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SPATIAL REALIZATION OF A LIE ALGEBRA AND THE BAR CONSTRUCTION

YVES FÉLIX AND DANIEL TANRÉ

ABSTRACT. We prove that the spatial realization of a rational complete Lie algebra L , concentrated in degree 0, is isomorphic to the simplicial bar construction on the group, obtained from the Baker-Campbell-Hausdorff product on L .

INTRODUCTION

In [4], we construct a cosimplicial differential graded complete Lie algebra (henceforth cdgl) (\mathcal{L}_\bullet, d) , in which (\mathcal{L}_1, d) is the Lawrence-Sullivan model of the interval introduced in [7]. As in the work of Sullivan ([9]) for differential commutative graded algebras, the existence of this cosimplicial object gives an adjoint pair of functors between the category cdgl of cdgl 's and the category \mathbf{Sset} of simplicial sets, see [4] or [2]. In this work, we focus on one of them, the spatial realization functor,

$$\langle - \rangle : \text{cdgl} \rightarrow \mathbf{Sset},$$

defined by $\langle L \rangle = \text{Hom}_{\text{cdgl}}(\mathcal{L}_\bullet, L)$ for $L \in \text{cdgl}$. (Let us also notice that $\langle L \rangle$ is isomorphic to the nerve of L , a deformation retract of the Getzler-Hinich realization, see [1], [8].)

More precisely, we are interested in the realization $\langle L \rangle$ of a complete Lie algebra, L , concentrated in degree 0 and (thus) with the differential 0. In this case, a group structure can be defined on the set L from the Baker-Campbell-Hausdorff formula. We denote by $\exp L$ this group. The realization $\langle L \rangle$ is an Eilenberg-MacLane space of type $K(\pi, 1)$, see [5]. The purpose of this work is the determination of $\langle L \rangle$ up to isomorphism.

Main Theorem. *Let L be a complete differential graded Lie algebra, concentrated in degree 0. Then, its spatial realization $\langle L \rangle$ is isomorphic to the simplicial bar construction on $\exp L$.*

Let us emphasize that for an Eilenberg-MacLane space $K(G, 1)$, of associated Lie algebra L , then the realizations of the $A_{PL}(K(G, 1))$ of Sullivan ([9]) and $\text{MC}_*(L)$ of Getzler ([6]) are weakly equivalent to the simplicial bar resolution. In our setting, the simplicial set $\text{Hom}(\mathcal{L}_\bullet, L)$ is isomorphic to the simplicial bar resolution. With [2], we know that spaces more general than a $K(G, 1)$ admits a cdgl model L . For them, the

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simplicial set $\text{Hom}(\mathcal{L}_\bullet, L)$ appears as a natural extension of the bar construction. We will come back on this point in a forthcoming work.

In Section 1, we recall basic background on **cdgl** and our construction \mathfrak{L}_\bullet . Section 2 consists of the proof of the Main theorem.

1. SOME REMINDERS

We first recall the construction of the cosimplicial cdgl \mathfrak{L}_\bullet . Let V be a finite dimensional graded vector space. The completion of the free graded Lie algebra on V , $\mathbb{L}(V)$, is the inverse limit,

$$\widehat{\mathbb{L}}(V) = \varprojlim_n \mathbb{L}(V)/\mathbb{L}^{\geq n}(V),$$

where $\mathbb{L}^{\geq n}(V)$ is the ideal generated by the Lie brackets of length $\geq n$. We call $\widehat{\mathbb{L}}(V)$ the free complete graded Lie algebra on V .

As a graded Lie algebra, \mathfrak{L}_n is the free complete graded Lie algebra on the rational vector space generated by the elements $a_{i_0 \dots i_k}$ of degree $|a_{i_0 \dots i_k}| = k - 1$, with $0 \leq i_0 < \dots < i_k \leq n$. We denote by $\text{ad}_{i_0 \dots i_k}$ the Lie derivation $[a_{i_0 \dots i_k}, -]$. The cdgl \mathfrak{L}_n satisfies the following properties.

