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ON HYPERBOLIC DIMENSION GAP FOR ENTIRE FUNCTIONS

VOLKER MAYER AND MARIUSZ URBAŃSKI

ABSTRACT. Polynomials and entire functions whose hyperbolic dimension is strictly smaller than the Hausdorff dimension of their Julia set are known to exist but in all these examples the latter dimension is maximal, i.e. equal to two. In this paper we show that there exist hyperbolic entire functions f having Hausdorff dimension of the Julia set $\text{HD}(\mathcal{J}_f) < 2$ and hyperbolic dimension $\text{HypDim}(f) < \text{HD}(\mathcal{J}_f)$.

1. INTRODUCTION

In this paper we consider some relations between the Hausdorff dimension $\text{HD}(\mathcal{J}_f)$ and the hyperbolic dimension $\text{HypDim}(f)$ of the Julia set \mathcal{J}_f of an entire function $f : \mathbb{C} \rightarrow \mathbb{C}$, where \mathbb{C} , as usually, denotes the complex plane. More precisely, we show the following.

Theorem 1.1. *There exist hyperbolic entire functions f in the Eremenko–Lyubich class \mathcal{B} such that*

$$\text{HypDim}(f) < \text{HD}(\mathcal{J}_f) < 2.$$

The concept $\text{HypDim}(f)$ of hyperbolic dimension has been introduced by Shishikura in [17]. Given an entire function $f : \mathbb{C} \rightarrow \mathbb{C}$ it is defined to be the supremum of Hausdorff dimensions of all hyperbolic sets of f . We recall that a set $X \subset \mathbb{C}$ is hyperbolic if it is compact, forward-invariant under f and if there exist $c > 0$ and $\kappa > 1$ such that

$$|(f^n)'(z)| \geq c\kappa^n \quad \text{for every } z \in X \text{ and all } n \geq 1.$$

It is immediate from this definition that $X \subset \mathcal{J}_f$. For hyperbolic polynomials the whole Julia set is a hyperbolic set, whence there is no difference between the

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hyperbolic dimension and the Hausdorff dimension of the Julia set. In general, since hyperbolic sets of f are subsets of the Julia set of f , we have that

$$(1.1) \quad \text{HypDim}(f) \leq \text{HD}(\mathcal{J}_f).$$

Examples of entire functions with strict inequality are known ([19], [21]). Quite recently Avila-Lyubich [1, 2] showed that there exist Feigenbaum polynomials having this property. But in all known examples with strict inequality in (1.1) the Hausdorff dimension of the Julia set is maximal, i.e. equal to two and Avila-Lyubich mention that for arbitrary polynomials f with $\text{HD}(\mathcal{J}_f) < 2$ one should have equality. Here we show that this is not the case for entire functions even inside the Eremenko-Lyubich class \mathcal{B} consisting of all entire functions having a bounded set of finite singularities.

In order to prove Theorem 1.1 we first need good candidates of entire functions whose Julia sets have Hausdorff dimension less than two. The first such examples were provided by Gwyneth Stallard during 1990's. The interested reader can find an overview in her survey in [16]. These examples are entire functions having one single logarithmic tract over infinity; see Section 2.1 for the definition of the singularities of entire functions, in particular of logarithmic tract. As nowadays it is well known, the geometry of such a tract or the growth of the function in the tract influences the size of the Julia set. Particularly interesting for the present work is her family of intermediate growth in [18]. The growth does depend on a parameter $p > 0$ and these functions are defined by the formula

$$(1.2) \quad E(z) := \frac{1}{2i\pi} \int_L \frac{\exp(e^{(\log \xi)^{1+p}})}{\xi - z} d\xi,$$

where L is the boundary of the region

$$(1.3) \quad G = \left\{ x + iy \in \mathbb{C} : |y| < \frac{\pi x}{(1+p)(\log x)^p}, x > 3 \right\},$$

oriented in the clockwise direction, for $z \in \mathbb{C} \setminus \overline{G}$ and by analytic continuation for $z \in \overline{G}$. Appropriate details of such analytic extension are given in Section 2.2.. The reader should have in mind that this function is close to

$$(1.4) \quad f(z) = \exp(e^{(\log z)^{1+p}}) \quad \text{for } z \in G$$

and is bounded elsewhere. Here $(\log z)^{1+p}$ is defined so that it gives real values for real $z > e$.

Consider then the family $(\mathbf{E}_l : \mathbb{C} \rightarrow \mathbb{C})_{l \in \mathbb{C}}$ defined by the formula

$$\mathbf{E}_l(z) := E(z - l).$$

Shifting in this way the function E by a large $l > 0$ makes the logarithmic tract is backward invariant and yields $J_{\mathbf{E}_l} \subset G$. Consequently, only the dynamics of \mathbf{E}_l in G , the domain on which \mathbf{E}_l is close to the function \mathbf{f}_l , given by the formula

$$\mathbf{f}_l(z) := f(z - l),$$

is relevant for our purposes. The details of this and the definition of the Julia set in the present setting are given in Section 2.1.

Fact 1.2 (Stallard [20]). *Let $p > 0$. All the functions \mathbf{E}_l , $l \in \mathbb{C}$, belong to the Eremenko–Lyubich class \mathcal{B} and there exists a constant $C_p > 0$ such that for all real $l > C_p$ we have that*

$$\text{HD}(J_{\mathbf{E}_l}) = 1 + \frac{1}{1+p} < 2.$$

In the present note we analyze the hyperbolic dimension of these functions. In fact, we first work with the functions \mathbf{f}_l and then transfer the results to the globally defined entire functions \mathbf{E}_l .

