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SAMPLE AVERAGE APPROXIMATION FOR STOCHASTIC PROGRAMMING WITH EQUALITY CONSTRAINTS

THOMAS LEW*, RICCARDO BONALLI[†], AND MARCO PAVONE*

Abstract. We revisit the sample average approximation (SAA) approach for general non-convex stochastic programming. We show that applying the SAA approach to problems with expected value equality constraints does not necessarily result in asymptotic optimality guarantees as the sample size increases. To address this issue, we relax the equality constraints. Then, we prove the asymptotic optimality of the modified SAA approach under mild smoothness and boundedness conditions on the equality constraint functions. Our analysis uses random set theory and concentration inequalities to characterize the approximation error from the sampling procedure. We apply our approach to the problem of stochastic optimal control for nonlinear dynamical systems subject to external disturbances modeled by a Wiener process. We verify our approach on a rocket-powered descent problem and show that our computed solutions allow for significant uncertainty reduction.

1. Introduction. Sampling-based algorithms for stochastic programming have been successfully applied to a plethora of applications such as portfolio optimization [24], resource management [16], and controller design [10], among many others [13]. In particular, the sample average approximation (SAA) is a popular approach that consists of replacing the expected value of cost and constraints functions of the original stochastic program with empirical averages from a random sample. The resulting deterministic problem is then numerically solved via standard optimization techniques. The SAA approach is applicable to general non-convex stochastic programs, including with chance constraints [24] and risk measures [26].

Many works have investigated the asymptotic convergence properties of the SAA approach [9, 24, 28, 34]. The most fundamental property is consistency: as the sample size increases, solutions to the approximated problems should solve the original stochastic program. This consistency result is key to justifying the use of the SAA. However, these previous works only consider expected value inequality constraints. Considering equality constraints is crucial in many applications. For instance, stochastic optimal control problems in robotics and aerospace often include an expected value equality constraint on the final state of the system [6, 8]. By assuming that inequality constraints are strictly feasible, the results in [9, 24, 28, 34] do not apply to stochastic programs with expected value equality constraints, see Remarks 2.1 and 5.1. Problems with equality constraints are tackled in [15] by combining the SAA with the penalty method. However, the theoretical convergence properties of the SAA approach with strict equality constraints enforcement remain poorly understood.

In this work, we show that the SAA approach can be applied to stochastic programs with both expected value equality and inequality constraints and yield asymptotic optimality guarantees under mild assumptions. The key insight consists of relaxing the empirical equality constraints. By considering equality constraints, our analysis generalizes previous results in the stochastic programming literature [9, 24, 28, 34] and applies to pathological stochastic programs with inequality constraints, see Remark 5.1. We rely on mild smoothness and boundedness assumptions on the functions defining the problem that can be verified prior to running the SAA approach.

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Our analysis relies on random set theory and concentration inequalities. Specifically, we prove the almost-sure convergence of compact random level sets under a high-probability uniform error bound of the level set functions (Lemma 3.2). Also, we derive a concentration inequality (Proposition 3.5) that applies to classes of functions that are only Hölder continuous (A2), in contrast to prior work that assumes the Lipschitz continuity of the function class [4, 17, 27, 33]. We believe these two intermediate results are of independent interest and will find other applications.

We apply the approach to the problem of stochastic optimal control for nonlinear systems characterized by a stochastic differential equation. Solving the general nonconvex formulation remains challenging, see [5, 7, 8, 25]. Hamilton-Jacobi-Bellman approaches [18] optimize over closed-loop policies and thus suffer from the curse of dimensionality due to the discretization of the state space. In contrast, we optimize over open-loop control trajectories which allows for efficient numerical resolution. This approach is common in the literature [5, 6, 8] and allows for feedback by recursively solving the open-loop problem over time [22]. We verify the approach on a Mars rocket-powered descent control problem and obtain solutions that minimize both the fuel consumption and the variance of the final state error.

The paper is organized as follows. In **Section 2**, we define the stochastic programming problem with expectation equality constraints, propose a sampling-based approximation of the problem, and state our assumptions and main asymptotic optimality result (Theorem 2.1). In **Section 3**, we prove Theorem 2.1. The analysis requires two intermediate steps. First, in **Section 3.1**, we reformulate the problem and the sampling-based approximation using random compact sets. With this reformulation, we derive asymptotic optimality guarantees of the approach under convergence assumptions on the cost and feasible set approximations (Theorem 3.1). Finally, we derive sufficient conditions for the convergence of the random level sets that define the feasible equality constraints sets (Lemma 3.2). Second, in **Section 3.2**, we derive a concentration inequality (Proposition 3.6) that implies the satisfaction of a high-probability uniform error bound required in Lemma 3.2 under mild assumptions (A2-3). In **Section 4**, we extend the analysis to problems with inequality constraints. In **Section 5**, we apply the analysis to stochastic optimal control problems. In **Section 6**, we provide numerical results validating the proposed approach and compare with prior work. We conclude in **Section 7** and discuss directions of future work.

2. Problem formulation and main result. For clarity of exposition, we defer the analysis of problems with inequality constraints to Section 4. Let $(\Omega, \mathcal{G}, \mathbb{P})$ be a probability space, $n, d \in \mathbb{N}$, $\mathcal{U} \subset \mathbb{R}^d$ be a compact set, $f : \mathcal{U} \times \Omega \rightarrow \mathbb{R}$ and $h : \mathcal{U} \times \Omega \rightarrow \mathbb{R}^n$ be two Carathéodory functions¹. We consider the stochastic optimization problem

$$\mathbf{P} : \inf_{u \in \mathcal{U}} \mathbb{E}[f(u, \omega)] \quad \text{s.t.} \quad \mathbb{E}[h(u, \omega)] = 0$$

and assume that \mathbf{P} is feasible.

We consider the sample average approximation (SAA) that consists of replacing the expectations in \mathbf{P} with approximations from a sample of independent and identically distributed (i.i.d.) realizations $\omega^i \in \Omega$. Let $(\bar{\Omega}, \bar{\mathcal{G}}, \bar{\mathbb{P}})$ denote the product probability space (where $\bar{\Omega} = (\Omega \times \Omega \times \dots)$, see Section A for details) and $\bar{\omega} = (\omega^1, \dots) \in \bar{\Omega}$ denote a sample, such that each $\omega^i \in \Omega$ corresponds to an i.i.d. realization from

¹The cost function $f : \mathcal{U} \times \Omega \rightarrow \mathbb{R}$ is Carathéodory if $f(u, \cdot)$ is \mathcal{G} -measurable for all $u \in \mathcal{U}$ and $f(\cdot, \omega)$ is continuous \mathbb{P} -almost-surely, and similarly for the equality constraints function h .

$(\Omega, \mathcal{G}, \mathbb{P})$. Given a sample size $N \in \mathbb{N}$, we define the empirical estimates

$$(2.1) \quad \begin{aligned} f_N : \mathcal{U} \times \bar{\Omega} &\rightarrow \mathbb{R} & h_N : \mathcal{U} \times \bar{\Omega} &\rightarrow \mathbb{R}^n \\ (u, \bar{\omega}) &\mapsto \frac{1}{N} \sum_{i=1}^N f(u, \omega^i) & (u, \bar{\omega}) &\mapsto \frac{1}{N} \sum_{i=1}^N h(u, \omega^i). \end{aligned}$$

Given a sequence of strictly positive scalars $(\delta_N)_{N \in \mathbb{N}}$, we consider the sampled problem

$$\mathbf{SP}_N(\bar{\omega}) : \inf_{u \in \mathcal{U}} f_N(u, \bar{\omega}) \quad \text{s.t.} \quad \|h_N(u, \bar{\omega})\| \leq \delta_N.$$

For any $N \in \mathbb{N}$ and $\bar{\omega} \in \bar{\Omega}$, we denote by (f_0^*, S_0^*) and by $(f_N^*(\bar{\omega}), S_N^*(\bar{\omega}))$ the optimal value and set of solutions to \mathbf{P} and to $\mathbf{SP}_N(\bar{\omega})$, respectively.

The first objective of this work is to provide reasonable sufficient assumptions on f and h that can be checked prior to applying the SAA approach and allow guaranteeing that $\bar{\mathbb{P}}$ -almost-surely, (1) $\mathbf{SP}_N(\bar{\omega})$ is feasible for a sample size N large enough, and (2) solutions to $\mathbf{SP}_N(\bar{\omega})$ converge to optimal solutions to \mathbf{P} as N increases. Relaxing the constraints of the sampled problem $\mathbf{SP}_N(\bar{\omega})$ is a critical algorithmic choice to yield the aforementioned guarantees.

Remark 2.1 (Infeasibility of the standard SAA) Directly applying the standard SAA approach to \mathbf{P} without relaxing the equality constraints by δ_N does not necessarily yield almost-sure asymptotic feasibility and optimality guarantees, as we show next. Let $\mathcal{U} = \Omega = [-1, 1]$, $\mathcal{G} = \mathcal{B}(\Omega)$, and \mathbb{P} be the Lebesgue measure normalized to Ω , so that ω is uniformly distributed over Ω . Consider the two problems

$$\mathbf{P}_1 : \inf_{u \in \mathcal{U}} |u| \quad \text{s.t.} \quad \mathbb{E}[(u + \omega, u - \omega)] = (0, 0),$$

$$\mathbf{P}_2 : \inf_{u \in \mathcal{U}} |u| \quad \text{s.t.} \quad \mathbb{E}[u^2 + \omega] = 0,$$

where \mathbf{P}_1 is convex and \mathbf{P}_2 has only one constraint. The solution to \mathbf{P}_1 and \mathbf{P}_2 is $u^* = 0$. However, the two sampled problems with strict equality constraint enforcement

$$\inf_{u \in \mathcal{U}} |u| \quad \text{s.t.} \quad \begin{pmatrix} u + \frac{1}{N} \sum_{i=1}^N \omega^i \\ u - \frac{1}{N} \sum_{i=1}^N \omega^i \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

$$\inf_{u \in \mathcal{U}} |u| \quad \text{s.t.} \quad u^2 + \frac{1}{N} \sum_{i=1}^N \omega^i = 0$$

are infeasible with $\bar{\mathbb{P}}$ -probability 1 and $\frac{1}{2}$, respectively, for all sample sizes $N \in \mathbb{N}$. The problem comes from the lack of constraint qualification condition [28], see also [2, Condition A] and the hypotheses in [9, Section 5.2.5.2] which do not hold in this example, see Figure 1. Instead of making such assumptions that could potentially be restrictive, in this work, we assume that the equality constraints functions h are smooth and bounded (see (A2-3)) and relax the equality constraints.

By selecting the right dependency of δ_N on N , we show that the proposed stochastic programming approach yields asymptotic feasibility and optimality guarantees under assumptions of integrability and smoothness on f and of smoothness and boundedness on h .

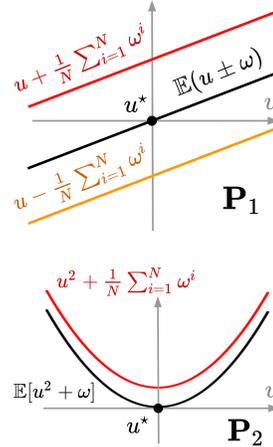


Fig. 1: Two infeasible examples for the standard SAA approach.

(A1) There exists a \mathcal{G} -measurable function $d : \Omega \rightarrow \mathbb{R}$ such that $\mathbb{E}[d(\omega)] < \infty$ and $|f(u, \omega)| \leq d(\omega)$ for all $u \in \mathcal{U}$ \mathbb{P} -almost surely.

(A1) \mathbb{P} -almost-surely, the map $u \mapsto f(u, \omega)$ is α -Hölder continuous for some exponent $\alpha \in (0, 1]$ and Hölder constant $L(\omega)$ such that

$$|f(u_1, \omega) - f(u_2, \omega)| \leq L(\omega) \|u_1 - u_2\|_2^\alpha \quad \text{for all } u_1, u_2 \in \mathcal{U}.$$

(A2) \mathbb{P} -almost-surely, the map $u \mapsto h(u, \omega)$ is α -Hölder continuous for some exponent $\alpha \in (0, 1]$ and Hölder constant $M(\omega)$ satisfying $\mathbb{E}[M(\omega)^2] < \infty$, such that $\|h(u_1, \omega) - h(u_2, \omega)\| \leq M(\omega) \|u_1 - u_2\|_2^\alpha$ for all $u_1, u_2 \in \mathcal{U}$.

(A3) \mathbb{P} -almost-surely, $\sup_{u \in \mathcal{U}} \|h(u, \omega)\| \leq \bar{h}$ for some constant $0 \leq \bar{h} < \infty$.

(A1-3) are standard assumptions that are mild and can be checked prior to applying the proposed sampling-based approach. Assuming α -Hölder continuity for $\alpha < 1$ in (A1) and (A2) is weaker than assuming that the maps $u \mapsto (f, g)(u, \omega)$ are Lipschitz continuous, as is typically done in the literature [4, 14, 17, 27, 33]. Making these weaker assumptions is necessary to apply our analysis to optimal control problems for dynamical systems described with a stochastic differential equation, see Section 5 and (5.3). By multiplying h with a smooth cutoff function, (A3) is always satisfied. We further discuss assumptions (A1-3) (see Remark 3.3) in the next section. Below, we state the main result of this paper.

THEOREM 2.1 (Asymptotic optimality of the SAA approach for problems with expectation equality constraints). *Assume that the functions f and h satisfy (A1), (A2), and (A3). Given any constants $\epsilon \in (0, \frac{1}{2})$ and $C > 0$, define the sequence $(\delta_N)_{N \in \mathbb{N}}$ as*

$$(2.2) \quad \delta_N = CN^{-(\frac{1}{2}-\epsilon)}.$$

Then, $\bar{\mathbb{P}}$ -almost-surely, there exists a subsequence $\{N_k(\bar{\omega})\}_{k \in \mathbb{N}}$ such that

$$f_{N_k(\bar{\omega})}^*(\bar{\omega}) \rightarrow f_0^* \quad \text{and} \quad \mathbb{D}(S_{N_k(\bar{\omega})}^*(\bar{\omega}), S_0^*) \rightarrow 0 \quad \text{as } k \rightarrow \infty$$

with the distance function \mathbb{D} between the solution sets $S_N^*(\bar{\omega})$ and S_0^* defined in (3.3). If in addition, f satisfies (A1), then $f_N^*(\bar{\omega}) \rightarrow f_0^*$ and $\mathbb{D}(S_N^*(\bar{\omega}), S_0^*) \rightarrow 0$ as $N \rightarrow \infty$ $\bar{\mathbb{P}}$ -almost-surely.

This result implies that $\bar{\mathbb{P}}$ -almost-surely, \mathbf{SP}_N is feasible for a sample size N large enough and solutions to \mathbf{SP}_N converge (up to a subsequence) to solutions to \mathbf{P} as N increases. If in addition, the cost function f is α -Hölder continuous under (A1), then the entire sequence of solutions to \mathbf{SP}_N converges to solutions to \mathbf{P} .

The remainder of this paper consists of proving Theorem 2.1 (Section 3), generalizing it to problems with inequality constraints (Section 4), and applying the proposed approach to stochastic optimal control problems (Sections 5 and 6).

3. Proof of the main asymptotic optimality result (Theorem 2.1).

3.1. Reformulation with random sets and convergence analysis. We first observe that the feasible set of \mathbf{SP}_N is a random compact set and reformulate \mathbf{P} and \mathbf{SP}_N accordingly. Then, we derive convergence guarantees under sufficient conditions.

