau(y)^{-1}(g\nu_P)^y$ and $\bar{m}_{K \cap Q}$ are measures on Q/P . Furthermore,

$$\left\| \frac{d\tau(y)^{-1}(g\nu_P)^y}{d\bar{m}_{K \cap Q}} \right\|_\infty = \left\| \frac{d(g\nu_P)^y}{d(\tau(y) \cdot \bar{m}_K)^y} \right\|_\infty \leq \left\| \frac{d(g\nu_P)}{d(\tau(y) \cdot \bar{m}_K)} \right\|_\infty \left\| \frac{d(\tau(y) \cdot \bar{m}_K)}{d(g\nu_P)} \right\|_\infty.$$

Similar calculation applies to $\|dm_0/d\beta_g\|_\infty$. Therefore Assumption **(B)** implies that $\|d\beta_g/dm_0\|_\infty$ and $\|dm_0/d\beta_g\|_\infty$ are bounded by a finite constant depending on $y, r, g, \|d\nu_P/d\bar{m}_K\|_\infty$ and $\|d\bar{m}_K/d\nu_P\|_\infty$.

Take a Furstenberg discretization random walk κ_F on F such that $(\partial\mathbb{H}, m_0)$ is F -isomorphic to the Poisson boundary of (F, κ_F) . Denote by ν the κ_F -harmonic measure on ∂F . We apply Proposition 9.10 to β_e and β_g , viewed as measures on ∂F through the F -measurable isomorphism of Proposition 10.4. This gives locally constant uniform approximations for $\alpha_{x,e}$ and $\alpha_{x,g}$ on

$S_{y'}$, where $\alpha_{x,g} = (\zeta_H)_* \beta_g$. Since the corresponding Radon-Nikodym derivatives are bounded, Proposition 8.3 applies to show that the map

$$\begin{aligned} (y, r) \times \text{Tree}_{F, H_0} &\rightarrow \mathbb{R} \\ x = (y, r, H) &\mapsto H_{\alpha_{x,g} \| \alpha_{x,e}}(\mathcal{P}_{x,n}) \end{aligned}$$

is continuous with respect to the Chabauty topology. \square

We deduce lower semi-continuity of the entropy of the Poisson bundle (Z, λ_ρ^μ) .

Corollary 10.8. *Let μ be a step distribution on G satisfying (B). The map $\rho \mapsto h_\mu(Z, \lambda_\rho^\mu)$ is lower semi-continuous, where ρ is in the space of ergodic F -invariant measures on Tree_{F, H_0} , equipped with the weak*-topology.*

Proof. The KL-divergence $D(\alpha_{x,g} \| \alpha_{x,e})$ is the increasing limit of $H_{\alpha_{x,g} \| \alpha_{x,e}}(\mathcal{P}_{x,n})$, where the latter is continuous on $S_{y'}$ by Proposition 10.7. Then by Lemma 8.2, we have that the map $p \mapsto h_\mu(M, \alpha_p)$ is lower semi-continuous. By the identification in Proposition 10.5, the Poisson bundle (Z, λ_ρ^μ) is G -measurable isomorphic to the induced end-compactification bundle (M, α_p) . Thus $h_\mu(Z, \lambda_\rho) = h_\mu(M, \alpha_p)$, the statement follows. \square

10.4 Entropy realization for $SL(d, \mathbb{R})$ and its lattices

Denote by Δ simple roots of $G = SL(d, \mathbb{R})$ and let $I \subseteq \Delta$. Recall that we list $\Delta - I = \{i_1, \dots, i_{\ell-1}\}$ in increasing order, and associated with I is the partition $d = d_1 + \dots + d_\ell$, where $d_j = i_j - i_{j-1}$, $i_0 = 0$ and $i_\ell = d$. Suppose there is a k is such that $d_k = 2$. Then $i_k - 1 \in I$ and it corresponds to a rank 1 factor of L_I . Write $I' = I - \{i_{k-1}\}$.

Proposition 10.9. *In the setting of Theorem 1.8, the interval $[h_\mu(G/P_I), h_\mu(G/P_{I'})]$ is contained in the Furstenberg entropy spectrum $\text{EntSp}(S, \mu)$.*

Proof. Let $Q = P_I$. Let ρ be an ergodic IRS of the free group F and (X, η_ρ) be the induced G -system in (28), then (X, η_ρ) is a relative-measure preserving extension of $(G/Q, \nu_Q)$. The G -action on (X, η_ρ) is ergodic by general properties of inducing (see e.g. [Zim78] or [Zim84, Prop 4.2.19]). For $\ell \in \mathbb{N}$, $p \in [0, 1]$, let $H_0 = K_\ell$ and $\rho = \rho_{\ell,p}$ be the F -ergodic IRS supported on Tree_{F, K_ℓ} constructed by Bowen as reviewed in Subsection 9.4. We now verify that a lattice Γ in G also acts ergodically on $(X, \eta_{\rho_{\ell,p}})$. First note that when $p = 0$ or 1 , G acts transitively on $\text{supp} \eta_{\rho_{\ell,p}}$, the system is of the form G/H for some noncompact closed subgroup H and the corresponding measure is in the quasi-invariant measure class. By Moore ergodicity (see e.g., [Zim84, Thm 2.2.6]), H acts ergodically on $(G/\Gamma, \bar{m})$, it follows that Γ acts ergodically on $(X, \eta_{\rho_{\ell,p}})$ for $p \in \{0, 1\}$. Next for $p \in (0, 1)$, we have by construction that $F \curvearrowright (\text{Tree}_F, \rho_{\ell,p})$ is a weakly mixing extension of a finite transitive system. Then as a G -system, $(X, \eta_{\rho_{\ell,p}})$ is a relative measure-preserving, relative weakly mixing extension of a homogeneous system of the form G/H_1 equipped with a quasi-invariant measure, where H_1 is not compact. It remains as a weakly mixing extension when viewed as Γ -systems (see e.g., [BF14, Prop 2.2 (iv)]), it follows that $\Gamma \curvearrowright (X, \eta_{\rho_{\ell,p}})$ is ergodic.

The assumption on μ guarantees that (B) is satisfied. Let $(Z, \lambda_{\ell,p})$ be the μ -Poisson bundle over the standard system $(X, \eta_{\rho_{\ell,p}})$ with $x \mapsto L_x = \text{Stab}_G(x)$. Apply Corollary 6.3 in the case where μ is in B_∞ class, and Corollary 6.4 when μ is supported on a lattice Γ , we have that $p \mapsto h_\mu(Z, \lambda_{\ell,p})$

is upper semi-continuous. Combined with the lower semi-continuity statement in Corollary 10.8, it follows that $p \mapsto h_\mu(Z, \lambda_{\ell,p})$ is continuous on $[0, 1]$. In the proof of Proposition 10.8, we have identified $(Z, \lambda_{\ell,p})$ with an induced end-compactification bundle $(M, \alpha_{\ell,p})$.

For $p = 1$, the IRS $\rho_{\ell,1} = \delta_{\{e\}}$ is supported on the trivial subgroup. The bundle M is simply $G/Q \times_\beta (\Omega_0 \times_\sigma \partial F)$. By definition of p_k , the Q -space $\Omega_0 \times_\sigma \partial F$ is isomorphic to $SL(2, \mathbb{R})/P(2, \mathbb{R})$ and so to $Q/P_{I'}$. It follows that the bundle M is G -isomorphic to $G/Q \times_\beta Q/P_{I'} = G/P_{I'}$. This shows $h_\mu(Z, \lambda_{\ell,1}) = h_\mu(G/P_{I'})$.

