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APPROXIMATION DIFFUSION FOR THE NONLINEAR SCHRÖDINGER EQUATION WITH A RANDOM POTENTIAL

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Abstract: We prove that the stochastic Nonlinear Schrödinger (NLS) equation is the limit of NLS equation with random potential with vanishing correlation length. We generalize the perturbed test function method to the context of dispersive equations. Apart from the difficulty of working in infinite dimension, we treat the case of random perturbations which are not assumed uniformly bounded.

1. Introduction and main result

We study in this work the limit of Non-Linear Schrödinger Equation with randomness. More precisely, we consider the following problem

$$i\partial_t X^\varepsilon(t) = -\Delta X^\varepsilon(t) + \lambda|X^\varepsilon(t)|^{2\sigma}X^\varepsilon(t) + \frac{1}{\varepsilon}X^\varepsilon(t)m(t/\varepsilon^2), \tag{1.1}$$

on the domain $\mathbb{R}^d$, with regular initial data. Such an equation occurs in many situations, for instance in optical fibers dynamics (see [18], [25]). More generally, the Nonlinear Schrödinger equation is an equation describing wave propagation in a nonhomogeneous dispersive medium and random effects often enter the description via a potential. Here we consider such a random potential which depends on time $t \geqslant 0$ and space $x \in \mathbb{R}^d$ with a scaling of the form $\frac{1}{\varepsilon}m(\frac{t}{\varepsilon^2}, x)$.

Under such a scaling, we are in the situation of approximation-diffusion. The random term formally converges to a spatially dependent white noise in time and we expect to obtain a white noise driven stochastic partial differential equation at the limit. Such stochastic non-linear Schrödinger equations are used in the physics literature and have been mathematically studied, for example in a conservative version by Debussche and de Bouard ([7], [8], [9] ...). Barbu, Röckner and Zhang proved in [1] well posedness results in both conservative and nonconservative cases thanks to rescaling transformations, while Brzezniak and Millet studied in [4] the stochastic Nonlinear Schrödinger equation on a two-dimensional manifold. We can also cite [20] and [21] where the authors study the one dimensional L^2-critical and supercritical cases for Nonlinear Schrödinger equation with spatially correlated noise and space time white noise. We believe that it is important to prove that these equations are limits of equations with realistic noises such as (1.1).

Our basic tool is the Perturbed Test Function method. This method provides an elegant way for approximation-diffusion problems; it was first introduced by Papanicolaou, Stroock and Varadhan [22] in a finite dimensional case, and one can find many applications in the book of Fouque, Garnier, Papanicolaou and Solna [17]. Generalizations in infinite dimension have been recently developped,

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for instance in [10], [11] or [14]. Our aim is to develop this method in the context of the Nonlinear Schrödinger equation with a random potential. The difficulty is that the fundamental object for this method is the infinitesimal generator of the Markov process associated to (1.1), which is a complicated object since we work in infinite dimension. Besides the Perturbed Test Function method is based on compactness arguments which are not trivial in the case of (1.1) whose space variable lives in the full space $\mathbb{R}^d$. We overcome this latter problem by working in a weighted space.

In [1], the main tool to study the stochastic Nonlinear Schrödinger equation is the rescaling transform. This changes the stochastic equation containing a spatially dependent white noise into a Nonlinear Schrödinger with continuous in time random terms. The counterpart is that the Laplace operator is replaced by a linear differential operator with varying coefficient. It is not clear how to use the scaling transform for our study since these linear operators would depend on ε. Moreover, the rescaling transform is useful only for purely multiplicative noise. Our arguments immediately extend to more general equations, for instance if $\frac{1}{\varepsilon}X(t)m(\frac{t}{\varepsilon^2})$ is replaced by $\frac{1}{\varepsilon}\Lambda(X(t))m(\frac{t}{\varepsilon^2})$ for a Nonlinear operator of Nemitsky type Λ with sublinear growth such that $\Lambda(z)\bar{z}$ is real valued. For the sake of clarity we decided to restrict to the case $\Lambda(z) = z$.

The driving noise m is an ergodic Markov process with values in a Sobolev space to be described below. In all the articles mentioned above on approximation-diffusion problem for partial differential equations, this noise is assumed to be uniformly bounded. This is important to get a priori estimates. In this article, we introduce a new argument which allows to replace this assumption by a much more satisfactory one: we only assume that m has sufficiently many finite moments. This introduces several difficulties. In particular the correctors cannot be bounded uniformly in ε. We use a stopping time argument which, together with a control of the growth of stationary process, allows to obtain a bound on the correctors. Another difficulty is to obtain bounds on the solutions of (1.1) uniform in ε. In particular, the control of the energy is very delicate. Our idea has been used and improved in a recent work on approximation diffusion for kinetic equations (see [24]).

We work with solutions in the Sobolev space $H^1(\mathbb{R}^d)$ and assume that the non linear term is subcritical:

$$0 \leqslant \sigma < \frac{2}{d-2},\ d \geq 3 \text{ or } 0 \leqslant \sigma,\ d = 1,\ 2. \tag{1.2}$$

Moreover, we work with global solutions and, when $\lambda < 0$, we need a further assumption. Namely, we assume

$$\sigma < \frac{2}{d},\ \text{if } \lambda < 0. \tag{1.3}$$

Our main result can be stated informally as follows. Precise assumptions are given below.

Theorem 1.1. *For all $T > 0$ and $0 < \zeta \leqslant 1$, assume that X_0^ε converges in law to X_0 in the space Σ^ζ, see (2.1), then for all $s \in [0, 1)$ the $C([0,T]; H^s(\mathbb{R}^d))$-valued process X^ε, solution of (1.1) with initial data X_0^ε converges in law in $C([0,T], H^s(\mathbb{R}^d))$ to X solution of*

$$idX = \Big(-\Delta X + \lambda |X|^{2\sigma} X - \frac{i}{2} F X \Big) dt + X Q^{1/2} dW \tag{1.4}$$

with initial data X_0, where F, Q are defined respectively by (2.26) and (2.36) below, and W is a cylindrical Wiener process on $L^2(\mathbb{R}^d)$.

The last two terms in (1.4) actually correspond to the Stratonovitch noise $X \circ Q^{\frac{1}{2}} dW$. The covariance operator Q is described below and is explicit in terms of the correlation of m, see (2.24) and (2.36) below. As mentioned above, we could consider a more general noise. Let $\Lambda : \mathbb{C} \to \mathbb{C}$

be a map with sublinear growth such that $\Lambda(z)\bar{z} \in \mathbb{R}$ for all $z \in \mathbb{C}$. If in (1.1), the random term $\frac{1}{\varepsilon}X(t)m(\frac{t}{\varepsilon^2})$ is replaced by $\frac{1}{\varepsilon}\Lambda(X(t))m(\frac{t}{\varepsilon^2})$, the limit equation would be the stochastic Nonlinear Schrödinger with the Stratonovitch noise $\Lambda(X) \circ dW$.

The article is organized as follows. We first introduce the notations and state preliminary results on the Schrödinger equation and on the driving process m. In Section 3, we adapt Kato's method (see [5]) to get global existence of the process X^ε. The estimates are obtained by classical manipulations of the equation and blow up when $\varepsilon \to 0$. To avoid this we adapt the perturbed test function method to our problem in Section 4 and 6.1. This enables us to get both tightness of the process and the expression of the infinitesimal generator of the limit. In Section 5 we use all the results proved in Section 3 and Skorohod Theorem (see [2]) to prove the weak convergence of X^ε to X. Finally, Appendix A is devoted to details about an example of process m that can be considered in (1.1) and to proofs of technical estimates.

In this article $\mathcal{C}$, C or c denote constants whose value may change from one line to the other and which unless explicitely stated are independent of ε or of the smoothing parameter δ introduced below. They may eventually depend on other parameters such as T, d, σ, R or η and if needed we may precise the dependences by denoting for example $\mathcal{C}_T$.

2. Preliminaries and Main result

2.1. **Notations.** Throughout this paper, for $p \geq 1$, we denote by $L^p(\mathbb{R}^d, ; \mathbb{C})$ the Lebesgue space of p integrable $\mathbb{C}$-valued functions on $\mathbb{R}^d$, endowed with the usual norm. For $p = 2$, $(\cdot, \cdot)$ is the inner product of $L^2(\mathbb{R}^d; \mathbb{C})$ given by

$$(f, g) = Re \int_{\mathbb{R}^d} f(x)\bar{g}(x)dx.$$

For $s \in \mathbb{R}$, we use the usual Sobolev space $H^s := H^s(\mathbb{R}^d; \mathbb{C})$ of tempered distributions $u \in \mathcal{S}'(\mathbb{R}^d)$ such that $\langle\xi\rangle^s \hat{u}(\xi) \in L^2(\mathbb{R}^n; \mathbb{C})$, where $\langle\xi\rangle = \sqrt{1 + |\xi|^2}$, endowed with the usual norm

$$\|u\|_{H^s} = \|\langle\xi\rangle^s \hat{u}(\xi)\|_{L^2},$$

where $\hat{u}$ denotes the Fourier transform.

For $m \in \mathbb{N}$ and $p \geq 1$, we also use the standard Sobolev spaces $W^{m,p}(\mathbb{R}^d; \mathbb{C})$ consisting of functions which are in $L^p(\mathbb{R}^d; \mathbb{C})$ as well as their derivatives up to order m. It is classical that $H^m(\mathbb{R}^d; \mathbb{C}) = W^{m,2}(\mathbb{R}^d; \mathbb{C})$.

We also consider the same spaces for $\mathbb{R}$-valued functions, they are denoted by the same symbols with $\mathbb{C}$ replaced by $\mathbb{R}$. When there is no ambiguity, we omit $\mathbb{R}^d$ and $\mathbb{C}$ or $\mathbb{R}$. For instance, we simply write H^1 for $H^1(\mathbb{R}^d; \mathbb{C})$.

In this article, the following weighted Sobolev spaces are particularly useful. We define for $0 < \zeta \leqslant 1$,

$$\Sigma^\zeta = \left\{u \in H^1(\mathbb{R}^d; \mathbb{C}); \langle x\rangle^\zeta u \in L^2\right\} \tag{2.1}$$

endowed with the norm $\|u\|_{\Sigma^\zeta} = \|u\|_{H^1} + \left\|\langle x\rangle^\zeta u\right\|_{L^2}$. Such weighted spaces are commonly used in the theory of the deterministic Schrödinger equation. The group $S(t)$ introduced below has some smoothing effect in these spaces for instance ([5], section 2.5). They are also useful to study finite time blow-up ([5], section 6.5).

Given a Banach space E, $C(E)$ (resp. $C^k(E)$) denotes the space of real valued continuous (resp. C^k) functions on E. And $C_b(E)$ (resp. $C_b^k(E)$) is the space of bounded continuous functions (resp.

C^k bounded as well as their derivatives up to order k). Finally $C_P(E)$ is the subset of $C(E)$ of functions with polynomial growth:

$$C_P = \{\psi \in C(E), \exists\, \kappa > 0,\, p \geq 0 : \forall\, n \in E, |\psi(n)| \leqslant \kappa(1 + \|n\|_E)^p\}\,, \tag{2.2}$$

When H, K are Hilbert spaces, we denote by $\mathcal{L}(H,K)$ the space of linear operators from H to K. If $H = K$, we simply write $\mathcal{L}(H)$. We also denote by $\mathcal{L}_2(H,K)$ the space of Hilbert-Schmidt operators from H to K.

Given a Hilbert space H endowed with a scalar product $(\cdot,\cdot)$ and a Banach space E continuously embedded in H, we use the common abuse of notations for the duality between E and E':

$$\langle \ell, x\rangle_{E',E} = (\ell, x),\ \ell \in E',\ x \in E.$$

We denote by $S(t)$ the group associated to the linear homogeneous equation and defined by $S(t) = e^{it\Delta}$. The solution of (1.1) is taken in the mild sense.

As already mentioned we assume that (1.2), (1.3) hold so that we are able to prove global well posedness of (1.1) in $H^1(\mathbb{R}^d;\mathbb{C})$ (see [5] and Proposition 3.1 below).

The energy of $u \in H^1(\mathbb{R}^d;\mathbb{C})$ is denoted by $\mathcal{H}(u)$ and is given by

$$\mathcal{H}(u) = \frac{1}{2}\|\nabla u\|_{L^2}^2 + \frac{\lambda}{2\sigma+2}\|u\|_{L^{2\sigma+2}}^{2\sigma+2}. \tag{2.3}$$

By Sobolev embeddings, we know that $H^1(\mathbb{R}^d;\mathbb{C}) \subset L^p(\mathbb{R}^d;\mathbb{C})$ for $p \in [2, \frac{2d}{d-2}]$ when $d \geq 3$, $p \in [2,\infty)$ when $d = 2$ and $p \in [2,\infty]$ when $d = 1$. Thus, by (1.2), this is a well defined quantity for $u \in H^1(\mathbb{R}^d;\mathbb{C})$.

In order to justify the computations when getting energy estimates, we may need a regularization procedure, so we choose ρ a mollifier, namely a function which satisfies

$$\rho \in C^\infty(\mathbb{R}^d,\mathbb{R}),\ \rho \geq 0,\ \int_{\mathbb{R}^d}\rho(x)dx = 1 \text{ and } \mathrm{supp}\rho \subset B(0,1). \tag{2.4}$$

Finally for $\delta > 0$, we define $\rho_\delta(x) = \frac{1}{\delta^n}\rho\left(\frac{x}{\delta}\right)$ and $u \star v$ stands for the convolution of u and v, when it makes sense.

2.2. **The random process m.** We assume that m is a centered, càdlàg, stochastically continuous, stationary $E = H^{s_0}(\mathbb{R}^d,\mathbb{R})$-valued process, for $s_0 > \frac{d}{2} + 3$ so that $E \hookrightarrow W^{3,\infty}(\mathbb{R}^d,\mathbb{R})$ and E is an algebra, on a probability space $(\Omega,\mathcal{F},\mathbb{P})$ adapted to a filtration $(\mathcal{F}_t)_{t\in\mathbb{R}}$ (see [23] for the basic theory of Hilbert space valued Markov processes). In particular that $m(t,x)$ is real valued. We also define the rescaled process

$$m^\varepsilon(t) = m(t/\varepsilon^2),\ t \geq 0, \tag{2.5}$$

which is centered stationary $E = H^s$-valued process $(\mathcal{F}_t^\varepsilon)_{t\in\mathbb{R}}$-adapted, where $\mathcal{F}_t^\varepsilon = \mathcal{F}_{t/\varepsilon^2}$.

Note that, as it is often the case in the study of partial differential equations, functions depending on space and time are seen as functions depending on time with values in a space of spatially dependent functions. That is, in the case of the process m, we use the identification $m(t,x) = m(t)(x)$. With this in mind, the rescaling above may be writen:

$$m^\varepsilon(t,x) = m(t/\varepsilon^2, x),\ t \geq 0,\ x \in \mathbb{R}^d.$$

The process m is supposed to be an homogeneous Markov process. We denote by $(P_t)_{t\geq 0}$ the transition semigroup associated to m, $\mathcal{M}$ its infinitesimal generator. For simplicity, we assume that

there exists a Markov process $(\tilde{m}(t,n))_{t\geq 0, n\in E}$ on $(\Omega, \mathcal{F}, \mathbb{P})$ adapted to the filtration $(\mathcal{F}_t)_{t\in\mathbb{R}}$ such that $\tilde{m}(t, m(0)) = m(t)$ and, for bounded borelian function φ on E, $P_t\varphi(n) = \mathbb{E}(\varphi(\tilde{m}(t,n)))$, $t \geq 0$.[1]

Recall that a borelian function φ on E is in the domain of the infinitesimal generator $\mathcal{M}$ if for $n \in E$

$$\varphi(\tilde{m}(t,n)) - \int_0^t \mathcal{M}\varphi(\tilde{m}(s,n))ds$$

is an integrable martingale. It is in general difficult to describe completely the domain of an infinitesimal generator. Here we only require that sufficiently many function are in the range of the generator. More precisely, we assume that there exist sets $\mathcal{P}_\mathcal{M}$ and $\mathcal{D}_\mathcal{M}$ included in $C_P(E)$ such that $\mathcal{D}_\mathcal{M}$ included in the domain of $\mathcal{M}$ and $\mathcal{P}_\mathcal{M}$ is included in the range of $\mathcal{M}$. For $\psi \in \mathcal{P}_\mathcal{M}$, we assume that there exists $\mathcal{M}^{-1}\psi \in \mathcal{D}_\mathcal{M}$ such that for $n \in E$

$$\mathcal{M}^{-1}\psi(\tilde{m}(t,n)) - \int_0^t \psi(\tilde{m}(s,n))ds \tag{2.6}$$

is an integrable martingale. In other words, $\mathcal{M}^{-1}\psi$ is in the domain of $\mathcal{M}$ and $\mathcal{M}\mathcal{M}^{-1}\psi = \psi$. This is an ergodicity assumption on m. Below we require that some specific functions are in $\mathcal{P}_\mathcal{M}$

We assume that $\tilde{m}$ has a unique invariant measure ν, which is the invariant law of m. Clearly, our setting requires that

$$\mathbb{E}_\nu[\psi] = \int_E \psi(n)d\nu(n) = 0, \qquad \psi \in \mathcal{P}_\mathcal{M}. \tag{2.7}$$

We need that $\mathcal{P}_M$ contains sufficiently many functions. Let us define for u, $v \in H^1$:

$$\Psi_1^{u,v}(n) = (un, v). \tag{2.8}$$

We assume that for each u, $v \in H^1$, $\Psi_1^{u,v} \in \mathcal{P}_M$ and there exist $L_1 : E \to E$ continuous such that

$$\mathcal{M}^{-1}\Psi_1^{u,v}(n) = (uL_1(n), v). \tag{2.9}$$

Informally, this says:

$$L_1(n) = \mathcal{M}^{-1}n. \tag{2.10}$$

Then we define for u, $v \in H^1$:

$$\Psi_2^{u,v}(n) = (unL_1(n), v) - \mathbb{E}_\nu(unL_1(n), v), \tag{2.11}$$

$$\Psi_3^{u,v}(n) = (u\nabla n, v\nabla L_1(n)) - \mathbb{E}_\nu(u\nabla n, v\nabla L_1(n)), \tag{2.12}$$

and assume that for each u, $v \in H^1$, $i = 2, 3$, $\Psi_i^{u,v} \in \mathcal{P}_M$ and there exist $L_2 : E \to E$ and $L_3 : E \to H^{s-1}$ continuous such that

$$\mathcal{M}^{-1}\Psi_i^{u,v}(n) = (uL_i(n), v). \tag{2.13}$$

Again, informally this may be written as:

$$L_2(n) = \mathcal{M}^{-1}(nL_1(n) - \mathbb{E}_\nu(nL_1(n))) = \mathcal{M}^{-1}(n\mathcal{M}^{-1}n - \mathbb{E}_\nu(n\mathcal{M}^{-1}n)) \tag{2.14}$$

and

$$\begin{aligned} L_3(n) &= \mathcal{M}^{-1}(\nabla n \cdot \nabla L_1(n) - \mathbb{E}_\nu(\nabla n \cdot \nabla L_1(n))) \\ &= \mathcal{M}^{-1}(\nabla n \cdot \nabla \mathcal{M}^{-1}n - \mathbb{E}_\nu(\nabla n \cdot \nabla \mathcal{M}^{-1}n)) \end{aligned} \tag{2.15}$$

[1] We do not really need this and could require the existence of $\tilde{m}$ such that $\tilde{m}(t, m(0)) = m(t)$ only in distribution. We think that this slightly stronger assumption allows to lighten the definitions and proofs below. Note that, with this assumption, we have a process $(\tilde{m}(t,n))_{t\geq 0}$ for all $n \in E$, even if m does not visit the whole space E.

We need to invert a further function of n. For u, $v \in H^1$, we define

$$\Psi_4^{u,v}(n) = (un, v)(uL_1(n), v) - \mathbb{E}_\nu(un, v)(uL_1(n), v) \tag{2.16}$$

and assume that for each u, $v \in H^1$, $\Psi_4^{u,v} \in \mathcal{P}_M$ and there exists $L_4 : E \to L^\infty(\mathbb{R}^d \times \mathbb{R}^d)$ continuous such that

$$\mathcal{M}^{-1}\Psi_4^{u,v}(n) = Re \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} u(x)u(y)L_4(n)(x,y)\bar{v}(x)\bar{v}(y)dxdy. \tag{2.17}$$

Informally:

$$L_4(n)(x,y) = \mathcal{M}^{-1}(n(x)L_1(n)(y) - \mathbb{E}_\nu n(x)L_1(n)(y)) \text{ and } L_2(n)(x) = L_4(n)(x,x). \tag{2.18}$$

In previous articles on approximation-diffusion for PDEs, the process is assumed to be almost surely bounded: $\|m(t)\|_E \leq K$ for all $t \geq 0$, $\mathbb{P}$ $a.s.$ for some deterministic constant K. This boundedness assumption is replaced here by the weaker assumption that there exist $\gamma > 6$ and a constant $\mathcal{C}$ such that the following estimate holds

$$\mathbb{E}\left(\sup_{t\in[0,1]} \|m(t)\|_E^\gamma\right) \leqslant \mathcal{C}. \tag{2.19}$$

By stationarity of m, this implies that for all $k \in \mathbb{N}$:

$$\mathbb{E}\left(\sup_{t\in[k,k+1]} \|m(t)\|_E^\gamma\right) \leqslant \mathcal{C}. \tag{2.20}$$

Note that this implies:

$$\int_E \|n\|_E^\gamma d\nu < \infty$$

and

$$\sup_{t\in[0,T]} \|m(t)\|_E < \infty, \ a.s.,$$

for any $T \geq 0$.