The key point is to employ the thermodynamic formalism of [11] and, in particular, the Bowen’s Formula from this paper that determines hyperbolic dimension. We will see that $\lim_{l \rightarrow \infty} \text{HypDim}(\mathbf{E}_l) = 1$ which clearly implies that $\text{HypDim}(\mathbf{E}_l) < \text{HD}(J_{\mathbf{E}_l})$ provided that $l > C_p$ is large enough.

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1.1. Notation. We use standard notation such as $\mathbb{D}(z, r)$ for the open disk in \mathbb{C} with center $z \in \mathbb{C}$ and radius $r > 0$. When the center is the origin, we also use the simplified notation

$$\mathbb{D}_r := \mathbb{D}(0, r).$$

The complement of its closure will be denoted by

$$\mathbb{D}_r^* := \mathbb{C} \setminus \overline{\mathbb{D}_r}.$$

Frequently we deal with half-spaces. Let

$$\mathcal{H}_s := \{z \in \mathbb{C} : \Re z > s\} \quad , \quad s \geq 0.$$

When $s = 0$, then we also write \mathcal{H} for \mathcal{H}_0 .

Many constants, especially those in Fact 2.3, depend on the parameter p of the definitions of the functions E and f . However, this will be fixed throughout the whole paper and we may ignore it.

We say that

$$A \leq B$$

for non-negative real expressions A and B if and only if there exists a positive constant C independent of variable parameters involved in A and B such that $A \leq CB$. We then say that $A \geq B$ if and only if $B \leq A$. Finally, $A \asymp B$ if and only if $A \leq B$ and $B \leq A$.

2. SINGULARITIES, MODELS AND APPROXIMATING ENTIRE FUNCTIONS

2.1. General definitions. Iversen's classification of singularities is explained in length in [9], see also [4]. An entire function $g : \mathbb{C} \rightarrow \mathbb{C}$ can have only two types of singular values. Firstly, a point $b \in \hat{\mathbb{C}}$ is a *critical value* of g if and only if $b = g(c)$ for some $c \in \mathbb{C}$ with $g'(c) = 0$. Secondly, a complex number $b \in \hat{\mathbb{C}}$ is an *asymptotical value* of g if and only if there exists a continuous function $\gamma : [0, +\infty) \rightarrow \mathbb{C}$ such that

$$\lim_{t \rightarrow +\infty} \gamma(t) = \infty \text{ and } \lim_{t \rightarrow +\infty} f(\gamma(t)) = b.$$

In this latter case for every $r > 0$ there exists an unbounded connected component Ω_r of $g^{-1}(\mathbb{D}(b, r))$ such that

$$\Omega_{r'} \subset \Omega_r$$

whenever $r' < r$ and

$$\bigcap_{r>0} \Omega_r = \emptyset.$$

Such a choice of components is called an asymptotic tract over b and it is called *logarithmic tract* in the case when the map $g : \Omega_r \rightarrow \mathbb{D}(b, r) \setminus \{b\}$ is a universal covering for some $r > 0$. The set of singular values of f is proved to consist of all critical and asymptotic values of f . Its intersection with \mathbb{C} will be denoted by $S(g)$.

We consider functions belonging to the Eremenko–Lyubich class \mathcal{B} that consists of all entire functions g for which $S(g)$ is a bounded set. These functions are also called of *bounded type*. If $g \in \mathcal{B}$, then there exists $r > 0$ such that $S(g) \subset \mathbb{D}_r$. Then $g^{-1}(\mathbb{D}_r^*)$ consists of mutually disjoint unbounded Jordan domains Ω_r with real analytic boundaries such that $g : \Omega_r \rightarrow \mathbb{D}_r^*$ is a covering map (see [8]). Thus, an entire function g in class \mathcal{B} has only logarithmic singularities over infinity. As we already mentioned it, the connected components of $g^{-1}(\mathbb{D}_r^*)$ are called *tracts* or, more precisely, *logarithmic tracts*. Then there exist all holomorphic branches of the logarithm of g restricted to Ω_r . Fix one of them and denote it by τ . So,

$$(2.1) \quad g|_{\Omega_r} = \exp \circ \tau,$$

where

$$\varphi = \tau^{-1} : \mathcal{H}_{\log r} \rightarrow \Omega_r$$

is a conformal homeomorphism. In addition, φ extends continuously to ∞ and $\varphi(\infty) = \infty$.

Keeping this notation, if we restrict g to the tracts over infinity then it is now standard, especially since the appearance of the papers [5, 6] by Chris Bishop, to call the map

$$g|_{g^{-1}(\mathbb{D}_r^*)} : g^{-1}(\mathbb{D}_r^*) \rightarrow \mathbb{D}_r^*$$

a model function. We will see that the functions considered in our current paper have only one single tract over infinity. This is the reason why we use the following simplified definition of a model function. This is in the spirit of the definition in [14], see [5, 6] for the general one.

