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Reduced-Order Observer Design for Time-Delay Systems using Partial Pole Placement

Ahlem Sassi¹ and Islam Boussaada^{2,3} and Silviu-Iulian Niculescu³

Abstract—This paper develops a functional reduced-order observer for a class of Linear Time-Invariant (LTI) systems with time-delay in the state. The proposed observer provides exponentially stable estimation errors. The unbiasedness conditions on the nominal part of the error dynamics of the reduced-order observer were given by employing an algebraic framework. Under such conditions, we propose the use of a partial pole placement of the error dynamical equations to handle sufficient conditions for the existence of the proposed observer. Specifically, we make use of the *multiplicity-induced-dominancy* property of the characteristic function corresponding to the system's error. The performance and the effectiveness of the developed observer is highlighted through two examples: first, the estimation of the HIV-1 infection dynamics, and second, the improvement of the environmental performance through the study of a diesel engine system.

I. INTRODUCTION

Delay-differential equations are one of the simplest ways to represent time heterogeneity. Transportation, propagation and communication represent the simplest phenomena where the delay appears naturally. As expected, due to its infinite dimensional character, the presence of time delay in a system model renders the control/observer design more difficult. However, it should be mentioned that if such a delay is properly taken into account, it may bring advantages in controlling system dynamics as emphasized by [1] [2].

Over the past decades, several significant advances have been done in stability analysis and feedback control of time-delay systems, [3], [4] and the references therein. In the sequel, we are interested on the observation of time-delay systems under the assumption that the delay is known, and, in particular, reduced-order observers. Roughly speaking, since the number of state variables in a reduced-order observer is less than the order of the considered system, the reduced-order observer is parsimonious, often a desirable engineering quality. But, in addition, a reduced-order observer may have better properties than a full-order observer, especially with respect to robustness of a control system which uses an observer to implement the control algorithm in an “observer-based” control design. Reduced-order observers are well recognized for having faster convergence rates and lower computational burden, as the only state variables to be

estimated are the ones which are actually not measured (see, for example, [5], [6], [7], [8], [9]).

It is well known that the aim of observer design is to reconstruct the trajectory of states of a dynamic system. An observer is an auxiliary system such that the error between the observer state and the system state decreases asymptotically towards zero. As a result, stability analysis tools are crucial in the design of observers. Among the existing methods, eigenvalue-based methods have become popular in the stabilization of linear time-invariant (LTI) time-delay systems [10] and recently in observers design [11], [12]. A recent pole placement approach, called *Partial Poles Placement* (PPP), has been introduced in [13]. It derives from two properties called respectively *multiplicity-induced-dominancy* (MID) and *coexistent-real-roots-inducing-dominancy* (CRRID), see for instance [14].

In a previous work [11], we propose the design of a full-order observer using the pole placement approach based on the MID property. Here, we propose a more general observer with reduced order to estimate a function of the states or the unmeasured ones. For the design, we call the *generic multiplicity-induced-dominancy* (GMID) which is a more general result for *Delay Differential Equations* (DDE) of order n with delayed polynomial term of order $n - 1$, which allows us to avoid assumption of the real rootedness of the non-delayed polynomial corresponding to the characteristic function. To the best of the authors' knowledge, such a construction represents a novelty in the open literature.

In this paper, we investigate the design of a reduced order observer for delay systems which employs the pole placement approach, for stabilization of the error systems involving a known time-delay. Some prerequisites and the problem formulation are presented in Section II. In Section III, the main results on the existence of the observer and stability analysis of the error system are given. In Section IV, the proposed observer is applied to the linear model of HIV-1 infection dynamics and the diesel engine which both involves known time-delay. Some concluding remarks end the paper. The notations are standard.

II. PREREQUISITES AND PROBLEM STATEMENT

Consider a generic dynamical system with a single delay described by the DDE:

$$\dot{\xi}(t) = A\xi(t) + A_d\xi(t - \tau), \quad (1)$$

under appropriate initial conditions, where $\xi \in \mathbb{R}^n$ is the state vector, τ is a positive constant delay. The matrices: $A \in \mathbb{R}^{n \times n}$ and $A_d \in \mathbb{R}^{n \times n}$ are known constant matrices.