We need some control on the growth of the functions L_i introduced above. We assume that there exist $\eta < \gamma/2 - 1$ and $p \in \mathbb{N}$ such that:

$$\begin{aligned} \|L_i(n)\|_E &\leqslant c(1 + \|n\|_E)^\eta, \ n \in E, \ i = 1, 2, \\ \|L_3(n)\|_{H^{s-1}} &\leqslant c(1 + \|n\|_E)^\eta, n \in E, \\ \|L_4(n)\|_{L^\infty(\mathbb{R}^d\times\mathbb{R}^d)} &\leqslant c(1 + \|n\|_E)^\eta, \ n \in E. \end{aligned} \tag{2.21}$$

In many situations, $\mathcal{M}^{-1}\psi$ is given by:

$$\mathcal{M}^{-1}\psi(n) = -\int_0^\infty P_t\psi(n)dt,$$

and, in particular, for each $n \in E$, $P_t\psi(n)$ is an integrable function: $t \mapsto P_t f(n) \in L^1(0,\infty)$. Here we do not need this but we simply use the assumption that there exists $\lambda \in L^1(0,\infty)$ such that:

$$\|\mathbb{E}(\tilde{m}(t,n))\|_E \leqslant \lambda(t)(1 + \|n\|_E)^{\gamma-1} \tag{2.22}$$

and that L_1 is given by

$$(w, L_1(n)) = -\int_0^\infty \mathbb{E}\left[(w, \tilde{m}(t,n))\right] dt, \quad w \in E'. \tag{2.23}$$

We also use the following object in next section. Note that assumptions (2.19) and (2.22) imply that, for $x, y \in \mathbb{R}^d$, $t \mapsto \mathbb{E}(m(0)(x)m(t)(y))$ is integrable over $[0, \infty)$, so that we can define:

$$k(x,y) = \int_0^\infty \mathbb{E}(m(0)(x)m(t)(y) + m(t)(x)m(0)(y))dt \text{ for } x, y \in \mathbb{R}^d. \tag{2.24}$$

It is not difficult to check that $k \in W^{3,\infty}(\mathbb{R}^d \times \mathbb{R}^d) \cap H^1(\mathbb{R}^d \times \mathbb{R}^d)$. We assume that

$$k(\cdot,\cdot) : x \mapsto k(x,x) \in W^{1,1}(\mathbb{R}^d), \quad \Delta_1 k(\cdot,\cdot) : x \mapsto \Delta_1 k(x,x) \in L^1(\mathbb{R}^d), \tag{2.25}$$

where Δ_1 denotes the Laplace operator with respect to the first variable of k.

We use the notation

$$F(x) = k(x,x). \tag{2.26}$$

Thanks to (2.22), it can be seen that $F \in E$.

We end this section with the following Lemma. It is similar to [6, Lemma 15.4.4] and is fundamental to avoid the uniform boundedness assumption on the process m.

Lemma 2.1. *For all $T > 0$, and $\varepsilon > 0$, and $\alpha > \frac{2}{\gamma}$*

$$\mathbb{P}\left(\sup_{t\in[0,T]} \|m^\varepsilon(t)\|_E \geq \varepsilon^{-\alpha} \right) \to 0,$$

when ε goes to 0.

Proof. For $k \in \mathbb{N}$, denote by η_k the random variable

$$\eta_k := \sup_{t\in[k,k+1]} \|m(t)\|_E ,$$

then by the Markov inequality and (2.20) we have for all δ and k

$$\mathbb{P}(\eta_k \geq k^\delta) \leqslant \frac{\mathbb{E}\,(\eta_k^\gamma)}{k^{\delta\gamma}} \leqslant \frac{\mathcal{C}}{k^{\delta\gamma}},$$

and choosing δ such that $\delta\gamma > 1$, we have

$$\sum_{k\geq 0} \mathbb{P}(\eta_k \geq k^\delta) < \infty,$$

so we get by the Borel-Cantelli lemma that for $\mathbb{P}$ almost every $\omega \in \Omega$, there exists $k_0(\omega)$ such that $\eta_k \leq k^\delta$ for $k \geq k_0(\omega)$. It follows that, for $t \geq k_0(\omega)$, $\|m(t)\|_E \leq t^\delta$ and

$$\mathbb{P} - a.s., \forall t \in \mathbb{R}^+, \|m(t)\|_E \leqslant Z_1 + |t|^\delta,$$

with the random variable Z_1 defined by $Z_1(\omega) = \sup_{s\in[0,k_0(\omega)]} \|m(s\|_E$. Finally, since

$$\left\{ \sup_{t\in[0,T]} \|m^\varepsilon(t)\|_E \geq \varepsilon^{-\alpha} \right\} \subset \left\{ Z_1 + \left|\frac{T}{\varepsilon^2}\right|^\delta \geq \varepsilon^{-\alpha} \right\}$$

and the probability of the right-hand side event goes to 0, when $\varepsilon \to 0$, under the condition $\alpha > 2\delta$, this ends the proof. □

We naturally define the $(\mathcal{F}_t^\varepsilon)$-stopping time τ^ε by

$$\tau^\varepsilon = \inf\{t \in [0,T], \|m^\varepsilon(t)\|_E \geq \varepsilon^{-\alpha}\}, \tag{2.27}$$

with the convention that $\tau^\varepsilon = T$ when this set is empty.

Note it is possible that τ^ε is equal to 0, this is the case when $\|m^\varepsilon(0)\|_E \geq \varepsilon^{-\alpha}$. Otherwise, when $\tau^\varepsilon > 0$, we have $\sup_{t\in[0,\tau^\varepsilon]} \|m^\varepsilon(t)\|_E \leq \varepsilon^{-\alpha}$.

We will use many times that $\mathbb{P}(\tau^\varepsilon < T) \leqslant \mathbb{P}(\sup_{t\in[0,T]} \|m^\varepsilon(t)\|_E \geq \varepsilon^{-\alpha})$, which together with Lemma 2.1 yields

$$\mathbb{P}(\tau^\varepsilon < T) \to_{\varepsilon\to 0} 0. \tag{2.28}$$

2.3. **An example.** An example of assumptions on the process m that can be checked in practice and are sufficient to satisfy all the above hypotheses is the following:

(1) For every $n \in E$, $(\tilde{m}(t,n))_{t\geqslant 0}$ is a stochastically continuous Markov process associated to the semigroup $(P_t)_{t\geqslant 0}$.
(2) $(P_t)_{t\geqslant 0}$ is Feller.
(3) For every $k \in \mathbb{N}$ and $n \in E$, $\mathbb{E}[\|\tilde{m}(t,n)\|_E^k]$ is finite and there exist $C_k > 0, \ell_k \in \mathbb{N}$ such that for every $t \in [0,T]$

$$\mathbb{E}[\|\tilde{m}(t,n)\|_E^k] \leqslant C_k(1+\|n\|_E)^{\ell_k}, \qquad n \in E. \tag{2.29}$$

(4) There exist $\kappa > 0, \gamma > 0, k_0 \in \mathbb{N}$ such that for any $n_1, n_2 \in E$ we can construct a coupling $(m^1(t), m^2(t))$ of $(\tilde{m}(t,n_1), \tilde{m}(t,n_2))_{t\geqslant 0}$ such that

$$\mathbb{P}\left(m^1(t) \neq m^2(t)\right) \leqslant \kappa\left(1+\|n_1\|_E + \|n_2\|_E\right)^{k_0} e^{-\gamma t}. \tag{2.30}$$

By coupling, we mean that the law of $(m_1(t))_{t\geq 0}$ (resp. $(m_2(t))_{t\geq 0}$ is the same as $(\tilde{m}(t,n_1))_{t\geq 0}$ (resp. $(\tilde{m}(t,n_2))_{t\geq 0}$). It follows classically from this last assumption that $(P_t)_{t\geqslant 0}$ has a unique invariant measure ν which is exponentially mixing. We finally need

(5) ν is centered: $\displaystyle\int_E n\nu(dn) = 0$, and for any $k \in \mathbb{N}$

$$\int_E \|n\|_E^k \,\nu(dn) < \infty. \tag{2.31}$$

We then take m as a stationary process with law ν such that $\tilde{m}(t, m(0)) = m(t)$.

Define

$$\mathcal{P}_\mathcal{M} = \left\{\psi \in C_P(E), \int_E \psi(n)d\nu(n) = 0\right\}.$$

For $\psi \in \mathcal{P}_\mathcal{M}$, we may define

$$\mathcal{M}^{-1}\psi(n) = -\int_0^\infty P_t\psi(n)dt. \tag{2.32}$$

Our assumptions imply that $\mathcal{M}^{-1}\psi \in C_P(E)$ and that (2.6) is an integrable martingale. We define

$$\mathcal{D}_\mathcal{M} = \left\{\mathcal{M}^{-1}\psi, \psi \in \mathcal{P}_\mathcal{M}\right\}$$

.

Clearly for any $\mathcal{M}^{-1}\psi \in \mathcal{D}_\mathcal{M}$, we may take $\mathcal{M}\mathcal{M}^{-1}\psi = \psi$ and $L_1, \cdots, L_4$ can be constructed and follow our assumptions.

Let us now construct a process satisfying 1 to 5. We consider the following stochastic equation in a Hilbert space H: for $t > 0, x \in H$

$$\begin{cases} dX_t = (AX_t + G(X_t))dt + \sigma dW_t, \\ X_0 = x, \end{cases} \tag{2.33}$$

where A is an unbounded operator on $D(A)$ dense in H, invertible with a compact inverse, which generates an analytic semigroup $(e^{tA})_{t\geqslant 0}$ and such that

$$\langle Ax, x\rangle \leqslant -\lambda_1 \|x\|_H^2, \qquad x \in D(A), \tag{2.34}$$

and

$$\left\|e^{tA}\right\|_{\mathcal{L}(H)} \leqslant Me^{-\eta t}, \qquad t > 0,$$

for some $k > 0, M > 0, \eta > 0$. Assume moreover that, for all $t > 0$, e^{tA} is Hilbert-Schmidt on H and

$$\left\|e^{tA}\right\|_{\mathcal{L}_2(H)} \leqslant Lt^{-\gamma}, \qquad t > 0,$$

with $L > 0, \gamma \in [0, \frac{1}{2})$.

Remark that if $H = L^2(\mathbb{R}^d)$, $A = -(-\Delta + |x|^2)^r$ where $-\Delta$ is the Laplace operator, then A satisfies our assumptions provided that $r > d$.

The nonlinear term G can be chosen in various ways, for simplicity we assume that it is Lipschitz bounded. The covariance of the noise σ is an invertible operator on H. It has been proved in [12] that (2.33) defines a Markov process in H. We consider a continuous linear invertible operator $\Lambda : H \to E$ and take

$$\tilde{m}(t, n) = \Lambda\left(X(t, \Lambda^{-1}n + \bar{x}) - \bar{x}\right), \tag{2.35}$$

where $\bar{x} = \displaystyle\int_H x\nu(dx)$, ν being the invariant measure of X_t, and, for $x \in H$, $X(t, x)$ being the unique solution of (2.33).

When $H = L^2(\mathbb{R}^d)$ and $A = -(-\Delta + |x|^2)^r$, one may consider $\Lambda = (-\Delta + |x|^2)^\gamma$. Then, Λ maps H into E and (2.25) is satisfied for $\gamma > d + 1$. At the end of section 2.4, we introduce a further assumption which is satisfied for $\gamma \geq 2d$. More generally, Λ can be the solution map associated to an elliptic equation of sufficiently high order and containing a confining potential.

The conditions (1), (2) and (4) follow from corresponding properties on X_t proved in [12]. Concerning (3) and (5), they may be proved by classical computations based on the change of unknown

$$Y(t) = X(t) - \int_0^t e^{A(t-s)}\sigma dW_s$$

in (2.33) and energy estimates. In Appendix A we give details and prove that the process $m(t, n)$ satisfies conditions 1 to 5 and that these conditions are indeed sufficient to construct $L_1, \cdots, L_4$ as above.

Remark 2.1. *We could also build an example based on a Markov chain in E as in* [13].

2.4. **The covariance operator.** Let Q be the linear operator defined by

$$Qf(y) = Re \int_{\mathbb{R}^d} k(x, y)f(x)dx \text{ for } f \in H^1_{\mathbb{C}}. \tag{2.36}$$

Since $k \in H^1(\mathbb{R}^d \times \mathbb{R}^d)$, Q maps $H^1_{\mathbb{C}}$ into itself.

The following lemma, whose proof can be found in [15], is useful to prove that Q is non-negative.

Lemma 2.2 (Wiener-Kintchine)**.** *Let $\big(x(t)\big)_{t\in\mathbb{R}}$ be a real-valued stationary and centered process, we set $C(t) = \mathbb{E}(x(t)x(0))$ and assume C is integrable on $\mathbb{R}$. Defining*

$$\hat{x}_T(\nu) = \int_{-T}^{T} x(t)e^{-i\nu t}dt$$

we have

$$\int_{\mathbb{R}} C(\tau)e^{i\tau\nu}d\tau = \lim_{T\to+\infty}\frac{1}{2T}\mathbb{E}\Big|\hat{x}_T(\nu)\Big|^2.$$

We now use the last assumptions on m and $\mathcal{P}_{\mathcal{M}}$ to show the following properties:

Proposition 2.1. *The operator Q has finite trace on L^2, is self-adjoint and non-negative: $(Qf,f)\geq 0$ for all $f\in L^2$. Moreover the following identities hold*

$$k(x,y) = \int_{\mathbb{R}} \mathbb{E}(m(t)(x)m(0)(y))dt, \tag{2.37}$$

$$\int_0^\infty \mathbb{E}(m(t)(x)m(0)(y))dt = -\int_E n(y)L_1(n)(x)d\nu(n), \tag{2.38}$$

where ν is the law of $m(0)$.

Proof. We have assumed that $k(\cdot,\cdot)\in L^1(\mathbb{R}^d)$ so $Tr(Q)<\infty$, and k is obviously symmetric. Moreover the stationarity of m yields

$$\begin{aligned}\int_0^\infty \mathbb{E}\Big(m(0)(x)m(t)(y)\Big)dt &= \int_0^\infty \mathbb{E}\Big(m(-t)(x)m(0)(y)\Big)dt\\ &= \int_{-\infty}^0 \mathbb{E}\Big(m(t)(x)m(0)(y)\Big)dt,\end{aligned}$$

and we easily get (2.37). Let us now prove the positivity of Q. We define $x(t)=(m(t),f)$, which is centered stationary process, and denote by $C(t)=\mathbb{E}(x(t)x(0))$ its correlation function. Then definitions (2.36) and (2.37) of k and Q and Lemma 2.2 yield

$$(Qf,f) = \int_{\mathbb{R}} C(\tau)d\tau = \lim_{T\to+\infty}\frac{1}{T}\mathbb{E}\Big|\hat{x}_T(0)\Big|^2 \geq 0.$$

Finally, we write thanks to (2.23) with $w=\delta_x\in E'$:

$$\begin{aligned}-\int_E n(y)L_1(n)(x)d\nu(n) &= -\mathbb{E}\Big(m(0)(y)L_1(m(0))(x)\Big)\\ &= \mathbb{E}\Big(m(0)(y)\int_0^\infty \mathbb{E}\left(\tilde{m}(t,m(0))(x)|\mathcal{F}_0\right)dt\Big)\\ &= \int_0^\infty \mathbb{E}(m(t)(x)m(0)(y))dt,\end{aligned}$$

this proves (2.38) □

Thanks to Proposition 2.1, we may define the operator $Q^{1/2}$ which is Hilbert-Schmidt on L^2. Let us denote by q its kernel. Then, we have:

$$k(x,y) = \int_{\mathbb{R}^d} q(x,z)q(z,y)dz.$$

We need a little more smoothness on this operator. We have seen that $F\in E$, therefore

$$F\in W^{1,1}\cap W^{1,\infty}, \tag{2.39}$$

with F defined in (2.26). We also know that $Q^{1/2}$ is Hilbert-Schmidt from L^2 to H^1 since:

$$\|Q^{1/2}\|^2_{\mathcal{L}_2(L^2,H^1)} \leqslant \int_{\mathbb{R}^d} k(x,x)+|\Delta_1 k(x,x)|dx.$$

We assume that it is also γ-radonifying (see [3] or [7] for the definition) from $L^2(\mathbb{R}^d)$ to $W^{1,p}(\mathbb{R}^d)$ for any $p \geq 2$. It is shown in [3] that this amounts to assume that $q \in W^{1,p}(\mathbb{R}^d; L^2(\mathbb{R}^d))$. This is true for instance if $x \mapsto \Delta_1^\beta k(x,x) \in L^1(\mathbb{R}^d)$ for $\beta \geq d$ as can be seen from Sobolev embeddings since $Q^{1/2}$ is γ-radonifying from $L^2(\mathbb{R}^d)$ to $W^{1,p}(\mathbb{R}^d)$ for any $p \geq 2$ when it is Hilbert-Schmidt from $L^2(\mathbb{R}^d)$ to $H^{d/2}(\mathbb{R}^d)$.

Under these properties on $Q^{1/2}$ and F, it is shown in [8] that (1.4) has a unique global solution in H^1.

2.5. **Main result.** We state here our main result, where we recall all the assumptions needed on the process m.

Theorem 2.1. *Let $d \in \mathbb{N}$, $\lambda \in \mathbb{R}$, $0 \leqslant s < 1$, and let $\sigma \in \mathbb{R}$ such that*

$$0 \leqslant \sigma < \frac{2}{d-2}, d \geqslant 3 \text{ or } 0 \leqslant \sigma, d = 1, 2$$

and

$$\sigma < \frac{2}{d} \text{ if } \lambda < 0.$$

Consider the randomly perturbed Nonlinear Schrödinger equation

$$i\partial_t X^\varepsilon(t) = -\Delta X^\varepsilon(t) + \lambda |X^\varepsilon(t)|^{2\sigma} X^\varepsilon(t) + \frac{1}{\varepsilon} X^\varepsilon(t) m(t/\varepsilon^2),$$

where m is a centered $(\mathcal{F}_t)$-adapted, stationary $E = H^s(\mathbb{R}^d)$-valued homogeneous Markov process for which there exist $\gamma > 6$ and $C > 0$ such that

$$\mathbb{E}\left[\sup_{t\in[0,1]} \|m(t)\|_E^\gamma\right] \leqslant C.$$

Assume furthermore that (2.21),(2.22), (2.23) and (2.25) are satisfied. Finally, consider F and Q respectively defined in (2.26), (2.36), with k defined in (2.24) which can be written as

$$k(x,y) = \int_{\mathbb{R}^d} q(x,z) q(z,y) dz$$

where q is the kernel of the operator $Q^{\frac{1}{2}}$. Assume that $q \in W^{1,p}(\mathbb{R}^d, L^2(\mathbb{R}^d))$ for any $p \geqslant 1$. Then for all $T > 0$ and $0 < \zeta \leqslant 1$, assuming that X_0^ε converges in law to X_0 in Σ^ζ defined in (2.1), the $C([0,T], H^s)$-valued process X^ε solution of the above Schrödinger equation with initial data X_0^ε converges in law in $C([0,T], H^s(\mathbb{R}^d))$ to X solution of

$$idX_t = \Big(-\Delta X_t + \lambda |X_t|^{2\sigma} X_t - \frac{i}{2} F X_t \Big) dt - X_t Q^{1/2} dW_t$$

with initial condition X_0 and where W_t is a cylindrical Wiener process on $L^2(\mathbb{R}^d)$.

The proof of this theorem is detailed in section 5. We first study equation (1.1) and the generator of the couple $(X^\varepsilon, m^\varepsilon)$. This allows to introduce the perturbed test function method which is the key tool to obtain tightness and prove the main result.

3. The equation

3.1. **The Cauchy problem.** This subsection is devoted to prove the following proposition. It states the existence of X^ε and provides a bound in H^1. This bound is obtained by standard arguments but is not uniform in ε. A uniform bound is obtained with more sophisticated tools below.

Proposition 3.1. *Given $X_0^\varepsilon \in H^1$, then almost surely, there exists a unique mild solution X^ε of (1.1) with initial data X_0^ε which lies in $C([0,T], H^1(\mathbb{R}^d) \cap L^r(0,T; W^{1,2\sigma+2}(\mathbb{R}^d))$, $r = \frac{4(\sigma+1)}{n\sigma}$, for any T. Moreover, we have $\mathcal{C}_T$, $\mathcal{C}_{T,\varepsilon}$ such that $\mathbb{P}-a.s.$ for $0 \leqslant t \leqslant T$:*

$$\begin{aligned} &\|X^\varepsilon(t)\|_{L^2} = \|X_0^\varepsilon\|_{L^2}\,, \\ &\|\nabla X^\varepsilon(t)\|_{L^2} \leqslant \mathcal{C}_T e^{\mathcal{C}_{T,\varepsilon}} \left(\mathcal{H}(X_0^\varepsilon) + \sup_{s\in[0,t]} \|m^\varepsilon(s)\|_E^2 \, \|X_0^\varepsilon\|_{L^2}^2 + c_\lambda \|X_0^\varepsilon\|_{L^2}^{p_d} \right), \end{aligned} \tag{3.1}$$

with $p_d = (2\sigma + 2 - \sigma d)\frac{2}{2-\sigma d}$ and $c_\lambda = \lambda^-$ is the negative part of λ: $c_\lambda = -\lambda$ if $\lambda < 0$ and $c_\lambda = 0$ for $\lambda \geq 0$. Finally if X_0^ε is $\mathcal{F}_0$-measurable then the process X^ε is $(\mathcal{F}_t^\varepsilon)$-adapted.

Proof. We solve (1.1) pathwise and consider the equation

$$i\partial_t u^\varepsilon = -\Delta u^\varepsilon + \lambda |u^\varepsilon|^{2\sigma} u^\varepsilon + \frac{1}{\varepsilon} m^\varepsilon(t,\omega) u^\varepsilon.$$

We use here Kato's method (see [5, Theorem 4.4.1]) to get existence and uniqueness of $X^\varepsilon(\omega) = u^\varepsilon$ for a small enough T_0. We recall that X^ε is the fixed-point of ϕ defined for $u \in \mathcal{E}_T$ by:

$$\phi(u)(t) = S(t)X_0^\varepsilon(\omega) - i\lambda \int_0^t S(t-s)|u(s)|^{2\sigma}u(s)ds - \frac{i}{\varepsilon}\int_0^t S(t-s)u(s)m^\varepsilon(s)ds \tag{3.2}$$

in the space

$$\begin{aligned} \mathcal{E}_T = \{ & u \in L^\infty([0,T_0]; H^1(\mathbb{R}^d)) \cap L^r(0,T_0; W^{1,2\sigma+2}(\mathbb{R}^d)); \\ & \|u\|_{L^\infty(0,T_0;H^1(\mathbb{R}^d))} \leqslant M, \|u\|_{L^r(0,T_0;W^{1,2\sigma+2}(\mathbb{R}^d))} \leqslant M\} \end{aligned}$$

endowed with the distance

$$d(u,v) = \|u-v\|_{L^\infty(0,T_0;L^2(\mathbb{R}^d))} + \|u-v\|_{L^r(0,T_0;L^{2\sigma+2}(\mathbb{R}^d))}\,.$$

Kato's method proves that ϕ is a contraction for well chosen M, T_0, and allows to get the continuity in time of the solution.

Since ϕ given by (3.2) maps $(\mathcal{F}_t^\varepsilon)$-adapted processes onto $(\mathcal{F}_t^\varepsilon)$-adapted processes, and that X^ε is obtained by iterating ϕ, this gives that X^ε is $(\mathcal{F}_t^\varepsilon)$-adapted.

Moreover, it is easy to verify that $\|X^\varepsilon(t,\omega)\|_{L^2} = \|X_0^\varepsilon\|_{L^2}$. To prove the bound on the gradient, we use the energy $\mathcal{H}$ defined in (2.3) and a regularization is necessary to justify the computations.