Definition 2.1. *A model is any holomorphic map*

$$g = e^\tau : \Omega_r \rightarrow \mathbb{D}_r^*,$$

where

- (1) $r \in [1, +\infty)$,
- (2) Ω_r is a simply connected unbounded domain in \mathbb{C} , called a tract, such that $\partial\Omega_r$ is a connected subset of \mathbb{C}
and
- (3) $\tau : \Omega_r \rightarrow \mathcal{H}_{\log r}$ is a conformal homeomorphism fixing infinity; the latter more precisely meaning that

$$\tau(z) \rightarrow \infty \text{ as } z \rightarrow \infty.$$

The tract Ω_r may or may not intersect the disk \mathbb{D}_r . The later case has important dynamical consequences.

Definition 2.2. *If f is a model or an entire function of bounded type and if there exists $r > 0$ such that*

$$(2.2) \quad S(f) \subset \mathbb{D}_r \quad \text{and} \quad \overline{f^{-1}(\mathbb{D}_r^*)} \subset \mathbb{D}_r^*,$$

then f is called of disjoint type.

If f is such a disjoint type model or entire function, then the *Julia set* of f is defined to be

$$\mathcal{J}_f := \{z \in \mathbb{D}_r^* : f^n(z) \in \mathbb{D}_r^* \text{ for all } n \geq 1\}.$$

For disjoint type entire functions this definition coincides with the usual one, see Proposition 2.2 in [15].

2.2. Elementary properties of the functions E and f . We now discuss some elementary properties of the functions introduced in the introduction and we examine how they behave with respect to the above definitions.

To start with, we recall that these functions have been introduced and studied by Stallard and her paper [18, Section 3] lists elementary properties of E and f . We recall now some necessary facts from this paper.

Let's denote:

$$G_{x_0, \kappa} := \left\{ z = x + iy \in \mathbb{C} : x > x_0 \text{ and } |y| < \kappa \frac{\pi x}{(1+p)(\log x)^p} \right\}$$

and abbreviate $G_{x_0} = G_{x_0, 1}$ so that the set G of (1.3) is $G_3 = G_{3, 1}$. For any integer $n \geq 3$ let σ_{n+1} be the boundary of the open set $G \setminus \overline{G_{n+1}}$. The orientation of σ_{n+1} and of all following boundary curves are always understood in the clockwise direction. Cauchy's Integral Formula shows that

$$\frac{1}{2i\pi} \int_{\sigma_{n+1}} \frac{f(\xi)}{\xi - z} d\xi = 0 \quad \text{for every } z \notin \overline{G}.$$

Therefore, still for $z \notin \overline{G}$,

$$E(z) = \frac{1}{2i\pi} \int_{\partial G} \frac{f(\xi)}{\xi - z} d\xi = \frac{1}{2i\pi} \int_{\partial G_{n+1}} \frac{f(\xi)}{\xi - z} d\xi.$$

It thus follows that the right hand side integral gives the holomorphic extension of E to the domain $\mathbb{C} \setminus \overline{G_{n+1}}$.

Consider now an arbitrary point $z \in G \setminus \overline{G_{n+1}}$. Then, Cauchy's Residue Theorem shows that

$$\begin{aligned} E(z) &= \frac{1}{2i\pi} \int_{\partial G_{n+1}} \frac{f(\xi)}{\xi - z} d\xi \\ &= -\frac{1}{2i\pi} \int_{\partial(G_n \setminus \overline{G_{n+1}})} \frac{f(\xi)}{\xi - z} d\xi + \frac{1}{2i\pi} \int_{\partial G_n} \frac{f(\xi)}{\xi - z} d\xi \\ &= f(z) + \frac{1}{2i\pi} \int_{\partial G_n} \frac{f(\xi)}{\xi - z} d\xi. \end{aligned}$$

Starting with this observation, one can get the following fact which is contained in Lemma 3.1 in [18] along with its proof.

Fact 2.3. *Let \check{L}, \hat{L} be the boundary of $G_{D+1, \frac{5}{6}}, G_{D-1, \frac{7}{6}}$ respectively. Then there exist constants $C, D > 3$ such that the following hold.*

(1) *If $z \notin G_D$ then*

$$|E(z)| \leq C$$

and

if $z \in G_D$ then

$$|E(z) - f(z)| \leq C \text{ as well as } |E'(z) - f'(z)| \leq C.$$

(2)

$$E(z) = \frac{1}{2\pi i} \int_{\tilde{L}} \frac{f(t)}{t-z} dt \quad \text{for } z \notin G_D$$

and

$$E(z) = f(z) + \frac{1}{2\pi i} \int_{\hat{L}} \frac{f(t)}{t-z} dt \quad \text{for } z \in G_D.$$

(3) If $z \in G_{D, \frac{7}{6}} \setminus \text{Int}(G_{D, \frac{5}{6}})$ then

$$|f(z)| \leq \exp\left(-\frac{1}{2}e^{\frac{1}{2}(\log \Re z)^{1+p}}\right).$$

Item (1) from this Fact 2.3 shows that $f^{-1}(\mathbb{D}_r^*) \subset G_D$ for every $r > 2C$. Elementary estimates, based on the explicit representation of f , show that $f^{-1}(\mathbb{D}_r^*)$ is a simply connected unbounded domain in \mathbb{C} . It turns out that the same is true for the approximating entire function E ; details can be found in Proposition 2.2 of [14]. Thus, we have the following.