3.1.1. Reformulating \mathbf{P} and \mathbf{SP}_N with random sets. We denote by \mathcal{K} the family of compact subsets of \mathbb{R}^d , by $\mathcal{B}(\mathcal{K})$ the σ -algebra generated by the myopic topology on \mathcal{K} [21, 23], and define $\mathcal{K}' = \mathcal{K} \setminus \{\emptyset\}$. For any $N \in \mathbb{N}$, we define

$$(3.1) \quad f_0 : \mathcal{U} \rightarrow \mathbb{R} : u \mapsto \mathbb{E}[f(u, \omega)], \quad h_0 : \mathcal{U} \rightarrow \mathbb{R}^n : u \mapsto \mathbb{E}[h(u, \omega)],$$

f_N, h_N in (2.1), and the constraints sets

$$(3.2a) \quad C_0 = \{u \in \mathcal{U} : h_0(u) = 0\}$$

$$(3.2b) \quad C_N : \bar{\Omega} \rightarrow \mathcal{K} : \bar{\omega} \mapsto \{u \in \mathcal{U} : \|h_N(u, \bar{\omega})\| \leq \delta_N\}.$$

Since h is Carathéodory, $\mathcal{U} \in \mathcal{K}$, and \mathbf{P} is feasible, $C_0 \in \mathcal{K}'$. Also, C_N is a random compact set [3, 21, 23], i.e., a measurable map from $(\bar{\Omega}, \mathcal{G})$ to $(\mathcal{K}, \mathcal{B}(\mathcal{K}))$, see Appendix B. We will prove under mild conditions that $C_N(\bar{\omega}) \neq \emptyset$ for N large enough $\bar{\mathbb{P}}$ -almost-surely (i.e., that $\mathbf{SP}_N(\bar{\omega})$ is feasible), see Lemma 3.2 and Proposition 3.6.

Then, \mathbf{P} and $\mathbf{SP}_N(\bar{\omega})$ can be written as

$$\mathbf{P} : \inf_{u \in C_0} f_0(u) \quad \text{and} \quad \mathbf{SP}_N(\bar{\omega}) : \inf_{u \in C_N(\bar{\omega})} f_N(u, \bar{\omega}).$$

3.1.2. General asymptotic convergence analysis. We use the previous reformulation to prove a general convergence result under sufficient appropriate convergence of the approximated cost function f_N and of the random constraints set C_N . To describe the convergence of the solutions sets, for any $A, B \in \mathcal{K}'$, we define

$$(3.3) \quad \mathbb{D}(A, B) = \sup_{x \in A} \inf_{y \in B} \|x - y\|$$

and $\mathbb{D}(A, B) = \infty$ if A or B is empty. Although \mathbb{D} is not a metric, it satisfies the triangle inequality and the property that $\mathbb{D}(A, B) = 0$ implies $A \subseteq B$. We will use this last property to analyze the solution sets of \mathbf{P} and \mathbf{SP}_N .

We describe the convergence of the constraints sets $C_N(\bar{\omega})$ using the Hausdorff metric $d_H : \mathcal{K} \times \mathcal{K} \rightarrow [0, \infty)$ defined for any $A, B \in \mathcal{K}$ as

$$(3.4) \quad d_H(A, B) = \max(\mathbb{D}(A, B), \mathbb{D}(B, A)).$$

When considering optimization problems, it is generally only possible to prove convergence with respect to \mathbb{D} ; it is generally not possible to prove that $d_H(S_N^*(\bar{\omega}), S_0^*) \rightarrow 0$ as $N \rightarrow \infty$, see e.g. [32]. Assuming the uniform convergence of the cost function and constraints sets, we prove that up to a subsequence, $\mathbb{D}(S_N^*(\bar{\omega}), S_0^*) \rightarrow 0$ as $N \rightarrow \infty$ $\bar{\mathbb{P}}$ -almost-surely. We represent the distance $\mathbb{D}(S_N^*(\bar{\omega}), S_0^*)$ in Figure 2 and its limit.

THEOREM 3.1. *Assume that the costs and constraints in \mathbf{P} and \mathbf{SP}_N satisfy*

1. *(Uniform convergence of cost) $\bar{\mathbb{P}}$ -almost-surely,*

$$(3.5) \quad \sup_{u \in \mathcal{U}} |f_N(u, \bar{\omega}) - f_0(u)| \rightarrow 0 \quad \text{as} \quad N \rightarrow \infty.$$

2. *(Myopic convergence of constraint set) $\bar{\mathbb{P}}$ -almost-surely,*

$$(3.6) \quad d_H(C_N(\bar{\omega}), C_0) \rightarrow 0 \quad \text{as} \quad N \rightarrow \infty.$$

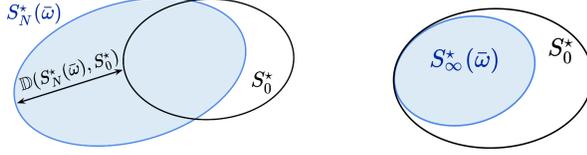


Fig. 2: \mathbb{D} -distance between the solution sets S_0^* and $S_N^*(\bar{\omega})$ to \mathbf{P} and to $\mathbf{SP}_N(\bar{\omega})$, respectively. If $\mathbb{D}(S_N^*(\bar{\omega}), S_0^*) \rightarrow 0$ as $N \rightarrow \infty$ as in Theorem 3.1, then the limit of the solutions sets $S_N^*(\bar{\omega})$ is a subset of the solution set S_0^* . Thus, in the limit as the sample size N increases, solutions $u_N^*(\bar{\omega}) \in S_N^*(\bar{\omega})$ to $\mathbf{SP}_N(\bar{\omega})$ are optimal solutions to \mathbf{P} .

Then, $\bar{\mathbb{P}}$ -almost-surely, there exists a subsequence $\{N_k(\bar{\omega})\}_{k \in \mathbb{N}}$ such that

$$(3.7a) \quad f_{N_k(\bar{\omega})}^*(\bar{\omega}) \rightarrow f_0^* \quad \text{as } k \rightarrow \infty,$$

$$(3.7b) \quad \mathbb{D}(S_{N_k(\bar{\omega})}^*(\bar{\omega}), S_0^*) \rightarrow 0 \quad \text{as } k \rightarrow \infty.$$

If in addition, f satisfies $\widehat{\text{(A1)}}$, then $f_N^*(\bar{\omega}) \rightarrow f_0^*$ and $\mathbb{D}(S_N^*(\bar{\omega}), S_0^*) \rightarrow 0$ as $N \rightarrow \infty$ $\bar{\mathbb{P}}$ -almost-surely.

The proof is provided in Appendix C. It is inspired from [28] but follows from different assumptions on the constraints sets. Specifically, in (3.6), we make a general assumption on the convergence of the constraints sets that allows considering equality constraints. By assuming that $d_H(C_N(\bar{\omega}), C_0) \rightarrow 0$ as $N \rightarrow \infty$ $\bar{\mathbb{P}}$ -almost-surely, Theorem 3.1 assumes that the sampled problems \mathbf{SP}_N are always feasible for a sample size N large enough.

The uniform convergence of f_N to f_0 in (3.5) holds under (A1), see e.g. [9, Theorem 5.40] and [28, Theorem 7.48]. (A1) and $\widehat{\text{(A1)}}$ are standard assumptions that are often satisfied in practice, see Section 5. The difficulty consists of providing sufficient conditions for the $\bar{\mathbb{P}}$ -almost-sure convergence of the random feasible level sets in (3.6). We derive such conditions in the next section.

3.1.3. Convergence of random level sets. The feasible sets of \mathbf{SP}_N are random compact sets defined as the zero level sets of the empirical estimates h_N . Next, we derive sufficient conditions for the $\bar{\mathbb{P}}$ -almost-sure convergence of these sets.

LEMMA 3.2 (Almost-sure convergence of random level sets). *Let $(\epsilon_N)_{N \in \mathbb{N}}$ and $(\beta_N)_{N \in \mathbb{N}}$ be two sequences such that $\epsilon_N \rightarrow 0$ as $N \rightarrow \infty$, $\sum_{N=1}^{\infty} \beta_N < \infty$, and*

$$(3.8) \quad \bar{\mathbb{P}} \left(\sup_{u \in \mathcal{U}} \|h_N(u, \bar{\omega}) - h_0(u)\| \leq \epsilon_N \right) \geq 1 - \beta_N \quad \text{for all } N \in \mathbb{N}.$$

Let $(\delta_N)_{N \in \mathbb{N}}$ be a sequence of scalars such that

$$\delta_N > 0 \text{ for all } N \in \mathbb{N}, \quad \delta_N \rightarrow 0 \text{ as } N \rightarrow \infty, \quad \frac{\epsilon_N}{\delta_N} \rightarrow 0 \text{ as } N \rightarrow \infty.$$

Assume that there exists $u^* \in \mathcal{U}$ such that $h_0(u^*) = 0$. Define the compact set $C_0 \subseteq \mathcal{U}$ and the sequence of random compact sets $C_N : \bar{\Omega} \rightarrow \mathcal{K}$ as in (3.2).

Then, $\bar{\mathbb{P}}$ -almost-surely, $C_N(\bar{\omega}) \neq \emptyset$ for all $N \geq N(\bar{\omega})$ with $N(\bar{\omega})$ large enough and

$$(3.9) \quad d_H(C_N(\bar{\omega}), C_0) \rightarrow 0 \text{ as } N \rightarrow \infty.$$

The proof is provided in Appendix B and uses a result that provides sufficient conditions for the myopic convergence of random compact sets [23, Proposition 1.7.23]. Thanks to Lemma 3.2, the condition (3.6) in Theorem 3.1 can be satisfied using a high-probability uniform bound (3.8) on the error $h_N(u, \bar{\omega}) - h_0(u)$.

3.2. Proving (3.8) via concentration inequalities. In this section, we use concentration inequalities to derive the high-probability uniform error bound (3.8) for the sampling-based approximation. We define the parametric function class

$$(3.10) \quad \mathcal{H} \triangleq \{h(u, \cdot) : \Omega \rightarrow \mathbb{R}^n, u \in \mathcal{U}\}.$$

To derive bounds of the form of (3.8), concentration inequalities exploit the structure of the considered function class. In this work, we make the smoothness and boundedness assumptions (A2-3). Assuming α -Hölder continuity for $\alpha < 1$ in (A2) is weaker than assuming that the map $u \mapsto h(u, \omega)$ is Lipschitz continuous, as is typically done in the literature [4, 17, 27, 33]. Making this weaker assumption is necessary to apply our approach and analysis to stochastic optimal control problems for nonlinear dynamical systems described with a stochastic differential equation, see Section 5.

Remark 3.3. Markov's inequality and a standard symmetrization argument give

$$(3.11) \quad \bar{\mathbb{P}} \left(\sup_{u \in \mathcal{U}} \left\| \frac{1}{N} \sum_{i=1}^N h(u, \omega^i) - \mathbb{E}[h(u, \omega)] \right\| \geq \epsilon \right) \leq \frac{2}{\epsilon} R_N(\mathcal{H})$$

for any $\epsilon > 0$ (see [33, (4.17-4.18)] and Appendix D), where $R_N(\mathcal{H})$ is defined in (D.1) and denotes the Rademacher complexity of the function class \mathcal{H} . Standard bounds for reasonable function classes give $R_N(\mathcal{H}) \leq CN^{-\frac{1}{2}}$ for some constant C [27, 33]. However, combining such bounds with Markov's inequality does not provide a bound that is tight enough, i.e., a bound satisfying (3.8) with $\sum_{N=1}^{\infty} \beta_N < \infty$. Obtaining tighter bounds requires making stronger assumptions on the function class \mathcal{H} . In this work, these stronger assumptions are (A2) and (A3).

3.2.1. Bounded differences. McDiarmid's inequality [27, Lemma 26.4], also known as the bounded differences inequality [33, Corollary 2.21], is a well-known result that allows deriving tight concentration inequalities for bounded function classes. Bounds resulting from this result are tighter than naively combining Markov's inequality with a symmetrization argument, see Remark 3.3. The following result is standard, see [33, (4.16)].

COROLLARY 3.4. *Assuming (A3), for any $N \in \mathbb{N}$, $\delta > 0$, and $\beta_N^\delta = \exp(-\frac{N\delta^2}{2h^2})$,*

$$\sup_{u \in \mathcal{U}} \left\| \frac{1}{N} \sum_{i=1}^N h(u, \omega^i) - \mathbb{E}[h(u, \omega)] \right\| \geq \bar{\mathbb{E}} \left[\sup_{u \in \mathcal{U}} \left\| \frac{1}{N} \sum_{i=1}^N h(u, \omega^i) - \mathbb{E}[h(u, \omega)] \right\| \right] + \delta$$

with $\bar{\mathbb{P}}$ -probability less than β_N^δ .

3.2.2. Dudley's entropy integral. In this section, we bound the expectation term $\bar{\mathbb{E}} \sup_u \left\| \frac{1}{N} \sum_i h(u, \omega^i) - \mathbb{E}h(u, \omega) \right\|$ in Corollary 3.4. To do so, since the parameter set \mathcal{U} is compact, we use Dudley's entropy integral (see [33, Theorem 5.22] and Theorem D.3). This concentration inequality exploits the tight tails of sub-Gaussian processes and covering arguments via chaining. We combine this result with the smoothness assumption (A2) of the equality constraints functions $h(u, \cdot) \in \mathcal{H}$ to obtain a tight concentration inequality for our problem setting.

PROPOSITION 3.5 (Concentration for α -Hölder continuous function classes). *Assume that h satisfies (A2) for some exponent $\alpha \in (0, 1]$ and Hölder constant $M(\omega)$. Let $D \triangleq 2 \sup_{u \in \mathcal{U}} \|u\|_2$, $C = 32$, $u_0 \in \mathcal{U}$, and Σ_0 denote the covariance matrix of $h(u_0, \cdot)$. Then,*

$$\bar{\mathbb{E}} \left[\sup_{u \in \mathcal{U}} \left\| \frac{1}{N} \sum_{i=1}^N h(u, \omega^i) - \mathbb{E}[h(u, \omega)] \right\| \right] \leq \frac{1}{\sqrt{N}} \left(\frac{8CD^{\frac{\alpha+1}{2}} d^{\frac{1}{2}} n^{\frac{3}{2}} \mathbb{E}[M^2]^{\frac{1}{2}}}{\alpha^{\frac{1}{2}}} + \text{Trace}(\Sigma_0)^{\frac{1}{2}} \right).$$

The proof is provided in Appendix D.3. In contrast to existing work in the literature that assumes Lipschitz continuity [4, 17, 27, 33], Proposition 3.5 also gives a bound for function classes that are only α -Hölder continuous for values of $\alpha < 1$, see (A2).

3.2.3. Concentration using bounded differences and Dudley's bound.

Using Corollary 3.4 and Proposition 3.5, we obtain a bound of the form of (3.8).

PROPOSITION 3.6 (Concentration for α -Hölder bounded function classes). *Assume that h satisfies (A2) and (A3). With the notations of Proposition 3.5, define the constant $\tilde{C} = \left(8CD^{\frac{\alpha+1}{2}} d^{\frac{1}{2}} n^{\frac{3}{2}} \mathbb{E}[M^2]^{\frac{1}{2}} \alpha^{-\frac{1}{2}} + \text{Trace}(\Sigma_0)^{\frac{1}{2}} \right)$. Given any $\epsilon > 0$, define the two sequences $(\epsilon_N)_{N \in \mathbb{N}}$ and $(\beta_N)_{N \in \mathbb{N}}$ by*

$$(3.12) \quad \epsilon_N = 2\tilde{C}N^{-\frac{1}{2} + \frac{\epsilon}{2}} \quad \text{and} \quad \beta_N = \exp\left(-\frac{\tilde{C}^2}{2h^2} N^\epsilon\right).$$

Then,

$$\bar{\mathbb{P}} \left(\sup_{u \in \mathcal{U}} \left\| \frac{1}{N} \sum_{i=1}^N h(u, \omega^i) - \mathbb{E}[h(u, \omega)] \right\| \leq \epsilon_N \right) \geq 1 - \beta_N \quad \text{for all } N \in \mathbb{N}.$$

The proof is provided in Appendix D.4. We obtain the following corollary.