We denote now by $X^\varepsilon_\delta = \rho_\delta \star X^\varepsilon(\omega) \in H^k$ for all $k \in \mathbb{N}$. And we compute $\mathcal{H}(\rho_\delta \star X^\varepsilon(t))$:

$$\begin{aligned}
&\frac{d\mathcal{H}(\rho_\delta \star X^\varepsilon(t))}{dt} = \mathcal{H}'(X^\varepsilon_\delta(t)).(\rho_\delta \star \partial_t X^\varepsilon) \\
&= Re\left(\int_{\mathbb{R}^d} (-\Delta X^\varepsilon_\delta + \lambda |X^\varepsilon_\delta|^{2\sigma} X^\varepsilon_\delta)\overline{\partial_t X^\varepsilon_\delta} dx\right) \\
&= Re\left(\int_{\mathbb{R}^d} (-\Delta X^\varepsilon_\delta + \lambda |X^\varepsilon_\delta|^{2\sigma} X^\varepsilon_\delta)(-i\Delta\overline{X^\varepsilon_\delta} + i\lambda\rho_\delta \star (|X^\varepsilon|^{2\sigma}\overline{X^\varepsilon} + \frac{i}{\varepsilon}\rho_\delta \star (\overline{X^\varepsilon} m^\varepsilon)))dx\right) \\
&= Re\left(\int_{\mathbb{R}^d} -\Delta X^\varepsilon_\delta (i\lambda\rho_\delta \star (|X^\varepsilon|^{2\sigma}\overline{X^\varepsilon}) + \frac{i}{\varepsilon}\rho_\delta \star (\overline{X^\varepsilon} m^\varepsilon))dx\right) \\
&\quad + Re\left(\int_{\mathbb{R}^d} \lambda |X^\varepsilon_\delta|^{2\sigma} X^\varepsilon_\delta (-i\Delta\overline{X^\varepsilon_\delta} + i\lambda\rho_\delta \star (|X^\varepsilon|^{2\sigma}\overline{X^\varepsilon}) + \frac{i}{\varepsilon}\rho_\delta \star (\overline{X^\varepsilon} m^\varepsilon))dx\right).
\end{aligned}$$

Since $\nabla X^\varepsilon \in L^\infty(0,T_0;L^2) \cap L^r(0,T_0;L^{2\sigma+2}))$ we get that $\nabla X^\varepsilon_\delta \to \nabla X^\varepsilon$ in $L^r(0,T_0;L^{2\sigma+2})$ and similarly $\rho_\delta \star (|X^\varepsilon|^{2\sigma}X^\varepsilon)$ converges to $|X^\varepsilon|^{2\sigma}X^\varepsilon$ in $L^{r'}(0,T_0;W^{1,\frac{2\sigma+2}{2\sigma+1}})$ as δ goes to 0. The other terms are treated similarly and after integration in time, and $\delta \to 0$ we obtain

$$\mathcal{H}(X^\varepsilon(t)) = \mathcal{H}(X^\varepsilon_0) - \frac{1}{\varepsilon}\int_0^t Im \int_{\mathbb{R}^d} \nabla X^\varepsilon(s).\nabla m^\varepsilon(s,\omega)\overline{X^\varepsilon(s)} ds$$

which leads to, using the Cauchy-Schwarz and Young inequalities

$$\mathcal{H}(X^\varepsilon(t)) \leqslant \mathcal{H}(X^\varepsilon_0) + \frac{1}{2\varepsilon}\int_0^t \|m^\varepsilon(s)\|_E^2 \|X^\varepsilon_0\|_{L^2}^2 + \|\nabla X^\varepsilon(s)\|_{L^2}^2 ds. \tag{3.3}$$

This implies (3.1) when $\lambda \geq 0$. For $\lambda < 0$, we use the Gagliardo-Nirenberg and Young inequalities and have for $u \in H^1$ and $\sigma < 2/d$

$$\frac{1}{2\sigma+2}\|u\|_{L^{2\sigma+2}}^{2\sigma+2} \leqslant \mathcal{C}\|u\|_{L^2}^{2\sigma+2-\sigma d}\|\nabla u\|_{L^2}^{\sigma d} \leqslant \frac{1}{4}\|\nabla u\|_{L^2}^2 + \mathcal{C}\|u\|_{L^2}^{p_d}. \tag{3.4}$$

Thus condition (1.3) implies that the energy provides a control on the L^2 norm of the gradient. In fact, the energy bound (3.3), the Gronwall lemma and the conservation of the L^2 norm yield the result. Global existence follows from (3.1) and the conservation of the L^2 norm.

□

3.2. **Persistence in the spaces Σ^ζ.** The result given in this subsection is a modification of [5, Theorem 6.5.1].

Proposition 3.2. *Assume $X^\varepsilon_0 \in \Sigma^\zeta$ for some $0 < \zeta \leqslant 1$ then $X^\varepsilon \in C([0,T],\Sigma^\zeta)$ almost surely for any $T > 0$. More precisely there exists $\mathcal{C}_T > 0$ a deterministic constant such that*

$$\left\|X^\varepsilon(t)\langle x\rangle^\zeta\right\|_{L^2}^2 \leqslant \mathcal{C}_T\left(\left\|X^\varepsilon_0\langle x\rangle^\zeta\right\|_{L^2}^2 + \int_0^t \|\nabla X^\varepsilon(s)\|_{L^2}^2 ds\right). \tag{3.5}$$

Remark 3.1. *Since the constant $\mathcal{C}_T$ does not depend on ε, this result shows that if we are able to prove a bound on the H^1 norm of the solution uniform in ε, a uniform bound in ε on the weighted norm follows immediately.*

Proof. Recall that we denote by $\langle x\rangle = \sqrt{1+|x|^2}$. Given $\delta \in (0,1)$, denote by φ_δ the function

$$\varphi_\delta(u) = \int_{\mathbb{R}^d} \langle x\rangle^{2\zeta}|u(x)|^2 e^{-\delta\langle x\rangle} dx.$$

Note that:

$$(X^\varepsilon(t), -i|X^\varepsilon(t)|^2X^\varepsilon(t)) = Re\left(\int_{\mathbb{R}^d} -i\overline{X^\varepsilon(t)}|X^\varepsilon(t)|^2X^\varepsilon(t)dx\right) = 0,$$

and similarly, since m^ε is real valued, $(X^\varepsilon(t), -im^\varepsilon(t)X^\varepsilon(t)) = 0$. It follows:

$$\begin{aligned}\frac{d}{dt}\varphi_\delta(X^\varepsilon(t)) &= 2(X^\varepsilon(t), \partial_t X^\varepsilon(t)\langle x\rangle^{2\zeta}e^{-\delta\langle x\rangle}) \\ &= 2(X^\varepsilon(t), i\Delta X^\varepsilon(t)\langle x\rangle^{2\zeta}e^{-\delta\langle x\rangle}) \\ &= 2Im\left(\int_{\mathbb{R}^d}\overline{X^\varepsilon(t)}\nabla X^\varepsilon(t).\nabla\left(\langle x\rangle^{2\zeta}e^{-\delta\langle x\rangle}\right)dx\right) + 2Im\left(\int_{\mathbb{R}^d}\nabla\overline{X^\varepsilon(t)}.\nabla X^\varepsilon(t)\langle x\rangle^{2\zeta}e^{-\delta\langle x\rangle}dx\right).\end{aligned}$$

Note that the last term vanishes. Since X^ε is sufficiently smooth, no regularization argument in the computation above is needed.

We have $\nabla(\langle x\rangle^{2\zeta}e^{-\delta\langle x\rangle}) = (2\zeta\langle x\rangle^{2\zeta-2} - \delta\langle x\rangle^{2\zeta-1})e^{-\delta\langle x\rangle}x$ and since $|x| \leq \langle x\rangle$:

$$|\nabla(\langle x\rangle^{2\zeta}e^{-\delta\langle x\rangle})| \leq (2\zeta\langle x\rangle^{2\zeta-1} + \delta\langle x\rangle^{2\zeta})e^{-\delta\langle x\rangle}.$$

It follows:

$$\begin{aligned}\varphi_\delta(X^\varepsilon(t)) &= \varphi_\delta(u_0^\varepsilon) \\ &\quad + 2Im\left(\int_0^t\int_{\mathbb{R}^d}\bar{X}^\varepsilon(t)\nabla X^\varepsilon(t).\left((2\zeta\langle x\rangle^{2\zeta-2} - \delta\langle x\rangle^{2\zeta-1})e^{-\delta\langle x\rangle}x\right)\right)dx \\ &\leq \varphi_\delta(u_0^\varepsilon) \\ &\quad + 2\left(\int_0^t\int_{\mathbb{R}^d}\left|X^\varepsilon(t)\langle x\rangle^\zeta e^{-\frac{\delta}{2}\langle x\rangle^{2\zeta}}\right|\left|\nabla X^\varepsilon(t)\right|\left(2\zeta\langle x\rangle^{\zeta-1} + \delta\langle x\rangle^\zeta\right)e^{-\frac{\delta}{2}\langle x\rangle}\right)dx.\end{aligned} \tag{3.6}$$

Since $(2\zeta t^{\zeta-1} + \delta t^\zeta)e^{-\frac{\delta}{2}t} \leq 2(1+e^{-1})$ for $\zeta \leq 1$, $t \geqslant 1$, $\delta \leqslant 1$, and $\varphi_\delta(u_0^\varepsilon) \leqslant \left\|u_0\langle x\rangle^\zeta\right\|_{L^2}^2$ we deduce

$$\varphi_\delta(X^\varepsilon(t)) \leqslant \left\|u_0\langle x\rangle^\zeta\right\|_{L^2}^2 + \mathcal{C}\int_0^t\|\nabla X^\varepsilon(s)\|_{L^2}\sqrt{\varphi_\delta(X^\varepsilon(s))}ds$$

which, thanks to the Young inequality and the Gronwall lemma, leads to

$$\varphi_\delta(X^\varepsilon(t)) \leqslant \mathcal{C}_T\left(\left\|u_0\langle x\rangle^\zeta\right\|_{L^2}^2 + \int_0^t\|\nabla X^\varepsilon(s)\|_{L^2}^2\,ds\right).$$

Finally, by Fatou's lemma, letting $\delta \downarrow 0$:

$$\left\|X^\varepsilon(t)\langle x\rangle^\zeta\right\|_{L^2}^2 \leqslant \mathcal{C}_T\left(\left\|u_0\langle x\rangle^\zeta\right\|_{L^2}^2 + \int_0^t\|\nabla X^\varepsilon(s)\|_{L^2}^2\,ds\right)$$

which is exactly (3.5). Then by (3.1), we see that the right-hand side of (3.5) is bounded, which implies $X^\varepsilon \in L^\infty([0,T],\Sigma^\zeta)$. Since we already know that it is continuous with values in H^1, we deduce $X^\varepsilon \in C_w([0,T],\Sigma^\zeta)$, where the subscript w indicates weak continuity in time. Letting $\delta \downarrow 0$ in the first part of (3.6) we see that

$$\left\|X^\varepsilon(t)\langle x\rangle^\zeta\right\|_{L^2}^2 = \left\|u_0\langle x\rangle^\zeta\right\|_{L^2}^2 + 4\zeta Im\int_0^t\left(\int_{\mathbb{R}^d}X^\varepsilon(s)\langle x\rangle^{2\zeta-2}\nabla X^\varepsilon(s).xdx\right)ds$$

and since the right-hand side is continuous, we get the strong continuity: $X^\varepsilon \in C([0,T],\Sigma^\zeta)$ □

3.3. **Generator of $(X^\varepsilon, m^\varepsilon)$.** In order to determine the law of the limiting process X, we need to identify the generator of X^ε. Clearly, X^ε is not a Markov process, because its increments depend on m^ε, but the couple $(X^\varepsilon, m^\varepsilon)$ is a Markov process, since m is.

We compute now the generator $\mathcal{L}^\varepsilon$ of the process $(X^\varepsilon, m^\varepsilon)$. We are able to compute this generator acting on functions such as in the next definition. There are many such functions. In particular, we can choose functions independent on n: $(u,h)^\ell$ for $\ell = 1, 2$, $h \in H^1$ which allow to characterize the dynamic or a diffusion process. When we apply the generator to these functions, other functions of the form $(uL_i(n), h)$, $i = 1, 2$ are needed, the correctors, and thanks to the assumptions on L_i these also are good test function in the sense of the definition below. We also use the energy $\mathcal{H}$ and associated correctors to obtain H^1 bounds independent on ε.

We consider functions depending on $(u, n) \in H^1_{\mathbb{C}} \times E$, for such function φ we denote by $D\varphi$ its differential with respect to the variable u.

Definition 3.1 (Good test function)**.** *We say that $\Psi : H^1_{\mathbb{C}} \times E \longrightarrow \mathbb{R}, (u, n) \mapsto \Psi(u, n)$ is a good test function if the following holds:*

- *Ψ is continuously differentiable with respect to u, the differential is denoted by $D\Psi$.*
- *Ψ is subpolynomial in u and n; $\exists\, C_\Psi, \exists\, p_1 \in \mathbb{N}, \exists\, \gamma_1 < \gamma, \forall (u, n) \in H^1 \times E$,*
$$|\Psi(u, n)| \leqslant C_\Psi(1 + \|u\|^{p_1}_{H^1})(1 + \|n\|^{\gamma_1}_E).$$
- *$(u, n) \mapsto D\Psi(u, n)$ is continuous from $H^1 \times E$ to $\mathcal{L}(H^1_{\mathbb{C}}; \mathbb{C})$.*
- *$D\Psi(u, n)$ is continuous with respect to the H^{-1} norm and satisfies the following subpolynomial bound in u: $\exists\, C_\Psi, \exists\, p_2 \in \mathbb{N}, \exists\, \gamma_2 < \gamma - 1, \forall (u, h, n) \in H^1 \times H^{-1} \times E$,*
$$|D\Psi(u, n).(h)| \leqslant C_\psi(1 + \|u\|_{H^1})^{p_2}(1 + \|n\|^{\gamma_2}_E) \|h\|_{H^{-1}}.$$
- *$\forall u \in H^1_{\mathbb{C}}, \Psi(u, .) \in \mathcal{D}_{\mathcal{M}}$*
- *$\mathcal{M}\Psi$ is continuous and subpolynomial in u and n; $\exists\, C_\Psi, \exists\, p_3 \in \mathbb{N}, \exists\, \gamma_3 < \gamma, \forall (u, n) \in H^1 \times E$,*
$$|\mathcal{M}\Psi(u, n)| \leqslant C_\Psi(1 + \|u\|^{p_3}_{H^1})(1 + \|n\|^{\gamma_3}_E).$$

With these good test functions, we may identify the generator $\mathcal{L}^\varepsilon$ of the Markov process $(X^\varepsilon, m^\varepsilon)$. For the definition of the predictable quadratic variation of a martingale, we refer to [19] and recall that it coincides with the quadratic variation when the martingale is continuous.

Proposition 3.3. *For a good test function φ , the infinitesimal generator $\mathcal{L}^\varepsilon$ of $(X^\varepsilon, m^\varepsilon)$ is given by the formula:*

$$\begin{aligned}\mathcal{L}^\varepsilon\varphi(u, n) =& \frac{1}{\varepsilon^2}\mathcal{M}\varphi(u, n)\\ &+ \frac{1}{\varepsilon}(D\varphi(u, n).(-iun))\\ &+ D\varphi(u, n).(i\Delta u - i\lambda|u|^{2\sigma}u)\end{aligned}$$

for $u \in H^1_{\mathbb{C}}$, $n \in E$. More precisely, if $\mathbb{E}\,\|X^\varepsilon_0\|^p_{H^1} < \infty$ for all p:

$$M_\varphi(t) = \varphi(X^\varepsilon(t), m^\varepsilon(t)) - \varphi(X^\varepsilon_0, m^\varepsilon(0)) - \int_0^t \mathcal{L}^\varepsilon\varphi(X^\varepsilon(s), m^\varepsilon(s))ds$$

is a càdlàg and integrable $(\mathcal{F}^\varepsilon_t)$ zero-mean martingale. If furthermore γ_1, $\gamma_2 + 1$, $\gamma_3 < \gamma/2$ and φ^2 is also a good test function, its continuous quadratic variation is given by:

$$\langle M_\varphi, M_\varphi\rangle_t = \int_0^t (\mathcal{L}^\varepsilon\varphi^2 - 2\varphi\mathcal{L}^\varepsilon\varphi)(X^\varepsilon(s), m^\varepsilon(s))ds. \tag{3.7}$$

Proof. Let $t \geq s \geq 0$ and $s = t_0 < ... < t_n = t$ be a subdivision of $[s,t]$ with $sup_i|t_{i+1} - t_i| = \bar{\delta}$. Given g a $\mathcal{F}_s^\varepsilon$-measurable and bounded function, we have

$$\mathbb{E}\left[(\varphi(X^\varepsilon(t), m^\varepsilon(t)) - \varphi(X_0^\varepsilon, m^\varepsilon(0)))\, g\right] = \mathbb{E}\left[\left(\int_s^t \mathcal{L}^\varepsilon \varphi(X^\varepsilon(\sigma), m^\varepsilon(\sigma))d\sigma\right) g\right] + I + II,$$

where

$$\begin{aligned} I =& \sum_{i=0}^{n-1} \mathbb{E}\left[(\varphi(X^\varepsilon(t_{i+1}), m^\varepsilon(t_{i+1})) - \varphi(X^\varepsilon(t_i), m^\varepsilon(t_{i+1}))\right. \\ &- \int_{t_i}^{t_{i+1}} D\varphi(X^\varepsilon(s), m^\varepsilon(s)).(i\Delta X^\varepsilon(s) - i\lambda|X^\varepsilon(s)|^{2\sigma}X^\varepsilon(s) - \frac{i}{\varepsilon}X^\varepsilon(s)m^\varepsilon(s))ds\ )\, g] \\ =& \mathbb{E}\left[\left(\int_0^t i_{\bar{\delta}}(s)ds\right) g\right] \end{aligned}$$

with

$$i_{\bar{\delta}}(s) = -\sum_{i=0}^{n-1} \mathbf{1}_{[t_i,t_{i+1}]}(s)\left(D\varphi(X^\varepsilon(s), m^\varepsilon(t_{i+1})) - D\varphi(X^\varepsilon(s), m^\varepsilon(s))\right).\frac{dX^\varepsilon}{dt}(s),$$

and

$$\begin{aligned} II =& \sum_{i=0}^{n-1} \mathbb{E}\left[(\varphi(X^\varepsilon(t_i), m^\varepsilon(t_{i+1})) - \varphi(X^\varepsilon(t_i), m^\varepsilon(t_i))\right. \\ &- \frac{1}{\varepsilon^2}\int_{t_i}^{t_{i+1}} \mathcal{M}\varphi(X^\varepsilon(s), m^\varepsilon(s))ds\ )\, g] \\ =& \mathbb{E}\left[\left(\int_0^t ii_{\bar{\delta}}(s)ds\right) g\right]. \end{aligned}$$

To treat $i_{\bar{\delta}}$, we write $\frac{dX^\varepsilon}{dt}(\sigma) = i\Delta X^\varepsilon(s) - i|X^\varepsilon)(s)|^{2\sigma}X^\varepsilon(s) - i\frac{1}{\varepsilon}X^\varepsilon(s)m^\varepsilon(s)$ and since $D\varphi$ is regularizing and subpolynomial, we have

$$\left|D\varphi(X^\varepsilon(s), m^\varepsilon(s)) \cdot \left(i\Delta X^\varepsilon + \frac{X^\varepsilon m^\varepsilon}{\varepsilon}(s)\right)\right| \leqslant C_{\varepsilon,\varphi}(1 + \|X^\varepsilon(s)\|_{H^1}^{p_2+1})(1 + \|m^\varepsilon(s)\|_E^{\gamma_2+1}). \tag{3.8}$$

Also, since $\sigma < \frac{2}{n-2}$, we know that H^1 is continuously embedded in $L^{2\sigma+2}$ and by duality $L^{\frac{2\sigma+2}{2\sigma+1}}$ is continuously embedded in H^{-1}. We deduce:

$$\begin{aligned} |D\varphi(X^\varepsilon, n) \cdot \left(|X^\varepsilon|^{2\sigma}X^\varepsilon\right)| \quad &\leqslant C_{\varepsilon,\varphi}(1 + \|X^\varepsilon(s)\|_{H^1}^{p_2})(1 + \|m^\varepsilon(s)\|_E^{\gamma_2})\left\||X^\varepsilon|^{2\sigma}X^\varepsilon\right\|_{L^{\frac{2\sigma+2}{2\sigma+1}}} \\ &\leqslant C_{\varepsilon,\varphi}(1 + \|X^\varepsilon(s)\|_{H^1}^{p_2})(1 + \|m^\varepsilon(s)\|_E^{\gamma_2})\,\|X^\varepsilon\|_{H^1}^{2\sigma+1} \end{aligned}$$

which gives, thanks to (3.1) and (2.19), the uniform (in (s,ω)) integrability of $i_{\bar{\delta}}$ Moreover, since X^ε is almost surely continuous and m^ε is stochastically continous, $D\varphi(X^\varepsilon(s), m^\varepsilon(t_{i+1})) - D\varphi(X^\varepsilon(s), m^\varepsilon(s))$ converges to 0 when $\bar{\delta} \to 0$ in probability, so I converges to 0, as $\bar{\delta}$ tends to 0.

We claim that

$$ii_{\bar{\delta}}(s) = \frac{1}{\varepsilon^2}\sum_{i=0}^{n-1} \mathbf{1}_{[t_i,t_{i+1}]}\left(\mathcal{M}\varphi(X^\varepsilon(t_i), m^\varepsilon(s)) - \mathcal{M}\varphi(X^\varepsilon(s), m^\varepsilon(s))\right).$$

Indeed, for $u \in H^1$, $\varphi(u,\cdot) \in \mathcal{D}_{\mathcal{M}}$ (see Definition 3.1) and

$$M^\varepsilon(u,t) = \varphi(u, m^\varepsilon(t)) - \varphi(u, m(0)) - \int_0^t \frac{1}{\varepsilon^2} \mathcal{M}\varphi(u, m^\varepsilon(s))ds$$

is a $(\mathcal{F}_t^\varepsilon)$-martingale. As $X^\varepsilon(t_i)$ and g are $\mathcal{F}_{t_i}^\varepsilon$-measurable, we get

$$\begin{aligned}\mathbb{E}\left([\varphi(X^\varepsilon(t_i), m^\varepsilon(t_{i+1})) - \ \varphi(X^\varepsilon(t_i), m^\varepsilon(t_i))]\, g\right)& \\ = \frac{1}{\varepsilon^2}\mathbb{E}\left(\left[\int_{t_i}^{t_{i+1}} \mathcal{M}\varphi(X^\varepsilon(t_i), m^\varepsilon(s))ds\right] g\right).&\end{aligned}$$

Since $\mathcal{M}\varphi$ satisfies the polynomial bound in Definition 3.1 we get the uniform integrability of $ii_{\bar\delta}$. The convergence in probability to 0 comes from the continuity of $\mathcal{M}\varphi$.