Fact 2.4. *Let C be given by Fact 2.3. Then, there exists $r_0 > 4C$ such that*

$$S(E) \subset \mathbb{D}_{r_0/2}$$

and for every $r \geq r_0/2$, both sets $E^{-1}(\mathbb{D}_r^*)$ and $f^{-1}(\mathbb{D}_r^*)$ are simply connected unbounded domains in \mathbb{C} contained in G_D . They will be respectively denoted by

$$\Omega_{E,r} := E^{-1}(\mathbb{D}_r^*) \quad \text{and} \quad \Omega_{f,r} := f^{-1}(\mathbb{D}_r^*).$$

From now on fix any

$$(2.3) \quad r \geq r_0/2,$$

where r_0 comes from Fact 2.4. Then the map $f : \Omega_{f,r} \rightarrow \mathbb{D}_r^*$ is of the form $f(z) = e^{\tau(z)}$ with $\tau : \Omega_{f,r} \rightarrow \mathcal{H}_{\log r}$ given by

$$\tau(z) := \exp((\log z)^{1+p}).$$

We have to know what the inverse conformal homeomorphism $\varphi = \tau^{-1} : \mathcal{H}_{\log r} \rightarrow \Omega_{f,r}$ looks like. Indeed, a straightforward calculation gives

$$(2.4) \quad \varphi(\xi) = \exp\left((\log \xi)^{\frac{1}{1+p}}\right),$$

where \log is the principal branch of logarithm again, i.e. determined by the requirement that $\log 1 = 0$.

In conclusion,

$$(2.5) \quad f|_{\Omega_{f,r}} = e^\tau : \Omega_{f,r} \rightarrow \mathbb{D}_r^*$$

is a model as defined in Definition 2.1; fact 2.3 explains how the entire function E approximates this model.

Lemma 2.5. *There exists a constant $K \geq 1$ such that*

$$\frac{1}{K} \leq \frac{|\varphi'(\xi + iy)|}{|\varphi'(\xi)|} \leq K$$

for every ξ with $\Re(\xi) \geq \log r_0$ and every $0 \leq y \leq 2\pi$.

Proof. The statement follows from Koebe's Distortion Theorem since the conformal map $\varphi = \mathcal{H}_{\log r_0} \rightarrow \Omega_{f,r_0}$ is in fact defined on the half space $\mathcal{H}_{\log(r_0/2)}$. \square

2.3. Disjoint Type Versions of E_l and f_l . Given any $l \in \mathbb{C}$, the functions $\mathbf{f}_l = f \circ T_l$ and $\mathbf{E}_l = E \circ T_l$, where T_l is the translation $z \mapsto z - l$, have been defined in the introduction. We have that $\mathbf{E}_l \in \mathcal{B}$ since it is known, see [20], that $E \in \mathcal{B}$.

Obviously,

$$(2.6) \quad \Omega_{\mathbf{f}_l,r} := \mathbf{f}_l^{-1}(\mathbb{D}_r^*) = f^{-1}(\mathbb{D}_r^*) + l = \Omega_{f,r} + l,$$

and also

$$(2.7) \quad \Omega_{\mathbf{E}_l,r} = \mathbf{E}_l^{-1}(\mathbb{D}_r^*) = E^{-1}(\mathbb{D}_r^*) + l.$$

By Fact 2.4, for all $r \geq r_0/2$ and $l \in [0, +\infty)$, all these tracts are contained in respective sets $G_D + l$. So, setting

$$(2.8) \quad l_r := \max\{0, r - D\},$$

we have that

$$(2.9) \quad \Omega_{\mathbf{f}_l,r}, \Omega_{\mathbf{E}_l,r} \subset \mathbb{D}_r^*$$

for all $r \geq r_0/2$ and all $l \geq l_r$. Consequently, all the functions $\mathbf{f}_l, \mathbf{E}_l, l \geq l_r$, are of disjoint type and for their Julia sets we have that

$$(2.10) \quad \mathcal{J}_{\mathbf{f}_l}, \mathcal{J}_{\mathbf{E}_l} \subset \mathbb{D}_r^*$$

for all $r \geq r_0/2$ and all $l \geq l_r$.

Recall that for the model f we have the expression (2.5). The analogous expression for \mathbf{f}_l is

$$(2.11) \quad \mathbf{f}_l|_{\Omega_{\mathbf{f}_l,r}} = e^{\tau_l} : \Omega_{\mathbf{f}_l,r} \rightarrow \mathbb{D}_r^*$$

where $\tau_l(z) = \tau(z - l)$ so that the inverse of τ_l is

$$(2.12) \quad \varphi_l = \varphi + l : \mathcal{H}_{\log r} \rightarrow \Omega_{\mathbf{f}_l,r}$$

where φ is still the conformal map defined by (2.4)

3. THERMODYNAMICAL FORMALISM

Our ultimate goal is to determine the hyperbolic dimension of the functions \mathbf{E}_l which, under certain conditions, can be done by employing the methods of thermodynamic formalism. The hyperbolic dimension is then given by the zero of the topological pressure, the fact that goes back to Bowen [7]. In the present context, namely for disjoint type models and entire functions of bounded type, such a theory has been developed in [11].

Let $\mathcal{C}_b(\mathbb{D}_r^*)$ be the vector space of all complex-valued bounded continuous functions defined on \mathbb{D}_r^* . Endowed with the supremum norm, it becomes a Banach space.