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It is well known that the asymptotic behavior of the solutions of (1) is determined from its spectrum (see, e.g. [15]), that is the set of the (characteristic) roots of the associated characteristic function (denoted $\Delta(s, \tau)$ in the sequel).

The characteristic function $\Delta : \mathbb{C} \times \mathbb{R}_+ \rightarrow \mathbb{C}$ corresponding to (1) writes as follows:

$$\Delta(s, \tau) = \det(sI_n - A - A_d e^{-\tau s}) \quad (2)$$

A generic result on the location of spectral values corresponding to (2) is given by the following:

Proposition 1: [16] If s is a characteristic root of system (1), then it satisfies

$$|s| \leq \|A + A_d e^{-\tau s}\|_2 \quad (3)$$

This result combined with the triangular inequality provides a generic envelope curve around the characteristic roots corresponding to system (1).

In the sequel, we are interested in the study of a class of LTI delay systems characterized by the quasipolynomial function of the form:

$$\Delta(s, \tau) = P_0(s) + P_1(s)e^{-\tau s} \quad (4)$$

where $\deg(P_0) > \deg(P_1)$. The question of analytical characterization of its rightmost root will be a key in the resolution of our problem.

A. Partial pole placement in delay systems

One of the most natural and classical ways to stabilize a dynamical LTI delay system is to select the free parameters of the controller in order to choose the location of finitely many roots while also guaranteeing that the *dominant root*¹ is among the chosen ones. This has been the subject of several recent works, such as [17], [14], [13]. Contrary to the strategy of FPP used, e.g., in [18], the controllers designed calling these approaches do not make the closed-loop system finite-dimensional, instead, control its rightmost spectral value. These methods also extend to some partial differential equations, as detailed, for example, in [19]. In particular, we shall use the MID property as described in [20]. For LTI systems including delays in their model representation, spectral methods can be used to understand the asymptotic behavior of solutions by considering the roots of some characteristic function (see, e.g., [21], [22], [15], [23], [24], [25]) which, for (1), is the function $\Delta : \mathbb{C} \rightarrow \mathbb{C}$ defined for $s \in \mathbb{C}$ by

$$\Delta(s) = s^n + \sum_{k=0}^{n-1} a_k s^k + e^{-s\tau} \sum_{k=0}^m \alpha_k s^k. \quad (5)$$

More precisely, the exponential behavior of solutions of (5) is given by the real number $\gamma_0 = \sup\{\Re(s)/s \in \mathbb{C}, \Delta(s) = 0\}$, called the *spectral abscissa* of Δ , in the sense that, for every $\varepsilon > 0$, there exists $C > 0$ such that, for every solution e of (10), one has $|e(t)| \leq C e^{(\gamma_0 + \varepsilon)t} \max_{\theta \in [-\tau, 0]} |e(\theta)|$ [21]. Moreover, all solutions of (5) converge exponentially to 0 if

and only if $\gamma_0 < 0$. An important difficulty in the analysis of the asymptotic behavior of (5) is that, contrary to the delay-free case, the corresponding characteristic function Δ has infinitely many roots.

Theorem 1 ([26]): Consider the qua-sipolynomial Δ given by (5). Let $s_0 \in \mathbb{R}$ be a root of Δ with maximal multiplicity i.e. $M(s_0) = \deg(\Delta)$, then,

- 1) (Retarded) If $m < n$, then s_0 is a strictly dominant root of Δ .
- 2) (Neutral) If $m = n$, then s_0 is a dominant root of Δ and, for every other complex root s of Δ , one has $\Re(s) = s_0$.

Notice that the GMID consists in forcing a root to reach its maximal multiplicity, which does not allow any degree of freedom in assigning s_0 . In order to allow for some additional freedom when assigning s_0 , one can relax such a constraint by forcing the root s_0 to have a multiplicity lower than the maximal.