COROLLARY 3.7. *Assume that h satisfies (A2) and (A3). Define \tilde{C} as in Proposition 3.6. Let $\epsilon > 0$, $\beta \in (0, 1)$, and $N \in \mathbb{N}$ be such that $N \geq \epsilon^{-2}(\tilde{C} + \bar{h}(2 \log(1/\beta))^{\frac{1}{2}})^2$. Then,*

$$\bar{\mathbb{P}} \left(\sup_{u \in \mathcal{U}} \left\| \frac{1}{N} \sum_{i=1}^N h(u, \omega^i) - \mathbb{E}[h(u, \omega)] \right\| \leq \epsilon \right) \geq 1 - \beta.$$

By selecting $\epsilon \in (0, \frac{1}{2})$ in Proposition 3.6, the sequences $(\epsilon_N)_{N \in \mathbb{N}}$ and $(\beta_N)_{N \in \mathbb{N}}$ satisfy $\epsilon_N \rightarrow 0$ as $N \rightarrow \infty$ and $\sum_{N=1}^{\infty} \beta_N < \infty$. We use this fact in the next section.

3.3. Proof of Theorem 2.1. By combining the concentration inequalities derived in this section with the results from Section 3.1, we prove Theorem 2.1.

Proof of Theorem 2.1 (asymptotic optimality of the SAA approach). First, the cost function f satisfies the uniform law of large numbers in (3.5) thanks to (A1), see e.g. [9, Theorem 5.40] and [28, Theorem 7.48]. Second, Proposition 3.6 gives

$$\bar{\mathbb{P}} \left(\sup_{u \in \mathcal{U}} \|h_N(u, \bar{\omega}) - h_0(u)\| \leq \epsilon_N \right) \geq 1 - \beta_N \quad \text{for all } N \in \mathbb{N}$$

for $\epsilon_N = 2\tilde{C}N^{-\frac{1}{2} + \frac{\epsilon}{2}}$ and $\beta_N = \exp\left(-\frac{\tilde{C}^2}{2h^2} N^\epsilon\right)$ for some constant \tilde{C} . Since $\epsilon \in (0, \frac{1}{2})$, $\frac{\epsilon_N}{\delta_N} = \frac{2\tilde{C}}{C} N^{\frac{\epsilon}{2} - \epsilon} \rightarrow 0$ as $N \rightarrow \infty$ and $\sum_{N=1}^{\infty} \beta_N < \infty$. Thus, by Lemma 3.2,

$$d_{\text{H}}(C_N(\bar{\omega}), C_0) \rightarrow 0 \quad \text{as } N \rightarrow \infty$$

$\bar{\mathbb{P}}$ -almost-surely. The conclusion follows from Theorem 3.1. \square

4. Extension to problems with expected value inequality constraints.

We generalize our analysis to account for $q \in \mathbb{N}$ expected value inequality constraints. We consider a Carathéodory function $g : \mathcal{U} \times \Omega \rightarrow \mathbb{R}^q$ and the stochastic program

$$\mathbf{P}^g : \inf_{u \in \mathcal{U}} \mathbb{E}[f(u, \omega)] \quad \text{s.t.} \quad \mathbb{E}[g(u, \omega)] \leq 0, \quad \mathbb{E}[h(u, \omega)] = 0.$$

We define the true expectation function and the empirical estimate

$$(4.1a) \quad g_0 : \mathcal{U} \rightarrow \mathbb{R}^q, \quad u \mapsto \mathbb{E}[g(u, \omega)]$$

$$(4.1b) \quad g_N : \mathcal{U} \times \bar{\Omega} \rightarrow \mathbb{R}^q, \quad (u, \bar{\omega}) \mapsto \frac{1}{N} \sum_{i=1}^N g(u, \omega^i).$$

Following [24, 28], we consider the following first set of assumptions on the inequality constraints function g .

(A4a) There exists a \mathcal{G} -measurable function $e : \Omega \rightarrow \mathbb{R}$ such that $\mathbb{E}[e(\omega)] < \infty$ and $\|g(u, \omega)\| \leq e(\omega)$ for all $u \in \mathcal{U}$ \mathbb{P} -almost surely.

(A4b) There exists a solution $u^* \in S_0^*$ such that for any $\epsilon > 0$, there exists $u \in \mathcal{U}$ such that $\|u - u^*\| \leq \epsilon$ and $g_0(u) < 0$.

(A4a) implies the \mathbb{P} -almost-sure uniform convergence $\sup_{u \in \mathcal{U}} |g_N(u, \bar{\omega}) - g_0(u)| \rightarrow 0$ as $N \rightarrow \infty$. (A4b) is a standard constraints qualification condition [24, 28] that guarantees the \mathbb{P} -almost-sure feasibility of the sampled problems defined below for a sample size N large enough. (A4b) also guarantees the existence of a sequence of solutions to the sampled problems that converges to u^* as the sample size N increases. Given a sequence of scalars $(\delta_N)_{N \in \mathbb{N}}$, we consider the sampled problems

$$\mathbf{SP}_N^g(\bar{\omega}) : \inf_{u \in \mathcal{U}} f_N(u, \bar{\omega}) \quad \text{s.t.} \quad g_N(u, \bar{\omega}) \leq 0, \quad \|h_N(u, \bar{\omega})\| \leq \delta_N.$$

Following (A2) and (A3), we propose the alternative assumption $\widehat{\text{(A4)}}$ to tackle problems for which (A4b) does not hold.

$\widehat{\text{(A4)}}$ g satisfies (A2) and (A3): there exists an exponent $\alpha \in (0, 1]$, a Hölder constant $M(\omega)$ with $\mathbb{E}[M^2] < \infty$, and a bound $0 \leq \bar{g} < \infty$ such that \mathbb{P} -almost-surely,

1. $\|g(u_1, \omega) - g(u_2, \omega)\| \leq M(\omega) \|u_1 - u_2\|_2^\alpha$ for all $u_1, u_2 \in \mathcal{U}$, and
2. $\sup_{u \in \mathcal{U}} \|g(u, \omega)\| \leq \bar{g}$.

Remark 5.1. As represented in Figure 3, the stochastic program with $\mathcal{U} = [-1, 1]^2$, $f(u, \omega) = u_1$, and $g = (g_1, g_2)$ with $g_1(u, \omega) = u_2 - u_1 \max(u_1, 0)$ and $g_2(u, \omega) = -u_2 - u_1 \max(u_1, 0)$ has the unique optimal solution $u^* = (-1, 0)$. However, for $\epsilon \leq 1$, there exists no $u \in \mathcal{U}$ that is strictly feasible ($g_0(u) < 0$) and ϵ -close to the solution u^* , so this problem does not satisfy (A4b). However, this problem satisfies $\widehat{\text{(A4)}}$. Compared to (A4b), $\widehat{\text{(A4)}}$ represents smoothness and boundedness assumptions on the functions defining the problem that are simple to verify prior to applying the SAA approach.

Given $(\delta_N)_{N \in \mathbb{N}}$ and $\delta_N = (\delta_N, \dots, \delta_N) \in \mathbb{R}^q$, we define the sampled problem

$$\widehat{\mathbf{SP}}_N^g(\bar{\omega}) : \inf_{u \in \mathcal{U}} f_N(u, \bar{\omega}) \quad \text{s.t.} \quad g_N(u, \bar{\omega}) \leq \delta_N, \quad \|h_N(u, \bar{\omega})\| \leq \delta_N.$$

The next result generalizes Theorem 2.1 to problems with inequality constraints.

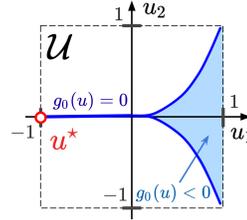


Fig. 3: Example that does not satisfy (A4b): there exists no $u \in \mathcal{U}$ that satisfies $g_0(u) < 0$ and $\|u - u^*\| \leq 1$.

THEOREM 4.1 (Asymptotic optimality of the SAA approach). *Assume that the functions f and h satisfy (A1), (A2), and (A3). Given any constants $\epsilon \in (0, \frac{1}{2})$ and $C > 0$, define the sequence $(\delta_N)_{N \in \mathbb{N}}$ as*

$$\delta_N = CN^{-(\frac{1}{2}-\epsilon)}.$$

For any $N \in \mathbb{N}$ and $\bar{\omega} \in \bar{\Omega}$, denote by (f_0^*, S_0^*) , $(f_N^*(\bar{\omega}), S_N^*(\bar{\omega}))$, and $(\widehat{f}_N^*(\bar{\omega}), \widehat{S}_N^*(\bar{\omega}))$ the optimal value and set of solutions to \mathbf{P}^g , $\mathbf{SP}_N^g(\bar{\omega})$, and $\widehat{\mathbf{SP}}_N^g(\bar{\omega})$, respectively.

If g satisfies (A4), then, $\bar{\mathbb{P}}$ -a.s., there exists a subsequence $\{N_k(\bar{\omega})\}_{k \in \mathbb{N}}$ such that

$$f_{N_k(\bar{\omega})}^*(\bar{\omega}) \rightarrow f_0^* \quad \text{and} \quad \mathbb{D}(S_{N_k(\bar{\omega})}^*(\bar{\omega}), S_0^*) \rightarrow 0 \quad \text{as} \quad k \rightarrow \infty$$

and, if f also satisfies (A1), then the entire sequence converges as $N \rightarrow \infty$.

If g satisfies (A4), then, $\bar{\mathbb{P}}$ -a.s., there exists a subsequence $\{N_k(\bar{\omega})\}_{k \in \mathbb{N}}$ such that

$$\widehat{f}_{N_k(\bar{\omega})}^*(\bar{\omega}) \rightarrow f_0^* \quad \text{and} \quad \mathbb{D}(\widehat{S}_{N_k(\bar{\omega})}^*(\bar{\omega}), S_0^*) \rightarrow 0 \quad \text{as} \quad k \rightarrow \infty$$

and, if f also satisfies (A1), then the entire sequence converges as $N \rightarrow \infty$.

Proof. If g satisfies (A4), then the result follows from minor modifications to the proof of Theorem 3.1. We refer to Appendix C for details.

Second, we assume that g satisfies (A4) and define the new feasible sets

$$(4.2a) \quad \tilde{C}_0 = \{u \in \mathcal{U} : h_0(u) = 0, g_0(u) \leq 0\}$$

$$(4.2b) \quad \tilde{C}_N : \bar{\Omega} \rightarrow \mathcal{K} : \bar{\omega} \mapsto \{u \in \mathcal{U} : \|h_N(u, \bar{\omega})\| \leq \delta_N, g_N(u, \bar{\omega}) \leq \delta_N\}.$$

By following the proof of Lemma 3.2, we can show that $\bar{\mathbb{P}}$ -almost-surely, $\tilde{C}_N(\bar{\omega}) \neq \emptyset$ for all $N \geq N(\bar{\omega})$ with $N(\bar{\omega})$ large enough and $d_H(\tilde{C}_N(\bar{\omega}), \tilde{C}_0) \rightarrow 0$ as $N \rightarrow \infty$. The conclusion then follows from Theorem 3.1.

Indeed, to show that $\tilde{C}_N(\bar{\omega}) \neq \emptyset$ for $N \geq N(\bar{\omega})$, we define the infeasibility event

$$\tilde{B}_N = \left\{ \bar{\omega} \in \bar{\Omega} : \inf_{u \in \mathcal{U}} \max(\|h_N(u, \bar{\omega})\|, g_{N,1}(u, \bar{\omega}), \dots, g_{N,q}(u, \bar{\omega})) > \delta_N \right\} \subseteq \tilde{B}_N^h \cup \tilde{B}_N^g$$

where $\tilde{B}_N^h = \{\bar{\omega} \in \bar{\Omega} : \|h_N(u^*, \bar{\omega})\| > \delta_N\}$ and $\tilde{B}_N^g = \{\bar{\omega} \in \bar{\Omega} : g_N(u^*, \bar{\omega}) > \delta_N\}$ with $u^* \in S_0^*$. Then, by Boole's inequality, $\bar{\mathbb{P}}(\tilde{B}_N) \leq \bar{\mathbb{P}}(\tilde{B}_N^h) + \bar{\mathbb{P}}(\tilde{B}_N^g)$. By following the proof of Lemma 3.2, thanks to (A2-3), (A4) and Proposition 3.6, we obtain that $\sum_{N=1}^{\infty} \bar{\mathbb{P}}(\tilde{B}_N) < \infty$ so that $\tilde{C}_N(\bar{\omega}) = \emptyset$ only finitely many times $\bar{\mathbb{P}}$ -a.s. The proof that $d_H(\tilde{C}_N(\bar{\omega}), \tilde{C}_0) \rightarrow 0$ as $N \rightarrow \infty$ with $\bar{\mathbb{P}}$ -probability one also follows with minor adaptations of the proof of Lemma 3.2. \square

5. Application to stochastic optimal control. We apply our methodology to general stochastic control problems with expected value inequality and equality constraints. Let $T > 0$ be a finite time horizon and $n, m \in \mathbb{N}$ be state and control dimensions. We consider as optimization parameter an open-loop control trajectory $\mathbf{u} = (u_0, \dots, u_{S-1}) \in \mathbb{R}^{Sm}$ with $S \in \mathbb{N}$ stepwise-constant controls u_s taking values in a compact set $U \subset \mathbb{R}^m$. Thus, the control space $\mathcal{U} \cong U \times \dots \times U \subset \mathbb{R}^{Sm} = \mathbb{R}^d$ is finite-dimensional and any control $u \in \mathcal{U} \subset L^2([0, T]; U)$ takes the form

$$(5.1) \quad u(t) = \sum_{s=0}^{S-1} u_s \mathbb{1}_{[\frac{sT}{S}, \frac{(s+1)T}{S})}(t) \quad \text{where } u_s \in U \text{ for all } s = 0, \dots, S-1$$

and satisfies $\|u\|_{L^2([0,T];U)}^2 = \frac{T}{S} \|\mathbf{u}\|_2^2$. Thus, in the remainder of this section, we implicitly identify \mathcal{U} with U^S and their corresponding norms. This finite-dimensional piecewise-constant control representation is common in the literature [1, 18], as it allows for simple numerical implementation. Note that by increasing S , any function in $L^2([0,T];U)$ can be approximated arbitrarily well by a function in \mathcal{U} .

We model the uncertain dynamical system using a stochastic differential equation (SDE). Let $b \in C^0(\mathbb{R}^n \times U; \mathbb{R}^n)$ be a drift function and $\sigma \in C^0(\mathbb{R}^n \times U; \mathbb{R}^{n \times n})$ be a diffusion coefficient. We assume the following regularity condition:

(A5) There exists a bounded constant $K \geq 0$ such that for all $x, y \in \mathbb{R}^n$ and $u, v \in U$,

$$\|b(x, u) - b(y, v)\| + \|\sigma(x, u) - \sigma(y, v)\| \leq K(\|x - y\| + \|u - v\|).$$

By requiring additional appropriate assumptions (see [35] and [12]), generalizing our analysis to time-varying uncertain drift and diffusion coefficients (i.e., b, σ depend on time $t \in [0, T]$ and on uncertain parameters $\omega \in \Omega$) is straight-forward.

Let $(\Omega, \mathcal{G}, \mathcal{F}, \mathbb{P})$ be a filtered probability space with W the standard n -dimensional Brownian motion and $L^2_{\mathcal{F}}(\Omega; C([0, T]; \mathbb{R}^n))$ the space of square-integrable \mathcal{F} -adapted stochastic processes with continuous sample paths taking values in \mathbb{R}^n . Given a control $u \in \mathcal{U}$ and an initial condition $x_0 \in \mathbb{R}^n$, we define the SDE

$$(5.2) \quad dx_u(t) = b(x_u(t), u(t))dt + \sigma(x_u(t), u(t))dW_t, \quad t \in [0, T], \quad x_u(0) = x_0.$$

Under (A5), for every $u \in \mathcal{U}$, (5.2) has a unique solution $x_u \in L^2_{\mathcal{F}}(\Omega; C([0, T]; \mathbb{R}^n))$ up to stochastic indistinguishability.