Let $g := \mathbf{f}_l$ or $g := \mathbf{E}_l$. Given $t > 0$, the transfer operator for the map g and for the parameter t , acting on a function $h \in \mathcal{C}_b(\mathbb{D}_r^*)$, is defined by the formula

$$(3.1) \quad \mathcal{L}_{g,t}h(w) := \sum_{g(z)=w} |g'(z)|_1^{-t} h(z) \quad \text{for every } w \in \mathbb{D}_r^*,$$

where

$$|g'(z)|_1 := \frac{|g'(z)|}{|g(z)|} |z|$$

is the logarithmic derivative of g evaluated at the point z .

We are to find out for which parameters $t > 0$ the following two crucial properties hold:

$$(3.2) \quad \|\mathcal{L}_{g,t}\mathbb{1}\|_\infty < +\infty \quad \text{and} \quad \lim_{w \rightarrow \infty} \mathcal{L}_{g,t}\mathbb{1}(w) = 0.$$

Indeed, since our map g is of disjoint type, once (3.2) is verified then, following [11, Section 8], we deduce that the whole thermodynamic formalism, along with all its applications obtained in [11], holds. Especially Bowen's Formula does. This formula involves topological pressure which for the disjoint type map g is given at a parameter $t \in (0, +\infty)$ by the formula

$$(3.3) \quad P(g, t) = \lim_{n \rightarrow \infty} \frac{1}{n} \log \mathcal{L}_{g,t}^n \mathbb{1}(w),$$

where $w \in \mathbb{D}_r^*$ is any arbitrarily chosen point. The limit exists and is independent of w because of Theorem 8.1 in [11] which ultimately goes back to Lemma 5.8 and Corollary 5.18 in [10].

3.1. Estimates for the Transfer operators of the Model Functions \mathbf{f}_l .

Proposition 3.1. *Let $\mathcal{L}_{\mathbf{f}_l,t}$ be the transfer operator of \mathbf{f}_l , $l \geq 0$, with a parameter $t > 0$. Fix $r \geq r_0$. Let $w_0 \in \mathbb{D}_r^*$. Then*

$$\mathcal{L}_{\mathbf{f}_l,t}\mathbb{1}(w_0) < \infty \quad \text{if and only if} \quad t > 1.$$

Moreover, if $t > 1$ then (3.2) holds for $g = \mathbf{f}_l$.

Proof. Having $w_0 \in \mathbb{D}_r^*$ and $t > 0$, let us start exactly as in the proof of Theorem 4.1 in [11]. If $z_l \in \mathbf{f}_l^{-1}(w_0)$ then, using (2.11), the logarithmic derivative can be expressed as follows:

$$|\mathbf{f}'_l(z_l)|_1 = |\tau'_l(z_l)z_l| = \frac{|\varphi_l(\xi)|}{|\varphi'_l(\xi)|} = |(\log \varphi_l)'(\xi)|^{-1}$$

where $\xi = \tau_l(z_l)$ and where $\varphi_l = \varphi + l$ is the map of (2.12). Notice that $\xi = u + iv$ does not depend on l , where $u = \log |w_0|$. From this, together with Lemma 2.5, we get that

$$\mathcal{L}_{\mathbf{f}_l, t} \mathbb{1}(w_0) = \sum_{\exp(\xi) = w_0} |(\log \varphi_l)'(\xi)|^t = \int_{\mathbb{R}} |(\log \varphi_l)'(\log |w_0| + iv)|^t dv.$$

Now, since $\varphi_l = \varphi + l$ and since we have the explicit expression (2.4) for φ , we can calculate as follows:

$$|(\log \varphi_l)'(\xi)| = \left| \frac{\varphi(\xi)}{\varphi(\xi) + l} \right| \frac{1}{1+p} \frac{1}{|\xi| |\log \xi|^{\frac{p}{1+p}}} \asymp \left| \frac{\varphi(\xi)}{\varphi(\xi) + l} \right| \frac{1}{|\xi| (\log |\xi|)^{\frac{p}{1+p}}}.$$

since $\arg(\xi) \in (-\pi/2, \pi/2)$. Therefore,

$$(3.4) \quad \mathcal{L}_{\mathbf{f}_l, t} \mathbb{1}(w_0) \asymp \int_{\mathbb{R}} \left| \frac{\varphi(\xi)}{\varphi(\xi) + l} \right|^t \frac{1}{|\xi|^t (\log |\xi|)^{\frac{tp}{1+p}}} dv.$$

Since $\lim_{|v| \rightarrow +\infty} \varphi(\log |w_0| + iv) = \infty$, we have that

$$\frac{2}{3} \leq \left| \frac{\varphi(\xi)}{\varphi(\xi) + l} \right| \leq 2$$

whenever $|v| = |\Im(\xi)|$ is sufficiently large. Thus we get from (3.4) that $\mathcal{L}_{\mathbf{f}_l, t} \mathbb{1}(w_0)$ is finite if and only if $t > 1$.