Next for $p = 0$, as in the proof of Theorem 1.10 at the end of Section 9, we have the weak* convergence $\rho_{\ell,0} \rightarrow \kappa = \frac{1}{2} \sum_{i=1}^2 \delta_{A_i}$, where A_i is the normal closure of the cyclic subgroup $\langle a_i \rangle$ of F . Corollary 6.3 on upper semi-continuity gives

$$\limsup_{\ell \rightarrow \infty} h_\mu(Z, \lambda_{\ell,0}) \leq h_\mu(Z, \lambda_\kappa) = \frac{1}{2} \sum_{i=1}^2 h_\mu(Z, \lambda_{\delta_{A_i}}).$$

We have that A_i is a normal subgroup of F with quotient $A_i \backslash F$ isomorphic to \mathbb{Z} . Since any random walk on \mathbb{Z} has trivial Poisson boundary, A_i acts ergodically on $(\partial \mathbb{H}, \bar{m}_{SO(2)})$, which is identified with Poisson boundary of κ_F -random walk on F . By the description in Proposition 10.5, we have that the Poisson bundle $(Z, \lambda_{\delta_{A_i}})$ has trivial fibers over the corresponding base system $(X, \eta_{\delta_{A_i}})$. In particular, $(Z, \lambda_{\delta_{A_i}})$ is a measure-preserving extension of $(G/P_I, \nu_{P_I})$. We conclude $\limsup_{\ell \rightarrow \infty} h_\mu(Z, \lambda_{\ell,0}) \leq h_\mu(G, P_I)$ and

$$\bigcup_{\ell \in \mathbb{N}} \{h_\mu(Z, \lambda_{\ell,p}) : p \in [0, 1]\} = (h_\mu(G, P_I), h_\mu(G, P_{I'}]).$$

□

Proof of Theorem 1.8. Combine Theorem 1.7 and Proposition 10.9. □

10.5 Interpretation in terms of Lyapunov exponents

Equip \mathbb{R}^n with the standard Euclidean norm $\|\cdot\|$ and $SL(d, \mathbb{R})$ with the operator norm $\|\cdot\|_{\text{op}}$. For a step distribution μ on $G = SL(d, \mathbb{R})$ satisfying the first moment condition $\int \log^+ \|g\|_{\text{op}} d\mu(g) < \infty$, by the Oseledets multiplicative ergodic theorem, there exists exponents $\lambda_1 > \lambda_2 > \dots > \lambda_k$ and for a.e. ω a flag $V^{\leq \lambda_k} \subset V^{\leq \lambda_{k-1}} \subset \dots \subset V^{\leq \lambda_1} = \mathbb{R}^d$.

When the harmonic measure ν_P is in the quasi-invariant measure class, it is well-known that the Furstenberg entropy $h_\mu(G/Q, \nu_Q)$ can be expressed in terms of the Lyapunov exponents, see [Led84] and references therein. Then in the setting of Theorem 1.8, the Furstenberg entropy spectrum is determined by the Lyapunov spectrum of the μ -random walk. We include the explicit formulae below for the convenience of the reader.

Proposition 10.10. *For the minimal parabolic subgroup P ,*

$$h_\mu(G/P, \nu_P) = \sum_{1 \leq i < j \leq d} \lambda_i - \lambda_j.$$

For a standard parabolic subgroup Q such that the corresponding flag in G/Q is of type (r_1, r_2, \dots, r_k) ,

$$h_\mu(G/Q, \nu_Q) = \sum_{\ell=1}^k \sum_{r_{\ell-1} < i \leq r_\ell, j > r_\ell} \lambda_i - \lambda_j = h_\mu(G/P, \nu_P) - \sum_{\ell=1}^k \sum_{r_{\ell-1} \leq i < j \leq r_\ell} \lambda_i - \lambda_j.$$

Sketch of proof. Given a parabolic subgroup $Q = P_I$, let $\bar{V} = \bar{V}_I$ be the corresponding opposite unipotent subgroup. denote by η the Haar measure on \bar{V} . Recall that the projection $G \rightarrow G/Q$ maps \bar{V} diffeomorphically onto a set of full \bar{m}_K -measure. Still denote by η the pushward of η to G/Q . Write $g = n_g \sigma_g k_g \in \bar{N}AK$ for the Iwasawa decomposition. Note that \bar{N} preserves the measure η on G/Q . By [NZ00, Proposition 1.15] we may change \bar{m}_K to the measure η in the Radon-Nikodym derivative and

$$\begin{aligned} th_\mu(G/Q, \bar{m}_K) &= - \int_G \int_{G/Q} \log \frac{dg^{-1} \cdot \bar{m}_K}{d\bar{m}_K}(x) d\bar{m}_K(x) d\mu^{(t)}(g) \\ &= - \int_G \int_{G/Q} \log \frac{dg^{-1} \cdot \eta}{d\eta}(x) d\bar{m}_K(x) d\mu^{(t)}(g) \\ &= - \int_G \int_{G/Q} \log \frac{d\sigma_g^{-1} n_g^{-1} \cdot \eta}{d\eta}(x) d\bar{m}_K(x) d\mu^{(t)}(g) \\ &= - \int_G \int_{G/Q} \log \frac{d\sigma_g^{-1} \eta}{d\eta}(x) d\bar{m}_K(x) d\mu^{(t)}(g). \end{aligned}$$

Denote a μ -random walk by $(\omega_t)_{t=0}^\infty$. By the almost sure convergence [Led84, Theorem 2.6] and the equivalence between Iwasawa and polar decompositions [GR81, Corollary 2.8], it is known that $\frac{1}{t} \log \sigma_{\omega_t}$ converges almost surely to the deterministic diagonal matrix Λ with Lyapunov exponents $(\lambda_1, \lambda_2, \dots, \lambda_d)$, repeated with multiplicity when the spectrum is not simple. So an appropriate i, j entry of the corresponding matrix in \bar{V} is essentially multiplied by $e^{t(\lambda_i - \lambda_j)}$. The formula follows. \square

A The Nevo-Zimmer operation on functions

In this section we review some arguments from [NZ02a] for the use in Section 3. Let G be a lsc group and H be a closed subgroup of G . Throughout this section we assume that an ergodic G -system (X, ν) has the following structure: there is an H -invariant measure λ on X and a measure ν_0 on G/H such that the map

$$\begin{aligned} \xi_0 : (G \times_H X_0, \nu_0 \times \lambda) &\rightarrow (X, \nu) \\ [g, x_0] &\mapsto g.x_0 \end{aligned}$$

is a G -factor map, where $X_0 = \text{supp } \lambda \subseteq X$. We identify X_0 as a subset of $G \times_H X_0$ via $x_0 \mapsto [e, x_0]$.

A.1 Conditional expectation

Let s be an element in H . Denote by \mathcal{F}^s the sub- σ -field of (X_0, λ) which consists of s -invariant measurable sets, that is, $\mathcal{F}^s = \{A \in \mathcal{B}(X_0) : s.A = A\}$. Denote by $\mathbb{E}_\lambda[\cdot | \mathcal{F}^s] : L^2(X_0, \lambda) \rightarrow L^2(X_0, \lambda)$ the conditional expectation on the space of s -invariant functions.

Given a function $f \in C(X)$, lift it to a function \tilde{f} on $G \times X_0$ by $\tilde{f}(g, x_0) = f(g.x_0)$. With a fixed element $g \in G$, we view $\tilde{f}(g, \cdot)$ as a function in $L^2(X_0, \lambda)$ and take its conditional expectation given \mathcal{F}^s .

Lemma A.1. *Suppose for an element $g \in G$, for any $f \in C(X)$,*

$$\mathbb{E}_\lambda \left[\tilde{f}(g, \cdot) | \mathcal{F}^s \right] = \mathbb{E}_\lambda \left[\tilde{f}(e, \cdot) | \mathcal{F}^s \right] \quad \lambda\text{-a.e.}, \quad (32)$$

then the measure λ (viewed as a measure on X) is invariant under g .

Proof. By the definition of conditional expectation, we have that

$$\int_{X_0} \mathbb{E}_\lambda \left[\tilde{f}(g, \cdot) | \mathcal{F}^s \right] d\lambda = \int_{X_0} \tilde{f}(g, x_0) d\lambda(x_0) = \int_X f(g.x_0) d\lambda(x_0) = \int_X f(x) dg.\lambda(x). \quad (33)$$

In the second equality we used the fact that $X_0 = \text{supp}(\lambda)$. Then (32) implies $\int_X f(x) dg.\lambda(x) = \int_X f(x) d\lambda(x)$, since it holds for all $f \in C(X)$, $g.\lambda = \lambda$. \square

A.2 Dynamics of the $\langle s \rangle$ -action

When the $\langle s \rangle$ -action on G/H has certain contracting properties, in [NZ02a] it is shown that the resulting conditional expectation given \mathcal{F}^s factors through ξ_0 . We now describe the conditions.