The quadratic variation $\langle M, M\rangle$ is characterized by the property that $M^2(t) - \int_0^t \langle M, M\rangle(s)ds$ is a martingale. Let $s < t$ and $(t_i)_i$ be a subdivision of $[s,t]$ with $\sup_i |t_{i+1} - t_i| = \bar\delta$. Recall the following sequence of identity:

$$\begin{aligned}&\mathbb{E}(M^2(t) - M^2(s) - \int_0^t (\mathcal{L}^\varepsilon\varphi^2 - 2\varphi\mathcal{L}^\varepsilon\varphi)(X^\varepsilon(s), m^\varepsilon(s))ds \\ &= \sum_i \mathbb{E}(M^2(t_{i+1}) - M^2(t_i) - \int_{t_i}^{t_{i+1}} (\mathcal{L}^\varepsilon\varphi^2 - 2\varphi\mathcal{L}^\varepsilon\varphi)(X^\varepsilon(s), m^\varepsilon(s))ds \\ &= \sum_i \mathbb{E}((M(t_{i+1}) - M(t_i))^2 - \int_{t_i}^{t_{i+1}} (\mathcal{L}^\varepsilon\varphi^2 - 2\varphi\mathcal{L}^\varepsilon\varphi)(X^\varepsilon(s), m^\varepsilon(s))ds,\end{aligned} \tag{3.9}$$

where the last equality follows from

$$\mathbb{E}((M(t_{i+1}) - M(t_i))^2) = \mathbb{E}(M^2(t_{i+1}) - 2M(t_{i+1})M(t_i) + M^2(t_i))$$

and

$$\mathbb{E}(M(t_{i+1})M(t_i)) = \mathbb{E}(\mathbb{E}(M(t_{i+1})M(t_i)|\mathcal{F}_{t_i})) = \mathbb{E}(M^2(t_i)).$$

Hence, it suffices to show that the right hand side of (3.9) goes to zero as $\bar\delta \to 0$.

Let us write:

$$\begin{aligned}(M(t_{i+1}) - M(t_i))^2 =& \left(\varphi(X^\varepsilon(t_{i+1}), m^\varepsilon(t_{i+1})) - \varphi(X^\varepsilon(t_i), m^\varepsilon(t_i))\right. \\ &\left. - \int_{t_i}^{t_{i+1}} \mathcal{L}^\varepsilon\varphi(X^\varepsilon(\sigma), m^\varepsilon(\sigma))d\sigma\right)^2 \\ =&\varphi^2(X^\varepsilon(t_{i+1}), m^\varepsilon(t_{i+1})) \\ &- 2\varphi(X^\varepsilon(t_{i+1}), m^\varepsilon(t_{i+1}))\varphi(X^\varepsilon(t_i), m^\varepsilon(t_i)) + \varphi^2(X^\varepsilon(t_i), m^\varepsilon(t_i)) \\ &- 2(\varphi(X^\varepsilon(t_{i+1}), m^\varepsilon(t_{i+1})) - \varphi(X^\varepsilon(t_i), m^\varepsilon(t_i))) \int_{t_i}^{t_{i+1}} \mathcal{L}^\varepsilon\varphi(X^\varepsilon(\sigma), m^\varepsilon(\sigma))d\sigma \\ &+ \left(\int_{t_i}^{t_{i+1}} \mathcal{L}^\varepsilon\varphi(X^\varepsilon(\sigma), m^\varepsilon(\sigma))d\sigma\right)^2.\end{aligned}$$

Applying the above to φ^2 implies that the process

$$\tilde M(t) = \varphi^2(X^\varepsilon(t), m^\varepsilon(t)) - \varphi^2(X^\varepsilon(0), m^\varepsilon(0)) - \int_0^t \mathcal{L}^\varepsilon\varphi^2(X^\varepsilon(\sigma), m^\varepsilon(\sigma))d\sigma$$

is a martingale for the filtration generated by m^ε. We can now write

$$\begin{aligned}\sum_{i=0}^{n-1}(M(t_{i+1})-M(t_i))^2 &= \sum_{i=0}^{n-1}\tilde{M}(t_{i+1})-\tilde{M}(t_i)+\int_{t_i}^{t_{i+1}}\mathcal{L}^\varepsilon\varphi^2(X^\varepsilon(\sigma),m^\varepsilon(\sigma))d\sigma\\ &\quad-2\varphi(X^\varepsilon(t_i),m^\varepsilon(t_i))\Big(\varphi\big(X^\varepsilon(t_{i+1}),m^\varepsilon(t_{i+1})\big)-\varphi(X^\varepsilon(t_i),m^\varepsilon(t_i))\Big)\\ &\quad-2(\varphi(X^\varepsilon(t_{i+1}),m^\varepsilon(t_{i+1}))-\varphi(X^\varepsilon(t_i),m^\varepsilon(t_i)))\int_{t_i}^{t_{i+1}}\mathcal{L}^\varepsilon\varphi(X^\varepsilon(\sigma),m^\varepsilon(\sigma))d\sigma\\ &\quad+\left(\int_{t_i}^{t_{i+1}}\mathcal{L}^\varepsilon\varphi(X^\varepsilon(\sigma),m^\varepsilon(\sigma))d\sigma\right)^2\end{aligned}$$

and

$$\begin{aligned}&\sum_i\mathbb{E}((M(t_{i+1})-M(t_i))^2-\int_{t_i}^{t_{i+1}}(\mathcal{L}^\varepsilon\varphi^2-2\varphi\mathcal{L}^\varepsilon\varphi)(X^\varepsilon(s),m^\varepsilon(s))ds\\ &=2\sum_i\mathbb{E}\left(\int_{t_i}^{t_{i+1}}\varphi\mathcal{L}^\varepsilon\varphi(X^\varepsilon(s),m^\varepsilon(s))ds-\varphi(X^\varepsilon(t_i),m^\varepsilon(t_i))\right.\\ &\qquad\left.\times\left(\varphi\left(X^\varepsilon(t_{i+1}),m^\varepsilon(t_{i+1})\right)-\varphi(X^\varepsilon(t_i),m^\varepsilon(t_i))\right)\right)\\ &-2\sum_i\mathbb{E}\left((\varphi(X^\varepsilon(t_{i+1}),m^\varepsilon(t_{i+1}))-\varphi(X^\varepsilon(t_i),m^\varepsilon(t_i)))\int_{t_i}^{t_{i+1}}\mathcal{L}^\varepsilon\varphi(X^\varepsilon(\sigma),m^\varepsilon(\sigma))d\sigma\right)\\ &+\sum_i\mathbb{E}\left(\left(\int_{t_i}^{t_{i+1}}\mathcal{L}^\varepsilon\varphi(X^\varepsilon(\sigma),m^\varepsilon(\sigma))d\sigma\right)^2\right).\end{aligned}$$

Again, the inequalities (2.19) and (3.1) and uniform integrability can be used to prove that the three terms of the right hand side go to zero under the extra assumption that $\gamma_1,\ \gamma_2+1,\ \gamma_3<\gamma/2$. □

Remark 3.2. *This proof is not completely rigourous. Indeed, we have differentiated $\varphi(X^\varepsilon(t))$ with respect to t but we do not know whether X^ε is C^1 with values in H^1. This is easily overcome by a regularization argument as in the proof of Proposition 3.1: we replace φ by $\varphi_\delta=\varphi(\rho_\delta*\cdot)$ and let $\delta\to 0$ at the end of the proof.*

4. The perturbed test function method

4.1. **Correctors.** From the expression of $\mathcal{L}^\varepsilon\varphi(u,n)$, we see that negative powers of ε are present. The term of order -2 cancels if φ does not depend on n. Since we are interested only in the behaviour of X^ε when $\varepsilon\to 0$, it is natural to consider such functions. To treat the -1 order term we need to add correctors to φ. Assuming $\varphi_1,\ \varphi_2$ are good test functions we have

$$\begin{aligned}\mathcal{L}^\varepsilon(\varphi+\varepsilon\varphi_1+\varepsilon^2\varphi_2)(u,n) &= \frac{1}{\varepsilon^2}\mathcal{M}\varphi(u)\\ &\quad+\frac{1}{\varepsilon}\left(D\varphi(u).(-iun)+\mathcal{M}\varphi_1(u,n)\right)\\ &\quad+\mathcal{M}\varphi_2(u,n)+D\varphi_1(u,n).(-iun)+D\varphi(u).(i\Delta u-i\lambda|u|^{2\sigma}u)\\ &\quad+\varepsilon\left(D\varphi_2(u).(-iun)+D\varphi_1(u,n).(i\Delta u-i\lambda|u|^{2\sigma}u)\right)\\ &\quad+\varepsilon^2 D\varphi_2(u,n).(i\Delta u-i\lambda|u|^{2\sigma}u).\end{aligned}$$

Recall that the notation $D\varphi$ denotes the differential of φ with respect to u. Let us compute formally the correctors. As already mentionned, the -2 order term vanishes because we chose a function φ depending only on u. The first corrector is chosen so that the -1 order term cancels. It is formally given by:

$$\varphi_1(u,n) = \mathcal{M}^{-1}D\varphi(u)\cdot(iun) = D\varphi(u)\cdot(iu\mathcal{M}^{-1}n) = D\varphi(u)\cdot(iuL_1(n)),$$

with L_1 defined in (2.8), (2.9), (2.10).

The second corrector enables to identify the limit generator. The average with respect ot ν of the third line is given by

$$\begin{aligned}\mathcal{L}\varphi(u) =& D\varphi(u)\cdot(i\Delta u - i\lambda|u|^{2\sigma}u)\\ &+\mathbb{E}_\nu\left(D^2\varphi(u)\cdot(-iun, iu\mathcal{M}^{-1}n) + D\varphi(u)\cdot(un\mathcal{M}^{-1}n)\right)\\ =& D\varphi(u)\cdot(i\Delta u - i\lambda|u|^{2\sigma}u)\\ &+\mathbb{E}_\nu\left(D^2\varphi(u)\cdot(-iun, iuL_1(n)) + D\varphi(u)\cdot(unL_1(n))\right)\end{aligned} \tag{4.1}$$

and we choose φ_2 such that:

$$\mathcal{M}\varphi_2(u,n) + D\varphi_1(u,n).(-iun) + D\varphi(u).(i\Delta u - i\lambda|u|^{2\sigma}u) - \mathcal{L}\varphi(u) = 0.$$

In this way, we formally get $\mathcal{L}^\varepsilon(\varphi + \varepsilon\varphi_1 + \varepsilon^2\varphi_2)(u,n) \to \mathcal{L}\varphi(u)$ and we indeed identify the limit generator.

We do not need to justify rigorously the above computation for many functions. As we shall see below, for our purpose, it is sufficient to consider test functions of the form: $\varphi(u) = (u,h)^\ell$ for $h \in H^1$, $\ell = 1, 2$. It is clearly a good test function and satisfies all assumptions of Proposition 3.3.

Proposition 4.1 (First corrector)**.** *Let $\varphi(u) = (u,h)^\ell$ with $h \in H^1$, $\ell = 1,2$. Then there exists φ_1 a good test function such that:*

$$D\varphi(u).(-iun) + \mathcal{M}\varphi_1(u,n) = 0 \quad \forall u, n \in H^1_{\mathbb{C}} \times E.$$

Moreover, $\varphi_1 = \ell(u,h)^{\ell-1}(iuL_1(n), h)$ and satisfies all assumptions of Proposition 3.3.

Proof of Proposition 4.1. For any $k \in H^1$, $D\varphi(u)\cdot k = \ell(u,h)^{\ell-1}(k,h)$. Therefore:

$$\begin{aligned}\int_E D\varphi(u).(-iun)d\nu(n) =& \ell(u,h)^{\ell-1}\int_E(-iun,h)d\nu(n) = \ell(u,h)^{\ell-1}(-iu\int_E nd\nu(n), h)\\ =& 0\end{aligned}$$

where ν is the law of $m(t)$, which is centered. Thanks to our assumptions, φ_1 is given by

$$\varphi_1(u,n) = \ell(u,h)^{\ell-1}(iuL_1(n), h).$$

By (2.21), we easily see that this is a good test function and Proposition 3.3 applies. □

Note that thanks to (2.21), we have:

$$|\varphi_1(u,n)| \leqslant C(1+\|n\|_E)^\eta\|u\|^\ell_{H^1}\|h\|^\ell_{H^1}.$$

Moreover, for $k \in H^1$,

$$D\varphi_1(u,n)\cdot k = \ell(u,h)^{\ell-1}(ikL_1(n),h) + \ell(\ell-1)(k,h)(iuL_1(n),h).$$

We now compute the second corrector. For the test function $\varphi(u) = (u,h)^\ell$, we have

$$\begin{aligned}\mathcal{L}\varphi(u) =& \ell(u,h)^{\ell-1}(i\Delta u - i\lambda|u|^{2\sigma}u, h)\\ &+\mathbb{E}_\nu\left(\ell(\ell-1)(-iun,h)(iuL_1(n),h) + \ell(u,h)^{\ell-1}(unL_1(n),h)\right).\end{aligned}$$

The equation for φ_2 then writes:

$$\begin{aligned}\mathcal{M}\varphi_2(u,n) &= \ell(\ell-1)(iun,h)(iuL_1(n),h)+\ell(u,h)^{\ell-1}(unL_1(n),h))\\ &+\mathbb{E}_\nu\left(\ell(\ell-1)(-iun,h)(iuL_1(n),h)+\ell(u,h)^{\ell-1}(unL_1(n),h)\right).\end{aligned}\tag{4.2}$$

The following proposition is again a straigthforward application of our assumptions.

Proposition 4.2 (Second corrector)**.** *Let $\varphi(u)=(u,h)^\ell$ with $h\in H^1$, $\ell=1,2$. Then there exists φ_2 a good test function such that:*

$$\begin{aligned}\mathcal{M}\varphi_2(u,n) &= \ell(\ell-1)(iun,h)(iuL_1(n),h)+\ell(u,h)^{\ell-1}(unL_1(n),h))\\ &+\mathbb{E}_\nu\left(\ell(\ell-1)(-iun,h)(iuL_1(n),h)+\ell(u,h)^{\ell-1}(unL_1(n),h)\right).\end{aligned}$$

Moreover, φ_2^2 is also a good test function.

Proof. Let $\ell=1$, then the right hand side of (4.2) is of the form (2.11). It follows that φ_2 exists and by (2.13), (2.14) is given by:

$$\varphi_2(u,n)=-(uL_2(n),h).$$

Similarly, for $\ell=2$, the right hand side of (4.2) is of the form (2.16) and by (2.17), (2.18) is given by:

$$\varphi_2(u,n)=-2\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}u(x)u(y)L_4(n)(x,y)\bar{h}(x)\bar{h}(y)dxdy-2(u,h)(uL_2(n),h)$$

Thanks to (2.21), we have in both cases:

$$|\varphi_2(u,n)|\leqslant C(1+\|n\|_E)^\eta\|u\|_{H^1}^\ell\|h\|_{H^1}^\ell.$$

It follows that φ_2 and φ_2^2 are good test functions. □

Proposition 4.3 (Perturbed test-function method)**.** *Let $\varphi(u)=(u,h)^\ell$, where $h\in H^1$, $\ell=1,2$, and φ_1,φ_2 given by Propostions 4.1 and 4.2. For $\varepsilon\in(0,1)$, we define $\varphi^\varepsilon=\varphi+\varepsilon\varphi_1+\varepsilon^2\varphi_2$. Then φ^ε verifies for $u\in H^1$, $n\in E$:*

(1)

$$|\varphi^\varepsilon(u,n)-\varphi(u)|\leq C\varepsilon\|u\|_{H^1}^\ell\|h\|_{H^1}^\ell(1+\|n\|_E)^\eta.$$

(2)

$$|\mathcal{L}^\varepsilon\varphi^\varepsilon(u,n)-\mathcal{L}\varphi(u)|\leqslant C\varepsilon(1+\|u\|_{H^1}^{2\sigma+\ell})\|h\|_{H^1}^\ell(1+\|n\|_E)^{\eta+1}.\tag{4.3}$$

(3) The process

$$M_{\varphi^\varepsilon}(t)=\varphi^\varepsilon(X^\varepsilon(t),m^\varepsilon(t))-\varphi^\varepsilon(X_0^\varepsilon,m^\varepsilon(0))-\int_0^t\mathcal{L}^\varepsilon\varphi^\varepsilon(X^\varepsilon(s),m^\varepsilon(s))ds$$

is a càdlàg and integrable $(\mathcal{F}_t)$ zero-mean martingale.

Proof of Proposition 4.3. We treat the case $\ell=1$. The case $\ell=2$ is similar but lengthier. The first assertion clearly follows from the bound we have written above on φ_1 and φ_2.

By definition of $\mathcal{L}\varphi$ and φ^ε:

$$\begin{aligned}\mathcal{L}^\varepsilon\varphi^\varepsilon(u,n)-\mathcal{L}\varphi(u) =&\varepsilon\left(D\varphi_1(u,n).(i\Delta u-i\lambda|u|^{2\sigma}u)+\varepsilon D\varphi_2(u,n).(-iun)\right)\\ &+\varepsilon^2D\varphi_2(u,n).(i\Delta u-i\lambda|u|^{2\sigma}u).\end{aligned}$$

Recalling the expressions of φ_1, φ_2, we have

$$\begin{aligned}&\left|D\varphi_1(u,n).(i\Delta u - i\lambda|u|^{2\sigma}u) + \varepsilon D\varphi_2(u,n).(-iun)\right| \\ &\leqslant C|h|_{H^1}(|L_1(n)|_E\|i\Delta u - i\lambda|u|^{2\sigma}u\|_{H^{-1}} + \varepsilon|u|_{H^1}\|nL_2(n)\|_E)\end{aligned}$$

and

$$\begin{aligned}&\left|D\varphi_2(u,n).(i\Delta u - i\lambda|u|^{2\sigma}u)\right| \\ &\leqslant C|h|_{H^1}|L_2(n)|_E\|i\Delta u - i\lambda|u|^{2\sigma}u\|_{H^{-1}}.\end{aligned}$$

We have seen in the proof of Proposition 3.3 that

$$\|i\Delta u - i\lambda|u|^{2\sigma}u\|_{H^{-1}} \leq C(1+\|u\|_{H^1})^{2\sigma+1}.$$

The estimate of the second assertion follows easily. By Proposition 4.1, Proposition 4.2 we know that φ^ε is a good test function. Therefore Proposition 3.3 applies and the third point is also clear. □

4.2. **Tightness of the process X^ε.** In this subsection, we aim to obtain tightness of the family of stopped processes $(X^{\varepsilon,\tau^\varepsilon})_\varepsilon$ where $X^{\varepsilon,\tau^\varepsilon}(t) = X^\varepsilon(t\wedge\tau^\varepsilon)$. The definition (2.27) of τ^ε depends on α. We choose $\alpha < 2/\gamma$ such that $\alpha(\eta+1) < 1$.

The crucial ingredient used in previous works on diffusion-approximation in infinite dimension is an assumption on uniform boundedness of the driving process in the adequate functional space, which would be $L^\infty(0,T;E)$ in our case, w.r.t. ε (see [14] and [10]). Under our weaker assumptions, the result remains true provided we use the stopping time τ^ε. We will see that this is sufficient to conclude.

Proposition 4.4. *Assuming $X_0^\varepsilon \to X_0, \mathbb{P}-a.s.$ in the space Σ^ζ for some $0<\zeta\leqslant 1$ then the family of process $(X^{\varepsilon,\tau^\varepsilon})_{\varepsilon>0}$ is tight in $C([0,T],H^s)$ for $s<1$.*

This result strongly relies on the following a priori estimate.

Lemma 4.1. *Let $p\geq 1$. Assume that $\sup_{\varepsilon>0}\mathbb{E}\left(\|X_0^\varepsilon\|_{L^2}+\mathcal{H}(X_0^\varepsilon)\right)^p \leqslant \mathcal{C}_p$, and let X^ε be the solution of (1.1) with initial data X_0^ε and τ^ε the stopping time introduced in (2.27). Then for any stopping time $\tau\leqslant T$, there exists a constant $\mathcal{C}_p(T)$ depending on T and p but not on ε such that for $t\in[0,T]$:*

$$\mathbb{E}\left[\left\|X^{\varepsilon,\tau^\varepsilon}(\tau)\right\|_{H^1}^p\right] \leqslant \mathcal{C}_p(T). \tag{4.4}$$

The proofs of Lemma 4.1 and Proposition 4.4 are technical and are postponed to section 6.

We remark that this lemma, together with (3.5) and assuming $\mathbb{E}\left\|X_0^\varepsilon\langle x\rangle^\zeta\right\|_{L^2}^p \leqslant \mathcal{C}$ show that the following inequality holds

$$\mathbb{E}\left(\left\|X^{\varepsilon,\tau^\varepsilon}(\tau)\right\|_{\Sigma^\zeta}^p\right) \leqslant \mathcal{C}_p(T). \tag{4.5}$$

5. Proof of Theorem 1.1

The proof of Theorem 1.1 is divided into 3 steps. First we identify the SPDE associated to $\mathcal{L}$, then we prove the weak convergence of X^ε to X solution of (1.4), linked to $\mathcal{L}$, and finally we conclude using the uniqueness of the solution.