The uniform bound of $\|\mathcal{L}_{\mathbf{f}_l, t}\|_{\infty} < \infty$ also follows from (3.4). Indeed, let $w = e^{\xi} \in \mathbb{D}_r^*$. Then $z = \varphi(\xi) \in G_D$, whence $x = \Re(z) > 0$. Thus,

$$(3.5) \quad \left| \frac{\varphi(\xi)}{\varphi(\xi) + l} \right|^2 = \frac{x^2 + y^2}{(x+l)^2 + y^2} \leq 1.$$

It follows from this that

$$\mathcal{L}_{\mathbf{f}_l, t} \mathbb{1}(w) \leq \int_{\mathbb{R}} \frac{1}{|\xi|^t (\log |\xi|)^{\frac{tp}{1+p}}} dv = \frac{1}{2} \int_{\mathbb{R}} \frac{1}{(u^2 + v^2)^{\frac{t}{2}} (\log(u^2 + v^2))^{\frac{tp}{1+p}}} dv.$$

Since for every for $w \in \mathbb{D}_r^*$ we have $u \geq u_r = \log r$ it follows that

$$(3.6) \quad \sup_{w \in \mathbb{D}_r^*} \mathcal{L}_{\mathbf{f}_l, t} \mathbb{1}(w) \leq C := \int_{\mathbb{R}} \frac{1}{(u_r^2 + v^2)^{\frac{t}{2}} (\log(u_r^2 + v^2))^{\frac{tp}{1+p}}} dv < +\infty.$$

Finally, if $t > 1$ then $\delta = (t - 1)/2 > 0$, whence

$$(3.7) \quad \mathcal{L}_{\mathbf{f}_l, t} \mathbb{1}(w) \leq \frac{1}{u^\delta} \int_{\mathbb{R}} \frac{1}{|u_r + iv|^{1+\delta}} dv \leq \frac{1}{(\log |w|)^\delta}.$$

This shows that $\lim_{w \rightarrow \infty} \mathcal{L}_{\mathbf{f}_l, t} \mathbb{1}(w) = 0$. \square

The next result gives an estimate for the topological pressure. More precisely, it shows that for a given $t > 1$ the pressure $P(\mathbf{f}_l, t) < 0$ for all sufficiently large values of l .

Proposition 3.2. *Let $t > 1$. Fix $r \geq r_0$. Then, for every $\varepsilon > 0$ there exists $l_{\varepsilon, r, t} \geq l_r$ such that*

$$\mathcal{L}_{\mathbf{f}_l, t} \mathbb{1}(w) \leq \varepsilon \quad \text{for every } l \geq l_{\varepsilon, r, t} \text{ and every } w \in \mathbb{D}_r^*.$$

Proof. Let $t > 1$ and $\varepsilon > 0$. We are in the same situation as in the proof of Proposition 3.1. The first benefit we take out of this proof is that the convergence $\lim_{w \rightarrow \infty} \mathcal{L}_{\mathbf{f}_l, t} \mathbb{1}(w) = 0$ is uniform in $l \geq 0$; see (3.7). Therefore, there exists $r_\varepsilon \geq r$ such that

$$\mathcal{L}_{\mathbf{f}_l, t} \mathbb{1}(w) < \varepsilon \quad \text{whenever } |w| \geq r_\varepsilon \text{ and } l \geq 0.$$

Moreover, this proof shows that the integral

$$\int_{\mathbb{R}} \frac{1}{|\xi|^t (\log |\xi|)^{\frac{tp}{1+p}}} dv \quad \xi = u + iv,$$

converges uniformly for $u \geq u_r = \log r$. Therefore, there exists $V = V_{\varepsilon, t}$ such that

$$\int_{|v| \geq V} \frac{1}{|\xi|^t (\log |\xi|)^{\frac{tp}{1+p}}} dv \leq \frac{\varepsilon}{2} \quad \text{for every } u \geq u_r.$$

So, by invoking now (3.4) and (3.5), we conclude that it remains to estimate the integral

$$\int_{|v| < V} \left| \frac{\varphi(\xi)}{\varphi(\xi) + l} \right|^t \frac{1}{|\xi|^t (\log |\xi|)^{\frac{tp}{1+p}}} dv$$

from above by $\varepsilon/2$ for all $l \geq 0$ large enough and all $w \in \mathbb{D}_r^* \setminus \mathbb{D}_{r_\varepsilon}^*$. Here we used again the notation $\xi = u + iv$, $u = \log |w|$. Notice that all points ξ that appear in this integral belong to the compact set

$$K = \{\xi = u + iv : \log r \leq u \leq \log r_\varepsilon \text{ and } |v| \leq V\}.$$

Since $M := \sup_{\xi \in K} \{|\varphi(\xi)|\} < +\infty$, we have that

$$\left| \frac{\varphi(\xi)}{\varphi(\xi) + l} \right| \leq \frac{M}{l - M} \quad ,$$

for every $l > M$ and all $\xi \in K$. Thus,

$$\int_{|v| < V} \left| \frac{\varphi(\xi)}{\varphi(\xi) + l} \right|^t \frac{1}{|\xi|^t (\log |\xi|)^{\frac{tp}{1+p}}} dv \leq C \frac{M}{l - M} \leq \frac{\varepsilon}{2},$$

where $C \in (0, +\infty)$ is the constant coming from (3.6) and the last inequality was written assuming that l is large enough. \square

3.2. Behavior of the Transfer Operators for Entire functions \mathbf{E}_l . We now have sufficiently strong estimates for the transfer operators of the models f_l . Since ultimately we are after the entire functions \mathbf{E}_l , we have to carry over these estimates to the transfer operators of these functions \mathbf{E}_l . Since the entire functions approximate the models, i.e. since we have Fact 2.3, we are in a similar situation as in [12] where also the operators of some models and approximating entire functions have been compared. Following the approach of that paper we will prove the following.