Recall that $\text{Int}(s).g = sgs^{-1}$.

Assumption A.2. *Suppose we have an element $s \in H$, subgroups U, V of G and a normal subgroup W of H satisfying*

(i) *the map*

$$\begin{aligned} p : U \times V &\rightarrow G/H \\ (u, v) &\mapsto uvH \end{aligned}$$

takes $U \times V$ homeomorphically to a ν_0 -conull set in G/H , and moreover $p_(m_U \times m_V)$ is in the same measure class as ν_0 .*

(ii) *$\text{Int}(s)$ acts trivially on U , that is, s commutes with elements in U ,*

(iii) *$\text{Int}(s^{-1})$ acts as a contracting automorphism on V ; $\text{Int}(s)$ acts as a contracting automorphism on W .*

Note that (iii) implies that $U \cap V = \{e\}$ and (i) implies that $U \cap H = V \cap H = \{e\}$. Also note that since s is an element in H , the measure λ is invariant under s .

Example A.3. *For $G = SL(3, \mathbb{R})$, these assumptions are satisfied for example for*

$$H = P = \begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & 0 & * \end{pmatrix}, U = \begin{pmatrix} 1 & 0 & 0 \\ * & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, V = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ * & * & 1 \end{pmatrix}, W = \begin{pmatrix} 1 & 0 & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{pmatrix},$$

$$\text{and } s = \begin{pmatrix} e^{-t_1} & 0 & 0 \\ 0 & e^{-t_1} & 0 \\ 0 & 0 & e^{-t_2} \end{pmatrix}, \quad \text{for } t_1 < t_2.$$

When (i) - (iii) are satisfied, consider the following continuous map

$$\begin{aligned}\xi : U \times V \times X_0 &\rightarrow X \\ (u, v, x_0) &\mapsto uv.x_0,\end{aligned}$$

which is related to the map ξ_0 by $\xi(u, v, x_0) = \xi_0([uv, x_0])$. Since $\nu = \nu_0 * \lambda$, by condition (i) we have that the image $\xi(U \times V \times X_0)$ is a ν -conull set in X . Denote by $\tilde{\mathcal{L}}(X)$ the sub- σ -field of the Borel σ -field of $U \times V \times X_0$ that consists of (classes modulo null sets of) lifts by ξ of measurable subsets of X . Denote by $\tilde{L}^\infty(X) = \tilde{L}^\infty(X, \nu)$ the subspace of $L^\infty(U \times V \times X_0, m_U \times m_V \times \lambda)$ that consists of functions that are measurable with respect to $\tilde{\mathcal{L}}(X)$. In other words, $\tilde{L}^\infty(X)$ consists of lift functions in $L^\infty(X)$.

The following actions of the groups U and $\langle s \rangle$ on $U \times V \times X_0$ preserve the product structure:

$$\begin{aligned}u_1.(u, v, x_0) &= (u_1u, v, x_0), \quad u_1 \in U, \\ s.(u, v, x_0) &= (sus^{-1}, vs^{-1}, s.x_0) = (u, \text{Int}(s).v, s.x_0).\end{aligned}$$

The map ξ is equivariant under U and $\langle s \rangle$. Note also that since U commutes with $\langle s \rangle$, U preserves the s -invariant σ -field \mathcal{F}^s .

For a continuous function $f \in C(X)$, consider the composition $f \circ \xi$, which is a continuous function on $U \times V \times X_0$. The proof of [NZ02a, Proposition 7.1] applies verbatim to the current setting and implies that the conditional expectation $\mathbb{E}_\lambda \left[\tilde{f}(u, \cdot) | \mathcal{F}^s \right]$ is the limit of Cesaro averages under the $\langle s \rangle$ -action.

Proposition A.4. *Under conditions (i), (ii) and (iii), we have*

$$\lim_{N \rightarrow \infty} \frac{1}{N+1} \sum_{n=0}^N s^n.(f \circ \xi)(u, v, x_0) = \mathbb{E}_\lambda \left(\tilde{f}(u, \cdot) | \mathcal{F}^s \right)(x_0),$$

where the converge is a.s. and in L^2 . Moreover, the function $(u, v, x_0) \mapsto \mathbb{E}_\lambda \left(\tilde{f}(u, \cdot) | \mathcal{F}^s \right)(x_0)$ is in $\tilde{L}^\infty(X)$.

This property allows to define a map

$$\begin{aligned}\mathcal{E}_s : C(X) &\rightarrow \tilde{L}^\infty(X) \\ \mathcal{E}_s f(u, v, x_0) &= \lim_{N \rightarrow \infty} \frac{1}{N+1} \sum_{n=0}^N s^n.(f \circ \xi)(u, v, x_0) = \mathbb{E}_\lambda \left(\tilde{f}(u, \cdot) | \mathcal{F}^s \right)(x_0).\end{aligned}$$

We will refer to the map \mathcal{E}_s as the Nevo-Zimmer operation with (s, U, V, W) on $C(X)$. The resulting function does not depend on the v -coordinate, that is, $\mathcal{E}_s f(u, v, x_0) = \mathcal{E}_s f(u, e, x_0)$. Note the following immediate properties:

Lemma A.5. *In the setting above, we have that for $f \in C(X)$,*

- (i) $u.\mathcal{E}_s(f) = \mathcal{E}_s(u.f)$ for $u \in U$. It follows that $\int_X f du.\lambda = \int_{X_0} \mathcal{E}_s f(u, e, x_0) d\lambda(x_0)$.
- (ii) If $u \in U$ is such that $u.X_0 = X_0$, then $\mathcal{E}_s f(u, e, x_0) = \mathcal{E}_s f(e, e, u.x_0)$.

(iii) Suppose $h \in H$ is an element such that either $\text{Int}(s)$ or $\text{Int}(s^{-1})$ contracts h to e , then $\mathcal{E}_s f(u, v, h.x_0) = \mathcal{E}_s f(u, e, x_0)$.

Proof. (i). Since s is assumed to commute with U , we have that for $u \in U$, $us^n.(f \circ \xi) = s^n.((u.f) \circ \xi)$. The statement then follows from Proposition A.4.

(ii). This follows from $(f \circ \xi)(u, e, x_0) = (f \circ \xi)(e, e, u.x_0)$.

(iii). For a given $u \in U$, the function $\mathcal{E}_s f(u, e, \cdot)$ viewed as an element in $L^2(X_0, \lambda)$, is invariant under $\langle s \rangle$ in the sense that $\mathcal{E}_s f(u, e, x_0) = \mathcal{E}_s f(u, e, s.x_0)$. Apply the generalized Mautner lemma ([Mar91, Lemma 3.2]) to the unitary representation of H on $L^2(X_0, \lambda)$, we have that $\mathcal{E}_s f(u, e, \cdot)$ is invariant under h as well. \square

A.3 Three cases

Suppose (s, U, V, W) are such that conditions (i) - (iii) are satisfied. When we apply the Nevo-Zimmer operation \mathcal{E}_s , one of the following scenarios occurs. The first case is:

(I) The subgroup U preserves the measure λ .

As in the [NZ02a, Section 8], in the negation of (I), there are two situations to consider, depending on whether $\langle s \rangle \curvearrowright (X_0, \lambda)$ is ergodic.

(III) There exists a function $f \in C(X)$ and $u \in U$ such that $\int_X f d\lambda \neq \int_X f d(u.\lambda)$; and for m_U -a.e. $u' \in U$, the function $x_0 \mapsto \mathcal{E}_s f(u', e, x_0)$ is λ -constant.

The remaining case is

(II2) = $\neg(\text{I} \vee \text{III})$. There exists a function $f \in C(X)$ such that $\int_X f d\lambda \neq \int_X f d(u.\lambda)$ for some $u \in U$. Moreover, for every such f , there is a m_U -positive set of $u \in U$ where the function $x_0 \mapsto \mathcal{E}_s f(u, e, x_0)$ is not λ -constant.