B. Problem formulation

1) *Studied Class of systems:* The class of system investigated in this work is defined by the following state space model,

$$\dot{x}(t) = Ax(t) + A_d x(t - \tau) + Bu(t) \quad (6a)$$

$$y(t) = Cx(t) \quad (6b)$$

$$x(t) = \phi(t) \quad t \in [0, \tau] \quad (6c)$$

$$z(t) = Lx(t) \quad (6d)$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $y \in \mathbb{R}^p$ are the states vector, the control input and the measurements vector respectively. $z \in \mathbb{R}^r$ is the vector to be estimated where $r \leq n$. τ is a known constant delay. Finally, $\phi(t)$ is a continuous function defining the corresponding initial condition. The matrices: $A \in \mathbb{R}^{n \times n}$, $A_d \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$ and $C \in \mathbb{R}^{p \times n}$ are known constant matrices of appropriate dimensions. It is assumed that:

$$\text{rank}(L) = r \quad (7)$$

In order to estimate the states $z(t)$, we consider the following structure of functional observer

$$\dot{\eta}(t) = N\eta(t) + N_d\eta(t - \tau) + My(t) + M_d y(t - \tau) \quad (8a)$$

$$+ Fu(t) \quad (8b)$$

$$\hat{z}(t) = \eta(t) + Ey(t) \quad (8c)$$

where $\eta \in \mathbb{R}^r$. $\hat{z} \in \mathbb{R}^r$ is the estimate of z . The matrices N , M , N_d , M_d , F and E are unknown, which should be determined later such that the estimation of the states \hat{z} converge to its real values z .

2) *The error equation:* Let us note $e(t) = z(t) - \hat{z}(t) = \Psi x(t) - \eta(t)$, where $\Psi = L - EC$. The estimation error dynamics is described by the following system

$$\begin{aligned} \dot{e}(t) = & Ne(t) + N_d e(t - \tau) + (\Psi A - N\Psi - MC)x(t) \\ & + (\Psi A_d - N_d\Psi - M_d C)x(t - \tau) + (F - \Psi B)u(t) \end{aligned} \quad (9)$$

¹the rightmost characteristic root in the complex plane

Proposition 2: The dynamical system (8) represents a reduced functional observer for system (6), if and only if the dynamic error system given by

$$\dot{e}(t) = Ne(t) + N_d e(t - \tau) \quad (10)$$

is asymptotically stable and the following equation hold

$$\begin{aligned} F - \Psi B &= 0 \\ \Psi A - N\Psi - MC &= 0 \\ \Psi A_d - N_d\Psi - M_d C &= 0 \end{aligned}$$

Proof 1: Let us compute the dynamics of the estimation error, as follows:

$$\begin{aligned} \dot{e}(t) &= \Psi \dot{x}(t) - \dot{\eta}(t) \\ &= \Psi Ax(t) + \Psi A_d x(t - \tau) + \Psi Bu(t) - N\eta(t) \\ &\quad - N_d \eta(t - \tau) - My(t) - M_d y(t - \tau) - Fu(t) \end{aligned}$$

By adding and subtracting the term $\pm N\Psi x(t) \pm N_d\Psi x(t - \tau)$, system (9) is obtained. Thus, in order to ensure the unbiasedness of the proposed observer, the dynamics of the error estimation must be independent of the state vector x and the input vector u . So, we need to put the terms which do not depend on these errors equal to zero. The convergence of the proposed observer require the stability of the error system (10).□

The first step for the stability analysis of the estimation error is to guarantee the unbiasedness of system (9). In other words, we have to put the terms which do not depend on the error equal to zero. Thus, the matrix F had to be chosen as

$$F = \Psi B \quad (11)$$

and the following Sylvester equations must hold

$$\Psi A - N\Psi - MC = 0 \quad (12a)$$

$$\Psi A_d - N_d\Psi - M_d C = 0 \quad (12b)$$

Then, for the second step, we need to ensure the stability of the error dynamics, i.e. to choose the matrices N and N_d in order to guarantee the stability of the error dynamics.