Proof of Theorem 1.1. Step 1: Identification of the limiting generator

For $h \in H^{2-s}$ with $s < 1$, we use the functionals $\varphi(u) = (u, h)$ and φ^2 which are clearly good test functions. We now compute $\mathcal{L}\varphi$ and $\mathcal{L}\varphi^2$,

$$\begin{aligned}\mathcal{L}\varphi(u) &= D\varphi(u).(i\Delta u - i\lambda|u|^{2\sigma}u) + \mathbb{E}_\nu\left(\underbrace{D^2\varphi(-iun, iuL_1(n))}_{0} + D\varphi(u).(unL_1(n))\right)\\ &= \big(i\Delta u - i\lambda|u|^{2\sigma}u + \mathbb{E}_\nu unL_1(n), h\big)\\ &= \left(i\Delta u - i\lambda|u|^{2\sigma}u - \frac{1}{2}uF, h\right),\end{aligned}$$

where F is given by (2.26) (see (2.38)). In the case of φ^2, we again have

$$D\varphi^2(u).(i\Delta u - i\lambda|u|^{2\sigma}u) + \mathbb{E}_\nu\left(D\varphi^2(u).(un\mathcal{M}^{-1}n)\right) = D\varphi^2(u).(i\Delta u - i\lambda|u|^{2\sigma}u - \frac{1}{2}uF)$$

but now the term $\mathbb{E}_\nu D^2\varphi(-iun, iu\mathcal{M}^{-1}n)$ does not vanish,

$$\begin{aligned}\mathbb{E}_\nu D^2\varphi^2(-iun, iuL_1(n)) &= 2\int_E (-iun, h)(iuL_1(n), h)d\nu(n)\\ &= 2\mathbb{E}\Big(\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} m(0)(x)Re(iu(x)\bar{h}(x))\Big(\int_0^\infty m(t)(y)dt\Big)Re(iu(y)\bar{h}(y))dxdy\Big)\\ &\quad \text{since } m(t) \text{ is real-valued}\\ &= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} k(x,y)Re(iu(x)\bar{h}(x))Re(iu(y)\bar{h}(y))dxdy.\end{aligned}$$

Let us denote now by q the kernel of $Q^{1/2}$ where Q is given by (2.36), we have

$$k(x,y) = \int_{\mathbb{R}^d} q(x,z)q(z,y)dz$$

and

$$\begin{aligned}&\mathbb{E}_\nu D^2\varphi^2(-iun, iuL_1(n))\\ &\qquad = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} q(x,z)q(z,y)Re(iu(x)\bar{h}(x))Re(iu(y)\bar{h}(y))dzdxdy\\ &\qquad = \int_{\mathbb{R}^d}\big(iuq(.,z), h\big)\big(iuq(.,z), h\big)dz\\ &\qquad = \frac{1}{2}Tr\big(D^2\varphi^2.(iuQ^{1/2}, iuQ^{1/2})\big).\end{aligned}$$

Finally, we have

$$\mathcal{L}\varphi(u) = D\varphi(u).(i\Delta u - i\lambda|u|^{2\sigma}u - \frac{1}{2}uF) \tag{5.1}$$

and

$$\mathcal{L}\varphi^2(u) = D\varphi^2(u).(i\Delta u - i\lambda|u|^{2\sigma}u - \frac{1}{2}uF) + \frac{1}{2}Tr\big(D^2\varphi^2.(iuQ^{1/2}, iuQ^{1/2})\big). \tag{5.2}$$

Step 2: Convergence

Given $0 < \zeta \leqslant 1$, by Propostion 4.4, we have a subsequence of $(X^{\varepsilon,\tau^\varepsilon})_{\varepsilon>0}$, still denoted by $(X^{\varepsilon,\tau^\varepsilon})$, of law P^ε and a probability measure P on $C([0,T],H^s)$ such that

$$P^\varepsilon \to P \text{ weakly on } C([0,T],H^s).$$

Since $[0,T]$ is compact and H^s is separable, $C([0,T],H^s)$ is also separable, and by Skohorod theorem (see [2]) there exist a probability space $(\tilde\Omega,\tilde{\mathcal{F}},\tilde{\mathbb{P}})$ and random variables $\tilde X^\varepsilon, \tilde X$ on $\tilde\Omega$ with values in $C([0,T],H^s)$ such that

$$\tilde X^\varepsilon \to \tilde X \text{ in } C([0,T],H^s), \quad \tilde{\mathbb{P}} \text{ a.s. as } \varepsilon \to 0$$

and $\mathcal{L}(\tilde X^\varepsilon) = P^\varepsilon$ and $\mathcal{L}(\tilde X) = P$. For $h \in H^{2-s}$, we use the test function $\varphi(u) = (u,h)$ and φ_1, φ_2 the correctors given by Propositions 4.1 and 4.2. We define $\varphi^\varepsilon = \varphi + \varepsilon\varphi_1 + \varepsilon^2\varphi_2$, then by Proposition 4.3, the process

$$M(t) = \varphi^\varepsilon(X^\varepsilon(t), m^\varepsilon(t)) - \varphi^\varepsilon(X^\varepsilon(0), m^\varepsilon(0)) - \int_0^t \mathcal{L}^\varepsilon\varphi^\varepsilon(X^\varepsilon(s), m^\varepsilon(s))ds$$

is a martingale for the filtration $(\mathcal{F}^\varepsilon_t)_t$, and the stopped process $M(t\wedge\tau^\varepsilon)$ is also a martingale, that is for all $0 \leqslant s_1 \leqslant ... \leqslant s_n \leqslant s \leqslant t$ and $g \in C_b\big((H^s)^n\big)$

$$\mathbb{E}\Big(\big(\varphi^\varepsilon(X^\varepsilon(t\wedge\tau^\varepsilon), m^\varepsilon(t\wedge\tau^\varepsilon)) - \varphi^\varepsilon(X^\varepsilon(s\wedge\tau^\varepsilon), m^\varepsilon(s\wedge\tau^\varepsilon)) \\ - \int_{s\wedge\tau^\varepsilon}^{t\wedge\tau^\varepsilon} \mathcal{L}^\varepsilon\varphi^\varepsilon(X^\varepsilon(s'), m^\varepsilon(s'))ds'\big)g(X^\varepsilon(s_1\wedge\tau^\varepsilon), ..., X^\varepsilon(s_n\wedge\tau^\varepsilon))\Big) = 0.$$

Moreover, we easily have

$$\int_{s\wedge\tau^\varepsilon}^{t\wedge\tau^\varepsilon} \mathcal{L}^\varepsilon\varphi^\varepsilon(X^\varepsilon(s'), m^\varepsilon(s'))ds'\big)g(X^\varepsilon(s_1\wedge\tau^\varepsilon), ..., X^\varepsilon(s_n\wedge\tau^\varepsilon)) \\ = \int_{s\wedge\tau^\varepsilon}^{t\wedge\tau^\varepsilon} \mathcal{L}^\varepsilon\varphi^\varepsilon(X^{\varepsilon,\tau^\varepsilon}(s'), m^{\varepsilon\wedge\tau^\varepsilon}(s))ds'\big)g(X^{\varepsilon,\tau^\varepsilon}(s_1), ..., X^{\varepsilon,\tau^\varepsilon}(s_n)).$$

Then we get:

$$\mathbb{E}\Big(\big(\varphi(X^{\varepsilon,\tau^\varepsilon}(t)) - \varphi(X^{\varepsilon,\tau^\varepsilon}(s)) - \int_s^t \mathcal{L}\varphi(X^{\varepsilon,\tau^\varepsilon}(s'))ds'\big)g(X^{\varepsilon,\tau^\varepsilon}(s_1), ..., X^{\varepsilon,\tau^\varepsilon}(s_n))\Big) \\ = \mathbb{E}\Big[\Big(-\varepsilon\big(\varphi_1(X^{\varepsilon,\tau^\varepsilon}(t), m^\varepsilon(t\wedge\tau^\varepsilon)) - \varphi_1(X^{\varepsilon,\tau^\varepsilon}(s), m^\varepsilon(s\wedge\tau^\varepsilon))\big) \\ - \varepsilon^2\big(\varphi_2(X^{\varepsilon,\tau^\varepsilon}(t), m^\varepsilon(t\wedge\tau^\varepsilon)) - \varphi_2(X^{\varepsilon,\tau^\varepsilon}(s), m^\varepsilon(s\wedge\tau^\varepsilon))\big) \\ - \Big(\int_{s\wedge\tau^\varepsilon}^s \mathcal{L}\varphi(X^{\varepsilon,\tau^\varepsilon}(s'))ds' - \int_t^{t\wedge\tau^\varepsilon} \mathcal{L}\varphi(X^{\varepsilon,\tau^\varepsilon}(s'))ds' \\ - \int_{s\wedge\tau^\varepsilon}^{t\wedge\tau^\varepsilon} (\mathcal{L}^\varepsilon\varphi^\varepsilon - \mathcal{L}\varphi)(X^\varepsilon(s'), m^\varepsilon(s'\wedge\tau^\varepsilon))ds'\Big)g(X^{\varepsilon,\tau^\varepsilon}(s_1), ..., X^{\varepsilon,\tau^\varepsilon}(s_n))\Big] \\ = \mathbb{E}[T_1 + T_2 + T_3 + T_4].$$

We now use Proposition 4.3 and Lemma 4.1 and notice that $T_1 = T_2 = 0$ when $\tau^\varepsilon = 0$ to get the bound:

$$\begin{aligned}\mathbb{E}[T_1 + T_2] &\leq C\varepsilon\mathbb{E}\left(\mathbf{1}_{\tau^\varepsilon>0}\sup_{s'\in[0,T]}\left\|X^{\varepsilon,\tau^\varepsilon}(s')\right\|_{H^1}\|h\|_{H^1}\left(1+\sup_{s'\in[0,T]}\|m^\varepsilon(s'\wedge\tau^\varepsilon)\|_E\right)^\eta\right)\\ &\leq C\varepsilon^{1-\alpha\eta}.\end{aligned}$$

Similarly:

$$\mathbb{E}[T_4] \leq C\varepsilon^{1-\alpha(\eta+1)}.$$

By the embedding $H^1 \subset L^{2\sigma+2}$ (see (1.2)) and the Hölder inequality, we have

$$\begin{aligned}\left|\mathcal{L}\varphi(X^{\varepsilon,\tau^\varepsilon}(s'))\right| \leqslant& \|\nabla h\|_{L^2}\left\|\nabla X^{\varepsilon,\tau^\varepsilon}(s')\right\|_{L^2} + \|h\|_{L^{2\sigma+2}}\left\|X^{\varepsilon,\tau^\varepsilon}(s')\right\|_{L^{2\sigma+2}}^{2\sigma+1}\\ &+\|F\|_{L^\infty}\|h\|_{L^2}\left\|X^{\varepsilon,\tau^\varepsilon}(s')\right\|_{L^2}\\ \leqslant& \mathcal{C}\|h\|_{H^1}(1+\left\|X^{\varepsilon,\tau^\varepsilon}(s')\right\|_{H^1})^{2\sigma+1}.\end{aligned}$$

Lemma 4.1 and Hölder inequality yield

$$\begin{aligned}&\mathbb{E}\Big[\Big(\Big|\int_{s\wedge\tau^\varepsilon}^s \mathcal{L}\varphi(X^{\varepsilon,\tau^\varepsilon}(s'))ds'\Big)g(X^\varepsilon(s_1),...,X^\varepsilon(s_n))\Big|\Big)\Big]\\ &\quad\leqslant \mathcal{C}\|h\|_{H^1}\mathbb{E}\left[\int_s^{s\wedge\tau^\varepsilon}(1+\left\|X^{\varepsilon,\tau^\varepsilon}(s')\right\|_{H^1})^{2\sigma+1}ds'\right]\\ &\quad\leqslant \mathcal{C}\|h\|_{H^1}\left(\mathbb{E}\left[(s-s\wedge\tau^\varepsilon)^2\right]\right)^{1/2} := r_1(\varepsilon),\end{aligned}$$

with $r_1(\varepsilon)\to 0$ when $\varepsilon\to 0$ by Lemma 2.1 and the uniform integrability. Similarly we have

$$\mathbb{E}\left|\int_t^{t\wedge\tau^\varepsilon}\mathcal{L}\varphi(X^{\varepsilon,\tau^\varepsilon}(s'))ds'g(X^\varepsilon(s_1),...,X^\varepsilon(s_n))\right| \leqslant r_2(\varepsilon),$$

with $r_2(\varepsilon)\to 0$ when $\varepsilon\to 0$.

Finally, we obtain

$$\begin{aligned}&\left|\mathbb{E}\Big(\big(\varphi(X^{\varepsilon,\tau^\varepsilon}(t)) - \varphi(X^{\varepsilon,\tau^\varepsilon}(s)) - \textstyle\int_s^t \mathcal{L}\varphi(X^{\varepsilon,\tau^\varepsilon}(s')ds'\big)g(X^{\varepsilon,\tau^\varepsilon}(s_1),...,X^{\varepsilon,\tau^\varepsilon}(s_n))\Big)\right|\\ &\qquad\leqslant C\varepsilon^{1-(\eta+1)\alpha} + r_1(\varepsilon) + r_2(\varepsilon),\end{aligned} \tag{5.3}$$

where C does not depend on ε. Moreover, as $X^{\varepsilon,\tau^\varepsilon}$ and $\tilde{X}^\varepsilon$ have the same law, then (5.3) is also true by replacing $X^{\varepsilon,\tau^\varepsilon}$ by $\tilde{X}^\varepsilon$, and $\mathbb{P}$ by $\tilde{\mathbb{P}}$. Since φ, $\mathcal{L}\varphi$ and g are continuous from H^s to $\mathbb{R}$ (the continuity of $\mathcal{L}\varphi$ requires the continuity of the nonlinearity which is given by the embedding $H^s \subset L^{2\sigma+1}$), taking the limit $\varepsilon\to 0$, we get

$$\tilde{\mathbb{E}}\Big(\big(\varphi(\tilde{X}(t)) - \varphi(\tilde{X}(s)) - \int_s^t \mathcal{L}\varphi(\tilde{X}(s'))ds'\big)g(\tilde{X}(s_1),...,\tilde{X}(s_n))\Big) = 0. \tag{5.4}$$

Then the process

$$\tilde{M}(t) = \varphi(\tilde{X}(t)) - \varphi(\tilde{X}(0)) - \int_0^t \mathcal{L}\varphi(\tilde{X}(s))ds$$

is a martingale with respect to the filtration $\mathcal{G}_s$ generated by $\tilde{X}(s)$. Note that this martingale is continuous.

Similarly, we can pass to the limit $\varepsilon \to 0$ in the definition of the quadratic variation and obtain that the quadratic variation of $\tilde{M}$ is given by:

$$\langle \tilde{M}, \tilde{M} \rangle(t) = \int_0^t \big(\mathcal{L}(\varphi)^2 - 2\varphi \mathcal{L}\varphi\big)(\tilde{X}(s))ds.$$

Note that this step requires the use of the perturbed test function method applied to $\varphi^2(u) = (u,h)^2$.

From (5.1) and (5.2), we deduce

$$\begin{aligned}\big(\mathcal{L}(\varphi)^2 - 2\varphi\mathcal{L}\varphi\big)(u) =& 2(u,h)(i\Delta u - i\lambda|u|^{2\sigma}u - \frac{1}{2}uF, h) + Tr\big((iuQ^{1/2}, h)^2\big) \\ & - 2(u,h)\Big((i\Delta u - \frac{i}{2}uF, h)\Big) \\ =& Tr\big((iuQ^{1/2}, h)^2\big)\end{aligned}$$

The continuous H^{-1}-valued martingale

$$\mathbf{M}(t) = \tilde{X}(t) - \tilde{X}(0) - \int_0^t i\Delta\tilde{X}(s) - i|\tilde{X}(s)|^{2\sigma}\tilde{X}(s) - \frac{1}{2}\tilde{X}(s)F$$

has the quadratic variation

$$\int_0^t \Big(i\tilde{X}(s)Q^{1/2}\Big)\Big(i\tilde{X}(s)Q^{1/2}\Big)^* ds$$

then, using the martingale representation theorem (see [6]) and up to enlarging the probability space, there exists a cylindrical Wiener process W such that:

$$\mathbf{M}(t) = \int_0^t i\tilde{X}(s)Q^{1/2}dW(s).$$

Step 3: Uniqueness of the limit Note that $\tilde{X}^\varepsilon$ also satisifies (4.4). Letting $\varepsilon \to 0$, we deduce that $\tilde{X} \in L^\infty(0,T; L^p(\tilde{\Omega}; H^1(\mathbb{R}^d)))$ for any $p \geq 1$ is a martingale solution of (1.4). Using the integral form of (1.4), we see that it has paths in $L^\infty(0,T;H^1(\mathbb{R}^d))$. We know from [1] that under our assumptions that (1.4) has a unique solution with paths in $L^\infty(0,T;H^1(\mathbb{R}^d))$. This implies uniqueness in law for martingale solutions.

As P is the law of $\tilde{X}$, we deduce this is the law of the solution of (1.4). By uniqueness of the limit, we conclude that the whole sequence $(X^{\varepsilon,\tau^\varepsilon})$ converges in law to X, in the space of probability measure of $C([0,T],H^s)$.

Finally, we obviously have for $\delta > 0$,

$$\left\{\sup_{0\leqslant t\leqslant T}\left\|X^\varepsilon(t) - X^{\varepsilon,\tau^\varepsilon}(t)\right\|_{H^1} > \delta\right\} \subset \{\tau^\varepsilon < T\},$$

and together with Lemma 2.1 yields the convergence in probability of $X^\varepsilon - X^{\varepsilon,\tau^\varepsilon}$ to 0.

Using finally [2, Theorem 4.1], we obtain the weak convergence of X^ε to X in $C([0,T],H^s)$.

□

6. Technical proofs

6.1. **Proof of Lemma 4.1.** As seen in Section 3, a straight application of standard energy arguments gives a very bad dependance on ε. The idea is to use the perturbed test function. This mimics Itô formula which is used to get a priori estimates for the limit equation with white noise (see [8]).

If one tries to use similar arguments to those of Proposition 4.3 with the functional H, defined by (2.3), in place of linear or quadratic functional, this requires a lot of smoothness on H and the useless assumption ($\sigma \geq 1/2$). We proceed slightly differently. We first smooth the functional and take advantage of the various cancelations before constructing the correctors.

Proof of Lemma 4.1. We give the proof only for $p = 2$. The general case is not more complicated but is lengthier.

We consider the functional $\varphi_\delta(u) = \mathcal{H}(\rho_\delta \star u)$, where ρ_δ is the mollifier introduced at the end of Section 2.1. We claim that it is a good test function. Indeed, we have for $u, h \in H^1$

$$D\varphi_\delta(u) \cdot h = \left(-\Delta\rho_\delta \star u + \lambda|\rho_\delta \star u|^{2\sigma}(\rho_\delta \star u), \rho_\delta \star h\right), \tag{6.1}$$

and there is no difficulty to verify that φ_δ satisfies the condition in Definition 3.1. By Proposition 3.3, we know that the process

$$M^\varepsilon_\delta(t) = \varphi_\delta(X^\varepsilon(t)) - \varphi_\delta(X^\varepsilon_0) - \int_0^t \mathcal{L}^\varepsilon\varphi_\delta(X^\varepsilon(s), m^\varepsilon(s))ds,$$

is a martingale. It can be seen that $\varphi_\delta(u) \to \mathcal{H}(u)$ when $\delta \to 0$ and $u \in H^1$. Moreover, we have

$$\begin{aligned}
&\mathcal{L}^\varepsilon\varphi_\delta(X^\varepsilon(s), m^\varepsilon(s)) \\
&\quad = \frac{1}{\varepsilon}\left(-\Delta\rho_\delta \star X^\varepsilon(s) + \lambda|\rho_\delta \star X^\varepsilon(s)|^{2\sigma}(\rho_\delta \star X^\varepsilon), -i\rho_\delta \star (X^\varepsilon(s)m^\varepsilon(s))\right) \\
&\quad + \left(-\Delta\rho_\delta \star X^\varepsilon(s) + \lambda|\rho_\delta \star X^\varepsilon(s)|^{2\sigma}(\rho_\delta \star X^\varepsilon), i\rho_\delta \star \left(\Delta X^\varepsilon(s) - \lambda|X^\varepsilon(s)|^{2\sigma}X^\varepsilon(s)\right)\right),
\end{aligned}$$

and we proved in Proposition 3.1 that the second term of the right-hand side converges to 0, when $\delta \to 0$. Similarly we have when δ tends to 0

$$\begin{aligned}
|\rho_\delta \star X^\varepsilon(s)|^{2\sigma}X^\varepsilon(s) &\to |X^\varepsilon(s)|^{2\sigma}X^\varepsilon(s) \text{ in } L^{\frac{2\sigma+2}{2\sigma+1}}, \\
\rho_\delta \star X^\varepsilon(s)m^\varepsilon(s) &\to X^\varepsilon(s)m^\varepsilon(s) \text{ in } L^{2\sigma+2},
\end{aligned}$$

so that

$$\begin{aligned}
\left(|\rho_\delta \star X^\varepsilon(s)|^{2\sigma}X^\varepsilon(s), i\rho_\delta \star X^\varepsilon(s)m^\varepsilon(s)\right) &\to \left(|X^\varepsilon(s)|^{2\sigma}X^\varepsilon(s), iX^\varepsilon(s)m^\varepsilon(s)\right) \\
&= Re\left(\int_{\mathbb{R}^d} i|X^\varepsilon(s)|^{2\sigma+2}m^\varepsilon(s)dx\right) = 0,
\end{aligned}$$

since m is real valued. In the same way, we have

$$(-\Delta\rho_\delta \star X^\varepsilon(s), i\rho_\delta \star X^\varepsilon(s)m^\varepsilon(s)) \to (\nabla X^\varepsilon(s), iX^\varepsilon(s)\nabla m^\varepsilon(s)).$$

Finally, as δ converges to 0, $M^\varepsilon_\delta(t)$ converges almost surely to

$$M^\varepsilon(t) := \mathcal{H}(X^\varepsilon(t)) - \mathcal{H}(X^\varepsilon_0) - \int_0^t \frac{1}{\varepsilon}\left(\nabla X^\varepsilon(s), iX^\varepsilon(s)\nabla m^\varepsilon(s)\right) ds,$$

and since $M_\delta(t)$ is a $(\mathcal{F}^\varepsilon_t)$-martingale, from (3.1) and the dominated convergence theorem, we deduce immediately that $M^\varepsilon(t)$ is also a $(\mathcal{F}^\varepsilon_t)$-martingale.