Proposition 3.3. *There exist constants $\mathcal{K} \in [1, +\infty)$ and $r_1 \geq r_0$ such that for every $t > 1$, all $l \in \mathbb{C}$, and all $r \geq r_1$, we have that*

$$\frac{1}{\mathcal{K}^t} \leq \frac{\mathcal{L}_{\mathbf{E}_l, t} \mathbb{1}(w)}{\mathcal{L}_{f_l, t} \mathbb{1}(w)} \leq \mathcal{K}^t \quad \text{for all } w \in \mathbb{D}_r^*.$$

In our proof of Proposition 3.3 we adapt here the approach of [12], particularly Section 7 of that paper. We will show that [12, Lemma 7.3] holds in the present setting if $r \geq r_0$ is large enough. This will suffice. We first shall prove the following.

Fact 3.4. *For all sufficiently large $r \geq r_0$, say $r \geq r_1 \geq r_0$, we have that*

$$\frac{1}{2} \leq \frac{|\mathbf{E}_l(z)|}{|f_l(z)|} \leq 2 \quad \text{and} \quad \frac{1}{2} \leq \frac{|\mathbf{E}'_l(z)|}{|f'_l(z)|} \leq 2$$

for all $l \in \mathbb{C}$ and all $z \in \Omega_{f_l, r}$.

Proof. The first inequality is a direct consequence of item (1) in Fact 2.3 combined with the inequality $r \geq r_0 > 4C$ established in Fact 2.4.

In order to proof the second inequality we also start with item (1) in Fact 2.3. It gives

$$\left| \frac{|E'(z)|}{|f'(z)|} - 1 \right| \leq \frac{C}{|f'(z)|} \quad \text{for all } z \in \Omega_{f, r} \subset G_D.$$

This time we have to estimate $|f'(z)|$ and to show that there exists some $r \geq r_0$ such that

$$(3.8) \quad \frac{C}{|f'(z)|} \leq \frac{1}{2} \quad \text{for all } z \in \Omega_{f,r}.$$

Remember that $f(z) = e^{\tau(z)} = e^{\varphi^{-1}(z)}$ for every $z \in \Omega_{f,r}$. Thus,

$$f'(z) = \frac{f(z)}{\varphi'(\xi)} \quad \text{where } \xi = \varphi^{-1}(z) \in \mathcal{H}_{\log r}.$$

Obviously $|f(z)| > r$ but what about $|\varphi'(\xi)|$? From the formula (2.4) we get

$$\varphi'(\xi) = \frac{1}{1+p} \exp((\log \xi)^{\frac{1}{1+p}}) \frac{1}{(\log \xi)^{\frac{p}{1+p}} \xi}.$$

If $v := \log \xi$ then

$$(3.9) \quad |\varphi'(\xi)| \leq \left| \frac{\exp(v^{\frac{1}{1+p}})}{v^{\frac{p}{1+p}} e^v} \right| = \frac{\exp\left(\Re(v^{\frac{1}{1+p}} - v)\right)}{|v|^{\frac{p}{1+p}}}.$$

Since $\xi \in \mathcal{H}_{\log r}$, $\Re v > \log \log r$, and $|\Im v| < \pi/2$, so if we write $v = se^{i\alpha}$, then

$$s > \log \log r \quad \text{and} \quad |\alpha| < \frac{\pi/2}{\log \log r}.$$

Thus,

$$\Re(v^{\frac{1}{1+p}} - v) = -s \left(\cos \alpha - s^{-\frac{p}{1+p}} \cos\left(\frac{\alpha}{1+p}\right) \right) \leq -\frac{s}{2} \leq -\frac{\log \log r}{2}$$

provided r is sufficiently large. In this case we get from (3.9) that

$$|\varphi'(\xi)| \leq \frac{1}{\sqrt{\log r} (\log \log r)^{\frac{p}{1+p}}}.$$

This shows that (3.8) holds for all $r \geq r_0$ sufficiently large. Thus, 3.4 holds for f and E , i.e. if $l = 0$. It then holds for all $l \in \mathbb{C}$ because of (2.6). \square

Having established Fact 3.4, the proof of Proposition 7.4 in [12] applies word by word and shows that the required inequality in Proposition 3.3 holds.

4. PROOF OF THEOREM 1.1

As it was explained in the Introduction, it suffices to show that

$$(4.1) \quad \lim_{l \rightarrow \infty} \text{HypDim}(\mathbf{E}_l) = 1.$$

In order to do this fix $t > 1$. Fix also any $r \geq r_1$, for example $r = r_1$. By virtue of Proposition 3.2, we have that

$$\mathcal{L}_{f_l, t} \mathbb{1}(w) \leq \mathcal{K}^{-t}$$

for all $l \geq l_{\mathcal{K}^{-t}, r, t}$ and all $w \in \mathbb{D}_r^*$. So, by Proposition 3.3,

$$\mathcal{L}_{E_l, t} \mathbb{1}(w) \leq 1$$

for all $l \geq l_{\mathcal{K}^{-t}, r, t}$ and all $w \in \mathbb{D}_r^*$. In conjunction with (3.3), this gives that

$$P(E_l, t) \leq 0$$

for all $l \geq l_{\mathcal{K}^{-t}, r, t}$. So, if $X \subset \mathcal{J}_{E_l}$ is an arbitrary hyperbolic set for E_l , then

$$P(E_l|_X, t) \leq 0.$$

The supremum over all hyperbolic sets of the left hand side of this inequality is the hyperbolic pressure $P_{hyp}(E_l, t)$ of E_l evaluated at t . So, we have that

$$(4.2) \quad P_{hyp}(E_l, t) = \sup\{P(E_l|_X, t) : X \text{ is a hyperbolic set for } E_l\} \leq 0.$$