Case (III) is treated in [NZ02a, Proposition 9.2]. We briefly describe the argument to show existence of a nontrivial homogeneous factor in this case. Take a function $f \in C(X)$ as in the description of (III). Since $x_0 \mapsto \mathcal{E}_s f(u', e, x_0)$ is λ -constant, we have $\mathcal{E}_s f(u', e, x_0) = \int_{X_0} \mathcal{E}_s f(u', e, x_0) d\lambda(x_0) = \int f du'.\lambda$. Recall that we have the map $\xi : U \times V \times X_0 \rightarrow X$ where $\xi(u, v, x_0) = uv.x_0$; and the projection map $p : U \times V \times X_0 \rightarrow G/H$ given by $(u, v, x_0) \mapsto uvH$. Similar to the sub- σ -field $\tilde{\mathcal{L}}(X)$, denote by $\tilde{\mathcal{L}}(G/H)$ the sub- σ -field of the Borel σ -field of $U \times V \times X_0$ that consists of lifts by p of measurable subsets of G/H . Then the function $(u, v, x_0) \mapsto \mathcal{E}_s f(u, e, x_0) = \int f du.\lambda$ can be viewed as a non-constant function measurable with respect to the intersection of σ -fields $\tilde{\mathcal{L}}(X) \cap \tilde{\mathcal{L}}(G/H)$. Take the Mackey realization of $\tilde{\mathcal{L}}(X) \cap \tilde{\mathcal{L}}(G/H)$, we obtain a nontrivial common factor of (X, ν) and $(G/H, m_{G/H})$. Thus in this case we conclude that (X, ν) has a nontrivial homogeneous factor.

Case (II2) is treated by considerations of the Gauss map in [NZ02a]. By the second part of condition (iii), $\text{Int}(s)$ acts as a contracting automorphism on $W \triangleleft H$. As in the proof of [NZ02a, Proposition 4.1], apply the generalized Mautner lemma to the unitary representation of H on $L^2(X_0, \lambda)$, we have that s -fixed functions in $L^2(X_0, \lambda)$ are fixed by W as well. In $\mathcal{B}(X_0)$ take the W -invariant sub- σ -algebra:

$$\mathcal{B}_W(X_0) := \{A \in \mathcal{B}(X_0) : g.A = A \text{ for all } g \in W\}.$$

In particular, $\mathcal{F}^s \subseteq \mathcal{B}_W(X_0)$ by the Mautner lemma. Since W is assumed to be normal in H , we have that if $A \in \mathcal{B}_W(X_0)$, then for $h \in H$, $h.A \in \mathcal{B}_W(X_0)$ as well. Denote by X'_0 the Mackey realization

of the σ -algebra $\mathcal{B}_W(X_0)$, equipped with the measure λ' which corresponds to the restriction of λ to $\mathcal{B}_W(X_0)$. Then X'_0 is an H -space and W acts trivially on X'_0 . Note that by construction $X'_0 = \text{supp}\lambda'$.

As in [NZ02a, Section 6], take the space (X', ν') , which is the largest common G -factor of (X, ν) and $(G \times_H X'_0, \nu_0 \times \lambda')$

$$\begin{array}{ccc} G \times_H X_0 & \longrightarrow & X \\ \downarrow & & \downarrow \\ G \times_H X'_0 & \longrightarrow & X'. \end{array}$$

The existence of a function f as described in (II2) implies that the sub- σ -field \mathcal{F}^s is nontrivial, it follows that (X'_0, λ') and (X', ν') are nontrivial, see [NZ02a, Section 8, Case I]. Note the following property.

Lemma A.6. *In Case (II2), the subgroup U does not preserve the measure λ' , which is viewed as a measure on X' .*

Proof. Take a function $f \in C(X)$ such that $\int_X f d\lambda \neq \int_X f d(u.\lambda)$ for some $u \in U$. The function $x_0 \mapsto \mathcal{E}_s f(e, e, x_0)$, which is measurable with respect to \mathcal{F}^s , thus also $\mathcal{B}_W(X_0)$, can be viewed as the lift of a function $\phi \in L^2(X'_0, \lambda')$ to $L^2(X_0, \lambda)$. Suppose on the contrary $u.\lambda' = \lambda'$. Then in particular, $u.\text{supp}\lambda' = \text{supp}\lambda'$ which implies $u.X_0 = X_0$. Apply Lemma A.5, we have that $\mathcal{E}_s f(u, e, \cdot)$ is the lift of $u^{-1}.\phi$ to $L^2(X_0, \lambda)$. Then

$$\begin{aligned} \int_X f(x) d\lambda(x) &= \int_{X_0} \mathcal{E}_s f(e, e, x_0) d\lambda(x_0) = \int_{X'} \phi d\lambda' \\ &= \int_{X'} \phi(x') du.\lambda'(x') = \int_{X_0} \mathcal{E}_s f(u, e, x_0) d\lambda(x_0) = \int_X f(x) du.\lambda(x), \end{aligned}$$

which is a contradiction. \square

Since W acts trivially on X'_0 , we have that in the induced system $G \times_H X'_0$, a point stabilizer contains a conjugate of W . Passing to the factor X' , it follows that a stabilizer $\text{Stab}_G(x')$ contains a conjugate of W for $x' \in X'$.

B Mutual information and entropy formulae

Let μ be an admissible measure on an lcsc group G . Consider (B, ν_B) the Poisson boundary, and (X, η) a standard (G, μ) -system. Throughout this appendix, we consider an intermediate factor of their joining :

$$(X \times B, \eta \vee \nu_B) \xrightarrow{\psi} (Z, \lambda) \xrightarrow{\varrho} (X, \eta), \quad (34)$$

where the composition $\varrho \circ \psi$ is the natural coordinate projection $X \times B \rightarrow X$. The definition of the joining of two stationary systems was given in Section 5.1. See also [FG10] for a detailed treatment.

B.1 A consequence of Birkhoff's ergodic theorem

Let us write $\tilde{\psi}$ for the map $X \times G^{\mathbb{N}} \rightarrow (Z, \lambda)$ given by $\tilde{\psi}(x, \omega) = \psi(x, \text{bnd}(\omega))$.

Lemma B.1. *In the setting above, we have*

$$h_{\mu}(Z, \lambda) = \int_{X \times G^{\mathbb{N}}} \log \frac{d\omega_1 \lambda}{d\lambda} \left(\tilde{\psi}(x, \omega) \right) d(\eta \vee \mathbb{P}_{\mu})$$

and

$$h_{\mu}(X, \eta) = \int_{X \times G^{\mathbb{N}}} \log \frac{d\omega_1 \eta}{d\eta} (x) d(\eta \vee \mathbb{P}_{\mu}).$$

Proof. The Furstenberg entropy of (Z, λ) is defined as

$$\begin{aligned} h_{\mu}(Z, \lambda) &= \int_G \int_Z \log \frac{d\lambda}{dg^{-1} \cdot \lambda} (z) d\lambda(z) d\mu(g) = \int_G \int_Z \log \frac{dg\lambda}{d\lambda} (g.z) d\lambda(z) d\mu(g) \\ &= \int_G \int_{X \times G^{\mathbb{N}}} \log \frac{dg\lambda}{d\lambda} \left(g \cdot \tilde{\psi}(x, \tilde{\omega}) \right) d(\eta \vee \mathbb{P}_{\mu})(x, \tilde{\omega}) d\mu(g) \\ &= \int_G \int_{G^{\mathbb{N}}} \int_X \log \frac{dg\lambda}{d\lambda} \left(g \cdot \tilde{\psi}(x, \tilde{\omega}) \right) d\eta_{\tilde{\omega}}(x) d\mathbb{P}_{\mu}(\tilde{\omega}) d\mu(g). \end{aligned}$$