III. MAIN RESULTS

A. Resolution of the Sylvester equations

To ensure the unbiasedness of the observer, the Sylvester equations described by (12) should be verified.

Using the expression $\Psi = L - EC$, the Sylvester equations are rewritten as

$$LA = NL + KC + ECA \quad (13a)$$

$$LA_d = N_d L + K_d C + ECA_d \quad (13b)$$

where $K = M - NE$, and $K_d = M_d - N_d E$.

It's clear that these Sylvester equations (13) can be written in the following compact form

$$V = \bar{X}W \quad (14)$$

where V and W contain the known matrices such as

$$V = [LA, LA_d]$$

$$W^T = \begin{bmatrix} L & 0 \\ 0 & L \\ C & 0 \\ 0 & C \\ CA & CA_d \end{bmatrix}$$

and \bar{X} is the matrix of the unknown observer's matrices to be designed, described as follows

$$\bar{X} = [N, N_d, K, K_d, E]$$

Equation (14) admits a solution if and only if the following rank condition is satisfied

$$\text{rank}(W) = \text{rank}([V^T, W^T]^T) \quad (15)$$

and, by defining $\ell = 2(r + p) + p$, the general solution is given by

$$\bar{X} = VW^\dagger + Z(I_\ell - WW^\dagger) \quad (16)$$

where Z is an arbitrary matrix with appropriate dimension, which will be chosen in order to satisfy the convergence of the estimation errors and W^\dagger is any generalised inverse of W , which fulfils the following equation [27]

$$W = WW^\dagger W \quad (17)$$

Then, the observers matrices can be expressed through a single gain matrix Z as

$$\underbrace{\begin{bmatrix} N \\ N_d \\ K \\ K_d \\ E \end{bmatrix}}_{\bar{X}} = \underbrace{\begin{bmatrix} N_1 \\ N_{d1} \\ K_1 \\ K_{d1} \\ E1 \end{bmatrix}}_{VW^\dagger} + Z \underbrace{\begin{bmatrix} N_2 \\ N_{d2} \\ K_2 \\ K_{d2} \\ E2 \end{bmatrix}}_{I_\ell - WW^\dagger} \quad (18)$$

In other words:

$$N = VW^\dagger R_N + Z(I_\ell - WW^\dagger)R_N \quad (19a)$$

$$N_d = VW^\dagger R_{N_d} + Z(I_\ell - WW^\dagger)R_{N_d} \quad (19b)$$

$$K = VW^\dagger R_K + Z(I_\ell - WW^\dagger)R_K \quad (19c)$$

$$K_d = VW^\dagger R_{K_d} + Z(I_\ell - WW^\dagger)R_{K_d} \quad (19d)$$

$$E = VW^\dagger R_E + Z(I_\ell - WW^\dagger)R_E \quad (19e)$$

with

$$R_N^T = [I_r \quad 0_{r \times r} \quad 0_{p \times r} \quad 0_{p \times r} \quad 0_{p \times r}]$$

$$R_{N_d}^T = [0_{r \times r} \quad I_r \quad 0_{p \times r} \quad 0_{p \times r} \quad 0_{p \times r}]$$

$$R_K^T = [0_{r \times p} \quad 0_{r \times p} \quad I_p \quad 0_{p \times p} \quad 0_{p \times p}]$$

$$R_{K_d}^T = [0_{r \times p} \quad 0_{r \times p} \quad 0_{p \times p} \quad I_p \quad 0_{p \times p}]$$

$$R_E^T = [0_{r \times p} \quad 0_{r \times p} \quad 0_{p \times p} \quad 0_{p \times p} \quad I_p]$$