In Appendix E, we show that up to a modification, for any $\alpha \in (0, \frac{1}{2})$ and $u, v \in \mathcal{U}$, there exists a constant $C_\alpha(\omega)$ satisfying $\mathbb{E}[C_\alpha(\omega)^2] < \infty$ such that, \mathbb{P} -almost-surely,

$$(5.3) \quad \sup_{0 \leq t \leq T} \|x_u(t, \omega) - x_v(t, \omega)\| \leq C_\alpha(\omega) \|u - v\|_{L^2([0,T];U)}^\alpha.$$

In particular, up to a modification, \mathbb{P} -almost-surely, the map $u \mapsto x_u(t, \omega)$ is continuous and satisfies (A2).

Let $\ell \in C^0(\mathbb{R}^n \times U; \mathbb{R})$ be a running cost function, $\varphi \in C^0(\mathbb{R}^n; \mathbb{R})$ be a terminal cost function, and $G \in C^0(\mathbb{R}^n; \mathbb{R}^q)$, $H \in C^0(\mathbb{R}^n; \mathbb{R}^n)$ be two bounded constraints functions. We assume the following condition:

(A6) There exists a bounded constant $L \geq 0$ such that for all $x, y \in \mathbb{R}^n$ and $u, v \in U$,

$$\begin{aligned} \|H(x) - H(y)\| + \|G(x) - G(y)\| &\leq L\|x - y\|, & \|H(x)\| + \|G(x)\| &\leq L, \\ \|\ell(x, u) - \ell(y, v)\| + \|\varphi(x) - \varphi(y)\| &\leq L(\|x - y\| + \|u - v\|). \end{aligned}$$

We apply our approach to optimal control problems with state constraints. We define

$$\begin{aligned} \mathbf{OCP} : \quad & \inf_{u \in \mathcal{U}} \mathbb{E} \left[\int_0^T \ell(x_u(t, \omega), u(t)) dt + \varphi(x_u(T, \omega)) \right] \\ & \text{s.t. (5.2), } \mathbb{E} \left[\sup_{t \in [0, T]} G(x_u(t, \omega)) \right] \leq 0, \quad \mathbb{E}[H(x_u(T, \omega))] = 0 \end{aligned}$$

with the notation $\sup_t G(x_u(t, \omega)) \triangleq (\sup_t G_1(x_u(t, \omega)), \dots, \sup_t G_q(x_u(t, \omega)))$. We assume that **OCP** is feasible.

Since the map $u \mapsto x_u(t, \omega)$ is continuous \mathbb{P} -almost-surely, the maps $f : \mathcal{U} \times \Omega \rightarrow \mathbb{R}$,

$g : \mathcal{U} \times \Omega \rightarrow \mathbb{R}^q$, and $h : \mathcal{U} \times \Omega \rightarrow \mathbb{R}^n$ defined as

$$(5.4a) \quad f(u, \omega) = \int_0^T \ell(x_u(t, \omega), u(t)) dt + \varphi(x_u(T, \omega))$$

$$(5.4b) \quad g(u, \omega) = \left(\sup_{t \in [0, T]} G_1(x_u(t, \omega)), \dots, \sup_{t \in [0, T]} G_q(x_u(t, \omega)) \right)$$

$$(5.4c) \quad h(u, \omega) = H(x_u(T, \omega))$$

are Carathéodory (note that \mathbb{P} -almost-surely, x_u has continuous sample paths, so that g is well-defined). Thus, **OCP** takes the form of **P**. This justifies applying the SAA approach to approximately solve **OCP**.

We denote the product probability space by $(\bar{\Omega}, \bar{\mathcal{G}}, \bar{\mathbb{P}})$ (see Section A for details) and a sample by $\bar{\omega} = (\omega^1, \dots) \in \bar{\Omega}$. Given a number $N \in \mathbb{N}$ of realizations $\omega^i \in \Omega$ of the Brownian motion and a padding $\delta_N > 0$, we define the sampled problem

$$\begin{aligned} \mathbf{SOCP}_N(\bar{\omega}) : \inf_{u \in \mathcal{U}} \frac{1}{N} \sum_{i=1}^N \left(\int_0^T \ell(x_u(t, \omega^i), u(t)) dt + \varphi(x_u(T, \omega^i)) \right) \quad \text{s.t.} \quad (5.2), \\ \frac{1}{N} \sum_{i=1}^N \sup_{t \in [0, T]} G(x_u(t, \omega^i)) \leq \delta_N, \quad \left\| \frac{1}{N} \sum_{i=1}^N H(x_u(T, \omega^i)) \right\| \leq \delta_N \end{aligned}$$

where $\delta_N = (\delta_N, \dots, \delta_N) \in \mathbb{R}^q$. Since the control space $\mathcal{U} \cong \mathbb{R}^{S_m}$ is compact and finite-dimensional and the constraint functions G and H are smooth and bounded, we apply Theorem 4.1 and obtain the following consistency result.

THEOREM 5.1 (Asymptotic optimality of the SAA approach for stochastic optimal control). *Assume that **OCP** is feasible and that $(b, \sigma, \ell, H, G, \varphi, \ell)$ satisfy (A5) and (A6). Given any constant $\epsilon \in (0, \frac{1}{2})$ and $C > 0$, define the sequence $(\delta_N)_{N \in \mathbb{N}}$ as $\delta_N = CN^{-(\frac{1}{2} - \epsilon)}$. For any $N \in \mathbb{N}$ and $\bar{\omega} \in \bar{\Omega}$, denote by (f_0^*, S_0^*) and by $(f_N^*(\bar{\omega}), S_N^*(\bar{\omega}))$ the optimal value and set of solutions to **OCP** and to **SOCP** $_N(\bar{\omega})$, respectively. Then, $\bar{\mathbb{P}}$ -almost-surely, $f_N^*(\bar{\omega}) \rightarrow f_0^*$ and $\mathbb{D}(S_N^*(\bar{\omega}), S_0^*) \rightarrow 0$ as $N \rightarrow \infty$.*

Proof. First, the function f defined in (5.4) satisfies (A1) and $\widehat{\text{(A1)}}$. Indeed, let $f_1(u, \omega) = \int_0^T \ell(x_u(s, \omega), u(s)) ds$ and $f_2(u, \omega) = \varphi(x_u(T, \omega))$ so that $f = f_1 + f_2$. Next, we show that f_1 satisfies (A1). The analysis for f_2 is similar. We define the map $d : \Omega \rightarrow \mathbb{R} : \omega \mapsto \sup_{u \in \mathcal{U}} |f_1(u, \omega)|$ which satisfies $|f_1(u, \omega)| \leq d(\omega)$ for all $u \in \mathcal{U}$ \mathbb{P} -a.s. Also, $\mathbb{E}[d(\omega)] < \infty$. Indeed, for any $t \in [0, T]$ and $u_0 \in \mathcal{U}$,

$$\sup_{u \in \mathcal{U}} |\ell(x_u(t), u(t))| \leq |\ell(x_{u_0}(t), u_0(t))| + \sup_{u \in \mathcal{U}} |\ell(x_u(t), u(t)) - \ell(x_{u_0}(t), u_0(t))|$$

which is \mathbb{P} -almost-surely bounded by a finite constant thanks to (A6) and (5.3) (note that the constant $C_\alpha(\omega)$ is integrable and \mathcal{U} is compact). Thus, $\mathbb{E}[d(\omega)] < \infty$ so f satisfies (A1). f also satisfies $\widehat{\text{(A1)}}$ thanks to (A6) and (5.3).

Second, we show that g and h as defined in (5.4) satisfy (A2-3) and $\widehat{\text{(A4)}}$, respectively. Indeed, thanks to (A6), the maps g and h are bounded. g and h may not be Lipschitz continuous, but are α -Hölder continuous for any $\alpha \in (0, \frac{1}{2})$. Indeed, thanks to (5.3) and (A6), \mathbb{P} -almost-surely, for all $u, v \in \mathcal{U}$, $\|h(u, \omega) - h(v, \omega)\| \leq LC_\alpha(\omega) \|u - v\|_{L^2([0, T], U)}^\alpha$ for any $\alpha \in (0, \frac{1}{2})$ with $\mathbb{E}[C_\alpha(\omega)^2] < \infty$. To show that g is

also α -Hölder continuous, for any $j = 1, \dots, q$ and $u, v \in \mathcal{U}$, we write

$$\begin{aligned} & \left| \sup_{t \in [0, T]} G_j(x_u(t, \omega)) - \sup_{s \in [0, T]} G_j(x_v(s, \omega)) \right| \leq \\ & \leq \sup_{t \in [0, T]} |G_j(x_u(t, \omega)) - G_j(x_v(t, \omega))| \leq LC_\alpha(\omega) \|u - v\|_{L^2([0, T]; U)}^\alpha \end{aligned}$$

by (A6) and (5.3). We apply Theorem 4.1 and the conclusion follows. \square

6. Numerical results. We consider a Mars entry, descent, and landing (EDL) scenario where a spacecraft must safely deploy a rover at a landing site [29]. Specifically, we consider the rocket-powered descent phase where the system must precisely reach a position above a landing site on Mars. The state of the system is given as $x = (r, v, m) \in \mathbb{R}^7$ where $r = (r_x, r_y, r_z) \in \mathbb{R}^3$ denotes the position, $v \in \mathbb{R}^3$ the velocity, and $m \in \mathbb{R}$ the mass of the system. The control input is the thrust force $u = (u_x, u_y, u_z) \in \mathbb{R}^3$. We model the dynamics of the system with the SDE

$$(6.1) \quad dx_u(t) = \begin{bmatrix} v(t) \\ \frac{1}{m(t)}(\bar{T}u(t) - \gamma|v|v) - g \\ -\rho\|u(t)\| \end{bmatrix} dt + \begin{bmatrix} 0 \\ \frac{1}{m(t)}\text{diag}(\eta_0 + \eta_1 v^2) \\ 0 \end{bmatrix} dW_t$$

where $t \in [0, T]$ with $T = 60$ s, $\bar{T} = 16 \cdot 10^3$, $g = (0, 0, 3.71)$, $\rho = 8$, $\gamma = 1$, $\eta_0 = 10$, $\eta_1 = 0.2$, and $\text{diag}(y_j) \in \mathbb{R}^{3 \times 3}$ denotes the diagonal matrix with non-zero elements y_j for any $y = (y_1, y_2, y_3) \in \mathbb{R}^3$. With this model, we account for disturbances from external wind gusts and drag. The state dependency of the diffusion coefficient $\sigma(x)$ makes numerically solving this problem challenging. Prior work in the literature considers deterministic dynamics [1, 20, 30] or a diffusion coefficient that only depends on the mass $m(t)$ [11].

The system starts at $x_u(0) = x_0 = (300, 0, 1500, 5, 0, -75, 1800)$ and must reach the goal state $x_g = (0, 0, 100, 0, 0, -10)$ in expectation, which we enforce by defining the final state equality constraint function $H : \mathbb{R}^7 \rightarrow \mathbb{R}^6 : x \mapsto (r, v) - x_g$. We enforce a glide-slope inequality constraint at slope $\psi = 35^\circ$ defined by $G : \mathbb{R}^7 \rightarrow \mathbb{R}^4$, $x \mapsto (\tan(\psi)r_x - r_z, \tan(\psi)r_y - r_z, -\tan(\psi)r_x - r_z, -\tan(\psi)r_y - r_z)$.

We minimize the control effort $\ell(x_u(t), u(t)) = \|u(t)\|$ to reduce fuel consumption and the final dispersion $\varphi(x_u(T)) = \|H(x_u(T))\|_2^2$. This choice for φ minimizes the variance at the terminal state $x_u(T)$, since $\mathbb{E}[\varphi(x_u(T))] = \text{Trace}(\text{Cov}(x_u(T)))$ if $\mathbb{E}[H(x_u(T))] = 0$. Up to compositions with smooth cutoff functions, the functions φ and G, H satisfy (A6). The control space consists of bound and pointing constraints $U = \{u \in \mathbb{R}^3 : \underline{u} \leq \|u(t)\| \leq \bar{u}, \|u(t)\| \cos(\theta) - u_z(t) \leq 0\}$ with min/maximum thrusts $(\underline{u}, \bar{u}) = (0.3, 0.8)$ and pointing angle $\theta = 45^\circ$. We consider piecewise-constant controls with $S = 20$ switches. The complete formulation takes the form of **OCP**.

We discretize (6.1) using an Euler-Maruyama scheme with S timesteps and enforce the inequality constraint at these S nodes. The analysis of the discretization error is beyond the scope of this paper and left for future work, see e.g. [18]. We formulate **SOCP**($\bar{\omega}$) using $N = 20$ and $\delta_N = 10^{-5}$. We solve **SOCP**($\bar{\omega}$) via a standard direct method based on standard sequential convex programming. For further details, we refer to our open-sourced implementation at <https://github.com/StanfordASL/stochasticEDL>.

We present results in Figure 4. As a baseline, we solve the deterministic problem without uncertainty (i.e., $\eta_0 = \eta_1 = 0$). The corresponding optimal strategy minimizes fuel consumption ($m(T) = 1592$ kg). In contrast, when considering state-dependent uncertainty, one obtains a different trajectory that minimizes the magnitude of disturbances by reducing the velocity magnitude along the trajectory, albeit

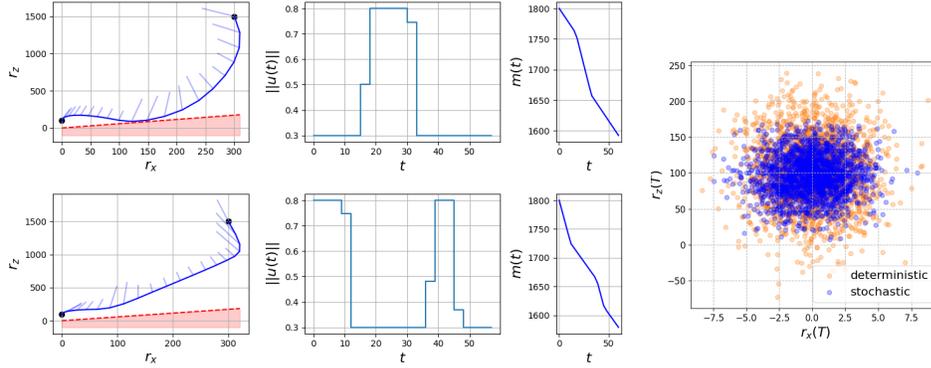


Fig. 4: Mars rocket-powered descent results. **Top-left:** trajectory obtained with a deterministic baseline that does not consider uncertainty. **Bottom-left:** trajectory obtained by solving $\text{SOCP}_N(\bar{\omega})$. **Right:** horizontal and vertical positions at the final time from Monte-Carlo simulations of (6.1) with the controls from the deterministic baseline and from $\text{SOCP}_N(\bar{\omega})$.

with larger fuel consumption ($m(T) = 1579$ kg). With 10^4 Monte-Carlo simulations of (6.1), we observe that this solution reduces the variance of the error at the final time, i.e., $\mathbb{E}[\|H(x_u(T))\|_2^2]$ is minimized. Indeed, the standard deviation of the final altitude $r_z(T)$ is only 28m for the proposed stochastic programming approach, versus 48m for the deterministic baseline. Thus, the trajectory computed by solving $\text{SOCP}(\bar{\omega})$ that considers uncertainty is potentially easier to track with a feedback controller.

7. Conclusions. We presented a simple modification to the SAA approach that makes the method applicable to a large class of stochastic programs with both expectation equality and inequality constraints. This modification consists of relaxing the equality constraints by a scalar δ_N that decreases to zero as the sample size N increases. Under mild assumptions, we proved the consistency of the approach by reformulating the problem using random compact sets and by analyzing the approximation error with concentration inequalities. We applied the analysis to challenging stochastic optimal control problems and demonstrated the benefits of the approach over deterministic methods on a Mars powered-descent control problem.

Exciting directions of future work include performing importance sampling to accelerate the algorithm and reduce the approximation error, deriving asymptotic convergence rates and finite-sample error bounds for the solution sets error $\mathbb{D}(S_N^*(\bar{\omega}), S_0^*)$, relaxing the assumption that $\mathcal{U} \cong \mathbb{R}^d$ to allow for optimization in more general function spaces, and considering risk functionals instead of expectations.