We define then the first corrector

$$\varphi_1(u,n) = (\nabla u, -iu\nabla L_1(n))\,. \tag{6.2}$$

This is not a good test function. For $\delta > 0$, we set $\varphi_{1,\delta}(u,n) = \varphi_1(\rho_\delta \star u, n)$. There is no difficulty to see that $\varphi_{1,\delta}$ is a good test function and then, we obtain by Proposition 3.3 that the process

$$\begin{aligned} M^\varepsilon_{1,\delta}(t) :=& \varphi_{1,\delta}(X^\varepsilon(t), m^\varepsilon(t)) - \varphi_{1,\delta}(X^\varepsilon(0), m^\varepsilon(0)) \\ &- \int_0^t \mathcal{L}^\varepsilon \varphi_{1,\delta}(X^\varepsilon(s), m^\varepsilon(s))ds, \end{aligned}$$

is a martingale. After computations, we have

$$\begin{aligned} \mathcal{L}^\varepsilon \varphi_{1,\delta}(u,n) =& \frac{1}{\varepsilon^2}\left(\nabla\rho_\delta\star u, -i\rho_\delta\star u\nabla n\right) \\ &+ \frac{1}{\varepsilon}\left[(\nabla\rho_\delta\star(un), \nabla L_1(n)\rho_\delta\star u) - (\nabla\rho_\delta\star u, n\nabla L_1(n)\rho_\delta\star u)\right] \\ &+ 2\left(\Delta\rho_\delta\star u, \nabla L_1(n)\nabla\rho_\delta\star u\right) + \left(\Delta\rho_\delta\star u, \rho_\delta\star u\Delta L_1(n)\right) \\ &- 2\lambda\left(\rho_\delta\star(|u|^{2\sigma}u), \nabla\rho_\delta\star u\nabla L_1(n)\right) - \lambda\left(\rho_\delta\star(|u|^{2\sigma}u), \rho_\delta\star u\Delta L_1(n)\right), \end{aligned}$$

after integration by parts, we get

$$\left(\Delta\rho_\delta\star u, \rho_\delta\star u\Delta L_1(n)\right) = -\left(\nabla\rho_\delta\star u, \nabla\rho_\delta\star u\Delta L_1(n)\right) - \left(\nabla\rho_\delta\star u, \rho_\delta u\nabla\Delta L_1(n)\right),$$

$$\left(\Delta\rho_\delta\star u, \nabla L_1(n)\nabla\rho_\delta\star u\right) = -\frac{1}{2}\left(\nabla\rho_\delta\star u, \nabla\rho_\delta\star u\Delta L_1(n)\right),$$

and

$$\left(\nabla\rho_\delta\star(|u|^{2\sigma}u), \rho_\delta\star u\nabla L_1(n)\right) = -\left(\rho_\delta\star(|u|^{2\sigma}u), \nabla\rho_\delta\star u\nabla L_1(n) + \rho_\delta\star u\Delta L_1(n)\right).$$

Finally, taking the limit $\delta\to 0$ and using

$$(2\sigma+2)\left(|u|^{2\sigma}u, \nabla u\nabla L_1(n)\right) = \left(\nabla|u|^{2\sigma+2}, \nabla L_1(n)\right) = -\left(|u|^{2\sigma+2}, \Delta L_1(n)\right),$$

we get that the process $M^\varepsilon_1(t)$ given by

$$\begin{aligned} M^\varepsilon_1(t) :=& \varphi_1(X^\varepsilon(t), m^\varepsilon(t)) - \varphi_1(X^\varepsilon(0), m^\varepsilon(0)) \\ &- \int_0^t \frac{1}{\varepsilon^2}\left(\nabla X^\varepsilon(s), -iX^\varepsilon(s)\nabla m^\varepsilon(s)\right) \\ &- \frac{1}{\varepsilon}\left[(X^\varepsilon(s)\nabla m^\varepsilon(s), X^\varepsilon\nabla L_1(m^\varepsilon(s)))\right] \\ &+ 2\left(\nabla X^\varepsilon(s), \nabla X^\varepsilon(s)\Delta L_1(m^\varepsilon(s))\right) + \left(\nabla X^\varepsilon, X^\varepsilon\nabla\Delta L_1(m^\varepsilon(s))\right) \\ &+ \lambda\frac{\sigma}{\sigma+1}\left(|X^\varepsilon(s)|^{2\sigma+2}, \Delta L_1(m^\varepsilon(s))\right)ds \end{aligned}$$

is a martingale.

We finally consider the C^1_P test function

$$\varphi_2(u,n) := \mathcal{M}^{-1}\left((u\nabla n, u\nabla L_1(n) - \mathbb{E}_\nu\left(u\nabla n, u\nabla L_1(n)\right))\right) = (uL_3(n), u),$$

where L_3 was defined in (2.12), (2.13), (2.15). Again,thanks to (2.21) and the assumptions on L_3, it is not difficult to check that φ_2 is a good test function.

Using Proposition 3.3, we know that

$$M^\varepsilon_2(t) := \varphi_2(X^\varepsilon(t), m^\varepsilon(t)) - \varphi_2(X^\varepsilon(0), m^\varepsilon(0)) - \int_0^t \mathcal{L}^\varepsilon\varphi_2(X^\varepsilon(s), m^\varepsilon(s))ds \tag{6.3}$$

is a $(\mathcal{F}_t^\varepsilon)$-martingale. After computation we obtain $\mathcal{L}^\varepsilon\varphi_2$:

$$\begin{aligned}\mathcal{L}^\varepsilon\varphi_2(u,n) =&\frac{1}{\varepsilon^2}\left(u(\nabla n\nabla L_1(n) - \mathbb{E}_\nu\nabla n\nabla L_1(n)),u\right)\\ &-2\left(i\nabla u.\nabla L_3(n),u\right)\end{aligned} \tag{6.4}$$

where we strongly used that n is real-valued.

Consequently, we know that the following process $\mathbf{M}^\varepsilon(t)$ is a martingale, with respect to the filtration $(\mathcal{F}_t^\varepsilon)$.

$$\begin{aligned}\mathbf{M}^\varepsilon(t) :=&M^\varepsilon(t) + \varepsilon M_1^\varepsilon(t) + \varepsilon^2 M_2^\varepsilon(t)\\ =&\mathcal{H}(X^\varepsilon(t)) - \mathcal{H}(X_0^\varepsilon) + \varepsilon\left(\varphi_1(X^\varepsilon(t),m^\varepsilon(t)) - \varphi_1(X_0^\varepsilon,m^\varepsilon(0))\right)\\ &+\varepsilon^2\left(\varphi_2(X^\varepsilon(t),m^\varepsilon(t)) - \varphi_2(X_0^\varepsilon,m^\varepsilon(0))\right)\\ &-\int_0^t\left(X^\varepsilon(s)(\mathbb{E}_\nu\nabla n\nabla L_1(m^\varepsilon(s)),X^\varepsilon(s)\right)\\ &+2\varepsilon\left(\nabla X^\varepsilon(s),\nabla X^\varepsilon(s)\Delta L_1(m^\varepsilon(s))\right) + \varepsilon\left(\nabla X^\varepsilon, X^\varepsilon\nabla\Delta L_1(m^\varepsilon(s))\right)\\ &+\varepsilon\frac{\lambda\sigma}{\sigma+1}\left(|X^\varepsilon(s)|^{2\sigma+2},\Delta L_1(m^\varepsilon(s))\right)\\ &-2\varepsilon^2\left(i\nabla X^\varepsilon(s)\nabla L_3(m^\varepsilon(s),X^\varepsilon(s)\right)ds.\end{aligned}$$

Hence, since τ^ε, given by (2.27), is a bounded $(\mathcal{F}_t^\varepsilon)$-stopping time, the process $\mathbf{M}(t\wedge\tau^\varepsilon)$ is a martingale. From the identity above, the L^2 conservation, (6.2) and (6.4) we have

$$\begin{aligned}\mathbb{E}[\mathcal{H}(X^{\varepsilon,\tau^\varepsilon}(t))] \leqslant \mathbb{E}\Big[&|\mathcal{H}(X_0^\varepsilon)| + \mathcal{C}\varepsilon\mathbf{1}_{\tau^\varepsilon>0}\left\|L_1(m^\varepsilon)\right\|_E\left\|X_0^\varepsilon\right\|_{L^2}\left(\left\|\nabla X^{\varepsilon,\tau^\varepsilon}(t)\right\|_{L^2} + \left\|\nabla X_0^\varepsilon\right\|_{L^2}\right)\\ &+\mathcal{C}\varepsilon^2\mathbf{1}_{\tau^\varepsilon>0}\left\|L_3(m^\varepsilon)\right\|_{H^\kappa}\left\|X_0^\varepsilon\right\|_{L^2}^2 + \mathbf{M}(t\wedge\tau^\varepsilon)\\ &+\mathcal{C}\int_0^{t\wedge\tau^\varepsilon}\mathbb{E}_\nu\left[\nabla n\nabla L_1(n)\right]\left\|X_0^\varepsilon\right\|_{L^2}^2 + \varepsilon\left\|L^1(m^\varepsilon)\right\|_E\left\|X^{\varepsilon,\tau^\varepsilon}(s)\right\|_{L^{2\sigma+2}}^{2\sigma+1}\\ &+\varepsilon\left\|L_1(m^\varepsilon)\right\|_E\left(\left\|\nabla X^{\varepsilon,\tau^\varepsilon}(s)\right\|_{L^2}^2 + \left\|\nabla X^{\varepsilon,\tau^\varepsilon}(s)\right\|_{L^2}\left\|X^{\varepsilon,\tau^\varepsilon}(s)\right\|_{L^2}\right)\\ &+\varepsilon^2\left(\left\|L_3(m^\varepsilon)\right\|_{H^\kappa}^2 + \left\|\nabla X^{\varepsilon,\tau^\varepsilon}(s)\right\|_{L^2}^2\right)\left\|X^{\varepsilon,\tau^\varepsilon}(s)\right\|_{L^2}ds\Big].\end{aligned}$$

We recall that $\|X^\varepsilon(t)\|_{L^2} = \|X_0^\varepsilon\|_{L^2}$ and that, when $\tau^\varepsilon > 0$, $\|m^\varepsilon(t\wedge\tau^\varepsilon)\|_E \leqslant \varepsilon^{-\alpha}$ according to Lemma 2.1. Hence using (2.21), (3.4),

$$\begin{aligned}
&\mathbb{E}\Big[\Big\|\nabla X^{\varepsilon,\tau^\varepsilon}(t)\Big\|_{L^2}^2\Big]\\
&\quad\leqslant 2\mathbb{E}\Big[|\mathcal{H}(X_0^\varepsilon)| + \mathcal{C}\varepsilon^{1-\alpha\eta}(\|X_0^\varepsilon\|_{L^2}\|\nabla X_0^\varepsilon\|_{L^2} + \Big\|\nabla X^{\varepsilon,\tau^\varepsilon}(t)\Big\|_{L^2}^2 + \|X_0^\varepsilon\|_{L^2}^2)\\
&\quad+\mathcal{C}\|X_0\|_{L^2}^{p_d} + \mathcal{C}\varepsilon^{2(1-\alpha\eta)}\Big\|X^{\varepsilon,\tau^\varepsilon}(t)\Big\|_{L^2}^2\\
&\quad+\mathcal{C}\int_0^{t\wedge\tau^\varepsilon}\mathbb{E}_\nu\left[\nabla n\nabla L_1(n)\right]\|X_0^\varepsilon\|_{L^2}^2 + \varepsilon^{1-\alpha\eta}(\|X_0^\varepsilon\|_{L^2}^2 + \Big\|\nabla X^{\varepsilon,\tau^\varepsilon}(s)\Big\|_{L^2}^2)\\
&\quad+\varepsilon^{1-\alpha\eta}(\Big\|\nabla X^{\varepsilon,\tau^\varepsilon}(s)\Big\|_{L^2}^2 + \|X_0^\varepsilon\|_{L^2}^{p_d}) + \varepsilon^2(1+\|X_0^\varepsilon\|_{L^2}^2 + \Big\|\nabla X^{\varepsilon,\tau^\varepsilon}(s)\Big\|_{L^2}^2)ds\Big],
\end{aligned}$$

then for ε small enough we get

$$\mathbb{E}\left[\Big\|\nabla X^{\varepsilon,\tau^\varepsilon}\Big\|_{L^2}^2\right] \leqslant \mathcal{C}(T,\|X_0^\varepsilon\|_{L^2}^{p_d},|\mathcal{H}(X_0^\varepsilon)|) + \mathcal{C}\varepsilon^{1-\alpha\eta}\int_0^{t\wedge\tau^\varepsilon}\mathbb{E}\left[\Big\|\nabla X^{\varepsilon,\tau^\varepsilon}(s)\Big\|_{L^2}^2\right]ds. \tag{6.5}$$

The Gronwall Lemma and the conservation of the L^2 norm give us that for any $t\in[0,T]$

$$\mathbb{E}\left[\Big\|X^{\varepsilon,\tau^\varepsilon}(t)\Big\|_{H^1}^2\right] \leqslant \mathcal{C}(T). \tag{6.6}$$

Then, taking τ a stopping time such that $\tau < T$, we can do the same computations as before and get a similar bound as (6.5):

$$\begin{aligned}
\mathbb{E}\left[\Big\|\nabla X^{\varepsilon,\tau^\varepsilon}(\tau)\Big\|_{L^2}^2\right] &\leqslant \mathcal{C}(T,\|X_0^\varepsilon\|_{L^2}^{p_d},|\mathcal{H}(X_0^\varepsilon)|) + \mathcal{C}\varepsilon^{1-\alpha}\mathbb{E}\left[\int_0^{\tau\wedge\tau^\varepsilon}\Big\|\nabla X^{\varepsilon,\tau^\varepsilon}(s)\Big\|_{L^2}^2 ds\right]\\
&\leqslant C(T,\|X_0^\varepsilon\|_{L^2}^{p_d},|\mathcal{H}(X_0^\varepsilon)|) + \mathcal{C}\int_0^T\mathbb{E}\left[\Big\|\nabla X^{\varepsilon,\tau^\varepsilon}(s)\Big\|_{L^2}^2\right]ds
\end{aligned}$$

and using the bound (6.6) we are able to conclude. □

6.2. **Proof of Proposition 4.4.** To prove the tightness of the sequence $(X^{\varepsilon,\tau^\varepsilon})_{\varepsilon\to0}$ in the space $C([0,T];H^s(\mathbb{R}^d))$, we use Aldous criterion, which can be found in [2, Theorem 16.10] in the finite dimensional case. In the case of an infinite dimensional separable space H, the hypothesis (16.22) in [2] has to be replaced. Let us state the criterion we use.

Proposition 6.1. *Let H be a complete separable space, and $(u^\varepsilon)_{\varepsilon>0}$ be a sequence of process on $[0,T]$ such that, for any $t\in[0,T]$, $u^\varepsilon(t)$ is H valued. Assume that*

(1)

$$\begin{aligned}
&\textit{For every } \eta>0, \textit{ for every } t\geqslant 0 \textit{ there exists a compact set } \Gamma_{\eta,t}\subset H, \textit{ such that}\\
&\inf_{\varepsilon>0}\mathbb{P}(u^\varepsilon(t)\in\Gamma_{\eta,t}) \geqslant 1-\eta,
\end{aligned} \tag{6.7}$$

(2)

(6.8) *For every $\lambda, \eta > 0$, there exist δ_0, ε_0 such that for $\delta < \delta_0$, and $\varepsilon < \varepsilon_0$, if τ is a stopping time then*

$$\mathbb{P}\left(\|u^\varepsilon(\tau+\delta) - u^\varepsilon(\tau)\|_H > \lambda\right) \leqslant \eta.$$

Then the sequence $(u^\varepsilon)_{\varepsilon>0}$ is tight in $C([0,T],H)$.

We can prove this result using the same proof as for [2, Theorem 16.10], but instead of hypothesis (16.22) we assume (a) in [16, Theorem 7.9].

Proof of Proposition 4.4. Lemma 4.1 and Proposition 3.2 ensure that $(X^{\varepsilon,\tau^\varepsilon})_{\varepsilon>0}$ satisfies (6.7) for $H = H^s(\mathbb{R}^d)$, $0 \leqslant s < 1$. Indeed, according to these two lemmas, we have

$$\mathbb{E}\left[\left\|X^{\varepsilon,\tau^\varepsilon}(t)\right\|_{\Sigma^\zeta}\right] \leqslant C(T),$$

with $C(T)$ independent on ε.

Thus for any $\eta > 0$ and $t \in [0,T]$:

$$\mathbb{P}\left(\left\|X^{\varepsilon,\tau^\varepsilon}(t)\right\|_{\Sigma^\zeta} > R\right) \leqslant \frac{C(T)}{R} \leqslant \eta$$

for R large enough independent on ε. Thus $X^{\varepsilon,\tau^\varepsilon}(t)$ lives in a bounded set of Σ^ζ with probability larger than $1-\eta$. It remains to prove that the embedding $\Sigma^\zeta \hookrightarrow H^s(\mathbb{R}^d)$ is compact. Let $(u_n)_{n\in\mathbb{N}}$ be a bounded sequence in Σ^ζ. Then by compact embedding and diagonal extraction, there exists a subsequence $(u_{n_k})_{k\in\mathbb{N}}$ which converges to $u \in L^2(B(0,R))$ for every $R > 0$. Now we compute:

$$\begin{aligned}\|u_{n_k} - u\|^2_{L^2(\mathbb{R}^d)} &\leqslant \|u_{n_k} - u\|^2_{L^2(B(0,R))} + \int_{B(0,R)^c} |u_{n_k}(x) - u(x)|^2 dx \\ &\leqslant \|u_{n_k} - u\|^2_{L^2(B(0,R))} + (1+R)^{-\zeta} \int_{B(0,R)^c} \langle x\rangle^\zeta \left(|u_{n_k}(x)|^2 + |u(x)|^2\right) dx\end{aligned}$$

for any $R > 0$, thus $(u_{n_k})_{k\in\mathbb{N}}$ converges to u in $L^2(\mathbb{R}^d)$, and is bounded in $H^1(\mathbb{R}^d)$, so that by interpolation the subsequence also converges to u in $H^s(\mathbb{R}^d), 0 \leqslant s < 1$.

We also need to prove that $X^{\varepsilon,\tau^\varepsilon}$ satisfies the second condition of Proposition 6.1. We first prove that (6.8) holds with H replaced by $H^{-\frac{1}{2}}(\mathbb{R}^d,\mathbb{C})$. For this we use the Perturbed Test Function method, and the equality:

$$\|u(\tau+\delta) - u(\tau)\|^2_{H^{-\frac{1}{2}}} = \|u(\tau+\delta)\|^2_{H^{-\frac{1}{2}}} - \|u(\tau)\|^2_{H^{-\frac{1}{2}}} - 2\langle u(\tau+\delta) - u(\tau), u(\tau)\rangle_{H^{-\frac{1}{2}}}.$$

First we apply the Perturbed Test Function method to $\varphi(u) = \|u\|^2_{H^{-\frac{1}{2}}}$. We compute $\mathcal{L}^\varepsilon\varphi$:

$$\mathcal{L}^\varepsilon\varphi(X^\varepsilon) = 2\langle(1-\Delta)^{-\frac{1}{2}}X^\varepsilon, i\Delta X^\varepsilon - i\lambda|X^\varepsilon|^{2\sigma}X^\varepsilon\rangle - \frac{2}{\varepsilon}\langle(1-\Delta)^{-\frac{1}{2}}X^\varepsilon, iX^\varepsilon m(\frac{t}{\varepsilon^2})\rangle.$$

We choose the first corrector φ_1 in order to cancel the term of negative order in ε, and recalling that $L_1(n) = \mathcal{M}^{-1}n$ we get

$$\varphi_1(u,n) = 2\langle(1-\Delta)^{-\frac{1}{2}}u, iuL_1(n)\rangle, \tag{6.9}$$

on which we have a bound using hypothesis (2.21):

$$|\varphi_1(u,n)| \leqslant 2\|u\|_{H^{-1}}\|u\|_{L^2}\|L_1(n)\|_E \leqslant 2\|u\|^2_{L^2}\|L_1(n)\|_E. \tag{6.10}$$

Now we apply the infinitesimal generator $\mathcal{L}^\varepsilon$ to $\varphi+\varepsilon\varphi_1$:

$$\begin{aligned}\mathcal{L}^\varepsilon\left(\varphi+\varepsilon\varphi_1\right)(X^\varepsilon,m^\varepsilon)=&2\langle(1-\Delta)^{-\frac{1}{2}}X^\varepsilon,i\Delta X^\varepsilon-i\lambda|X^\varepsilon|^{2\sigma}X^\varepsilon\rangle\\&+2\langle(1-\Delta)^{-\frac{1}{2}}X^\varepsilon,X^\varepsilon m^\varepsilon L_1(m^\varepsilon)\rangle\\&+2\langle(1-\Delta)^{-\frac{1}{2}}\left(X^\varepsilon m^\varepsilon\right),X^\varepsilon L_1(m^\varepsilon)\rangle\\&+\varepsilon D_u\varphi_1(i\Delta X^\varepsilon-i\lambda|X^\varepsilon|^{2\sigma}X^\varepsilon).\end{aligned}$$

Thus, treating the operator $(1-\Delta)^{-\frac{1}{2}}$ as a kernel operator with kernel k_Δ, we set

$$\begin{aligned}\varphi_2(u,n)=&Re\int_{\mathbb{R}^{2n}}\bar{u}(x)k_\Delta(x,y)u(y)L_4(n)(y,x)dxdy\\&-2\langle(1-\Delta)^{-\frac{1}{2}}u,uL_2(n)\rangle,\end{aligned}\tag{6.11}$$

where we recall that formally $L_2(n)=\mathcal{M}^{-1}\left(nL_1(n)-\mathbb{E}_\nu[nL_1(n)]\right)$. Finally, defining $\varphi^\varepsilon=\varphi+\varepsilon\varphi_1+\varepsilon^2\varphi_2$, we obtain that:

$$\begin{aligned}\mathcal{L}^\varepsilon\varphi^\varepsilon(X^\varepsilon,m^\varepsilon)=&2\langle(1-\Delta)^{-\frac{1}{2}}X^\varepsilon,i\Delta X^\varepsilon-i\lambda|X^\varepsilon|^{2\sigma}X^\varepsilon\rangle\\&+2\langle(1-\Delta)^{-\frac{1}{2}}X^\varepsilon,X^\varepsilon\mathbb{E}_\nu\left[nL_1(n)\right]\rangle\\&-2\mathbb{E}_\nu\left[\langle(1-\Delta)^{-\frac{1}{2}}\left(X^\varepsilon n\right),X^\varepsilon L_1(n)\rangle\right]\\&+\varepsilon\left(D_u\varphi_1(i\Delta X^\varepsilon-i\lambda|X^\varepsilon|^{2\sigma}X^\varepsilon)+D_u\varphi_2(-iX^\varepsilon m^\varepsilon)\right)\\&+\varepsilon^2D_u\varphi_2(i\Delta X^\varepsilon-i\lambda|X^\varepsilon|^{2\sigma}X^\varepsilon).\end{aligned}\tag{6.12}$$

First we get an estimate of $\varphi_2(u,n)$ in (6.11). We have

$$\langle(1-\Delta)^{-\frac{1}{2}}u,uL_2(n)\rangle\leqslant\|u\|_{H^{-1}}\|u\|_{L^2}\|L_2(n)\|_{L^\infty},$$

and for the other term in the right-hand side of (6.11) we use the assumptions made on L_4 to get the bound:

$$Re\int_{\mathbb{R}^{2n}}\bar{u}(x)k_\Delta(x,y)u(y)L_4(n)(y,x)dxdy\leqslant\|u\|_{H^{-1}}\|u\|_{L^2}\|L_4(n)\|_{L^\infty(\mathbb{R}^d\times\mathbb{R}^d)}.$$

Finally we obtain the following bound on φ_2:

$$\begin{aligned}|\varphi_2(u,n)|\leqslant&2\|u\|_{H^{-1}}\|u\|_{L^2}\left(\|L_2(n)\|_{L^\infty}+\|L_4(n)\|_{L^\infty(\mathbb{R}^d\times\mathbb{R}^d)}\right)\\\leqslant&2\|u\|_{L^2}^2\left(\|L_2(n)\|_{L^\infty}+\|L_4(n)\|_{L^\infty(\mathbb{R}^d\times\mathbb{R}^d)}\right),\end{aligned}\tag{6.13}$$

with L_2 and L_4 defined in (2.11), (2.13) ,(2.14) , (2.16), (2.17), (2.18).