B. Stability and attenuation analysis of the estimation error

Since the unbiasedness conditions given in section III-A are satisfied, the dynamic of the estimation error is written as in equation (10). Let us consider the estimation error dynamic written in the following form

$$\dot{e}(t) - Ne(t) - N_d e(t - \tau) = 0 \quad (20)$$

So that the characteristic function associated $\Delta : \mathbb{C} \times \mathbb{R}_+ \rightarrow \mathbb{C}$ is given by

$$\Delta(s, \tau) = |sI_n - N - N_d e^{-\tau s}| \quad (21)$$

Hence, the error system is characterised by the quasipolynomial function with the form of equation (4) where the polynomial P_0 is given by:

$$P_0(s) = |sI_n - N| = n_n s^n + n_{n-1} s^{n-1} + \dots + n_1 s + n_0$$

In order to check the stability of the estimation error, the following assumption is necessary:

Assumption 1: $\text{rank}(N_d) = 1$.

In a previous work [11] the intermediate MID property has been exploited through the result of [20] restricting the observer design to the following additional assumption:

Assumption 2: The polynomial P_0 to be realrooted.

The use of the GMID property allows to relax such an assumption by using the result from [26]. In the following theorem, we state sufficient condition that ensures the stability of the error estimation and thus the convergence of the proposed observer.

Theorem 2: Consider that Assumption 1 holds and let a negative s_0 be a multiple root of (21) with the maximal multiplicity. The error system (10) is exponentially stable with s_0 as decay rate and the estimated states vector converge to its real value.

Proof 2: Since condition 15 holds, the error dynamics is written under equation (10). Under the assumption 1, the quasipolynomial (21) is a particular case of (5). Thus, following Theorem 1, a root s_0 of Δ is necessarily dominant. So that, it corresponds to the exponential decay of (20). \square

IV. ILLUSTRATIVE EXAMPLES

In this section, two case studies are proposed and discussed.

A. HIV-1 Infection Model

In HIV-1 infection, the virus life cycle plays a crucial role in disease progression. The binding of a viral particle to a receptor on a target cell initiates a cascade of events that can ultimately lead to the target cell becoming productively infected, i.e. producing new virus. Some models [28], [29] assumed this process to occur instantaneously. In other words, it is assumed that as soon as virus contacts a target cell, the cell begins producing the virus. However, in reality there is a time delay between initial viral entry into a cell and subsequent viral production. This delay needs to be taken into account to accurately determine the half life of free virus from drug perturbation experiments. The intracellular delay is incorporated in the system model by assuming that the generation of virus producing cells at time t is due to the infection of target cells at time $(t - \tau)$, where τ is a constant delay. The HIV-1 dynamics is given by the following equations

$$\frac{dT^*(t)}{dt} = kT_0 V_I(t - \tau) - \delta T^*(t) \quad (22a)$$

$$\frac{dV_I(t)}{dt} = (1 - n_p)N\delta T^*(t) - cV_I(t) \quad (22b)$$

$$\frac{dV_{NI}(t)}{dt} = n_p N\delta T^*(t) - cV_{NI}(t) \quad (22c)$$

where T^* is the concentration of productively infected T-cells, V_I and V_{NI} represent the plasma concentrations of virions in the infectious pool (produced before the drug effect) and in the noninfectious pool (produced after the drug effect), respectively. For detailed study, refer to [28] on the experiment and the data.

A detailed list of variable is given below.

	Definition	Value
n_p	Protease inhibitor efficacy	0.7
k	Viral infectivity rate	$\frac{c}{NT_0}$
τ		0.91 days
T_0	Target T-cell concentration	408 cells.mm ⁻³
N	Bursting term for viral production after lysis	480 virions/cells
δ	Death rate of an infected T-cell	1,57 / day
c	Clearance rate of virus	4,3/day

To apply the proposed approach, it is convenient to transform system (22) into the form of system (6), where $x(t) = [T^*(t) \ V_I(t) \ V_{NI}(t)]^T$, and the matrices A and A_d are given by

$$A = \begin{bmatrix} -\delta & 0 & 0 \\ (1 - n_p)N\delta & -c & 0 \\ n_p N\delta & 0 & -c \end{bmatrix} \quad A_d = \begin{bmatrix} 0 & kT_0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad (23)$$