Now for $\omega = (\omega_1, \omega_2, \dots)$ in $G^{\mathbb{N}}$, set $T'\omega = (\omega_1^{-1}\omega_2, \omega_1^{-1}\omega_3, \dots)$. When ω has law \mathbb{P}_{μ} , then $(\omega_1, T'\omega)$ has the same law $\mu \times \mathbb{P}_{\mu}$ as $(g, \tilde{\omega})$. It follows that

$$\begin{aligned} h_{\mu}(Z, \lambda) &= \int_{G^{\mathbb{N}}} \int_X \log \frac{d\omega_1 \lambda}{d\lambda} \left(\omega_1 \cdot \tilde{\psi}(x, T'\omega) \right) d\eta_{T'\omega}(x) d\mathbb{P}_{\mu}(\omega) \\ &= \int_{G^{\mathbb{N}}} \int_X \log \frac{d\omega_1 \lambda}{d\lambda} \left(\tilde{\psi}(\omega_1 \cdot x, \omega') \right) d\eta_{\omega'}(\omega_1 \cdot x) d\mathbb{P}_{\mu}(\omega) \quad (\text{as } \eta_{\omega_1^{-1}\omega'} = \omega_1^{-1}\eta_{\omega'}) \\ &= \int_{G^{\mathbb{N}}} \int_X \log \frac{d\omega_1 \lambda}{d\lambda} \left(\tilde{\psi}(x, \omega) \right) d\eta_{\omega}(x) d\mathbb{P}_{\mu}(\omega). \end{aligned}$$

The second formula is proved similarly with $\varrho \circ \psi(x, \omega) = x$ in place of ψ . \square

Proposition B.2 (Consequence of Birkhoff's pointwise ergodic theorem). *Let (Z, λ) be an intermediate factor in (34). If (X, η) is an ergodic stationary system, then for $\eta \vee \mathbb{P}_{\mu}$ -a.e. $(x, \omega) \in X \times G^{\mathbb{N}}$, we have*

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log \frac{d\omega_n \lambda}{d\lambda} (\psi(x, \text{bnd}(\omega))) = h_{\mu}(Z, \lambda).$$

Proof. The telescoping argument is adapted from the proof of [Kai00, Lemma 4.2]. Given ω , let $g_k = \omega_{k-1}^{-1}\omega_k$ be its k -th increment, with $\omega_0 = id$. The Radon-Nikodym derivative can be rewritten as a product:

$$\begin{aligned} \frac{d\omega_n \lambda}{d\lambda} (\psi(x, \text{bnd}(\omega))) &= \prod_{k=1}^n \frac{d\omega_k \lambda}{d\omega_{k-1} \lambda} (\psi(x, \text{bnd}(\omega))) \\ &= \prod_{k=1}^n \frac{dg_k \lambda}{d\lambda} (\psi(\omega_{k-1}^{-1} \cdot x, \omega_{k-1}^{-1} \cdot \text{bnd}(\omega))) \quad (\text{by equivariance}) \\ &= \prod_{k=1}^n \frac{d(T^{k-1}(x, \omega))_1 \lambda}{d\lambda} (\tilde{\psi}(T^{k-1}(x, \omega))) \end{aligned}$$

where $T(x, \omega) = (\omega_1^{-1}x, (\omega_1^{-1}\omega_2, \omega_1^{-1}\omega_3, \dots))$ is the skew transformation defined in Section 5.1. Let $f(x, \omega) := \log \frac{d\omega_n \lambda}{d\lambda} (\tilde{\psi}(x, \omega))$. By Fact 5.1, we can apply Birkhoff's pointwise ergodic theorem to T and deduce almost surely,

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log \frac{d\omega_n \lambda}{d\lambda} (\psi(x, \text{bnd}(\omega))) = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n f(T^{k-1}(x, \omega)) = \int_{X \times G^{\mathbb{N}}} f(x, \omega) d\eta \vee \mathbb{P}_\mu = h_\mu(Z, \lambda).$$

□

B.2 KL-divergence and mutual information

We review the definition of KL-divergence, record a useful inequality, and recall several facts about mutual information due to Derriennic [Der85].

B.2.1 KL-divergence

Suppose α, β are probability measures on a measurable space (Ω, \mathcal{B}) and \mathcal{P}_n is a refining sequence of finite partitions whose union generates \mathcal{B} . Given a finite measurable partition \mathcal{P} of X , denote the relative entropy of \mathcal{P} with measure α with respect to β as

$$H_{\alpha \parallel \beta}(\mathcal{P}) := \sum_{A \in \mathcal{P}} \alpha(A) \log \frac{\alpha(A)}{\beta(A)}.$$

The Kullback-Leibler divergence of probability measures α and β can be defined by (see e.g., [Gra11, Corollary 7.3])

$$D(\alpha \parallel \beta) = \sup_n H_{\alpha \parallel \beta}(\mathcal{P}_n).$$

When α is absolutely continuous with respect to β , we have $D(\alpha \parallel \beta) = \int_M \log \left(\frac{d\alpha}{d\beta} \right) d\alpha$ by [Gra11, Lemma 7.4]. We refer to [Gra11, Chapter 7] for a detailed account.

In the proof of Proposition 8.3, we will use the following inequality, which is a consequence of the reverse Pinsker inequality in [Ver14, Theorem 7].

Lemma B.3. *Let $\alpha, \beta, \alpha', \beta'$ be probability measures on the space X and let \mathcal{P} be a finite partition of X . Denote by $C = C_{\alpha, \beta}(\mathcal{P}) = \max_{A \in \mathcal{P}} \{\alpha(A)/\beta(A)\}$. Then*

$$H_{\alpha \parallel \beta}(\mathcal{P}) - H_{\alpha' \parallel \beta'}(\mathcal{P}) \leq \log \left(\max_{A \in \mathcal{P}} \frac{\beta'(A)}{\beta(A)} \right) + 2C^{1/2} \max_{A \in \mathcal{P}} \left| 1 - \frac{\alpha'(A)}{\alpha(A)} \right|.$$

Proof. Write $M = M_{\alpha, \beta}$. The difference in relative entropies is

$$\begin{aligned} H_{\alpha \parallel \beta}(\mathcal{P}) - H_{\alpha' \parallel \beta'}(\mathcal{P}) &= \sum_{A \in \mathcal{P}} \alpha(A) \log \frac{\alpha(A)}{\beta(A)} - \alpha'(A) \log \frac{\alpha'(A)}{\beta'(A)} \\ &= \left(\sum_{A \in \mathcal{P}} \alpha(A) \log \frac{\alpha(A)}{\beta(A)} - \alpha'(A) \log \frac{\alpha'(A)}{\beta(A)} \right) + \sum_{A \in \mathcal{P}} \alpha'(A) \log \frac{\beta'(A)}{\beta(A)} \\ &= I + II. \end{aligned}$$

Rewrite I as

$$\begin{aligned} I &= \left(\sum_{A \in \mathcal{P}} \left(1 - \frac{\alpha'(A)}{\alpha(A)} \right) \alpha(A) \log \frac{\alpha(A)}{\beta(A)} \right) - H_{\alpha' \| \alpha}(\mathcal{P}) \\ &\leq \sum_{A \in \mathcal{P}} \left(1 - \frac{\alpha'(A)}{\alpha(A)} \right) \alpha(A) \log \frac{\alpha(A)}{\beta(A)}. \end{aligned}$$

Split the sum into two parts: A is in \mathcal{P}_+ (\mathcal{P}_- resp.) if $\alpha(A)/\beta(A) \geq 1$ (< 1 resp.). By the reverse Pinsker inequality we have

$$\sum_{A \in \mathcal{P}_+} \alpha(A) \log \frac{\alpha(A)}{\beta(A)} \leq \sqrt{C} d_{\text{TV}}(\alpha, \beta). \quad (35)$$

Note that [Ver14, Theorem 7] is stated as with $D(\alpha \| \beta)$ on the left-hand side in the inequality. For the convenience of the reader, we briefly repeat its proof here to show (35). For $1 \leq z \leq M$, $\frac{1-M^{-1}}{\log M} z \log z \leq z - 1$, summing over \mathcal{P}_+ , we have

$$\frac{1-M^{-1}}{\log M} \sum_{A \in \mathcal{P}_+} \beta(A) \frac{\alpha(A)}{\beta(A)} \log \frac{\alpha(A)}{\beta(A)} \leq \sum_{A \in \mathcal{P}_+} \beta(A) \left(\frac{\alpha(A)}{\beta(A)} - 1 \right) = d_{\text{TV}}(\alpha, \beta).$$

For $x \in (0, 1)$, $\sqrt{x} \leq (x-1)/\log x$, then the inequality (35) follows.