– \mathfrak{L}_0 is the free cdgl on a Maurer-Cartan element a_0 , that is:

$$\mathfrak{L}_0 = (\mathbb{L}(a_0), d), \quad da_0 = -\frac{1}{2}[a_0, a_0].$$

– $\mathfrak{L}_1 = (\widehat{\mathbb{L}}(a_0, a_1, a_{01}), d)$ is the Lawrence-Sullivan interval (see [7]), where a_0 and a_1 are Maurer-Cartan elements and

$$da_{01} = [a_{01}, a_1] + \frac{\text{ad}_{a_0}}{e^{\text{ad}_{a_0}} - 1}(a_1 - a_0).$$

– A model \mathfrak{L}_2 for the triangle has been described in [4] (see also [3]):

$$\mathfrak{L}_2 = (\widehat{\mathbb{L}}(a_0, a_1, a_2, a_{01}, a_{02}, a_{12}, a_{012}), d) \text{ with } d(a_{012}) = a_{01} * a_{12} * a_{02}^{-1} - [a_0, a_{012}].$$

Here $*$ denotes the Baker-Campbell-Hausdorff product defined for any pair of elements a, b of degree 0 by $a * b = \log(\exp^a \exp^b)$.

– Moreover these structures appear in each \mathfrak{L}_n : each vertex a_r is a Maurer-Cartan element, each triple (a_r, a_s, a_{rs}) is a Lawrence-Sullivan interval and each family $(a_r, a_s, a_t, a_{rs}, a_{rt}, a_{st}, a_{rst})$ is a triangle as above.

The family $(\mathfrak{L}_n)_{n \geq 0}$ forms a cosimplicial cdgl which allows the definition of the spatial realization of $L \in \mathbf{cdgl}$ by,

$$\langle L \rangle := \text{Hom}_{\mathbf{cdgl}}(\mathfrak{L}_\bullet, L).$$

The cofaces δ^i and the codegeneracies σ^i of the cosimplicial cdgl \mathfrak{L}_\bullet are defined by

$$\delta^i a_{i_0 \dots i_p} = a_{j_0 \dots j_p} \quad \text{with} \quad j_k = \begin{cases} i_k & \text{if } i_k < i, \\ i_k + 1 & \text{if } i_k \geq i, \end{cases} \quad (1.1)$$

$$\sigma^i a_{i_0 \dots i_p} = \begin{cases} 0 & \text{if } \{i, i+1\} \subset \{i_0, \dots, i_p\}, \\ a_{j_0 \dots j_p} & \text{otherwise, with } j_k = \begin{cases} i_k & \text{if } i_k \leq i, \\ i_k - 1 & \text{if } i_k > i. \end{cases} \end{cases} \quad (1.2)$$

2. PROOF OF THE MAIN THEOREM

The simplicial bar construction on a group G , is the simplicial set $B_\bullet G$ with set of n -simplices $B_n G = G^n$. Its elements are denoted $[g_1 | \dots | g_n]$, with $g_i \in G$. The faces d_i and degeneracies s_i of $B_\bullet G$ are defined as follows:

$$\begin{aligned} d_0[g_1 | \dots | g_n] &= [g_2 | \dots | g_n], \\ d_i[g_1 | \dots | g_n] &= [g_1 | \dots | g_i g_{i+1} | \dots | g_n], \quad \text{for } 0 < i < n, \\ d_n[g_1 | \dots | g_n] &= [g_1 | \dots | g_{n-1}]. \end{aligned} \tag{2.1}$$

The degeneracy s_i inserts the identity e of G in position i .

Let $L \in \mathbf{cdgl}$ be generated in degree 0 and $f: \mathfrak{L}_n \rightarrow L$ a morphism in \mathbf{cdgl} . For degree reasons, we have $f(a_{i_0 \dots i_k}) = 0$ if $k \neq 1$. Moreover, since f commutes the differential, from the definition of the differential in \mathfrak{L}_2 , we get

$$0 = df(a_{0rs}) = f(a_{0r}) * f(a_{rs}) * f(a_{0s})^{-1}.$$

Therefore, for any $r, s > 0$, we have

$$f(a_{rs}) = f(a_{0r})^{-1} * f(a_{0s}).$$

The map f being entirely defined by its values on the a_{0i} , we have a bijection

$$\Phi: \text{Hom}_{\mathbf{cdgl}}(\mathfrak{L}_n, L) \rightarrow L^n, \quad \text{defined by } f \mapsto (f(a_{01}), f(a_{02}), \dots, f(a_{0n})).$$