The level of virions circulating in plasma for both infectious and non infectious pool can be measured. So, we assume that the vector of measurement is given by

$$y(t) = Cx(t) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} x(t) = \begin{bmatrix} V_I(t) \\ V_I(t) + V_{NI}(t) \end{bmatrix}$$

and we propose to estimate the concentration of productively infected T-cells denoted T^* . In other words, we propose to estimate $z(t) = T^*(t)$, with $L = [1 \ 0 \ 0]$. For this aim, we consider an observer in the form (8). So the error dynamics is given by (9). For this application, the rank condition (15) holds. As the observer order is $r = 1$, so assumption 1 is already considered and just need to tune the matrix Z to obtain a stable error system. The quasipolynomial function is with the following form

$$\Delta(s, \tau) = (s - N) - e^{-\tau s} N_d \\ = (s - (N_1 + ZN_2)) - e^{-\tau s} (N_{d1} + ZN_{d2}) \quad (24)$$

Where $Z \in \mathbb{R}^{r \times \ell}$. The spectrum of the quasipolynomial is computed for a matrix $Z = [z_1 \ z_2 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0]$ and is shown in Figure 1. By setting the initial condition to $x(0) = [1 \ 180 \ 0]^T$ and $\eta(0) = [0]$ and with a suitable choice of the matrix $Z = [-1.57 \ 1 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0]$, the

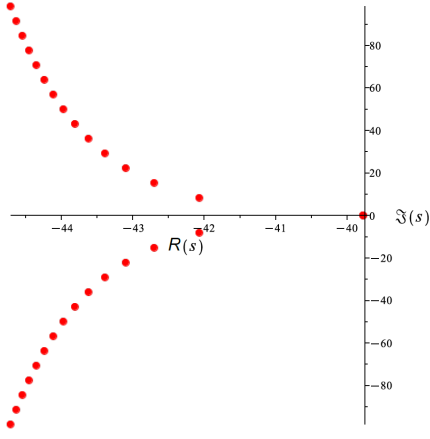


Fig. 1. Spectrum distribution of the quasipolynomials (24)

estimation of the concentration of productively infected T-cells $z(t)$ converges to its real values $T^*(t)$ (see Figure 2).

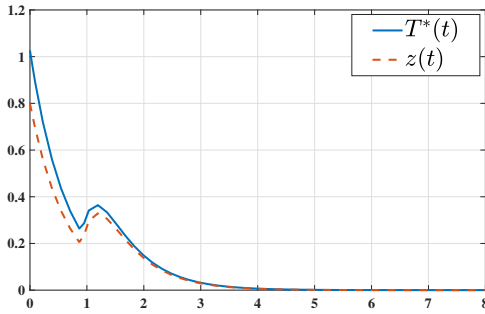


Fig. 2. Evolution of concentration of productively infected T-cells $T^*(t)$ (—) and its estimation $z(t)$ (- -).

B. Diesel Engine Model

A diesel engine with an exhaust gas recirculation (EGR) valve and a turbo-compressor with a variable geometry turbine (VGT) was modeled in [30]. The system is subjected to intake-to-exhaust transport delay. A LTI time-delay model is obtained by linearizing the original non-linear model under the assumption that the engine operates at a constant speed $N = 1500$ RPM. This model is described as

$$\dot{x}(t) = A_0x(t) + A_1x(t - \tau) + Bu(t) \quad (25a)$$

$$y(t) = Cx(t) = [0 \ 1 \ 0]x(t) \quad (25b)$$

$$\text{where } A_0 = \begin{bmatrix} -27 & 3.6 & 6 \\ 9.6 & -12.5 & 0 \\ 0 & 9 & -5 \end{bmatrix}, A_1 = \begin{bmatrix} 0 & 0 & 0 \\ 21 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

and $B = \begin{bmatrix} 0.26 & 0 \\ -0.9 & -0.8 \\ 0 & 0.18 \end{bmatrix}$ and τ is given as 0.06s at this particular operating point. The state variables are defined as intake manifold pressure (x_1), exhaust manifold pressure

(x_2), and compressor power (x_3). Furthermore, the system has two control inputs in which u_1 is an input for the EGR valve opening and u_2 is an input for the VGT mass flow rate.