Now, we want to use the Bowen's Formula of [3]. Theorem B of this paper applies to the functions E_l and states that the hyperbolic dimension of the set E_l is equal to

$$\text{HypDim}(\mathbf{E}_l) = \inf\{s > 0 : P_{hyp}(E_l, s) \leq 0\}.$$

Combined with (4.2), we thus get that

$$\text{HypDim}(\mathbf{E}_l) \leq t$$

for all $l \geq l_{\mathcal{K}^{-t}, r, t}$. So, the formula (4.1) is established and the proof of Theorem 1.1 is complete.

REFERENCES

- [1] Artur Avila and Mikhail Lyubich. Hausdorff dimension and conformal measures of Feigenbaum Julia sets. *J. Amer. Math. Soc.*, 21:305–383, 2008. 1
- [2] Artur Avila and Mikhail Lyubich. Lebesgue measure of Feigenbaum Julia sets. *Ann. of Math. (2)*, 195(1):1–88, 2022. 1
- [3] Krzysztof Barański, Krzysztofski, Bogusławska Karpińska and Anna Zdunik. Hyperbolic dimension of Julia sets of meromorphic maps with logarithmic tracts. *Int. Math. Res. Not. IMRN*, (4):615–624, 2009. 4
- [4] Walter Bergweiler and Alexandre Eremenko. Direct singularities and completely invariant domains of entire functions. *Illinois J. Math.*, 52(1):243–259, 2008. 2.1

- [5] Christopher J. Bishop. Models for the Eremenko-Lyubich class. *J. Lond. Math. Soc. (2)*, 92(1):202–221, 2015. 2.1
- [6] Christopher J. Bishop. Models for the Speiser class. *Proc. Lond. Math. Soc.*, (3) 114(5):765–797, 2017. 2.1
- [7] Rufus Bowen. Hausdorff dimension of quasicircles. *Inst. Hautes Études Sci. Publ. Math.*, (50):11–25, 1979. 3
- [8] A. Eremenko and M. Yu Lyubich. Dynamical properties of some classes of entire functions. *Annales de l’institut Fourier*, 42(4):989–1020, 1992. 2.1
- [9] J. Kotus and M. Urbański, Meromorphic Dynamics, Volume 2: Elliptic Functions with an Introduction to the Dynamics of Meromorphic Functions, 2.1
Cambridge University Press, series New Mathematical Monographs, 2023.
- [10] Volker Mayer and Mariusz Urbański. Thermodynamical Formalism and Multifractal Analysis for Meromorphic Functions of Finite Order, *Memoirs of AMS*. 203 (2010), no. 954. *Memoirs of AMS*, 203 (2010), no. 954. 3
- [11] Volker Mayer and Mariusz Urbański. Thermodynamic formalism and integral means spectrum of logarithmic tracts for transcendental entire functions. *Trans. Amer. Math. Soc.*, 373(11):7669–7711, 2020. 1, 3, 3, 3, 3.1
- [12] Volker Mayer and Anna Zdunik. The failure of Ruelle’s property for entire functions. *Advances in Mathematics*, 384:107723, 2021. 3.2, 3.2, 3.2
- [13] Feliks Przytycki and Mariusz Urbański, Conformal Fractals—Ergodic Theory Methods, *Cambridge University Press*, 2010.
- [14] Lasse Rempe-Gillen. Hyperbolic entire functions with full hyperbolic dimension and approximation by Eremenko-Lyubich functions. *Proc. Lond. Math. Soc. (3)*, 108(5):1193–1225, 2014. 2.1, 2.2
- [15] Lasse Rempe-Gillen. Arc-like continua, Julia sets of entire functions and Eremenko’s conjecture. *Preprint*, 2019. 2.1
- [16] Philip J. Rippon and Gwyneth M. Stallard, editors. *Transcendental Dynamics and Complex Analysis*. London Mathematical Society Lecture Note Series. Cambridge University Press, Cambridge, 2008. 1
- [17] Mitsuhiro Shishikura. The Hausdorff dimension of the boundary of the Mandelbrot set and Julia sets. *Ann. of Math. (2)*, 147(2):225–267, 1998. 1
- [18] Gwyneth M. Stallard. The Hausdorff dimension of Julia sets of entire functions iii. *Mathematical Proceedings of the Cambridge Philosophical Society*, 122(2):223–244, 1997. 1, 2.2
- [19] Gwyneth M. Stallard. The Hausdorff dimension of Julia sets of hyperbolic meromorphic functions. *Math. Proc. Cambridge Philos. Soc.*, 127(2):271–288, 1999. 1
- [20] Gwyneth M. Stallard. The Hausdorff dimension of Julia sets of entire functions. IV. *J. London Math. Soc. (2)*, 61(2):471–488, 2000. 1.2, 2.3
- [21] Mariusz Urbański and Anna Zdunik. The finer geometry and dynamics of the hyperbolic exponential family. *Michigan Math. J.*, 51(2):227–250, 2003. 1

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