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Appendix A. Sample space.

A.1. Product probability space. Let $(\Omega, \mathcal{G}, \mathbb{P})$ be a probability space. We define the product probability space $(\bar{\Omega}, \bar{\mathcal{G}}, \bar{\mathbb{P}})$ such that $\bar{\Omega} = \Omega^{\mathbb{N}}$ is the product space equipped with the product σ -algebra $\bar{\mathcal{G}} = \sigma(\{\pi_i^{-1}(B) : i \in \mathbb{N}, B \in \mathcal{G}\})$, where $\pi_i :$

$\bar{\Omega} \rightarrow \Omega : \bar{\omega} = (\omega^1, \dots) \mapsto \omega^i$, and $\bar{\mathbb{P}} : \bar{\mathcal{G}} \rightarrow [0, 1]$ is the product measure obtained via Kolmogorov's extension theorem [19, Theorem 6.3] such that for any $N \in \mathbb{N}$,

$$\bar{\mathbb{P}}(A_N) = \underbrace{(\mathbb{P} \otimes \dots \otimes \mathbb{P})}_{N \text{ times}}(A_N)$$

for any $A_N \in \mathcal{G}^N$, with the slight abuse of notation $\bar{\mathbb{P}}(A_N) \triangleq \bar{\mathbb{P}}((\pi_1, \dots, \pi_N)^{-1}(A_N))$.

Intuitively, each element ω^i of the sample $\bar{\omega} = (\omega^1, \dots)$ is interpreted as an i.i.d. realization of ω , since $\bar{\mathbb{P}}(\{\bar{\omega} \in \bar{\Omega} : \omega^i \in A\}) = \bar{\mathbb{P}}(\pi_i^{-1}(A)) = \mathbb{P}(A)$ for any $A \in \mathcal{G}$.

A.2. Sample space for stochastic optimal control. In Section 5, $(\Omega, \mathcal{G}, \mathbb{P})$ is the probability space that can be defined by setting $\Omega = C([0, T]; \mathbb{R}^n)$, $\mathcal{G} = \mathcal{B}(\Omega)$ the Borel σ -algebra for the metric topology for the norm $\|x\| = \sup_{t \in [0, T]} \|x(t)\|$, \mathbb{P} the Wiener measure [19, Section 2.2], and $W : \omega \mapsto \omega$ the canonical process that corresponds to Brownian motion with its canonical filtration \mathcal{F} . We denote by $(\bar{\Omega}, \bar{\mathcal{G}}, \bar{\mathbb{P}})$ the product probability space, by $\bar{\omega} = (\omega^1, \dots) \in \bar{\Omega}$ a sample, and by $L^2_{\mathcal{F}}(\Omega; C([0, T]; \mathbb{R}^n))$ the space of square-integrable \mathcal{F} -adapted stochastic processes with continuous sample paths taking values in \mathbb{R}^n .

For any $u \in \mathcal{U}$, the solution to the SDE in (5.2) satisfies $x_u \in L^2_{\mathcal{F}}(\Omega; C([0, T]; \mathbb{R}^n))$. Our results rely on the measurability of the maps f_N , g_N , and h_N defined from (5.4), which depend on the measurability of the maps

$$x_u^i : ([0, T] \times \bar{\Omega}, \mathcal{B}([0, T]) \otimes \bar{\mathcal{G}}) \rightarrow (\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n)), (t, \bar{\omega}) \mapsto x_u^i(t, \bar{\omega}) = x_u(t, \omega^i).$$

To prove that x_u^i is measurable, we define the measurable map

$$t \otimes W^i : ([0, T] \times \bar{\Omega}, \mathcal{B}([0, T]) \otimes \bar{\mathcal{G}}) \rightarrow ([0, T] \times \Omega, \mathcal{B}([0, T]) \otimes \mathcal{G}), (s, \bar{\omega}) \mapsto (s, \omega^i).$$

Then, the map x_u^i can be written as the composition $x_u^i = x_u \circ (t \otimes W^i)$. Since the stochastic process $x_u : ([0, T] \times \Omega, \mathcal{B}([0, T]) \otimes \mathcal{G}) \rightarrow (\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n)) : (t, \omega) \mapsto x_u(t, \omega)$ is measurable, x_u^i is measurable. By the continuity of the maps ℓ , φ , G , and H (to show the measurability of g , note that x_u has continuous sample paths), we conclude that the maps f_N , g_N , and h_N defined as in (2.1) and (4.1) from (5.4) are measurable.

Appendix B. Convergence of random compact sets and proof of Lemma

3.2. In this work, we use the myopic topology on the family \mathcal{K} of compact subsets of \mathbb{R}^d with its associated generated Borel σ -algebra $\mathcal{B}(\mathcal{K})$. A random compact set is a map $C : \Omega \rightarrow \mathcal{K}$ that is measurable, i.e., $\{\omega \in \Omega : C(\omega) \in \mathcal{Y}\} \in \mathcal{G}$ for any $\mathcal{Y} \in \mathcal{B}(\mathcal{K})$. We refer to [21, 23] for detailed treatments of random sets.

The convergence of random compact sets can be characterized using the following result [23, Proposition 1.7.23] (see also [21, Theorem 1-4-1] and [23, Theorem D.3]).

THEOREM B.1 (Convergence of random compact sets to a deterministic limit). *Let $(\Omega, \mathcal{G}, \mathbb{P})$ be a probability space, $C \in \mathcal{K}$, and $\{C_N\}_{N=1}^{\infty}$ be a sequence of random compact sets. Assume that \mathbb{P} -almost-surely, there exists a set $K_0 \in \mathcal{K}$ such that $C_N(\omega) \subset K_0$ for all $N \geq 1$, and that*

- For any $K \in \mathcal{K}$ such that $C \cap K = \emptyset$,

$$(C1) \quad \mathbb{P}(C_N \cap K \neq \emptyset \text{ infinitely often}) = \mathbb{P}\left(\bigcap_{M=1}^{\infty} \bigcup_{N=M}^{\infty} \{C_N \cap K \neq \emptyset\}\right) = 0.$$

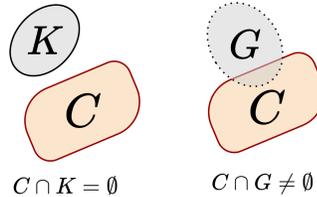


Fig. 5: Conditions of Theorem B.1.

- For any open subset $G \subset \mathbb{R}^n$ such that $C \cap G \neq \emptyset$,

$$(C2) \quad \mathbb{P}(C_N \cap G = \emptyset \text{ infinitely often}) = \mathbb{P} \left(\bigcap_{M=1}^{\infty} \bigcup_{N=M}^{\infty} \{C_N \cap G = \emptyset\} \right) = 0.$$

Then, \mathbb{P} -almost-surely, the sequence $\{C_N\}_{N=1}^{\infty}$ converges to C in the myopic topology.

We note that the random constraints sets $C_N(\bar{\omega})$ defined in (3.2b) are contained in the compact set \mathcal{U} . The requirement that $C_N(\bar{\omega}) \subset K_0 \in \mathcal{K}$ for all $N \geq 1$ is necessary: the sets $C_N = \{0, N\}$ belong to \mathcal{K} for all $N \in \mathbb{N}$ but do not converge in the myopic topology (although they converge to $\{0\}$ in the coarser Fell topology on the set of closed subsets of \mathbb{R}^d [23, Appendix D]).

We use Theorem B.1 to prove Lemma 3.2.

Proof of Lemma 3.2. First, we prove that $\bar{\mathbb{P}}$ -a.s., $C_N(\bar{\omega}) \neq \emptyset$ for all $N \geq N(\bar{\omega})$ with $N(\bar{\omega})$ large enough. We define the “infeasibility event”

$$B_N = \{\bar{\omega} \in \bar{\Omega} : \inf_{u \in \mathcal{U}} \|h_N(u, \bar{\omega})\| > \delta_N\} \subseteq \{\bar{\omega} \in \bar{\Omega} : \|h_N(u^*, \bar{\omega})\| > \delta_N\}.$$

Let \bar{N} be large enough so that $\delta_N > \epsilon_N$ for all $N \geq \bar{N}$. Then, for $N \geq \bar{N}$,

$$B_N \subseteq \{\bar{\omega} \in \bar{\Omega} : \|h_N(u^*, \bar{\omega})\| > \epsilon_N\} = \{\bar{\omega} \in \bar{\Omega} : \|h_N(u^*, \bar{\omega}) - h_0(u^*)\| > \epsilon_N\}.$$

Thus, $\bar{\mathbb{P}}(B_N) \leq \beta_N$ for $N \geq \bar{N}$ by (3.8) and $\sum_{N=1}^{\infty} \bar{\mathbb{P}}(B_N) < \infty$. By the first Borel-Cantelli lemma, $\bar{\mathbb{P}}$ -a.s., $\mathbf{P}_N(\bar{\omega})$ is infeasible and $C_N(\bar{\omega}) = \emptyset$ only finitely many times.

Second, we prove that the following two conditions hold:

- For any $K \in \mathcal{K}$ such that $C_0 \cap K = \emptyset$,

$$(C1) \quad \bar{\mathbb{P}}(C_N \cap K \neq \emptyset \text{ infinitely often}) = 0.$$

- For any open subset $h \subset \mathcal{U}$ such that $C_0 \cap h \neq \emptyset$,

$$(C2) \quad \bar{\mathbb{P}}(C_N \cap h = \emptyset \text{ infinitely often}) = 0.$$

- *Proof of (C1):* Let $K \in \mathcal{K}$ be such that $C_0 \cap K = \emptyset$.

Then, there exists some $\bar{\epsilon} > 0$ such that $\|h_0(u)\| \geq 2\bar{\epsilon}$ for all $u \in K$. Thus,

$$0 < 2\bar{\epsilon} \leq \|h_0(u)\| \stackrel{\text{a.s.}}{\leq} \|h_0 - h_N(\cdot, \bar{\omega})\|_{\infty} + \|h_N(u, \bar{\omega})\| \leq \epsilon_N + \|h_N(u, \bar{\omega})\|$$

for all $u \in K$ with $\bar{\mathbb{P}}$ -probability (w.p.) at least $1 - \beta_N$. Thus, w.p. at least $1 - \beta_N$,

$$(B.1) \quad \inf_{u \in K} \|h_N(u, \bar{\omega})\| \geq 2\bar{\epsilon} - \epsilon_N \geq \bar{\epsilon}$$

for N large enough so that $\epsilon_N \leq \bar{\epsilon}$. To conclude, define the “bad event”

$$B_N = \{\bar{\omega} \in \bar{\Omega} : C_N(\bar{\omega}) \cap K \neq \emptyset\} = \{\bar{\omega} \in \bar{\Omega} : \inf_{u \in K} \|h_N(u, \bar{\omega})\| \leq \delta_N\}.$$

Let N be large enough so that $\epsilon_N \leq \bar{\epsilon}$ and $\delta_N \leq \bar{\epsilon}$. Then, $\bar{\mathbb{P}}(B_N) \leq \beta_N$ by (B.1). Thus, $\sum_{N=1}^{\infty} \bar{\mathbb{P}}(B_N) < \infty$. We apply the first Borel-Cantelli lemma and conclude the proof of (C1).

- *Proof of (C2):* Let $G \subset \mathcal{U}$ be an open subset such that $C_0 \cap G \neq \emptyset$. Thus, there exists some $u \in G$ such that $h_0(u) = 0$. Then, with $\bar{\mathbb{P}}$ -probability at least $1 - \beta_N$, for N large enough so that $\epsilon_N \leq \delta_N$,

$$\|h_N(u, \bar{\omega})\| \leq \|h_0 - h_N(\cdot, \bar{\omega})\|_\infty + \|h_0(u)\| \leq \epsilon_N \leq \delta_N$$

so $u \in C_N(u, \bar{\omega})$. This implies that $\bar{\mathbb{P}}(C_N(\bar{\omega}) \cap G \neq \emptyset) \geq 1 - \beta_N$. To conclude, define the “bad event” $B_N = \{\bar{\omega} \in \bar{\Omega} : C_N(\bar{\omega}) \cap G = \emptyset\}$ so that $\bar{\mathbb{P}}(B_N) = 1 - \bar{\mathbb{P}}(C_N(\bar{\omega}) \cap G \neq \emptyset) \leq \beta_N$ for N large enough. Then, $\sum_{N=1}^{\infty} \bar{\mathbb{P}}(B_N) = \sum_{N=1}^{\infty} (1 - \bar{\mathbb{P}}(C_N(\bar{\omega}) \cap G \neq \emptyset)) < \infty$. We apply the first Borel-Cantelli lemma and conclude the proof of (C2).

Using (C1) and (C2), we apply Theorem B.1 (see [23, Proposition 1.7.23]) and conclude that the sequence of random compact sets $(C_N)_{N \in \mathbb{N}}$ converges to C_0 in the myopic topology $\bar{\mathbb{P}}$ -almost-surely. Since the myopic topology on \mathcal{K}' is equivalent to the topology on \mathcal{K}' generated by the Hausdorff metric on \mathcal{K}' [21, 23], we obtain (3.9). \square

Appendix C. Asymptotic optimality of the SAA approach.

C.1. Proof of Theorem 3.1: the case with equality constraints. We start with two lemmas.

LEMMA C.1. *Let $(\bar{\Omega}, \bar{\mathcal{G}}, \bar{\mathbb{P}})$ be a probability space, $\mathcal{U} \subset \mathbb{R}^d$ and $C_0 \subseteq \mathcal{U}$ be compact sets, $\{C_N\}_{N \in \mathbb{N}}$ be a sequence of random compact sets with $C_N : (\bar{\Omega}, \bar{\mathcal{G}}) \rightarrow (\mathcal{K}, \mathcal{B}(\mathcal{K}))$, and $\{u_N\}_{N \in \mathbb{N}}$ be a sequence of $\bar{\mathcal{G}}$ -measurable random variables taking values in \mathcal{U} . Assume that:*

- $\bar{\mathbb{P}}$ -almost-surely, $u_N(\bar{\omega}) \in C_N(\bar{\omega})$ for all $N \in \mathbb{N}$.
- $\bar{\mathbb{P}}$ -almost-surely, $\mathbb{D}(C_N(\bar{\omega}), C_0) \rightarrow 0$ as $N \rightarrow \infty$.

Then, $\bar{\mathbb{P}}$ -almost-surely, there exists $u(\bar{\omega}) \in C_0$ and a subsequence $\{N_k(\bar{\omega})\}_{k \in \mathbb{N}}$ such that $u_{N_k(\bar{\omega})}(\bar{\omega}) \rightarrow u(\bar{\omega})$ as $k \rightarrow \infty$.

Proof. We fix $\bar{\omega} \in \bar{\Omega}$ in a set of $\bar{\mathbb{P}}$ -measure one. Since $u_N(\bar{\omega}) \in C_N(\bar{\omega})$ for all $N \in \mathbb{N}$, $C_N(\bar{\omega}) \subset \mathcal{U}$ for all $N \in \mathbb{N}$, and $\mathcal{U} \subset \mathbb{R}^d$ is compact, there exists a subsequence $\{N_k(\bar{\omega})\}_{k \in \mathbb{N}}$ and a limit point $u(\bar{\omega}) \in \mathcal{U}$ such that $u_{N_k(\bar{\omega})}(\bar{\omega}) \rightarrow u(\bar{\omega})$ as $k \rightarrow \infty$.