The model includes intake-to-exhaust transport delay $\tau = 60$ ms when engine speed is 1500 RPM. Due to the time-delay assignable to the transport time of the gas from intake to exhaust manifold, the system is represented a DDE as in Equation (25). Since the observability matrix is full rank matrix, system (25) is observable and the corresponding quasipolynomial function is given by

$$\begin{aligned} \Delta(s, \tau) &= \det(sI_r - N - N_d e^{-\tau s}) \\ &= \det(sI_r - (N_1 + ZN_2)) - (N_{d1} + ZN_{d2})e^{-\tau s} \end{aligned} \quad (26)$$

where $r = 2$, $P_0(s)$ is with order 2 and $P_1(s)$ is with order 1. The matrix Z is tuned to get assumption 1.

For instance, choosing Z as :

$$Z = \begin{bmatrix} x & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & y & 0 & 0 & 0 & z \end{bmatrix}$$

one can define the rightmost root s_0 of (26), with a maximal multiplicity 4, as a function of the delay τ as shown in Figure 3. To show the convergence of the states estimation to

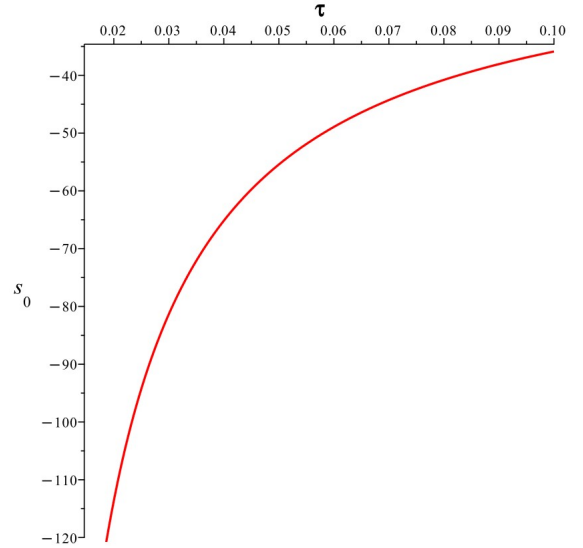


Fig. 3. The spectral abscissa of (26) as a function of the delay τ .

their real values, we choose Z with the following numerical values:

$$Z = \begin{bmatrix} -170 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -150 & 0 & 0 & 0 & 100 \end{bmatrix}$$

For the simulation, the initial states are chosen as $x(0) = [0.3 \ 0.2 \ -0.3]^T$ and $z(0) = [0 \ 0]^T$. As shown in Figure 4, the error $e(t)$ of the states and their estimation converge to zero in a short span of time. Thus, the observer reveals satisfactory performance.

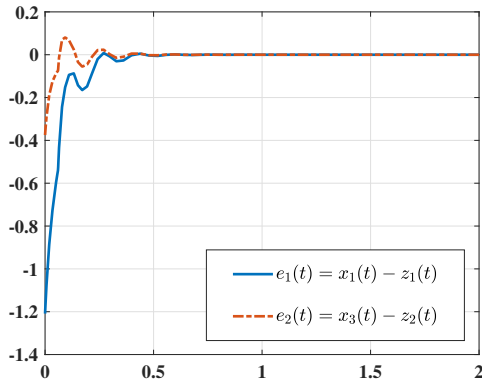


Fig. 4. Evolution of the errors between the system's states and their estimations.

V. CONCLUDING REMARKS

In this paper, a reduced-order observer was developed for a class of LTI systems described by delay-differential equations, including a known delay in the system state-space representation. By using a pole placement approach derived from the GMID property, we present some sufficient conditions for the existence of the observer, which guarantee that the estimation error converges asymptotically towards zero. The efficiency of the proposed method is illustrated through two examples.

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