Now we work on the different terms of the expression of $\mathcal{L}^\varepsilon\varphi^\varepsilon$ in (6.12). First we deal with the terms of order 0 in ε. Using integrations by parts and (3.4) we get

$$\langle(1-\Delta)^{-\frac{1}{2}}u,i\Delta u-i\lambda|u|^{2\sigma}u\rangle\leqslant\|u\|_{L^2}\|u\|_{H^1}+\|u\|_{L^2}\left(\|u\|_{L^2}^{p_d}+\|u\|_{H^1}^2\right).\tag{6.14}$$

Then we have

$$\begin{aligned}&\langle(1-\Delta)^{-\frac{1}{2}}u,u\mathbb{E}_\nu\left[nL_1(n)\right]\rangle\\&=-Re\int_{\mathbb{R}^d}(1-\Delta)^{-\frac{1}{2}}u(x)\bar{u}(x)\mathbb{E}\int_0^\infty m(0,x)m(s,x)dsdx\\&=\frac{1}{2}Re\int_{\mathbb{R}^d}(1-\Delta)^{-\frac{1}{2}}u(x)\bar{u}(x)k(x,x)dxdydz,\end{aligned}$$

where $k \in W^{3,\infty}(\mathbb{R}^d \times \mathbb{R}^d) \cap H^1(\mathbb{R}^d \times \mathbb{R}^d)$ is defined in (2.24). We get:

$$\langle (1-\Delta)^{-\frac{1}{2}} u, u\mathbb{E}_\nu \left[nL_1(n) \right] \rangle \leqslant \|k\|^2_{L^\infty(\mathbb{R}^d \times \mathbb{R}^d))} \|u\|^2_{L^2} . \tag{6.15}$$

We get the same estimate for the last term of order 0:

$$\mathbb{E}_\nu \left[\langle (1-\Delta)^{-\frac{1}{2}} (un), uL_1(n) \rangle \right] \leqslant \|k\|^2_{L^\infty(\mathbb{R}^d \times \mathbb{R}^d))} \|u\|^2_{L^2} . \tag{6.16}$$

Now we focus on the terms of order 1 in ε in (6.12). We recall the expression of the first corrector $\varphi_1(u,n) = 2\langle (1-\Delta)^{-\frac{1}{2}} u, iuL_1(n) \rangle$. Thus

$$\begin{aligned} D_u\varphi_1(i\Delta u - i\lambda|u|^{2\sigma}u) =& 2\langle (1-\Delta)^{-\frac{1}{2}} u, (\lambda|u|^{2\sigma}u - \Delta u)L_1(n) \rangle \\ &+ 2\langle (1-\Delta)^{-\frac{1}{2}} (\Delta u - \lambda|u|^{2\sigma}u), uL_1(n) \rangle. \end{aligned}$$

We use integrations by parts to deal with the terms involving Δu and we get that

$$\langle (1-\Delta)^{-\frac{1}{2}} u, \Delta u L_1(n) \rangle + \langle (1-\Delta)^{-\frac{1}{2}} \Delta u, uL_1(n) \rangle \leqslant C \|u\|_{L^2} \|u\|_{H^1} \|L_1(n)\|_E .$$

For the other terms, we use (3.4) to get

$$\begin{aligned} \langle (1-\Delta)^{-\frac{1}{2}} u, |u|^{2\sigma} uL_1(n) \rangle + & \langle (1-\Delta)^{-\frac{1}{2}} (|u|^{2\sigma}u), uL_1(n) \rangle \\ &\leqslant \|u\|_{L^2} \left(\|u\|^{p_d}_{L^2} + \|u\|^2_{H^1} \right) \|L_1(n)\|_{L^\infty} . \end{aligned}$$

Finally

$$|D_u\varphi_1(i\Delta u - i\lambda|u|^{2\sigma}u)| \leqslant C \|u\|_{L^2} \|u\|_{H^1} \|L_1(n)\|_E + \|u\|_{L^2} \left(\|u\|^{p_d}_{L^2} + \|u\|_{H^1} \right) \|L_1(n)\|_{L^\infty} . \tag{6.17}$$

It remains to control the terms coming from the introduction of the second corrector φ_2 written in (6.11). We have

$$\begin{aligned} \frac{1}{2} D_u\varphi_2(h) =& - \langle (1-\Delta)^{-\frac{1}{2}} h, uL_2(n) \rangle - \langle (1-\Delta)^{-\frac{1}{2}} u, hL_2(n) \rangle \\ =& Re \int_{\mathbb{R}^{2n}} h(x) k_\Delta(x,y) \bar{u}(y) \left(L_4(n)(x,y) + L_4(n)(y,x) \right) dxdy \end{aligned} \tag{6.18}$$

where k_Δ denotes the kernel of the operator $(1-\Delta)^{\frac{1}{2}}$ and $L_4(n)$ is defined in (2.18). We start by the estimate of $D_u\varphi_2(-iX^\varepsilon m^\varepsilon)$. We can easily bound the terms involving $L_2(n)$:

$$\begin{aligned} \langle (1-\Delta)^{-\frac{1}{2}} (iX^\varepsilon m^\varepsilon), uL_2(m^\varepsilon) \rangle + & \langle (1-\Delta)^{-\frac{1}{2}} X^\varepsilon, iX^\varepsilon m^\varepsilon L_2(m^\varepsilon) \rangle \\ &\leqslant C \|X^\varepsilon\|^2_{L^2} \|m^\varepsilon\|_{L^\infty} \|L_2(m^\varepsilon)\|_{L^\infty} . \end{aligned}$$

For the other term, compute:

$$\begin{aligned} Re \int_{\mathbb{R}^{2n}} h(x) k_\Delta(x,y) \bar{u}(y) (L_4(n)(x,y) + & L_4(n)(y,x)) dxdy \\ &\leqslant 2 \|h\|_{L^2} \|u\|_{H^{-1}} \|L_4(n)\|_{L^\infty(\mathbb{R}^d \times \mathbb{R}^d)} . \end{aligned} \tag{6.19}$$

We can use this bound to control the terms which do not involve $L_2(m^\varepsilon)$ in $D_u\varphi(-iX^\varepsilon m^\varepsilon)$:

$$\begin{aligned} Re \int_{\mathbb{R}^{2n}} & iX^\varepsilon(x) m^\varepsilon(x) k_\Delta(x,y) \bar{X}^\varepsilon(y) \left(L_4(m^\varepsilon)(x,y) + L_4(m^\varepsilon)(y,x) \right) dxdy \\ &\leqslant 2 \|L_4(m^\varepsilon)\|_{L^\infty(\mathbb{R}^d \times \mathbb{R}^d)} Re \int_{\mathbb{R}^d} i(1-\Delta)^{-\frac{1}{2}} \bar{X}^\varepsilon(x) X^\varepsilon(x) m^\varepsilon(x) dx \\ &\leqslant 2 \|L_4(m^\varepsilon)\|_{L^\infty(\mathbb{R}^d \times \mathbb{R}^d)} \|X^\varepsilon\|^2_{L^2} \|m^\varepsilon)\|_{L^\infty} , \end{aligned}$$

Finally we have

$$|D_u\varphi_2(-iX^\varepsilon m^\varepsilon)| \leqslant C\left\|X^\varepsilon\right\|_{L^2}^2 \left\|m^\varepsilon\right\|_{L^\infty}\left(\left\|L_2(m^\varepsilon)\right\|_{L^\infty} + \left\|L_4(m^\varepsilon\right\|_{L^\infty(\mathbb{R}^d\times\mathbb{R}^d)}\right). \tag{6.20}$$

It remains to control $D_u\varphi_2(i\Delta X^\varepsilon - i\lambda|X^\varepsilon|^{2\sigma}X^\varepsilon)$. We use the same computations as for $D_u\varphi_1(i\Delta X^\varepsilon - i\lambda|X^\varepsilon|^{2\sigma}X^\varepsilon)$ and the bounds (3.4) and (6.19) to get:

$$\begin{aligned}|D_u\varphi_2(i\Delta X^\varepsilon - i\lambda|X^\varepsilon|^{2\sigma}X^\varepsilon)| \leqslant & C\left\|X^\varepsilon\right\|_{L^2}\left\|L_2(m^\varepsilon)\right\|_{L^\infty}\left(\left\|X^\varepsilon\right\|_{L^2}^{p_d} + \left\|X^\varepsilon\right\|_{H^1}^2\right)\\ &+ C\left\|X^\varepsilon\right\|_{L^2}\left\|X^\varepsilon\right\|_{H^1}\left\|L_2(m^\varepsilon)\right\|_E\\ &+C\left\|L_4(m^\varepsilon)\right\|_{L^\infty(\mathbb{R}^d\times\mathbb{R}^d)}\left(\left\|X^\varepsilon\right\|_{L^2}\left\|X^\varepsilon\right\|_{H^1} + \left\|X^\varepsilon\right\|_{L^2}\left(\left\|X^\varepsilon\right\|_{L^2}^{p_d} + \left\|X^\varepsilon\right\|_{H^1}^2\right)\right).\end{aligned} \tag{6.21}$$

Finally, estimates (6.13), (6.17), (6.20), (6.21) coupled with Lemma 2.1, conservation of the L^2-norm and Lemma 4.1 give us for τ^ε the stopping time defined in (2.27):

$$\begin{aligned}\mathbb{E}\left[\mathbf{1}_{\tau^\varepsilon>\mathbf{0}}|\varphi(\mathbf{X}^{\varepsilon,\tau^\varepsilon}) - \varphi^\varepsilon(\mathbf{X}^{\varepsilon,\tau^\varepsilon},\mathbf{m}^{\varepsilon,\tau^\varepsilon})|\right] &\leqslant C(T)(\varepsilon^{1-\alpha}+\varepsilon^{2-2\alpha}),\\ \mathbb{E}\left[\mathbf{1}_{\tau^\varepsilon>\mathbf{0}}|\mathcal{L}^\varepsilon\varphi^\varepsilon(\mathbf{X}^{\varepsilon,\tau^\varepsilon},\mathbf{m}^{\varepsilon,\tau^\varepsilon})|\right] &\leqslant C(T)(1+\varepsilon^{1-\alpha}+\varepsilon^{2-2\alpha}),\end{aligned}$$

where $X^{\tau^\varepsilon}, m^{\tau^\varepsilon}$ denote the stopped process $X^{\tau^\varepsilon}(t) = X(t\wedge\tau^\varepsilon), m^{\tau^\varepsilon}(t) = m(t\wedge\tau^\varepsilon)$.

We use this last estimate and the fact that

$$\varphi^\varepsilon(X^{\varepsilon,\tau^\varepsilon}(t)) - \varphi^\varepsilon(X^{\varepsilon,\tau^\varepsilon}(0)) - \int_0^t \mathcal{L}^\varepsilon\varphi^\varepsilon(X^{\varepsilon,\tau^\varepsilon}(s))ds$$

is a martingale to compute for any stopping time τ:

$$\begin{aligned}&\mathbb{E}\Big[\left\|X^\varepsilon((\tau+\delta)\wedge\tau^\varepsilon)\right\|_{H^{-\frac12}}^2 . - \left\|X^\varepsilon(\tau\wedge\tau^\varepsilon)\right\|_{H^{-\frac12}}^2\Big]\\ &= \mathbb{E}\left[\varphi(X^\varepsilon((\tau+\delta)\wedge\tau^\varepsilon)) - \varphi(X^\varepsilon(\tau\wedge\tau^\varepsilon))\right]\\ &\leqslant \mathbb{E}\left[\varphi^\varepsilon(X^\varepsilon((\tau+\delta)\wedge\tau^\varepsilon)) - \varphi^\varepsilon(X^\varepsilon(\tau\wedge\tau^\varepsilon))\right] + C\varepsilon^{1-\alpha} + C\varepsilon^{2-2\alpha}\\ &\leqslant \mathbb{E}\left[\int_{\tau\wedge\tau^\varepsilon}^{(\tau+\delta)\wedge\tau^\varepsilon}\mathcal{L}^\varepsilon\varphi^\varepsilon(X^{\varepsilon,\tau^\varepsilon}(s),m^{\varepsilon,\tau^\varepsilon}(s))ds\right] + C\varepsilon^{1-\alpha} + C\varepsilon^{2-2\alpha}\\ &\leqslant C(T)\delta\left(1+\varepsilon^{1-\alpha}+\varepsilon^{2-2\alpha}\right) + C\varepsilon^{1-\alpha} + C\varepsilon^{2-2\alpha}.\end{aligned}$$

Finally, for δ and ε small enough we get:

$$\mathbb{E}\left[\left\|X^\varepsilon((\tau+\delta)\wedge\tau^\varepsilon)\right\|_{H^{-\frac12}}^2 - \left\|X^\varepsilon(\tau\wedge\tau^\varepsilon)\right\|_{H^{-\frac12}}^2\right] \leqslant \eta. \tag{6.22}$$

Then we apply the Pertubed Test Function method on $\varphi(u) = \langle u,h\rangle_{H^{-\frac12}}$ for a fixed function $h\in L^2(\mathbb{R}^d)$. Computations lead us to choose two correctors:

$$\begin{aligned}\varphi_1(u,n) =& \langle iu\mathcal{M}^{-1}n, (1-\Delta)^{-\frac12}h\rangle\\ =& \langle iuL_1(n), (1-\Delta)^{-\frac12}h\rangle,\\ \varphi_2(u,n) =& \langle u\mathcal{M}^{-1}\left(\mathbb{E}_\nu\left[nL_1(n)\right] - nL_1(n)\right), (1-\Delta)^{-\frac12}h\rangle\\ =& \langle uL_2(n), (1-\Delta)^{-\frac12}h\rangle,\end{aligned}$$

and the infinitesimal generator applied to $\varphi^\varepsilon = \varphi + \varepsilon\varphi_1 + \varepsilon^2\varphi_2$ is

$$\begin{aligned}\mathcal{L}^\varepsilon\varphi^\varepsilon(X^\varepsilon, m^\varepsilon) &= \langle i\Delta X^\varepsilon - i\lambda|X^\varepsilon|^{2\sigma}X^\varepsilon\rangle + \langle X^\varepsilon \mathbb{E}_\nu\left[nL_1(n)\right], (1-\Delta)^{-\frac{1}{2}}h\rangle \\ &+ \varepsilon\left(\langle(\lambda|X^\varepsilon|^{2\sigma}X^\varepsilon - \Delta X^\varepsilon)L_1(m^\varepsilon), (1-\Delta)^{-\frac{1}{2}}h\rangle - \langle iX^\varepsilon m^\varepsilon L_2(m^\varepsilon), (1-\Delta)^{-\frac{1}{2}}h\rangle\right) \\ &+ \varepsilon^2\langle(i\Delta X^\varepsilon - i\lambda|X^\varepsilon|^{2\sigma}X^\varepsilon)L_2(m^\varepsilon, (1-\Delta)^{-\frac{1}{2}}h\rangle.\end{aligned} \tag{6.23}$$

Here again there are several quantities which need to be bounded. We use similar computations as previously done for $\|u\|^2_{H^{-\frac{1}{2}}}$ and we obtain:

$$\begin{aligned}\mathbb{E}\left[\mathbf{1}_{\tau^\varepsilon>\mathbf{0}}|\varphi(\mathbf{X}^{\varepsilon,\tau^\varepsilon}) - \varphi^\varepsilon(\mathbf{X}^{\varepsilon,\tau^\varepsilon}, \mathbf{m}^{\varepsilon,\tau^\varepsilon})|\right] &\leqslant C(T)(\varepsilon^{1-\alpha} + \varepsilon^{2-2\alpha}), \\ \mathbb{E}\left[\mathbf{1}_{\tau^\varepsilon>\mathbf{0}}|\mathcal{L}^\varepsilon\varphi^\varepsilon(\mathbf{X}^{\varepsilon,\tau^\varepsilon}, \mathbf{m}^{\varepsilon,\tau^\varepsilon})|\right] &\leqslant C(T)(1 + \varepsilon^{1-\alpha} + \varepsilon^{2-2\alpha}),\end{aligned}$$

where $C(T)$ also depends on the L^2-norm of h. Now taking τ a stopping time and $h = X^\varepsilon(\tau\wedge\tau^\varepsilon)$, knowing that conditioning by $\mathcal{F}_{\tau\wedge\tau^\varepsilon}$

$$M_t := \varphi^\varepsilon(X^\varepsilon(t)) - \varphi^\varepsilon(X_0) - \int_0^t \mathcal{L}^\varepsilon\varphi^\varepsilon(X^\varepsilon(s))ds$$

is a martingale, we obtain

$$\begin{aligned}&\mathbb{E}\Big[\langle X^\varepsilon((\tau+\delta)\wedge\tau^\varepsilon) - X^\varepsilon(\tau\wedge\tau^\varepsilon), X^\varepsilon(\tau\wedge\tau^\varepsilon)\rangle_{H^{-\frac{1}{2}}}\Big] \\ &\quad= \mathbb{E}\left[\varphi(X^\varepsilon((\tau+\delta)\wedge\tau^\varepsilon)) - \varphi(X^\varepsilon(\tau\wedge\tau^\varepsilon))\right] \\ &\quad\leqslant \mathbb{E}\left[\varphi^\varepsilon(X^\varepsilon((\tau+\delta)\wedge\tau^\varepsilon)) - \varphi^\varepsilon(X^\varepsilon(\tau\wedge\tau^\varepsilon))\right] + C\varepsilon^{1-\alpha} + C\varepsilon^{2-2\alpha} \\ &\quad\leqslant \mathbb{E}\left[\mathbb{E}\left[\varphi^\varepsilon(X^\varepsilon((\tau+\delta)\wedge\tau^\varepsilon)) - \varphi^\varepsilon(X^\varepsilon(\tau\wedge\tau^\varepsilon)) \mid \mathcal{F}_{\tau\wedge\tau^\varepsilon}\right]\right] + C\varepsilon^{1-\alpha} + C\varepsilon^{2-2\alpha} \\ &\quad\leqslant \mathbb{E}\left[\mathbb{E}\left[\int_{\tau\wedge\tau^\varepsilon}^{(\tau+\delta)\wedge\tau^\varepsilon} \mathcal{L}^\varepsilon\varphi^\varepsilon(X^\varepsilon(s),)ds \mid \mathcal{F}_{\tau\wedge\tau^\varepsilon}\right]\right] \\ &\qquad+ \mathbb{E}\left[\mathbb{E}\left[M_{(\tau+\delta)\wedge\tau^\varepsilon} - M_{\tau\wedge\tau^\varepsilon} \mid \mathcal{F}_{\tau\wedge\tau^\varepsilon}\right]\right] + C\varepsilon^{1-\alpha} + C\varepsilon^{2-2\alpha} \\ &\quad\leqslant C(1 + \varepsilon^{1-\delta} + \varepsilon^{2-2\delta})\delta + C\varepsilon^{1-\alpha} + C\varepsilon^{2-2\alpha}.\end{aligned}$$

Thus for ε, δ small enough, we have

$$\mathbb{E}\left[\langle X^\varepsilon((\tau+\delta)\wedge\tau^\varepsilon) - X^\varepsilon(\tau\wedge\tau^\varepsilon), X^\varepsilon(\tau\wedge\tau^\varepsilon)\rangle_{H^{-\frac{1}{2}}}\right] \leqslant \eta. \tag{6.24}$$

Finally, gathering (6.22),(6.24) and using the Markov inequality, we get that for any stopping time τ and $R, \eta > 0$, for ε, δ small enough:

$$\mathbb{P}\left(\left\|X^{\varepsilon,\tau^\varepsilon}(\tau+\delta) - X^{\varepsilon,\tau^\varepsilon}(\tau)\right\|_{H^{-\frac{1}{2}}} > R\right) \leqslant \eta. \tag{6.25}$$

We then use an interpolation inequality to write for $s \in [0, 1)$:

$$\left\|X^{\varepsilon,\tau^\varepsilon}(\tau+\delta) - X^{\varepsilon,\tau^\varepsilon}(\tau)\right\|_{H^s} \leq c\left\|X^{\varepsilon,\tau^\varepsilon}(\tau+\delta) - X^{\varepsilon,\tau^\varepsilon}(\tau)\right\|^{2(1-s)/3}_{H^{-\frac{1}{2}}} \left\|X^{\varepsilon,\tau^\varepsilon}(\tau+\delta) - X^{\varepsilon,\tau^\varepsilon}(\tau)\right\|^{(1+2s)/3}_{H^1}.$$

It follows for any $M > 0$:

$$\begin{aligned}
&\mathbb{P}\left(\left\|X^{\varepsilon,\tau^\varepsilon}(\tau+\delta)-X^{\varepsilon,\tau^\varepsilon}(\tau)\right\|_{H^s} > R\right)\\
&\leq \mathbb{P}\left(\left\|X^{\varepsilon,\tau^\varepsilon}(\tau+\delta)-X^{\varepsilon,\tau^\varepsilon}(\tau)\right\|_{H^{-\frac{1}{2}}} > (Rc)^{3/2(1-s)}\, M^{-(1+2s)/2(1-s)}\right)\\
&+\mathbb{P}\left(\left\|X^{\varepsilon,\tau^\varepsilon}(\tau+\delta)-X^{\varepsilon,\tau^\varepsilon}(\tau)\right\|_{H^1} > M\right).
\end{aligned}$$

Thanks to Lemma 4.1 we can choose M such that the last term is smaller than $\eta/2$. Then, using (6.25), we can choose R such that the first term of the right hand is less than $\eta/2$. Thus Aldous criterion is satisfied, the process $(X^{\varepsilon,\tau^\varepsilon})_\varepsilon$ is tight in $\mathcal{C}([0,T],H^s(\mathbb{R}^d))$.