To conclude, we show that $u(\bar{\omega}) \in C_0$. By the triangle inequality, for any $k \in \mathbb{N}$,

$$\begin{aligned} \mathbb{D}(\{u(\bar{\omega})\}, C_0) &\leq \mathbb{D}(\{u(\bar{\omega})\}, \{u_{N_k(\bar{\omega})}(\bar{\omega})\}) + \mathbb{D}(\{u_{N_k(\bar{\omega})}(\bar{\omega})\}, C_{N_k(\bar{\omega})}) + \mathbb{D}(C_{N_k(\bar{\omega})}, C_0) \\ &= \|u(\bar{\omega}) - u_{N_k(\bar{\omega})}(\bar{\omega})\| + \mathbb{D}(C_{N_k(\bar{\omega})}, C_0). \end{aligned}$$

Taking the limit as $k \rightarrow \infty$, we obtain $\mathbb{D}(\{u(\bar{\omega})\}, C_0) = 0$. Thus, $u(\bar{\omega}) \in C_0$. \square

LEMMA C.2. *With the notations introduced in Theorem 3.1, assume that:*

- $\bar{\mathbb{P}}$ -almost-surely, $\mathbb{D}(C_0, C_N(\bar{\omega})) \rightarrow 0$ as $N \rightarrow \infty$.

Let $u^ \in S_0^*$ be any optimal solution to \mathbf{P} . Then, $\bar{\mathbb{P}}$ -almost-surely, there exists a sequence $\{u_N(\bar{\omega})\}_{N \in \mathbb{N}}$ such that $u_N(\bar{\omega}) \in C_N(\bar{\omega})$ for all $N \in \mathbb{N}$ and $u_N(\bar{\omega}) \rightarrow u^*$ as $N \rightarrow \infty$.*

Proof. We fix $\bar{\omega} \in \bar{\Omega}$ in a set of $\bar{\mathbb{P}}$ -measure one and $u^* \in S_0^* \subseteq C_0$. For any $\epsilon > 0$, there exists $N_\epsilon(\bar{\omega}) \in \mathbb{N}$ such that $\mathbb{D}(C_0, C_N(\bar{\omega})) < \epsilon$ for all $N \geq N_\epsilon(\bar{\omega})$. In particular, since $\mathbb{D}(C_0, \emptyset) = \infty$ by definition, $C_N(\bar{\omega})$ is not empty for $N \geq N_\epsilon(\bar{\omega})$.

Then, consider the sequence $\{u_N(\bar{\omega})\}_{N \in \mathbb{N}}$ defined for any $N \in \mathbb{N}$ as $u_N(\bar{\omega}) = \arg \inf_{u \in C_N(\bar{\omega})} \|u - u^*\|$. Note that $u_N(\bar{\omega}) \in C_N(\bar{\omega})$ for all $N \in \mathbb{N}$ since $C_N(\bar{\omega})$ is closed. Since $u^* \in C_0$, for all $N \geq N_\epsilon(\bar{\omega})$, $\|u_N(\bar{\omega}) - u^*\| = \mathbb{D}(\{u^*\}, C_N(\bar{\omega})) \leq \mathbb{D}(C_0, C_N(\bar{\omega})) < \epsilon$. Thus, the sequence $\{u_N(\bar{\omega})\}_{N \in \mathbb{N}}$ satisfies $\|u_N(\bar{\omega}) - u^*\| < \epsilon$ for all $N \geq N_\epsilon(\bar{\omega})$. As this statement holds for any $\epsilon > 0$, this concludes the proof. \square

We observe that Lemma C.1 requires that $\mathbb{D}(C_N(\bar{\omega}), C_0) \rightarrow 0$ and Lemma C.2 requires $\mathbb{D}(C_0, C_N(\bar{\omega})) \rightarrow 0$ as $N \rightarrow \infty$. To ensure that both assumptions hold, we require that $d_H(C_N(\bar{\omega}), C_0) \rightarrow 0$ in Theorem 3.1.

Proof of Theorem 3.1. We prove the two claims in (3.7) sequentially. Throughout, we fix $\bar{\omega} \in \bar{\Omega}$ in a set of \mathbb{P} -measure one.

Part 1: there exists a subsequence $\{N_k(\bar{\omega})\}_{k \in \mathbb{N}}$ such that $f_{N_k(\bar{\omega})}^*(\bar{\omega}) \xrightarrow[k \rightarrow \infty]{} f_0^*$.

Consider any sequence $\{u_N^*(\bar{\omega})\}_{N \in \mathbb{N}}$ of solutions to $\mathbf{SP}(\bar{\omega})$, such that $u_N^*(\bar{\omega}) \in S_N^*(\bar{\omega})$ and $f_N(u_N^*(\bar{\omega}), \bar{\omega}) = f_N^*(\bar{\omega})$. By Lemma C.1 (note that $u_N^*(\bar{\omega}) \in C_N(\bar{\omega})$ for all $N \in \mathbb{N}$ and $\lim_{N \rightarrow \infty} d_H(C_N(\bar{\omega}), C_0) = 0$ by assumption in (3.6)), there exists a limit point $u(\bar{\omega}) \in C_0$ and a subsequence $\{N_k(\bar{\omega})\}_{k \in \mathbb{N}}$ such that

$$(C.1) \quad u_{N_k(\bar{\omega})}^*(\bar{\omega}) \rightarrow u(\bar{\omega}) \quad \text{as } k \rightarrow \infty.$$

Next, we prove that for some $u^* \in S_0^*$, the subsequence $\{N_k(\bar{\omega})\}_{k \in \mathbb{N}}$ is such that

$$(C.2a) \quad \liminf_{k \rightarrow \infty} f_{N_k(\bar{\omega})}(u_{N_k(\bar{\omega})}^*(\bar{\omega}), \bar{\omega}) \geq f_0(u^*)$$

$$(C.2b) \quad \limsup_{k \rightarrow \infty} f_{N_k(\bar{\omega})}(u_{N_k(\bar{\omega})}^*(\bar{\omega}), \bar{\omega}) \leq f_0(u^*).$$

To show (C.2a), we first note that $\lim_{k \rightarrow \infty} f_{N_k(\bar{\omega})}(u_{N_k(\bar{\omega})}^*(\bar{\omega}), \bar{\omega}) = f_0(u(\bar{\omega}))$, since

$$\begin{aligned} & |f_{N_k(\bar{\omega})}(u_{N_k(\bar{\omega})}^*(\bar{\omega}), \bar{\omega}) - f_0(u(\bar{\omega}))| \\ & \leq \underbrace{|f_{N_k(\bar{\omega})}(u_{N_k(\bar{\omega})}^*(\bar{\omega}), \bar{\omega}) - f_0(u_{N_k(\bar{\omega})}^*(\bar{\omega}))|}_{\rightarrow 0 \text{ as } k \rightarrow \infty \text{ by (3.5)}} + \underbrace{|f_0(u_{N_k(\bar{\omega})}^*(\bar{\omega})) - f_0(u(\bar{\omega}))|}_{\rightarrow 0 \text{ as } k \rightarrow \infty \text{ (} f_0 \text{ continuous and (C.1))}}. \end{aligned}$$

Also, $f_0(u(\bar{\omega})) \geq f_0(u^*)$ for any $u^* \in S_0^*$, since $u(\bar{\omega}) \in C_0$. Combining the last two results, we obtain (C.2a).

Next, we show (C.2b). By Lemma C.2, there exists a sequence $\{u_N(\bar{\omega})\}_{N \in \mathbb{N}}$ such that $u_N(\bar{\omega}) \in C_N(\bar{\omega})$ for all $N \in \mathbb{N}$ and $\lim_{N \rightarrow \infty} u_N(\bar{\omega}) = u^*$ for any $u^* \in S_0^*$.

Thus, since $u_{N_k(\bar{\omega})}^*(\bar{\omega}) \in S_{N_k(\bar{\omega})}^*(\bar{\omega})$ and $u_{N_k(\bar{\omega})}(\bar{\omega}) \in C_{N_k(\bar{\omega})}(\bar{\omega})$ for all $k \in \mathbb{N}$,

$$f_{N_k(\bar{\omega})}(u_{N_k(\bar{\omega})}^*(\bar{\omega}), \bar{\omega}) \leq f_{N_k(\bar{\omega})}(u_{N_k(\bar{\omega})}(\bar{\omega}), \bar{\omega}) - f_0(u_{N_k(\bar{\omega})}(\bar{\omega})) + f_0(u_{N_k(\bar{\omega})}(\bar{\omega})).$$

By uniform convergence in (3.5), $\lim_{k \rightarrow \infty} |f_{N_k(\bar{\omega})}(u_{N_k(\bar{\omega})}(\bar{\omega}), \bar{\omega}) - f_0(u_{N_k(\bar{\omega})}(\bar{\omega}))| = 0$. Also, $\lim_{k \rightarrow \infty} f_0(u_{N_k(\bar{\omega})}(\bar{\omega})) = f_0(u^*)$ since $\lim_{N \rightarrow \infty} u_N(\bar{\omega}) = u^*$ and f_0 is continuous. Combining the last results, we obtain

$$f_{N_k(\bar{\omega})}(u_{N_k(\bar{\omega})}^*(\bar{\omega}), \bar{\omega}) \leq f_{N_k(\bar{\omega})}(u_{N_k(\bar{\omega})}(\bar{\omega}), \bar{\omega}) \rightarrow f_0(u^*) \quad \text{as } k \rightarrow \infty$$

which shows (C.2b).

Together, (C.2a) and (C.2b) imply² that $\lim_{k \rightarrow \infty} f_{N_k(\bar{\omega})}(u_{N_k(\bar{\omega})}^*(\bar{\omega}), \bar{\omega}) = f_0(u^*)$. This concludes the proof of (3.7a) in Theorem 3.1.

Part 2: the subsequence $\{N_k(\bar{\omega})\}_{k \in \mathbb{N}}$ is such that $\mathbb{D}(S_{N_k(\bar{\omega})}^*(\bar{\omega}), S_0^*) \xrightarrow[k \rightarrow \infty]{} 0$.

By contradiction, assume that for some $\epsilon > 0$ and $\bar{\omega} \in \bar{\Omega}$, $\|u_{N_k(\bar{\omega})}^*(\bar{\omega}) - u^*\| \geq \epsilon$ for all $u^* \in S_0^*$ and all $k \in \mathbb{N}$. This implies that $\|u(\bar{\omega}) - u^*\| \geq \epsilon$ for all $u^* \in S_0^*$, since $\epsilon \leq \|u_{N_k(\bar{\omega})}^*(\bar{\omega}) - u^*\| \leq \|u_{N_k(\bar{\omega})}^*(\bar{\omega}) - u(\bar{\omega})\| + \|u(\bar{\omega}) - u^*\|$ for all $k \in \mathbb{N}$, so that $\|u(\bar{\omega}) - u^*\| = \lim_{k \rightarrow \infty} (\|u_{N_k(\bar{\omega})}^*(\bar{\omega}) - u(\bar{\omega})\| + \|u(\bar{\omega}) - u^*\|) \geq \epsilon$.

²Indeed, $f_0(u^*) \leq \liminf_{k \rightarrow \infty} f_{N_k(\bar{\omega})}(u_{N_k(\bar{\omega})}^*(\bar{\omega}), \bar{\omega}) \leq \limsup_{k \rightarrow \infty} f_{N_k(\bar{\omega})}(u_{N_k(\bar{\omega})}^*(\bar{\omega}), \bar{\omega}) \leq f_0(u^*)$.

Thus, $f_0(u(\bar{\omega})) > f_0(u^*)$. Since f_0 is continuous and $u_{N_k(\bar{\omega})}^*(\bar{\omega}) \rightarrow u(\bar{\omega})$, we obtain

$$(C.3) \quad \lim_{k \rightarrow \infty} f_0(u_{N_k(\bar{\omega})}^*(\bar{\omega})) = f_0(u(\bar{\omega})) > f_0(u^*).$$

However, $|f_0(u(\bar{\omega})) - f_0(u^*)| \leq |f_0(u(\bar{\omega})) - f_0(u_{N_k(\bar{\omega})}^*(\bar{\omega}))| + |f_{N_k(\bar{\omega})}(u_{N_k(\bar{\omega})}^*(\bar{\omega}), \bar{\omega}) - f_0(u^*)| + |f_0(u_{N_k(\bar{\omega})}^*(\bar{\omega})) - f_{N_k(\bar{\omega})}(u_{N_k(\bar{\omega})}^*(\bar{\omega}), \bar{\omega})|$ which goes to zero as $k \rightarrow \infty$ due to (3.5) and Part 1 (i.e., (3.7a)). Thus, $f_0(u(\bar{\omega})) = f_0(u^*)$, which contradicts (C.3). This concludes the proof of (3.7b) in Theorem 3.1.

Part 3: if f also satisfies $(\widehat{A1})$, then the full sequence converges. Under this additional assumption, \mathbb{P} -almost-surely, there exists $\alpha, L(\omega) > 0$ such that $|f(u_1, \omega) - f(u_2, \omega)| \leq L(\omega)\|u_1 - u_2\|^\alpha$ for all $u_1, u_2 \in \mathcal{U}$. In particular, $|f_N(u_1, \bar{\omega}) - f_N(u_2, \bar{\omega})| \leq L(\bar{\omega})\|u_1 - u_2\|^\alpha$. For conciseness, we drop dependencies on $(\omega, \bar{\omega})$.

It suffices to prove that $\lim_{N \rightarrow \infty} \mathbb{D}(S_N^*, S_0^*) = 0$. By contradiction, there exists $\bar{\epsilon} > 0$ such that for all $M \in \mathbb{N}$, there is $N_M \geq M$ with $\sup_{x \in S_{N_M}^*} \inf_{y \in S_0^*} \|x - y\| \geq \bar{\epsilon}$. Thus, there exists a sequence $\{u_M\}_{M \in \mathbb{N}}$ with $u_M \in S_{N_M}^*$ and $\inf_{y \in S_0^*} \|u_M - y\| \geq \bar{\epsilon}$ for all $M \in \mathbb{N}$.

Since $u_M \in \mathcal{U}$ for all $M \in \mathbb{N}$ and \mathcal{U} is compact, there exists a subsequence $\{u_{M_k}\}_{k \in \mathbb{N}}$ such that $u_{M_k} \rightarrow \bar{u}$ as $k \rightarrow \infty$ for some $\bar{u} \in C_0$ (thanks to (3.6)) with $\lim_{k \rightarrow \infty} |f_{N_{M_k}}(u_{M_k}) - f_0(\bar{u})| = 0$ (thanks to (3.5)). Since $\inf_{y \in S_0^*} \|u_M - y\| \geq \bar{\epsilon}$ for all $M \in \mathbb{N}$, we have $\bar{u} \in C_0 \setminus S_0^*$, so

$$\inf_{u \in C_0} f_0(u) < f_0(\bar{u}) = \lim_{k \rightarrow \infty} f_{N_{M_k}}(u_{M_k}) = \lim_{k \rightarrow \infty} \inf_{u \in C_{N_{M_k}}} f_{N_{M_k}}(u).$$

To conclude, it suffices to prove that $\lim_{N \rightarrow \infty} \inf_{u \in C_N} f_N(u) \leq \inf_{u \in C_0} f_0(u)$:

$$\begin{aligned} \inf_{u \in C_N} f_N(u) - \inf_{u \in C_0} f_0(u) &= \inf_{u \in C_N} f_N(u) - f_0(u^*) \quad (\text{for some } u^* \in C_0) \\ &= \inf_{u \in C_N} f_N(u) - f_N(u^*) + f_N(u^*) - f_0(u^*) \\ &\leq L \inf_{u \in C_N} \|u - u^*\|^\alpha + \sup_{u \in \mathcal{U}} |f_N(u) - f_0(u)| \\ &\leq Ld_H(C_N, C_0)^\alpha + \sup_{u \in \mathcal{U}} |f_N(u) - f_0(u)| \end{aligned}$$

which converges to zero as $N \rightarrow \infty$ thanks to (3.6) and (3.5). \square

C.2. Proof of Theorem 4.1: the case with equality and inequality constraints. We first extend Lemmas C.1 and C.2 to account for inequality constraints.