Since $H_{\alpha \| \beta}(\mathcal{P}) \geq 0$, it follows that $-\sum_{A \in \mathcal{P}_-} \alpha(A) \log \frac{\alpha(A)}{\beta(A)} \leq \sum_{A \in \mathcal{P}_+} \alpha(A) \log \frac{\alpha(A)}{\beta(A)}$. Plugging back in I , we have then

$$\begin{aligned} I &\leq \max_{A \in \mathcal{P}} \left| 1 - \frac{\alpha'(A)}{\alpha(A)} \right| \left(\sum_{A \in \mathcal{P}_+} \alpha(A) \log \frac{\alpha(A)}{\beta(A)} - \sum_{A \in \mathcal{P}_-} \alpha(A) \log \frac{\alpha(A)}{\beta(A)} \right) \\ &\leq 2\sqrt{C} \max_{A \in \mathcal{P}} \left| 1 - \frac{\alpha'(A)}{\alpha(A)} \right|. \end{aligned}$$

The second part is bounded by

$$II = \sum_{A \in \mathcal{P}} \alpha'(A) \log \frac{\beta'(A)}{\beta(A)} \leq \sum_{A \in \mathcal{P}} \alpha'(A) \log \left(\max_{A \in \mathcal{P}} \frac{\beta'(A)}{\beta(A)} \right) = \log \left(\max_{A \in \mathcal{P}} \frac{\beta'(A)}{\beta(A)} \right).$$

□

B.2.2 Mutual information

The mutual information of two random variables X and Y of laws $P(X)$ and $P(Y)$ is the KL-divergence of their joint law with respect to their product law:

$$I(X, Y) := D \left(P(X, Y) \parallel P(X) \otimes P(Y) \right).$$

Given a probability space $(\Omega, \mathcal{B}, \mathbb{P})$, a random variable $X : \Omega \rightarrow S$ and a sub- σ -field \mathcal{F} of \mathcal{B} , denote by $P(X|\mathcal{F})$ the conditional law of X given \mathcal{F} . The mutual information of X and \mathcal{F} is given by

$$I(X, \mathcal{F}) = \int_{\Omega} \int_S \log \frac{dP(X|\mathcal{F})}{dP(X)} dP(X|\mathcal{F}) d\mathbb{P}.$$

These formulae are consistent: denote by $\sigma(Y)$ the σ -field generated by Y , then $I(X, Y) = I(X, \sigma(Y))$.

Recall from Subsection 5.5 that $(\xi_n^x)_{n=0}^\infty$ is the Markov chain obtained from a Doob transformed trajectory of law $\mathbb{P}_\mu^{\pi(x)}$ by taking the quotient map $G \rightarrow L_x \backslash G$. Following [Der85], we consider the mutual information $I(\xi_1^x, \mathcal{T}_x)$ between the chain at time one and its tail σ -field \mathcal{T}_x . In the setting of Subsection 5.3, we collect some known facts regarding entropy and mutual information.

Lemma B.4. *Let (Z, λ) be a Poisson bundle over a standard system $(X, \eta) \xrightarrow{\pi} (Y, \nu)$ satisfying assumption **(S)**.*

(i) $I(\xi_1^x, \xi_n^x) \leq I(\mathbb{P}_{\mu,1}^{\pi(x)}, \mathbb{P}_{\mu,n}^{\pi(x)})$.

(ii) *Integrated over (Y, ν) , we have*

$$\int_Y I(\mathbb{P}_{\mu,1}^y, \mathbb{P}_{\mu,n}^y) d\nu(y) = I(\mathbb{P}_{\mu,1}, \mathbb{P}_{\mu,n}) - h_\mu(Y, \nu). \quad (36)$$

(iii) *The sequence $I(\xi_1^x, \xi_n^x)$ is non-increasing and*

$$I(\xi_1^x, \mathcal{T}_x) = \inf_n I(\xi_1^x, \xi_n^x) = \lim_n I(\xi_1^x, \xi_n^x). \quad (37)$$

(iv) *The mutual information can be written as*

$$\begin{aligned} I(\xi_1^x, \mathcal{T}_x) &= \int_{G^\mathbb{N}} \log \frac{d(\omega_1 \lambda)_x(\psi_x(\omega))}{d\lambda_x} d\mathbb{P}_\mu^{\pi(x)}(\omega) \\ &= \int_G \int_{Z_x} \log \frac{d(g\lambda)_x(z)}{d\lambda_x} d(g\lambda)_x(z) \varphi_g(\pi(x)) d\mu(g). \end{aligned}$$

Proof. (i). Recall that $\mathbb{P}_\mu^{\pi(x)}$ is the law of the Doob transformed random walk (or rather Markov chain) on $G^\mathbb{N}$ conditioned by $\{\beta_Y(\omega) = \pi(x)\}$. The law of (ξ_1^x, ξ_n^x) can be viewed as the restriction of the joint law $(\mathbb{P}_{\mu,1}^{\pi(x)}, \mathbb{P}_{\mu,n}^{\pi(x)})$ on $G \times G$ to the sub- σ -field of L_x -invariant subsets, that is, measurable sets $A \subseteq G \times G$ such that $L_x A = A$. It then follows from general properties of KL-divergence with respect to restrictions of measures, see e.g., [Gra11, Corollary 7.2].

(ii). Denote by P the joint law $(\mathbb{P}_{\mu,1}, \mathbb{P}_{\mu,n})$ and by Q the product law $\mathbb{P}_{\mu,1} \times \mathbb{P}_{\mu,n}$ of times 1 and n of a random walk trajectory of law \mathbb{P}_μ . Similarly for y in Y , denote by P^y the joint law $(\mathbb{P}_{\mu,1}^y, \mathbb{P}_{\mu,n}^y)$ and by Q^y the product law $\mathbb{P}_{\mu,1}^y \times \mathbb{P}_{\mu,n}^y$ for a Doob transformed trajectory of law \mathbb{P}_μ^y . We have $\frac{dP^y}{dQ^y}(g, h) = \frac{dP}{dQ}(g, h) \frac{1}{\varphi_g(y)}$. Thus

$$\begin{aligned} \int_Y I(\mathbb{P}_{\mu,1}^y, \mathbb{P}_{\mu,n}^y) d\nu(y) &= \int_Y \int_{G \times G} \log \frac{dP}{dQ} dP^y d\nu(y) - \int_Y \int_{G \times G} \log \varphi_g(y) dP^y(g, h) d\nu(y) \\ &= \int_{G \times G} \log \frac{dP}{dQ} dP - \int_Y \int_{G \times G} \log \varphi_g(y) d\mu(g) d\mu^{n-1}(g^{-1}h) \varphi_h(y) d\nu(y) \\ &= I(\mathbb{P}_{\mu,1}, \mathbb{P}_{\mu,n}) - \int_Y \int_G \log \varphi_g(y) d\mu(g) dg \cdot \nu(y) = I(\mathbb{P}_{\mu,1}, \mathbb{P}_{\mu,n}) - \int_Y \log \varphi_g(y) d\nu(y). \end{aligned}$$

In the last line we used the harmonicity of φ_h , which implies $\int_G \varphi_h(y) d\mu^{n-1}(g^{-1}h) = \varphi_g(y)$. We conclude by Lemma B.1.

(iii) By [Der85, Section III], this is a consequence of the fact that (ξ_n^x) has the Markov property, according to Proposition 5.7.

