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Synopsis of Diago-hexagons and diago-hexagonal sequences

by David Pouvreau¹ and Dominique Tournès²

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Let $\mathcal{P} = A_1A_2A_3A_4A_5A_6$ be a strictly convex hexagon. The intersections of its nine diagonals which are interior to \mathcal{P} form a « constellation » of fifteen or possibly only thirteen distinct points. The « major » diagonals (A_1A_4) , (A_2A_5) and (A_3A_6) of \mathcal{P} intersect in the vertices of a triangle (possibly reduced to one point) which is here called the *diagonal triangle* of \mathcal{P} : $T_1 = (A_1A_4) \cap (A_2A_5)$, $T_2 = (A_2A_5) \cap (A_3A_6)$ and $T_3 = (A_3A_6) \cap (A_1A_4)$. There are moreover two interesting subfamilies of diagonal intersections, both having the cardinal 6.

The convex envelope of the constellation is the strictly convex hexagon \mathfrak{D}_1 , the vertices of which $(D_k^{(1)})_{1 \leq k \leq 6}$ are defined (modulo 6) by $D_k^{(1)} = (A_kA_{k+2}) \cap (A_{k+1}A_{k+3})$. It will be said that \mathfrak{D}_1 is the *primary diago-hexagon* of \mathcal{P} and that for every $k \in \llbracket 1; 6 \rrbracket$, $D_k^{(1)}$ is the vertex of \mathfrak{D}_1 corresponding to A_k . And we will call *primary diago-hexagonal sequence* of \mathcal{P} the sequence of strictly convex hexagons $(\mathfrak{D}_n)_{n \in \mathbb{N}}$ obtained by the recurrent construction :

$$\left\{ \begin{array}{l} \mathfrak{D}_0 = \mathcal{P} \quad (\forall k \in \llbracket 1; 6 \rrbracket, D_k^{(0)} = A_k) \\ \text{For every } n \in \mathbb{N}, \mathfrak{D}_{n+1} = (D_k^{(n+1)})_{1 \leq k \leq 6} \text{ is the primary diago-hexagon of } \mathfrak{D}_n = (D_k^{(n)})_{1 \leq k \leq 6} \end{array} \right.$$

For every $n \in \mathbb{N}$, the hexagon \mathfrak{D}_n will then be called the n -th order primary diago-hexagon of \mathcal{P} .

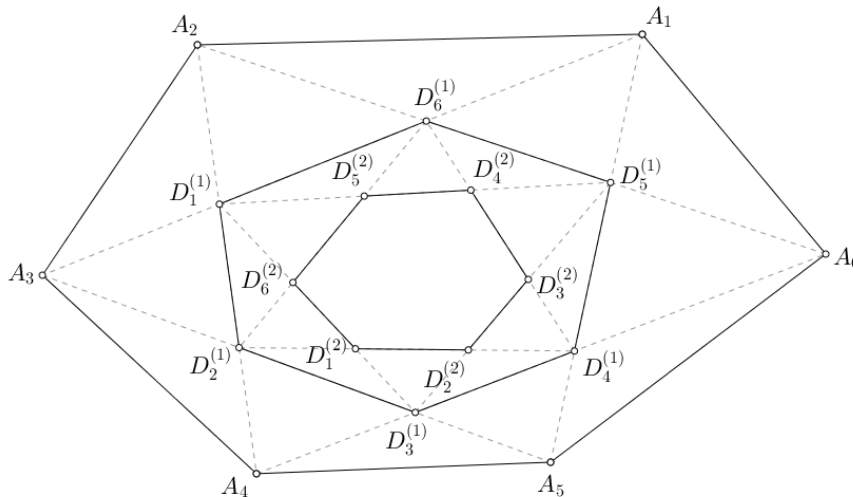


Figure 1 : Three first terms of a primary diago-hexagonal sequence

From the hexagon \mathcal{P} can also be obtained the strictly convex hexagon \mathcal{H}_1 , the vertices of which are the points $(H_k^{(1)})_{1 \leq k \leq 6}$ defined (modulo 6) by $H_k^{(1)} = (A_kA_{k+3}) \cap (A_{k+1}A_{k+5})$.