We now determine the image of the faces and degeneracies on $\text{Hom}_{\mathbf{cdgl}}(\mathfrak{L}_\bullet, L)$, induced from (1.1) and (1.2). For the face operators, as only the elements (a_{0r}) play a role, it suffices to consider,

$$\delta^i(a_{0r}) = \begin{cases} a_{0,r} & \text{if } r < i \\ a_{0,r+1} & \text{if } r \geq i \end{cases} \quad \text{for } i > 0, \quad \text{and } \delta^0(a_{0,r}) = a_{1,r+1}.$$

Let $f: \mathfrak{L}_n \rightarrow L$ be specified by $(f(a_{01}), \dots, f(a_{0n}))$. Then $d_0 f = f \circ \delta^0: \mathfrak{L}_{n-1} \rightarrow L$ is described by

$$\begin{aligned} d_0(f(a_{01}), \dots, f(a_{0(n-1)})) &= (f \circ \delta^0(a_{01}), \dots, f \circ \delta^0(a_{0(n-1)})) \\ &= (f(a_{12}), \dots, f(a_{1n})) \\ &= (f(a_{01})^{-1} * f(a_{02}), \dots, f(a_{01})^{-1} * f(a_{0n})). \end{aligned}$$

Therefore, the face operator d_0 on L^\bullet , induced from Φ , is

$$d_0(x_1, \dots, x_n) = (x_1^{-1} * x_2, \dots, x_1^{-1} * x_n).$$

Similar arguments give, for $i > 0$,

$$d_i(x_1, \dots, x_n) = (x_1, \dots, \hat{x}_i, \dots, x_n).$$

As for the degeneracies, starting from

$$\begin{aligned} \sigma^i(a_{0r}) &= \begin{cases} a_{0r} & \text{if } r \leq i \\ a_{0,r-1} & \text{if } r > i \end{cases} \quad \text{for } i > 0, \\ \sigma^0(a_{0r}) &= a_{0,r-1} \quad \text{if } r > 1, \quad \text{and } \sigma^0(a_{0,1}) = 0, \end{aligned}$$

we get

$$s_0(x_1, \dots, x_n) = (0, x_1, \dots, x_n)$$

and for $i > 0$,

$$s_i(x_1, \dots, x_n) = (x_1, \dots, x_i, x_i, \dots, x_n).$$

Now a straightforward and easy computation shows that the morphism

$$\Psi: \text{Hom}_{\text{cdgl}}(\mathfrak{L}_\bullet, L) \rightarrow B_\bullet(\exp L)$$

defined by

$$\Psi(f) = [f(a_{01})|f(a_{01})^{-1}f(a_{02})|f(a_{02})^{-1}f(a_{03})|\dots|f(a_{0(n-1)})^{-1}f(a_{0n})]$$

is an isomorphism of simplicial sets.

REFERENCES

- [1] Urtzi Buijs, Yves Félix, Aniceto Murillo, and Daniel Tanré, *The infinity Quillen functor, Maurer-Cartan elements and DGL realizations*, arXiv e-prints (2017), arXiv:1702.04397.
- [2] ———, *Homotopy theory of complete Lie algebras and Lie models of simplicial sets*, J. Topol. **11** (2018), no. 3, 799–825. MR 3989431
- [3] ———, *Symmetric Lie models of a triangle*, Fund. Math. **246** (2019), no. 3, 289–300. MR 3959254
- [4] ———, *Lie models of simplicial sets and representability of the Quillen functor*, Israel J. Math. **238** (2020), no. 1, 313–358. MR 4145802
- [5] Xue Zhi Cheng and Ezra Getzler, *Transferring homotopy commutative algebraic structures*, J. Pure Appl. Algebra **212** (2008), no. 11, 2535–2542. MR 2440265
- [6] Ezra Getzler, *Lie theory for nilpotent L_∞ -algebras*, Ann. of Math. (2) **170** (2009), no. 1, 271–301. MR 2521116
- [7] Ruth Lawrence and Dennis Sullivan, *A formula for topology/deformations and its significance*, Fund. Math. **225** (2014), no. 1, 229–242. MR 3205571
- [8] Daniel Robert-Nicoud, *Representing the deformation ∞ -groupoid*, Algebr. Geom. Topol. **19** (2019), no. 3, 1453–1476. MR 3954288
- [9] Dennis Sullivan, *Infinitesimal computations in topology*, Inst. Hautes Études Sci. Publ. Math. (1977), no. 47, 269–331 (1978). MR 0646078 (58 #31119)

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