LEMMA C.3. *Let $(\Omega, \mathcal{G}, \mathbb{P})$ be a probability space, $(\bar{\Omega}, \bar{\mathcal{G}}, \bar{\mathbb{P}})$ be the product probability space, $\mathcal{U} \subset \mathbb{R}^d$ and $C_0 \subseteq \mathcal{U}$ be compact sets, $\{C_N\}_{N \in \mathbb{N}}$ be a sequence of random compact sets $C_N : (\bar{\Omega}, \bar{\mathcal{G}}) \rightarrow (\mathcal{K}, \mathcal{B}(\mathcal{K}))$, and $\{u_N\}_{N \in \mathbb{N}}$ be a sequence of random variables $u_N : (\bar{\Omega}, \bar{\mathcal{G}}) \rightarrow (\mathcal{U}, \mathcal{B}(\mathcal{U}))$. Let $g : \mathcal{U} \times \Omega \rightarrow \mathbb{R}^q$ be a Carathéodory function and define g_0 and g_N as in (4.1). Assume that $\bar{\mathbb{P}}$ -almost-surely,*

- $u_N(\bar{\omega}) \in C_N(\bar{\omega})$ for all $N \in \mathbb{N}$.
- $g_N(u_N(\bar{\omega}), \bar{\omega}) \leq 0$ for all $N \in \mathbb{N}$.
- $\mathbb{D}(C_N(\bar{\omega}), C_0) \rightarrow 0$ as $N \rightarrow \infty$.
- g satisfies (A4a).

Then, $\bar{\mathbb{P}}$ -almost-surely, there exists $u(\bar{\omega}) \in C_0$ with $g_0(u(\bar{\omega})) \leq 0$ and a subsequence $\{N_k(\bar{\omega})\}_{k \in \mathbb{N}}$ such that $u_{N_k(\bar{\omega})}(\bar{\omega}) \rightarrow u(\bar{\omega})$ as $k \rightarrow \infty$.

Proof. First, $\bar{\mathbb{P}}$ -almost-surely, $\lim_{N \rightarrow \infty} \sup_{u \in \mathcal{U}} \|g_N(u, \bar{\omega}) - g_0(u)\| = 0$ from (A4a).

Consider the point $u(\bar{\omega}) \in C_0$ and the subsequence $\{N_k(\bar{\omega})\}_{k \in \mathbb{N}}$ from Lemma C.1 such that $u_{N_k(\bar{\omega})}(\bar{\omega}) \rightarrow u(\bar{\omega})$ as $k \rightarrow \infty$. Then, $g_0(u(\bar{\omega})) \leq 0$. Indeed,

$$\|g_N(u_N(\bar{\omega}), \bar{\omega}) - g_0(u(\bar{\omega}))\| \leq \|g_N(u_N(\bar{\omega}), \bar{\omega}) - g_0(u_N(\bar{\omega}))\| + \|g_0(u_N(\bar{\omega})) - g_0(u(\bar{\omega}))\|.$$

Thus, by the uniform convergence from (A4a) and the continuity of g_0 , by passing to the subsequence $N_k(\bar{\omega})$, we obtain that $g_0(u(\bar{\omega})) \leq 0$. \square

LEMMA C.4. *With the notations used in Theorem 4.1, assume that:*

- *h satisfies (A2-3).*
- *g satisfies (A4).*

Let $u^* \in S_0^*$ denote the optimal solution from (A4b). Then, $\bar{\mathbb{P}}$ -almost-surely, for $\tilde{N}_\epsilon(\bar{\omega}) \in \mathbb{N}$ large enough, there exists a sequence $\{u_N(\bar{\omega})\}_{N \in \mathbb{N}}$ such that $u_N(\bar{\omega}) \in C_N(\bar{\omega})$ and $g_N(u_N(\bar{\omega}), \bar{\omega}) \leq 0$ for all $N \geq \tilde{N}_\epsilon(\bar{\omega})$, and $u_N(\bar{\omega}) \rightarrow u^*$ as $N \rightarrow \infty$.

Proof. Given $\bar{\omega} \in \bar{\Omega}$, we consider the sequence $\{u_N(\bar{\omega})\}_{N \in \mathbb{N}}$ defined as

$$u_N(\bar{\omega}) = \underset{u \in (C_N(\bar{\omega}) \cap \{u \in \mathcal{U} : g_N(u, \bar{\omega}) \leq 0\})}{\arg \inf} \|u - u^*\|.$$

$\bar{\mathbb{P}}$ -almost-surely, there exists $N(\bar{\omega}) \in \mathbb{N}$ such that $u_N(\bar{\omega}) \neq \emptyset$ for all $N \geq N(\bar{\omega})$ (i.e., the intersection of $C_N(\bar{\omega})$ with $\{u \in \mathcal{U} : g_N(u, \bar{\omega}) \leq 0\}$ is not empty).

To prove this claim, we let $\epsilon \in (0, \frac{1}{2})$ and define β_N and ϵ_N as in (3.12) and δ_N as in (2.2). Then, we consider the sequence $\{v_{\epsilon_N}\}_{N \in \mathbb{N}}$ that satisfies $g_0(v_{\epsilon_N}) < 0$ and $\|v_{\epsilon_N} - u^*\|^\alpha < \epsilon_N$, which is well-defined thanks to (A4b).

Thanks to (A4a), $\bar{\mathbb{P}}$ -a.s., $\lim_{N \rightarrow \infty} \sup_{u \in \mathcal{U}} \|g_N(u, \bar{\omega}) - g_0(u)\| = 0$, so $\|g_N(v_{\epsilon_N}, \bar{\omega}) - g_0(v_{\epsilon_N})\| \rightarrow 0$ as $N \rightarrow \infty$. Thus, $\bar{\mathbb{P}}$ -a.s., $g_N(v_{\epsilon_N}, \bar{\omega}) \leq 0$ for all $N > \tilde{N}(\bar{\omega})$ with $\tilde{N}(\bar{\omega}) \in \mathbb{N}$ large enough. Also, since $h_0(u^*) = 0$ and thanks to (A2-3), Proposition 3.6, and Jensen's inequality, with $\bar{\mathbb{P}}$ -probability at least $1 - \beta_N$, $\|h_N(v_{\epsilon_N}, \bar{\omega})\| \leq \|h_N(v_{\epsilon_N}, \bar{\omega}) - h_0(v_{\epsilon_N})\| + \|h_0(v_{\epsilon_N}) - h_0(u^*)\| \leq (1 + \mathbb{E}[M(\omega)])\epsilon_N \leq \delta_N$ for all $N \geq \tilde{N}$ with $\tilde{N} \in \mathbb{N}$ large enough so that $(1 + \mathbb{E}[M(\omega)])\epsilon_N \leq \delta_N$ for all $N \geq \tilde{N}$ (note that $\epsilon_N/\delta_N \rightarrow 0$ as $N \rightarrow \infty$). Next, we define the infeasibility event $B_N = \{\bar{\omega} \in \bar{\Omega} : \|h_N(v_{\epsilon_N}, \bar{\omega})\| > \delta_N\}$. By the above, $\bar{\mathbb{P}}(B_N) \leq \beta_N$ for all $N \geq \tilde{N}$. By the first Borel-Cantelli Lemma, $\bar{\mathbb{P}}$ -a.s., B_N only happens finitely many times.

Thus, $\bar{\mathbb{P}}$ -a.s., $u_N(\bar{\omega}) \neq \emptyset$ for all $N \geq N(\bar{\omega})$ for $N(\bar{\omega}) \in \mathbb{N}$ large enough. Finally, $\|u_N(\bar{\omega}) - u^*\| \leq \|v_{\epsilon_N}(\bar{\omega}) - u^*\| \leq \epsilon_N$ for $N \geq N(\bar{\omega})$ with $\lim_{N \rightarrow \infty} \epsilon_N = 0$. \square

Proof of Theorem 4.1 if g satisfies (A4). By (A1) and (A2-3), $\sup_{u \in \mathcal{U}} |f_N(u, \bar{\omega}) - f_0(u)| \rightarrow 0$ and $d_H(C_N(\bar{\omega}), C_0) \rightarrow 0$ as $N \rightarrow \infty$ $\bar{\mathbb{P}}$ -a.s.. Also, by (A2-4) and Lemma C.4, $\bar{\mathbb{P}}$ -almost-surely, $\mathbf{SP}_N(\bar{\omega})$ is feasible for all $N \geq N(\bar{\omega})$ with $N(\bar{\omega})$ large enough.

The conclusion of the proof follows from the proof of Theorem 3.1 by replacing the use of Lemmas C.1 and C.2 with Lemmas C.3 and C.4. \square

Appendix D. Proving Propositions 3.5-3.6 with concentration inequalities. Section 3.2 leverages concentration inequalities, namely McDiarmid's inequality and Dudley's entropy integral. In this section, we prove Propositions 3.5-3.6.

D.1. Rademacher variables, Rademacher complexity, and symmetrization. To derive a uniform law of large numbers of the form of (3.8) for the function class \mathcal{H} in (3.10), we define the independent Rademacher variables³ $\epsilon_i : \bar{\Omega} \rightarrow \{-1, 1\}$

³To define $(\bar{\Omega}, \bar{\mathcal{G}}, \bar{\mathbb{P}})$ accounting for these random variables, one can start from the product space in Section 2 and redefine $\bar{\Omega} \leftarrow \bar{\Omega} \times \{-1, 1\}^{\mathbb{N}}$ and similarly for $\bar{\mathcal{G}}$ and $\bar{\mathbb{P}}$, see Appendix A.

so that $\bar{\mathbb{P}}(\epsilon_i = 1) = \bar{\mathbb{P}}(\epsilon_i = -1) = \frac{1}{2}$ (see, e.g., [27,33]) and the Rademacher complexity

$$(D.1) \quad R_N(\mathcal{H}) = \bar{\mathbb{E}} \left[\sup_{h_u \in \mathcal{H}} \left\| \frac{1}{N} \sum_{i=1}^N \epsilon_i h_u(\omega^i) \right\| \right].$$

By a standard symmetrization argument (see e.g. [33, (4.17)-(4.18)]),

$$(D.2) \quad \bar{\mathbb{E}} \left(\sup_{h_u \in \mathcal{H}} \left\| \frac{1}{N} \sum_{i=1}^N h_u(\omega^i) - \mathbb{E}[h_u(\omega)] \right\| \right) \leq 2R_N(\mathcal{H}).$$

Combining (D.2) with Markov's inequality gives (3.11) in Remark 3.3.

D.2. Dudley's entropy integral. Dudley's entropy integral [33, Theorem 5.22] is a concentration inequality exploiting the tight tails of sub-Gaussian processes and covering arguments via chaining. We recall basic concepts before stating the result.

DEFINITION D.1 (Sub-Gaussian process). *Let $(\Omega, \mathcal{G}, \mathbb{P})$ be a probability space and (Θ, ρ) be a metric space. An \mathbb{R} -valued stochastic process $\{X_\theta : \theta \in \Theta\}$ is sub-Gaussian with respect to the metric ρ on Θ if $\mathbb{E}[\exp(\lambda(X_{\theta_1} - X_{\theta_2}))] \leq \exp(\lambda^2 \rho(\theta_1, \theta_2)^2 / 2)$ for all $\lambda \in \mathbb{R}$ and all $\theta_1, \theta_2 \in \Theta$.*

Sub-Gaussian random variables satisfy the following properties.

1. If X_1, \dots, X_N are independent σ^2 -sub-Gaussian, $\sum_{i=1}^N X_i$ is $(N\sigma^2)$ -sub-Gaussian.
2. If X is σ^2 -sub-Gaussian and $a > 0$, then aX is $(a^2\sigma^2)$ -sub-Gaussian.
3. If $\underline{x} \leq X \leq \bar{x}$ almost surely and $\mathbb{E}[X] = 0$, then X is $((\bar{x} - \underline{x})^2 / 4)$ -sub-Gaussian.

DEFINITION D.2 (Covering). *Let (Θ, ρ) be a metric space and $\epsilon > 0$. A collection $\{\theta_1, \dots, \theta_N\}$ is called an ϵ -cover if $\min_i \rho(\theta, \theta_i) \leq \epsilon$ for all $\theta \in \Theta$. The ϵ -covering number of (Θ, ρ) is defined as $N(\Theta, \rho, \epsilon) = \inf \{N \in \mathbb{N} : \exists \text{ an } \epsilon\text{-cover } \{\theta_1, \dots, \theta_N\}\}$.*

The covering number of any compact set $\Theta \subset \mathbb{R}^d$ satisfies

$$(D.3) \quad N(\Theta, \|\cdot\|_2, \epsilon) \leq (1 + D/\epsilon)^d$$

with $D = 2 \sup_{\theta \in \Theta} \|\theta\|_2$, see e.g. [33, (5.9)]. Then, $N(\Theta, \|\cdot\|_2^\alpha, \epsilon^\alpha) \leq (1 + D/\epsilon)^\alpha$ for any $\alpha \in (0, 1]^4$. Indeed, the ϵ -cover $\{\theta_1, \dots, \theta_N\}$ for the Euclidean metric $\|\cdot\|_2$ that satisfies $\min \|\theta - \theta_i\|_2 \leq \epsilon$ for all $\theta \in \Theta$ also satisfies $\min \|\theta - \theta_i\|_2^\alpha \leq \epsilon^\alpha$ for all $\theta \in \Theta$.

THEOREM D.3 (Dudley's entropy integral). *Let $(\Omega, \mathcal{G}, \mathbb{P})$ be a probability space, (Θ, ρ) be a compact metric space, $\{X_\theta : \theta \in \Theta\}$ be an \mathbb{R} -valued sub-Gaussian process with respect to the metric ρ on Θ , $D = \sup_{\theta, \theta' \in \Theta} \rho(\theta, \theta')$, and $C = 32$. Then,*

$$\mathbb{E} \left[\sup_{\theta \in \Theta} X_\theta \right] \leq C \int_0^D \sqrt{\log(N(\Theta, \rho, u))} du.$$

Theorem D.3 allows obtaining bounds on $\mathbb{E}[\sup_{\theta \in \Theta} |X_\theta|]$ for stochastic processes X_θ that are symmetric (i.e., X_θ has the same distribution as $-X_\theta$):

$$(D.4) \quad \mathbb{E} \left[\sup_{\theta \in \Theta} |X_\theta| \right] \leq 2\mathbb{E} \left[\sup_{\theta \in \Theta} X_\theta \right] + \inf_{\bar{\theta} \in \Theta} \mathbb{E} [|X_{\bar{\theta}}|].$$

⁴Note that $\|\cdot\|^\alpha$ is a metric for any $\alpha \in (0, 1]$. Indeed, (1) $\|u - v\|^\alpha = 0 \iff u, v = 0$, (2) $\|u - v\|^\alpha = \|v - u\|^\alpha$, and (3) $\|u - w\|^\alpha \leq \|u - v\|^\alpha + \|v - w\|^\alpha$ since $(x + y)^\alpha \leq x^\alpha + y^\alpha$ for all $x, y \geq 0$ (proof: for y fixed, consider $f(x) = (x + y)^\alpha - x^\alpha - y^\alpha$. Note that $f(0) = 0$ and $f'(x) \leq 0$ for all $x, y \geq 0$ so $f(x)$ is nonincreasing. Thus, $f(x) \leq 0$ for all $x \geq 0$).

Indeed, by the triangle inequality⁵, if X_θ is a symmetric process, $\mathbb{E}[\sup_{\theta \in \Theta} |X_\theta|] \leq \mathbb{E}[\sup_{\theta, \tilde{\theta} \in \Theta} |X_\theta - X_{\tilde{\theta}}|] + \inf_{\tilde{\theta} \in \Theta} \mathbb{E}[|X_{\tilde{\theta}}|] = 2\mathbb{E}[\sup_{\theta \in \Theta} |X_\theta|] + \inf_{\tilde{\theta} \in \Theta} \mathbb{E}[|X_{\tilde{\theta}}|]$, see [31, Lemma 2.2.1]. The first term in (D.4) can be bounded using Theorem D.3.

D.3. Proof of Proposition 3.5. We first prove a concentration result for centered smooth function classes. We then use this result to prove Proposition 3.5.