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It will here be said that \mathcal{H}_1 is the *secondary diago-hexagon* of \mathcal{P} and that for every $k \in \llbracket 1; 6 \rrbracket$, $H_k^{(1)}$ is the vertex of \mathcal{H}_1 corresponding to A_k . It is then possible to define recurrently the *secondary diago-hexagonal sequence* of \mathcal{P} , as the sequence of strictly convex hexagons $(\mathcal{H}_n)_{n \in \mathbb{N}}$ obtained by :

$$\left\{ \begin{array}{l} \mathcal{H}_0 = \mathcal{P} \quad (\forall k \in \llbracket 1; 6 \rrbracket, H_k^{(0)} = A_k) \\ \text{For every } n \in \mathbb{N}, \mathcal{H}_{n+1} = (H_k^{(n+1)})_{1 \leq k \leq 6} \text{ is the secondary diago-hexagon of } \mathcal{H}_n = (H_k^{(n)})_{1 \leq k \leq 6} \end{array} \right.$$

For every $n \in \mathbb{N}$, the hexagon \mathcal{H}_n will be called the n -th order secondary diago-hexagon of \mathcal{P} .

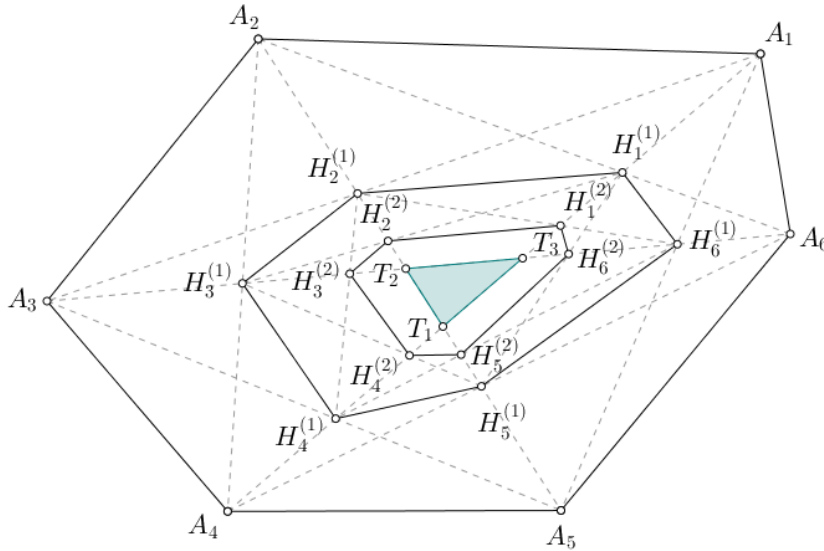


Figure 2 : Three first terms of a secondary diago-hexagonal sequence

The main results demonstrated in this paper by means of projective as well as algebraic and topological arguments, are the following ones :

Theorem 1

There exist a unique homography h such that $h(A_k) = D_k^{(2)}$ for every $k \in \llbracket 1; 6 \rrbracket$, hence such that $h(\mathcal{P}) = \mathcal{D}_2$. More generally, $h(\mathcal{D}_n) = \mathcal{D}_{n+2}$ for every $n \in \mathbb{N}$. h will be called the *diago-hexagonal homography* of \mathcal{P} .

Theorem 2

The major diagonals of \mathcal{P} are concurrent if, and only if its first order primary diago-hexagon is inscribable in a conic curve.

Theorem 3

The diago-hexagonal homography h of \mathcal{P} has a unique fixed point Ω interior to \mathcal{P} and the primary diago-hexagonal sequence $(\mathcal{D}_n)_{n \in \mathbb{N}}$ of \mathcal{P} converges toward Ω . Ω will be called the *diagonal center* of \mathcal{P} .

Theorem 4

If \mathcal{P} is inscribable in a conic curve, then each of its successive secondary diago-hexagons is inscribable in a conic curve.

Theorem 5

The secondary diago-hexagonal sequence of \mathcal{P} converges toward the diagonal triangle of \mathcal{P} .