□

Appendix A. Details about the example in section 2.3

Consider X_t solution of (2.33):

$$\left\{\begin{array}{l} dX_t = (AX_t + G(X_t))dt + \sigma dW_t \\ X_0 = x \end{array}\right., \tag{A.1}$$

where $t > 0$, and $x \in H$ with H a Hilbert space. Then we define

$$m(t,n) = \Lambda\left(X(t,\Lambda^{-1}n+\bar{x})-\bar{x}\right),$$

with $\Lambda : H \to E$ a continuous invertible operator and $\bar{x} = \int_H x\nu(dx)$ where ν is the invariant measure of X_t. In order to show that m satisfies the assumptions in E, it is sufficient to show that these assumptions are satisfied by X in H. It is straightforward that X_t is stochastically continuous, and it is proved in [12] that X_t satisfies the coupling assumption 4. We first prove that the moment of order 2 of X_t is bounded. Let us define

$$Z_t = \int_0^t e^{A(t-s)}\sigma dW_s, \qquad Y_t = X_t - Z_t, \quad t \geqslant 0. \tag{A.2}$$

Thanks to the assumptions made on A and σ we know that there exists $\alpha_0 > 0$ such that for any $k \in \mathbb{N}$,

$$\mathbb{E}\left[\sup_{t\in[0,T]} \|Z_t\|^k_{D((-A)^{\alpha_0}}\right] \leqslant C, \tag{A.3}$$

where $D((-A)^\alpha)$ is the domain of the operator $(-A)^\alpha$. Besides Y_t is solution of

$$\left\{\begin{array}{l} dY_t = (AY_t + G(Y_t+Z_t))dt \\ Y_0 = x \end{array}\right. . \tag{A.4}$$

Lemma A.1. *Let $x \in H$, and X_t the solution of equation (2.33) with initial data x. Then for any $k \in \mathbb{N}$ there exists a constant C_k depending on T and x such that*

$$\mathbb{E}\left[\sup_{t\in[0,T]} \|X_t\|^k_H\right] \leqslant C_k.$$

Proof. The proof is based on a classical energy estimate. We start from the equation satisfied by Y_t:

$$\partial_t Y_t = AY_t + G(Y_t+Z_t),$$

and take the scalar product with Y_t. We use the assumptions on A and the boundedness of G to obtain the inequality

$$\frac{1}{2}\partial_t \|Y_t\|_H^2 + \|Y_t\|_{D((-A)^{\frac{1}{2}})}^2 \leqslant M \|Y_t\|_H$$

which, using (2.34) for the right-hand side of the inequality, gives:

$$\frac{1}{2}\partial_t \|Y_t\|_H^2 + \frac{1}{2}\|Y_t\|_{D((-A)^{\frac{1}{2}})}^2 \leqslant \frac{M^2}{2\lambda_1}, \tag{A.5}$$

where λ_1 is the first eigenvalue of $-A$. Finally using (2.34) again:

$$\partial_t \|Y_t\|_H^2 + \lambda_1 \|Y_t\|_H^2 \leqslant \frac{M^2}{\lambda_1}.$$

Now multiply this inequality by $\|Y_t\|_H^{k-2}$ for $k \in \mathbb{N}$:

$$\begin{aligned}\frac{2}{k}\partial_t \|Y_t\|_H^k + \lambda_1 \|Y_t\|_H^k &\leqslant C \|Y_t\|_H^{k-2} \\ &\leqslant C_k + \frac{\lambda_1}{2}\|Y_t\|_H^k .\end{aligned}$$

A Grönwall argument allows to conclude that

$$\|Y_t\|_H^k \leqslant C_k + \|x\|_H^k e^{-\frac{k\lambda_1}{4}t}. \tag{A.6}$$

Then, coming back to the process X_t, we get

$$\mathbb{E}\left[\sup_{t\in[0,T]} \|X_t\|_H^k\right] \leqslant C_k \mathbb{E}\left[\sup_{t\in[0,T]} \left(\|Z_t\|_H^k + \|Y_t\|_H^k\right)\right] \leqslant C_k(x),$$

because of the bound that we just proved for $\|Y_t\|_H^2$ and because all moments of Z_t are bounded. □

Now we can start to prove that X_t satisfies the different assumptions 1 to 5. The following lemma ensures that X_t verifies 2.

Lemma A.2. *Let X_t the process which is solution of (2.33), and denote by P_t its transition semigroup. Then $(P_t)_{t\geqslant 0}$ is Feller.*

Proof. Let $x, y \in H$, denote by X_t^x and X_t^y the solutions of (2.33) with initial data x and y. Then we have

$$d(X_t^x - X_t^y) = (A(X_t^x - X_t^y) + G(X_t^x) - G(X_t^y))\, dt.$$

By classical energy estimate, using the assumptions on A and the Lipschitz assumption on G, we get by a Grönwall argument that

$$\|X_t^x - X_t^y\|_H \leqslant Ce^{Ct}\|x - y\|_H .$$

Thus for any $x \in H$, for any $t \geqslant 0$, the process X_t^y converges almost surely to X_t^x if y converges to x. Now let $\varphi \in C(H)$ with polynomial growth, and $(x_n)_{n\in\mathbb{N}}$ a sequence in H which converges to $x \in H$ as n goes to $+\infty$. The almost sure convergence proved above implies that $\varphi(X_t^{x_n}) - \varphi(X_t^x)$ converges almost surely to 0 in H as n goes to $+\infty$, and Lemma A.1 ensures uniform integrability, so that finally

$$\lim_{n\to+\infty} \|P_t\varphi(x_n) - P_t\varphi(x)\|_H = 0.$$

□

Now we are interested in the existence and uniqueness of an invariant measure for the process X_t.

Lemma A.3. *Let $x \in H$. Then P_t the transition semigroup of the process X_t solution of (2.33) with initial data x has a unique invariant measure denoted by ν.*

Proof. The existence of an invariant measure is based on the Krylov-Bogoliubov theorem. Here again we use the processes Y_t and Z_t defined in (A.2), and start from equation (A.5):

$$\frac{1}{2}\partial_t \|Y_t\|_H^2 + \frac{1}{2}\|Y_t\|^2_{D((-A)^{\frac{1}{2}})} \leqslant \frac{M^2}{2\lambda_1}.$$

If we integrate this inequality we get for $t > 0$:

$$\frac{1}{t}\|Y_t\|_H^2 + \frac{1}{t}\int_0^t \|Y_s\|^2_{D((-A)^{\frac{1}{2}})}\, ds \leqslant C + \frac{1}{t}\|Y_0\|_H^2,$$

so that, using (A.3) there exists a random variable κ with all moments finite such that

$$\frac{1}{t}\int_0^t \|(-A)^{\alpha_0} X_s\|_H\, ds \leqslant \kappa.$$

If we denote by $\mu_t = \frac{1}{t}\int_0^t \mathcal{L}(X_s)ds$ with $\mathcal{L}(X)$ the law of X solution of (A.1), we have

$$\mu_t\left(B^c_{D((-A)^{\alpha_0})}(0,R)\right) \leqslant \frac{C}{R}.$$

Thus the sequence $(\mu_t)_{t>0}$ is tight and by Krylov-Bogoliubov theorem there exists an invariant measure ν.

Now we prove the uniqueness of this invariant measure. It results from the coupling assumption 4 proved in [12] which gives for φ a continuous bounded function and $x, y \in H$:

$$\begin{aligned} |P_t\varphi(x) - P_t\varphi(y)| &= |\mathbb{E}\left[\varphi(X_t^x)\right] - \mathbb{E}\left[\varphi(X_t^y)\right]| \\ &= |\mathbb{E}\left[(\varphi(X_t^x) - \varphi(X_t^y))\,\mathbb{1}_{X_t^x \neq X_t^y}\right] \\ &\leqslant 2\|\varphi\|_0 (1 + \|x\|_H^2 + \|y\|_H^2) e^{-\gamma t}, \end{aligned} \tag{A.7}$$

where we used the Cauchy-Schwarz inequality in the last step of the computation and denoted by $\|\varphi\|_0$ the supremum of φ. Let ν_1, ν_2 be two invariant measures. Let $A \subset H$ and $\varphi = \mathbb{1}_{A \cap B_H(0,R)}$ for $R > 0$. The fact that ν_1 and ν_2 are both invariant measures gives:

$$|\nu_1(A \cap B_H(0,R)) - \nu_2(A \cap B_H(0,R))| = |\int_H P_t\varphi(x)\nu_1(dx) - \int_H P_t\varphi(y)\nu_2(dy)|$$

and according to the inequality coming from the coupling assumption we get:

$$|\nu_1(A \cap B_H(0,R)) - \nu_2(A \cap B_H(0,R))| \leqslant 2(1+R^2)e^{-\gamma t}$$

which goes to 0 as t goes to $+\infty$. However the left-hand side of this inequality does not depend on t, thus it is equal to 0, for any $R > 0$. Finally we get that for any $A \subset H$, $\nu_1(A) = \nu_2(A)$. □

It remains to prove that X_t satisfies assumptions 3 and 5. In this aim we decompose X_t and, using (A.6), we have for any $\varepsilon > 0$

$$\|X_t\|_H^k \leqslant (\|Z_t\|_H + \|Y_t\|_H)^k \leqslant C_{k,\varepsilon}\|Z_t\|_H^k + (1+\varepsilon)\|Y_t\|_H^k \leqslant C_{k,\varepsilon}(\|Z_t\|_H^k + 1) + (1+\varepsilon)\|x\|_H^k\, e^{-\frac{k\lambda_1}{4}t}.$$

which gives us that assumption (3) is satisfied. Indeed, all the moments of Z_t are bounded.

Given $t \geqslant t_0 \geqslant 0$, this inequality generalizes into:

$$\mathbb{E}\left[\|X_t\|_H^k \,|\mathcal{F}_{t_0}\right] \leqslant C + (1+\varepsilon)e^{-\frac{k\lambda_1}{k}t}\,\|X_{t_0}\|_H^k\,.$$

Choosing $\varepsilon > 0$ such that $\alpha = (1+\varepsilon)e^{-\frac{k\lambda_1}{4}} < 1$, we obtain, for $n \in \mathbb{N}$,

$$\mathbb{E}\left[\|X_n\|_H^k\right] \leqslant \frac{C}{1-\alpha} + \alpha^n \|x\|^k.$$

Under our assumptions, we know that the left hand side converges to $\int_H \|x\|_H^k\,\nu(dx)$. We deduce:

$$\int_H \|x\|_H^k\,\nu(dx) < \infty.$$

Finally we proved that X_t satisfies assumptions (1) to (5) (except the zero-mean property), and so does $m(t,n)$ according to its definition in (2.35).

Now we can work on the construction of the functionals $L_1, \cdots, L_4$. We state the following result, which ensures the existence of L_1.

Lemma A.4. *Let $u, v \in H^1(\mathbb{R}^d)$, and define $\varphi^{u,v}(n) = (un, v)$. Then if we define*

$$\mathcal{M}^{-1}\varphi^{u,v}(n) = -\int_0^\infty P_t\varphi^{u,v}(n)dt$$

there exists $L_1 : E \to E$ such that

$$\mathcal{M}^{-1}\varphi^{u,v}(n) = (uL_1(n), v), \quad \textit{and} \quad \|L_1(n)\|_E \leqslant C(1+\|n\|_E)^2.$$

Proof. The process $\varphi^{u,v}(n)$ is centered, thus we can write

$$\begin{aligned}|P_t\varphi^{u,v}(n)| &= |P_t\varphi^{u,v}(n) - \int_E P_t\varphi^{u,v}(\bar{n})\nu(d\bar{n})| \\ &\leqslant \int_E |P_t\varphi^{u,v}(n) - P_t\varphi^{u,v}(\bar{n})|\nu(d\bar{n})\end{aligned}$$

because ν is the invariant measure. Thus denoting by $(m^1_{n,\bar{n}}, m^2_{n,\bar{n}})$ a coupling of $(m(t,n), m(t,\bar{n}))$ satisfying (2.30), we have

$$\begin{aligned}|P_t\varphi^{u,v}(n)| &\leqslant \int_E |\mathbb{E}\left[\varphi^{u,v}(m(t,n)) - \varphi^{u,v}(m(t,\bar{n}))\right]|\nu(d\bar{n}) \\ &\leqslant \int_E |\mathbb{E}\left[\left(\varphi^{u,v}(m^1_{n,\bar{n}}(t)) - \varphi^{u,v}(m^2_{n,\bar{n}}(t))\right)\mathbb{1}_{m^1\neq m^2}\right]|\nu(d\bar{n}) \\ &\leqslant \int_E \mathbb{E}\left[\varphi^{u,v}(m^1_{n,\bar{n}}(t))^2 + \varphi^{u,v}(m^2_{n,\bar{n}}(t))^2\right]^{\frac{1}{2}}\mathbb{P}(m^1_{n,\bar{n}} \neq m^2_{n,\bar{n}})^{\frac{1}{2}}\nu(d\bar{n}).\end{aligned}$$

Denote by E' the dual space of E, then:

$$\begin{aligned}|P_t\varphi^{u,v}(n)| \leqslant C\,\|uv\|_{E'} \int_E \Bigg(&\mathbb{E}\left[\|m(t,n)\|_E^2\right]^{\frac{1}{2}} + \mathbb{E}\left[\|m(t,\bar{n}\|_E^2\right]^{\frac{1}{2}}\Bigg) \\ &\times (1+\|n\|_E^2 + \|\bar{n}\|_E^2)^{\frac{1}{2}}e^{-\frac{\gamma}{2}t}\nu(d\bar{n}).\end{aligned}$$

Besides we have

$$\begin{aligned}\mathbb{E}\left[\|m(t,n)\|_E^2\right] &= \mathbb{E}\left[\left\|\Lambda X(t,\Lambda^{-1}n+\bar{x})-\bar{x}\right\|_E^2\right]\\ &\leqslant C\left(1+\mathbb{E}\left[\left\|X(t,\Lambda^{-1}n+\bar{x})\right\|_H^2\right]\right)\\ &\leqslant C\left(1+\|n\|_E\right)^2\end{aligned}$$

according to assumption 3. Finally we get

$$|P_t\varphi^{u,v}(n)| \leqslant C\|uv\|_{E'}\left(1+\|n\|_E\right)^2 e^{-\frac{\gamma}{2}t}.$$

Thus $\mathcal{M}^{-1}\varphi^{u,v}(n)$ is well-defined and

$$|\mathcal{M}^{-1}\varphi^{u,v}(n)| \leqslant C\|uv\|_{E'}\left(1+\|n\|_E\right)^2. \tag{A.8}$$

We want now to show that we may choose $\mathcal{M}\mathcal{M}^{-1}\varphi^{u,v}=\varphi^{u,v}$. In other words we want

$$\mathcal{M}^{-1}\varphi^{u,v}(m(t,n)) - \int_0^t \varphi^{u,v}(m(s,n))ds$$

to be a martingale with respect to the filtration $(\mathcal{F}_t)_{t\geqslant 0}$. Let $r\leqslant t$. On the one hand:

$$\begin{aligned}&\mathbb{E}\left[-\int_0^\infty P_\tau\varphi^{u,v}(m(t,n))d\tau - \int_0^t \varphi^{u,v}(m(s,n))ds \mid \mathcal{F}_r\right]\\ &= -\mathbb{E}\left[\int_0^\infty \mathbb{E}\left[\varphi^{u,v}(m(t+\tau,n)) \mid \mathcal{F}_t\right]d\tau + \int_0^t \varphi^{u,v}(m(s,n))ds \mid \mathcal{F}_r\right]\\ &= -\mathbb{E}\left[\int_0^\infty \varphi^{u,v}(m(t+\tau,n))d\tau + \int_0^t \varphi^{u,v}(m(s,n))ds \mid \mathcal{F}_r\right]\\ &= -\mathbb{E}\left[\int_0^\infty \varphi^{u,v}(m(s,n))ds \mid \mathcal{F}_r\right].\end{aligned}$$

On the other hand we use the fact that $\int_0^r \varphi^{u,v}(m(s,n))ds \in \mathcal{F}_r$ to compute

$$\begin{aligned}&\mathcal{M}^{-1}\varphi^{u,v}(m(r,n)) - \int_0^r \varphi^{u,v}(m(s,n))ds\\ &= -\mathbb{E}\left[\int_0^\infty \varphi^{u,v}(m(r+\tau,n))d\tau + \int_0^r \varphi^{u,v}(m(s,n))ds \mid \mathcal{F}_r\right]\\ &= -\mathbb{E}\left[\int_0^\infty \varphi^{u,v}(m(s,n))ds \mid \mathcal{F}_r\right],\end{aligned}$$

thus the martingale property is satisfied.

Finally, for every $n\in E$ we have constructed a linear form on E': $uv \mapsto \mathcal{M}^{-1}\varphi^{u,v}(n)$. It is continuous:

$$|\mathcal{M}^{-1}\varphi^{u,v}(n)| \leqslant C(1+\|n\|_E)^2\|uv\|_{E'}.$$

Thus, since E is a reflexive space, there exists $L_1(n)\in E$ such that

$$\mathcal{M}^{-1}\varphi^{u,v}(n) = (L_1(n),uv) = (uL_1(n),v) \quad \text{and} \quad \|L_1(n)\|_E \leqslant C(1+\|n\|_E)^2.$$

□

So far we have proved that it is possible to construct $L_1(n)$ which satisfies our assumptions. Let us now construct the other functionals.

Lemma A.5. *Let $u, v \in H^1(\mathbb{R}^d)$, and define $\varphi^{u,v}(n) = (u(nL_1(n) - \mathbb{E}_\nu[nL_1(n)]), v)$. Then if we define*

$$\mathcal{M}^{-1}\varphi^{u,v}(n) = -\int_0^\infty P_t\varphi^{u,v}(n)dt$$

there exists $L_2 : E \to E$ such that

$$\mathcal{M}^{-1}\varphi^{u,v}(n) = (uL_2(n), v) \quad \textit{and} \quad \|L_2(n)\|_E \leqslant C(1 + \|n\|_E)^3.$$

Proof. We proceed as above and get

$$\begin{aligned}
&|P_t\varphi^{u,v}(n)| \\
&\leqslant C\|uv\|_{E'} \int_E \mathbb{E}\left[\left\|m(t,n)L_1(m(t,n)) - \mathbb{E}_\nu[nL^1(n)]\right\|_E^2\right]^{\frac{1}{2}} (1 + \|n\|^2 + \|\bar{n}\|_E^2)^{\frac{1}{2}} e^{-\frac{\gamma}{2}t} \\
&\quad + \mathbb{E}\left[\left\|m(t,\bar{n})L_1(m(t,\bar{n})) - \mathbb{E}_\nu[nL^1(n)]\right\|_E^2\right]^{\frac{1}{2}} (1 + \|n\|^2 + \|\bar{n}\|_E^2)^{\frac{1}{2}} e^{-\frac{\gamma}{2}t}\nu(d\bar{n}).
\end{aligned}$$

We use the fact that E is an algebra to estimate

$$\begin{aligned}
\mathbb{E}\left[\left\|m(t,n)L_1(m(t,n)) - \mathbb{E}_\nu[nL^1(n)]\right\|_E^2\right] &\leqslant C\left(1 + \mathbb{E}\left[\|m(t,n)\|_E^2 \|L_1(m(t,n))\|_E^2\right]\right) \\
&\leqslant C\left(1 + \|n\|_E\right)^4.
\end{aligned}$$

Finally we get

$$|P_t\varphi^{u,v}(n)| \leqslant C\|uv\|_{E'}\left(1 + \|n\|_E\right)^3 e^{-\frac{\gamma}{2}t},$$

and we can continue the proof exactly as for the construction on $L_1(n)$. Thus for any $n \in E$ there exists $L_2(n) \in E$ such that

$$\mathcal{M}^{-1}\varphi^{u,v}(n) = (uL_2(n), v) \quad \text{and} \quad \|L_2(n)\|_E \leqslant C(1 + \|n\|_E)^3.$$

□

We have the same result for L_3.

Lemma A.6. *Let $u, v \in H^1(\mathbb{R}^d)$, and define $\varphi^{u,v}(n) = (u(\nabla n \nabla L_1(n) - \mathbb{E}_\nu[\nabla n \nabla L_1(n)]), v)$. Then if we set*

$$\mathcal{M}^{-1}\varphi^{u,v}(n) = -\int_0^\infty P_t\varphi^{u,v}(n)dt$$

there exists $L_3 : E \to H^{s-1}$ such that

$$\mathcal{M}^{-1}\varphi^{u,v}(n) = (uL_3(n), v) \quad \textit{and} \quad \|L_3(n)\|_{H^{s-1}} \leqslant C(1 + \|n\|_E)^3.$$

Proof. We follow the same steps as for the two previous proofs, except that when we bound $\varphi^{u,v}$ we have

$$|\varphi^{u,v}(n)| \leqslant \|uv\|_{H^{1-s}} \|\nabla n \nabla L_1(n) - \mathbb{E}_\nu[\nabla n \nabla L_1(n)]\|_{H^{s-1}},$$

and we use the fact that $H^{s-1}(\mathbb{R}^d)$ is an algebra because $s > \frac{d}{2} + 3$ to write

$$|\varphi^{u,v}(n)| \leqslant C\|uv\|_{H^{1-s}}\left(\|n\|_E \|L_1(n)\|_E + C\right).$$

Thus the proof gives us for any $n \in E$ the existence of $L_3(n) \in H^{s-1}$ such that

$$\mathcal{M}^{-1}\varphi^{u,v}(n) = (uL_3(n), v) \quad \text{and} \quad \|L_3(n)\|_{H^{s-1}} \leqslant C(1 + \|n\|_E)^3.$$

□

Finally, we need to construct a last functional L_4, which is slightly different to the previous ones.

Lemma A.7. *Let $w \in L^1(\mathbb{R}^d \times \mathbb{R}^d)$, and define*

$$\varphi^w(n) = \int_{\mathbb{R}^{2n}} w(x,y)\left(n(x)L_1(n)(y) - \mathbb{E}_\nu\left[n(x)L_1(n)(y)\right]\right)dxdy.$$

Then if we set

$$\mathcal{M}^{-1}\varphi^w(n) = -\int_0^\infty P_t\varphi^w(n)dt$$

there exists $L_4 : E \to L^\infty(\mathbb{R}^d \times \mathbb{R}^d)$ such that

$$\mathcal{M}^{-1}\varphi^w(n) = \int_{\mathbb{R}^{2n}} w(x,y)L_4(n)(x,y)dxdy \quad \textit{and} \quad \|L_4(n)\|_{L^\infty(\mathbb{R}^d\times\mathbb{R}^d)} \leqslant C(1+\|n\|_E)^3.$$

Proof. The proof of this lemma is very similar to the proofs of the previous lemmas, but this time

$$|\varphi^w(n)| \leqslant \|w\|_{L^1(\mathbb{R}^d\times\mathbb{R}^d)}\left\|n(x)L_1(n)(y) - \mathbb{E}_\nu\left[n(x)L_1(n)(y)\right]\right\|_{E\times E}$$

because $E \hookrightarrow L^\infty(\mathbb{R}^d)$. Thus we get

$$|P_t\varphi^w(n)| \leqslant C\|w\|_{L^1(\mathbb{R}^d\times\mathbb{R}^d)}(1+\|n\|_E)^3 e^{-\frac{\gamma}{2}t},$$

and using the same procedure as for the previous proofs for any $n \in E$ there exists $L_4(n) \in L^\infty(\mathbb{R}^d \times \mathbb{R}^d)$ such that

$$\mathcal{M}^{-1}\varphi^w(n) = \int_{\mathbb{R}^{2n}} w(x,y)L_4(n)(x,y)dxdy \quad \text{and} \quad \|L_4(n)\|_{L^\infty(\mathbb{R}^d\times\mathbb{R}^d)} \leqslant C(1+\|n\|_E)^3.$$

□

We have constructed the last functional in a slightly different way than in (2.18), but taking $w(x,y) = u(x)u(y)\bar{v}(x)\bar{v}(y)$ we recover the same expression for $\mathcal{M}^{-1}\varphi^w(n)$.

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