(iv). By [Der85, Section V] one has

$$I(\xi_1^x, \mathcal{T}_x) = \int_{(L_x \setminus G)^{\mathbb{N}}} \log \frac{d\overline{\mathbb{P}}_{\mu, x, \xi_1^x} | \mathcal{T}_x}{d\overline{\mathbb{P}}_{\mu, x, e} | \mathcal{T}_x} d\overline{\mathbb{P}}_{\mu, x, e}, \quad (38)$$

where the notation $Q|_{\mathcal{F}}$ denote the restriction of the measure Q to a sub- σ -field \mathcal{F} . By Corollary 5.10, we may replace in (38) the tail σ -field \mathcal{T}_x by the invariant σ -field \mathcal{I}_x of (ξ_n^x) . Lemma 5.8 gives that for any g in G ,

$$\overline{\mathbb{P}}_{\mu, x, g} | \mathcal{I}_x = \theta(x, \cdot)_* \overline{\mathbb{P}}_{\mu, x, g} = (g\lambda)_x.$$

Finally Lemma 5.6 gives $\theta(x, \cdot)_* \overline{\mathbb{P}}_{\mu, x, e} = \theta(x, \cdot)_* \vartheta_* \mathbb{P}_{\mu}^{\pi(x)} = \psi(x, \cdot)_* \mathbb{P}_{\mu}^{\pi(x)}$. It follows that (38) implies

$$I(\xi_1^x, \mathcal{T}_x) = \int_{G^{\mathbb{N}}} \log \frac{d(\omega_1 \lambda)_x}{d\lambda_x} (\psi(x, \omega)) d\mathbb{P}_{\mu}^{\pi(x)}(\omega).$$

The second equality follows by conditioning on the first step $\omega_1 = g$ of the Doob transformed random walk, recalling that $(g\lambda)_x$ is the hitting distribution in the fiberwise Poisson boundary Z_x . \square

B.2.3 Proof of Proposition 5.11

Proof of Proposition 5.11. Recall that (Z, λ) fits into $(X \times G^{\mathbb{N}}, \eta \vee \mathbb{P}_{\mu}) \xrightarrow{\psi} (Z, \lambda) \xrightarrow{\varrho} (X, \eta)$. By Lemma B.1, the Furstenberg entropy of (Z, λ) is

$$h(Z, \lambda) = \int_{G^{\mathbb{N}}} \int_X \log \frac{d\omega_1 \cdot \lambda}{d\lambda} (\psi(x, \omega)) d\eta_{\omega}(x) d\mathbb{P}_{\mu}(\omega).$$

Next we disintegrate λ over $(Z, \lambda) \xrightarrow{\varrho} (X, \eta)$ and denote it as $\lambda = \int_X \lambda_x d\eta(x)$. Then the Radon-Nikodym derivative can be written as

$$\frac{dg \cdot \lambda}{d\lambda} (\psi(x, \omega)) = \frac{dg \cdot \eta}{d\eta}(x) \frac{d(g \cdot \lambda)_x}{d\lambda_x} (\psi(x, \omega)).$$

Thus we have

$$h(Z, \lambda) = \int_{G^{\mathbb{N}}} \int_X \log \frac{d\omega_1 \cdot \eta}{d\eta}(x) d\eta_{\omega}(x) d\mathbb{P}_{\mu}(\omega) + \int_{G^{\mathbb{N}}} \int_X \log \frac{d(\omega_1 \lambda)_x}{d\lambda_x} (\psi(x, \omega)) d\eta_{\omega}(x) d\mathbb{P}_{\mu}(\omega) = \text{I} + \text{II}.$$

By Lemma B.1 and as $\pi : X \rightarrow Y$ is measure preserving, we have that $\text{I} = h_{\mu}(X, \eta) = h_{\mu}(Y, \nu)$. By Lemma 5.6, we have

$$\text{II} = \int_X \int_{G^{\mathbb{N}}} \log \frac{d(\omega_1 \lambda)_x}{d\lambda_x} (\psi(x, \omega)) d\mathbb{P}_{\mu}^{\pi(x)}(\omega) d\eta(x),$$

where \mathbb{P}_{μ}^y is the law of the Doob transformed random walk on G conditioned on $\{\beta_Y(\omega) = y\}$. By Lemma B.4 (iv), we have $\text{II} = \int_X I(\xi_1^x, \mathcal{T}_x) d\eta(x) = I(\xi_1, \mathcal{T} | X, \eta)$. \square

B.3 Entropy formulae and Shannon's theorem for countable groups

In this subsection we assume that G is a countable group. For a convolution μ -random walk $(\xi_n)_{n=0}^\infty$ on a discrete group, it is classical that

$$I(\xi_1, \xi_n) = H(\mu^{(n)}) - H(\mu^{(n-1)})$$

where $H(p) = \sum_{g \in G} p(g) \log p(g)$ is the Shannon entropy of the discrete probability measure p on G . In the case of a countable group G endowed with a finite entropy probability measure μ , we will obtain a bundle version of this formula, stated in Theorem 5.12.

B.3.1 Proof of Theorem 5.12

Let us denote

$$I(\xi_1, \xi_n | X) := \int_X I(\xi_1^x, \xi_n^x) d\eta(x) \quad \text{and} \quad H(\xi_n | X) := \int_X H(\xi_n^x) d\eta(x).$$

Lemma B.5. *Assume G is a countable group endowed with a probability measure μ of finite Shannon entropy. Let (X, η) be a standard system satisfying assumption (S). Then*

$$I(\xi_1, \xi_n | X) = H(\xi_n | X) - H(\xi_{n-1} | X).$$

The proof follows standard calculations as in [Der85, Section I].

Proof. The sequence $(\xi_n^x)_{n=0}^\infty$ is a Markov chain in the coset space $L_x \backslash G$, started at the identity coset L_x and with transition probabilities given by Proposition 5.7. By definition, we have

$$I(\xi_1^x, \xi_n^x) = \sum_{\xi_1^x} \sum_{\xi_n^x} \log \left(\frac{P_{\mu,x}^{n-1}(\xi_1^x, \xi_n^x)}{P_{\mu,x}^n(L_x, \xi_n^x)} \right) P_{\mu,x}^1(L_x, \xi_1^x) P_{\mu,x}^{n-1}(\xi_1^x, \xi_n^x)$$

We can split the logarithm to get $I(\xi_1^x, \xi_n^x) = I_x + \Pi_x$, where $I_x = H(\xi_n^x)$, so $\int_X I_x d\eta(x) = H(\xi_n | X)$. There remains to show $\int_X \Pi_x d\eta(x) = -H(\xi_{n-1} | X)$, where

$$\Pi_x = \sum_{s \in G} \mu(s) \frac{ds\nu}{d\nu}(\pi(x)) \sum_{\xi_n^x \in L_x \backslash G} \log(P_{\mu,x}^{n-1}(L_x s, \xi_n^x)) P_{\mu,x}^{n-1}(L_x s, \xi_n^x)$$

as the transition probabilities satisfy

$$P_{\mu,x}^1(L_x, \xi_1^x) = \sum_{\{s: \xi_1^x = L_x s\}} \mu(s) \frac{ds\nu}{d\nu}(\pi(x)).$$

Now by equivariance, see (12) in the proof of Proposition 5.7,

$$P_{\mu,x}^{n-1}(L_x s, L_x h) = P_{\mu, s^{-1}x}^{n-1}(L_{s^{-1}x}, L_{s^{-1}x} s^{-1} h).$$

This is the law of $\xi_{n-1}^{s^{-1}x}$, in other terms of the position at time $n-1$ of a coset trajectory in the fiber over $s^{-1}x$. We get

$$\Pi_x = - \sum_{s \in G} \mu(s) \frac{ds\nu}{d\nu}(\pi(x)) H(\xi_{n-1}^{s^{-1}x}).$$

Use the disintegration $\eta = \int_Y \eta^y d\nu(y)$ over the measure preserving extension $\pi : X \rightarrow Y$ to get

$$\int_X \Pi_x d\eta(x) = - \sum_{s \in G} \mu(s) \int_Y \int_{\pi^{-1}(y)} H\left(\xi_{n-1}^{s^{-1} \cdot x}\right) d\eta^y(x) \frac{ds\nu}{d\nu}(y) d\nu(y) = - \int_X H(\xi_{n-1}^x) d\eta(x).$$

□

Lemma B.5 implies the random walk entropy formula stated in Theorem 5.12:

Proof of Theorem 5.12. By Lemma B.4 (iii), we have

$$I(\xi_1, \mathcal{T}|X) = \int_X I(\xi_1^x, \mathcal{T}_x) d\eta(x) = \int_X \lim_{n \rightarrow \infty} I(\xi_1^x, \xi_n^x) d\eta(x) = \lim_{n \rightarrow \infty} \int_X I(\xi_1^x, \xi_n^x) d\eta(x)$$

as the convergence is dominated by the integrable function

$$H(\xi_1^x) = I(\xi_1^x, \xi_1^x) \geq \inf_n I(\xi_1^x, \xi_n^x) = \lim_{n \rightarrow \infty} I(\xi_1^x, \xi_n^x).$$

Then Proposition 5.11 and Lemma B.5 give the first formula. The second formula follows by Cesaro average since $H(\xi_n|X) = \sum_{k=1}^n H(\xi_k|X) - H(\xi_{k-1}|X)$ and $H(\xi_k|X) - H(\xi_{k-1}|X) = I(\xi_1, \xi_k|X)$ is a non-increasing sequence tending to $I(\xi_1, \mathcal{T}|X)$. □

B.3.2 Shannon's theorem

In the case of a countable group G , Kingman's subadditive ergodic theorem implies:

Proposition B.6 (Shannon's theorem). *Assume G is a countable group endowed with a probability measure μ of finite Shannon entropy. Let (Z, λ) be a standard system satisfying assumption (S). Then*

$$\lim_{n \rightarrow \infty} -\frac{1}{n} \log P_{\mu, x}^n(L_x, L_x \omega_n) = \lim_{n \rightarrow \infty} \frac{1}{n} H(\xi_n|X) = h_\mu(Z, \lambda) - h_\mu(X, \eta),$$

$\eta \vee \mathbb{P}_\mu$ -a.s. and in L^1 limits.

Proof. By Fact 5.1, the skew transformation

$$T : (x, (\omega_1, \omega_2, \dots)) \mapsto (\omega_1^{-1} \cdot x, (\omega_1^{-1} \omega_2, \omega_1^{-1} \omega_3, \dots))$$

is ergodic p.m.p. on the space $(X \times G^{\mathbb{N}}, \eta \vee \mathbb{P}_\mu)$. Consider the functions $f_n : X \times G^{\mathbb{N}} \rightarrow \mathbb{R}$ given by $f_n(x, \omega) = P_{\mu, x}^n(L_x, L_x \omega_n)$. Then

$$\begin{aligned} f_{n+m}(x, \omega) &\geq P_{\mu, x}^n(L_x, L_x \omega_n) P_{\mu, x}^m(L_x \omega_n, L_x \omega_{n+m}) \\ &= P_{\mu, x}^n(L_x, L_x \omega_n) P_{\mu, \omega_n^{-1} \cdot x}^m(L_{\omega_n^{-1} \cdot x}, L_{\omega_n^{-1} \cdot x} \omega_m) \text{ by (12)} \\ &= f_n(x, \omega) f_m(T^n(x, \omega)). \end{aligned}$$

Kingman's subadditive ergodic theorem applied to $-\log f_n$, gives a constant h with $\lim_{n \rightarrow \infty} -\frac{1}{n} \log f_n = h$, both $\eta \gamma \mathbb{P}_\mu$ -a.s. and in L^1 . Moreover, Lemma 5.6 gives

$$\begin{aligned} \int_{X \times G^\mathbb{N}} -\log f_n d\eta \gamma \mathbb{P}_\mu &= \int_Y \int_{\pi^{-1}(y)} \int_{G^\mathbb{N}} -\log (P_{\mu,x}^n(L_x, L_x \omega_n)) d\mathbb{P}_\mu^y(\omega) d\eta^y(x) d\nu(y) \\ &= \int_Y \int_{\pi^{-1}(y)} \sum_{\omega_n \in G} -\log (P_{\mu,x}^n(L_x, L_x \omega_n)) P_{\mu,x}^n(L_x, L_x \omega_n) d\eta^y(x) d\nu(y) \\ &= \int_X H(\xi_n^x) d\eta(x) = H(\xi_n|X). \end{aligned}$$

Therefore

$$h = \lim_{n \rightarrow \infty} \frac{1}{n} \int_{X \times G^\mathbb{N}} -\log f_n d\eta \gamma \mathbb{P}_\mu = \lim_{n \rightarrow \infty} \frac{1}{n} H(\xi_n|X) = h_\mu(Z, \lambda) - h_\mu(X, \eta).$$

by Theorem 5.12. □

Note that the proof above shows subadditivity of the sequence $n \mapsto H(\xi_n|X)$.

B.4 A proof of the ray criterion

Proof of Ray approximation Theorem 7.6. It suffices to show that $h(M, \bar{\lambda}) = h(Z, \lambda)$. Suppose on the contrary $h(Z, \lambda) - h(M, \bar{\lambda}) = \delta > 0$. Take $\epsilon = \frac{1}{3}\delta$, we will omit the reference to ϵ in the notation for A_n^ϵ in what follows. By Corollary 7.2, for any $p > 0$, there is a subset V of W_Ω with measure $\bar{\mathbb{P}}_\mu(V) > 1 - p$ and a constant $N = N_{\epsilon,p} < \infty$ such that for all $(x, L_x \omega) \in V$ and $n \geq N$, we have

$$P_{\mu,x}^{\zeta,n}(L_x, L_x \omega_n) \leq e^{-(\delta-\epsilon)n} = e^{-2n\epsilon}. \quad (39)$$

For short, let us denote $(x, \zeta) = \theta_M(x, L_x \omega)$. Take a union bound over time m in the expression of probability in (i), we have

$$\begin{aligned} \bar{\mathbb{P}}_\mu(\exists m \geq n : L_x \omega_m \in A_n(x, \zeta) | (x, \zeta) \in U) \\ \leq \frac{1}{\bar{\lambda}(U)} \bar{\mathbb{P}}_\mu(\exists m \geq n : L_x \omega_m \in A_n(x, \zeta), (x, \zeta) \in U, (x, L_x \omega) \in V) \\ \quad + \frac{1}{\bar{\lambda}(U)} \bar{\mathbb{P}}_\mu((x, \zeta) \notin V) \\ \leq \frac{p}{\bar{\lambda}(U)} + \sum_{m=n}^{\infty} \bar{\mathbb{P}}_\mu(L_x \omega_m \in A_n(x, \zeta), (x, L_x \omega) \in V | (x, \zeta) \in U). \end{aligned}$$

To estimate the conditional probabilities that appear as summands, we use the disintegration $\bar{\mathbb{P}}_\mu = \int_M \bar{\mathbb{P}}_{\mu,x}^\zeta d\bar{\lambda}(x, \zeta)$. We have for $(x, \zeta) \in U$ and $m \geq N$,

$$\begin{aligned} \bar{\mathbb{P}}_\mu(L_x \omega_m \in A_n(x, \zeta), (x, L_x \omega) \in V | (x, \zeta) \in U) \\ \leq \frac{1}{\bar{\lambda}(U)} \int_M \mathbf{1}_U(x, \zeta) \bar{\mathbb{P}}_{\mu,x}^\zeta(L_x \omega_m \in A_n(x, \zeta), (x, L_x \omega) \in V) d\bar{\lambda}(x, \zeta) \\ \leq |A_n(x, \zeta)| e^{-2m\epsilon} \quad \text{by (39)}. \end{aligned}$$

Condition (ii) gives the upper bound $|A_n(x, \zeta)| \leq e^{\frac{3}{2}n\epsilon}$ for large n . Then

$$\sum_{m=n}^{\infty} \bar{\mathbb{P}}_{\mu}(L_x \omega_m \in A_n(x, \zeta), (x, L_x \omega) \in V | (x, \zeta) \in U) \leq ce^{-\frac{1}{2}\epsilon n}.$$

Since p can be arbitrarily small and $\bar{\lambda}(U) > 0$, we get the limit

$$\lim_{n \rightarrow \infty} \bar{\mathbb{P}}_{\mu}(\exists m \geq n : L_x \omega_m \in A_n(x, \zeta) | (x, \zeta) \in U) = 0.$$

This contradicts the strictly positive limit in (i). □

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