PROPOSITION D.4 (Concentration for centered α -Hölder function class). *With the notations and assumptions of Proposition 3.5, define the centered function class $\tilde{\mathcal{H}} = \{h(u, \cdot) - h(u_0, \cdot)\}_{u \in \mathcal{U}}$. Then,*

$$\bar{\mathbb{E}} \left[\sup_{h_u \in \tilde{\mathcal{H}}} \left\| \frac{1}{N} \sum_{i=1}^N h_u(\omega^i) - \mathbb{E}[h_u(\omega)] \right\| \right] \leq \frac{1}{\sqrt{N}} \frac{8CD^{\frac{\alpha+1}{2}} d^{\frac{1}{2}} n^{\frac{3}{2}}}{\alpha^{\frac{1}{2}}} \sqrt{\mathbb{E}[M^2]}.$$

Proof. First, we denote $h(u, \omega) = (h^1(u, \omega), \dots, h^n(u, \omega)) \in \mathbb{R}^n$ and define the dimension-wise centered function classes $\tilde{\mathcal{H}}^j = \{h^j(u, \cdot) - h^j(u_0, \cdot) : \Omega \rightarrow \mathbb{R}, u \in \mathcal{U}\}$ with $j = 1, \dots, n$. We note that any $h_u^j \in \tilde{\mathcal{H}}^j$ satisfies (A2) with α -Hölder constant

$$(D.5) \quad M_j(\omega) = \sqrt{n}M(\omega)$$

since $\|a\|_1 \leq \sqrt{n}\|a\|_2$ for any $a \in \mathbb{R}^n$ and $|h^j(u_1, \omega) - h^j(u_2, \omega)| \leq \|h(u_1, \omega) - h(u_2, \omega)\|_1 \leq \sqrt{n}\|h(u_1, \omega) - h(u_2, \omega)\|_2$. Since $\|a\|_2 \leq \|a\|_1$ for any $a \in \mathbb{R}^n$,

$$(D.6) \quad \bar{\mathbb{E}} \left[\sup_{h_u \in \tilde{\mathcal{H}}} \left\| \frac{1}{N} \sum_{i=1}^N h_u(\omega^i) - \mathbb{E}[h_u] \right\| \right] \leq \sum_{j=1}^n \bar{\mathbb{E}} \left[\sup_{h_u^j \in \tilde{\mathcal{H}}^j} \left| \frac{1}{N} \sum_{i=1}^N h_u^j(\omega^i) - \mathbb{E}[h_u^j] \right| \right].$$

Next, we fix $j = 1, \dots, n$ and bound the right-hand-side term above dimension-wise. We define the \mathbb{R} -valued centered process Z_u^j and the empirical expectation $\mathbb{E}_N[M_j^2]$:

$$Z_u^j \triangleq \frac{1}{\sqrt{N}} \sum_{i=1}^N \epsilon_i (h^j(u, \omega^i) - h^j(u_0, \omega^i)), \quad \mathbb{E}_N[M_j^2] \triangleq \frac{1}{N} \sum_{i=1}^N M_j(\omega^i)^2.$$

When conditioned on $\bar{\omega}$, the process $\{Z_u^j : u \in \mathcal{U}\}$ is ρ^2 -sub-gaussian for $\rho(u_1, u_2)^2 = \mathbb{E}_N[M_j^2] \|u_1 - u_2\|_2^{2\alpha}$. Indeed, the Rademacher variables ϵ_i take values in $\{-1, 1\}$ with equal probability, so they are 1-sub-Gaussian. Thus, $\epsilon_i a$ is a^2 -sub-Gaussian for any $a \geq 0$, and since the maps $u \mapsto h^j(u, \omega^i)$ satisfy (A2),

$$\begin{aligned} \bar{\mathbb{E}} \left[\exp(\lambda(Z_{u_1}^j - Z_{u_2}^j)) \mid \bar{\omega} \right] &= \bar{\mathbb{E}} \left[\exp \left(\frac{\lambda}{\sqrt{N}} \sum_{i=1}^N \epsilon_i (h^j(u_1, \omega^i) - h^j(u_2, \omega^i)) \right) \mid \bar{\omega} \right] \\ &\leq \exp \left(\frac{\lambda^2}{2N} \sum_{i=1}^N (h^j(u_1, \omega^i) - h^j(u_2, \omega^i))^2 \right) \stackrel{(A2)}{\leq} \exp \left(\frac{\lambda^2}{2} \left(\frac{1}{N} \sum_{i=1}^N M_j(\omega^i)^2 \right) \|u_1 - u_2\|_2^{2\alpha} \right) \end{aligned}$$

where $\bar{\mathbb{E}}[A \mid \bar{\omega}]$ denotes the conditional expectation of the random variable A given $\bar{\omega} = (\omega^1, \dots)$, i.e., the expected value of A over the Rademacher variables ϵ_i .

In addition, Z_u^j is a symmetric process, i.e., Z_u^j has the same distribution as $-Z_u^j$ when conditioned on $\bar{\omega}$. Thus, if we take the expectation over the Rademacher variables ϵ_i , since $Z_{u_0}^j = 0$, we obtain

$$(D.7) \quad \bar{\mathbb{E}} \left[\sup_{u \in \mathcal{U}} |Z_u^j| \mid \bar{\omega} \right] \stackrel{(D.4)}{\leq} 2\bar{\mathbb{E}} \left[\sup_{u \in \mathcal{U}} Z_u^j \mid \bar{\omega} \right] + \inf_{\tilde{u} \in \mathcal{U}} \bar{\mathbb{E}} \left[|Z_{\tilde{u}}^j| \mid \bar{\omega} \right] = 2\bar{\mathbb{E}} \left[\sup_{u \in \mathcal{U}} Z_u^j \mid \bar{\omega} \right].$$

⁵By the triangle inequality, we have $\sup_{\theta} |X_\theta| = \sup_{\tilde{\theta}} \inf_{\theta} |X_\theta| \leq \sup_{\theta} \inf_{\tilde{\theta}} (|X_\theta - X_{\tilde{\theta}}| + |X_{\tilde{\theta}}|) \leq \sup_{\theta} \inf_{\tilde{\theta}} (\sup_{\tilde{\theta}} (|X_\theta - X_{\tilde{\theta}}|) + |X_{\tilde{\theta}}|) \leq \sup_{\theta, \tilde{\theta}} |X_\theta - X_{\tilde{\theta}}| + \inf_{\tilde{\theta}} |X_{\tilde{\theta}}|$.

Moreover, $Z_u^j/\sqrt{\mathbb{E}_N[M_j^2]}$ is $\|u_1 - u_2\|_2^{2\alpha}$ -sub-Gaussian when conditioned on $\bar{\omega}$. Thus, we apply Dudley's entropy integral (Theorem D.3) and obtain

$$\begin{aligned} \mathbb{E} \left[\sup_{u \in \mathcal{U}} |Z_u^j| \mid \bar{\omega} \right] &\stackrel{(D.7)}{\leq} 2\mathbb{E} \left[\sup_{u \in \mathcal{U}} Z_u^j \mid \bar{\omega} \right] = 2\sqrt{\mathbb{E}_N[M_j^2]} \mathbb{E} \left[\sup_{u \in \mathcal{U}} \frac{Z_u^j}{\sqrt{\mathbb{E}_N[M_j^2]}} \mid \bar{\omega} \right] \\ &\leq 2C\mathbb{E}_N[M_j^2]^{\frac{1}{2}} \int_0^D \sqrt{\log N(\mathcal{U}, \|\cdot\|_2^\alpha, u)} du \stackrel{(D.3)}{\leq} 2C\mathbb{E}_N[M_j^2]^{\frac{1}{2}} \int_0^D \sqrt{d \log \left(1 + \frac{D}{u^{\frac{1}{\alpha}}} \right)} du \\ &\leq 2C\mathbb{E}_N[M_j^2]^{\frac{1}{2}} \int_0^D \sqrt{d \log \left(\left(1 + \frac{D^\alpha}{u} \right)^{\frac{1}{\alpha}} \right)} du = 2C\mathbb{E}_N[M_j^2]^{\frac{1}{2}} \int_0^D \sqrt{d \frac{1}{\alpha} \log \left(1 + \frac{D^\alpha}{u} \right)} du \\ &\leq 2C\mathbb{E}_N[M_j^2]^{\frac{1}{2}} \int_0^D \sqrt{\frac{dD^\alpha}{\alpha u}} du = \frac{4CD^{\frac{\alpha+1}{2}} d^{\frac{1}{2}}}{\alpha^{\frac{1}{2}}} \mathbb{E}_N[M_j^2]^{\frac{1}{2}} \end{aligned}$$

since $(1 + \frac{D}{u^{1/\alpha}})^\alpha \leq 1 + (\frac{D}{u^{1/\alpha}})^\alpha$ (see footnote 4) and $\log(1+t) \leq t$ for $t > -1$. Finally, by Jensen's inequality (J) and since $M_j(\omega) = \sqrt{n}M(\omega)$ (D.5),

$$\begin{aligned} \mathbb{E} \left[\sup_{h_u \in \tilde{\mathcal{H}}} \left\| \frac{1}{N} \sum_{i=1}^N h_u(\omega^i) - \mathbb{E}[h_u(\omega)] \right\| \right] &\stackrel{(D.6)}{\leq} \sum_{j=1}^n \mathbb{E} \left[\sup_{h_u^j \in \tilde{\mathcal{H}}^j} \left\| \frac{1}{N} \sum_{i=1}^N h_u^j(\omega^i) - \mathbb{E}[h_u^j] \right\| \right] \\ &\stackrel{(D.2)}{\leq} \sum_{j=1}^n 2R_N(\tilde{\mathcal{H}}^j) = \sum_{j=1}^n \frac{2}{\sqrt{N}} \mathbb{E} \left[\sup_{u \in \mathcal{U}} |Z_u^j| \right] = \sum_{j=1}^n \frac{2}{\sqrt{N}} \mathbb{E} \left[\mathbb{E} \left[\sup_{u \in \mathcal{U}} |Z_u^j| \mid \bar{\omega} \right] \right] \\ &\leq \sum_{j=1}^n \frac{2}{\sqrt{N}} \frac{4CD^{\frac{\alpha+1}{2}} d^{\frac{1}{2}}}{\alpha^{\frac{1}{2}}} \mathbb{E} \left[\sqrt{\mathbb{E}_N[M_j^2]} \right] \stackrel{(J)}{\leq} \frac{1}{\sqrt{N}} \sum_{j=1}^n \frac{8CD^{\frac{\alpha+1}{2}} d^{\frac{1}{2}}}{\alpha^{\frac{1}{2}}} \sqrt{\mathbb{E}[\mathbb{E}_N[M_j^2]]} \\ &\stackrel{(D.5)}{=} \frac{1}{\sqrt{N}} \frac{8CD^{\frac{\alpha+1}{2}} d^{\frac{1}{2}} n^{\frac{3}{2}}}{\alpha^{\frac{1}{2}}} \sqrt{\mathbb{E}[M^2]} \end{aligned}$$

which concludes the proof. \square

Proposition D.4 treats the centered function class $\tilde{\mathcal{H}} = \{h(u, \cdot) - h(u_0, \cdot)\}_{u \in \mathcal{U}}$. Proposition 3.5 follows by extending this result to the function class \mathcal{H} in (3.10).

Proof of Proposition 3.5. By the triangle inequality,

$$\begin{aligned} \left\| \frac{1}{N} \sum_{i=1}^N h(u, \omega^i) - \mathbb{E}[h(u, \omega)] \right\| &\leq \left\| \frac{1}{N} \sum_{i=1}^N (h(u, \omega^i) - h(u_0, \omega^i)) - \mathbb{E}[(h(u, \omega) - h(u_0, \omega))] \right\| \\ &\quad + \left\| \frac{1}{N} \sum_{i=1}^N h(u_0, \omega^i) - \mathbb{E}[h(u_0, \omega)] \right\|. \end{aligned}$$

With the centered function class $\tilde{\mathcal{H}} = \{h(u, \cdot) - h(u_0, \cdot)\}_{u \in \mathcal{U}}$ defined in Proposition D.4 and the function class $\mathcal{H} = \{h(u, \cdot)\}_{u \in \mathcal{U}}$ defined in (3.10), we obtain

$$\sup_{h_u \in \mathcal{H}} \left\| \frac{1}{N} \sum_{i=1}^N h_u(\omega^i) - \mathbb{E}[h_u] \right\| \leq \sup_{\tilde{h}_u \in \tilde{\mathcal{H}}} \left\| \frac{1}{N} \sum_{i=1}^N \tilde{h}_u(\omega^i) - \mathbb{E}[\tilde{h}_u] \right\| + \left\| \frac{1}{N} \sum_{i=1}^N h_{u_0}(\omega^i) - \mathbb{E}[h_{u_0}] \right\|.$$

where we write $h_{u_0} = h(u_0, \cdot) \in \mathcal{H}$.

Proposition D.4 bounds the expected value of the first term.

To bound the expected value of the second, by Jensen's inequality,

$$\begin{aligned} \mathbb{E} \left[\left\| \frac{1}{N} \sum_{i=1}^N h(u_0, \omega^i) - \mathbb{E}[h(u_0, \omega)] \right\|_2 \right] &\leq \mathbb{E} \left[\left\| \frac{1}{N} \sum_{i=1}^N h(u_0, \omega^i) - \mathbb{E}[h(u_0, \omega)] \right\|_2^2 \right]^{\frac{1}{2}} \\ &= \text{Trace} \left(\text{Cov} \left(\frac{1}{N} \sum_{i=1}^N h(u_0, \omega^i) \right) \right)^{\frac{1}{2}} = \text{Trace} \left(\frac{1}{N} \text{Cov}(h(u_0, \cdot)) \right)^{\frac{1}{2}} = \frac{1}{\sqrt{N}} \text{Trace}(\Sigma_0)^{\frac{1}{2}}. \end{aligned}$$

Combining Proposition D.4 and this bound gives the result. \square

D.4. Proof of Proposition 3.6. We use Corollary 3.4 and Proposition 3.5.

Proof of Proposition 3.6. By Corollary 3.4, for any $\delta > 0$, with $\bar{\mathbb{P}}$ -probability less than $\exp\left(-\frac{N\delta^2}{2\bar{h}^2}\right)$,

$$\sup_{u \in \mathcal{U}} \left\| \frac{1}{N} \sum_{i=1}^N h(u, \omega^i) - \mathbb{E}[h(u, \omega)] \right\| \geq \bar{\mathbb{E}} \left[\sup_{u \in \mathcal{U}} \left\| \frac{1}{N} \sum_{i=1}^N h(u, \omega^i) - \mathbb{E}[h(u, \omega)] \right\| \right] + \delta.$$

Also, note that the boundedness of $h(u_0, \omega)$ in (A3) implies that Σ_0 is finite. Then,

$$\bar{\mathbb{E}} \left[\sup_{u \in \mathcal{U}} \left\| \frac{1}{N} \sum_{i=1}^N h(u, \omega^i) - \mathbb{E}[h(u, \omega)] \right\| \right] \leq \frac{1}{\sqrt{N}} \tilde{C}$$

by Proposition 3.5. Thus, combining the previous two results,

$$(D.8) \quad \bar{\mathbb{P}} \left(\sup_{u \in \mathcal{U}} \left\| \frac{1}{N} \sum_{i=1}^N h(u, \omega^i) - \mathbb{E}[h(u, \omega)] \right\| \geq \frac{\tilde{C}}{\sqrt{N}} + \delta \right) \leq \exp\left(-\frac{N\delta^2}{2\bar{h}^2}\right).$$

Let $\epsilon > 0$ and $\delta = \tilde{C}N^{-\frac{1}{2} + \frac{\epsilon}{2}}$. Then, $\left(\tilde{C}N^{-\frac{1}{2}} + \tilde{C}N^{-\frac{1}{2} + \frac{\epsilon}{2}}\right) \leq 2\tilde{C}N^{-\frac{1}{2} + \frac{\epsilon}{2}}$, so

$$\bar{\mathbb{P}} \left(\sup_{u \in \mathcal{U}} \left\| \frac{1}{N} \sum_{i=1}^N h(u, \omega^i) - \mathbb{E}[h(u, \omega)] \right\| \geq 2\tilde{C}N^{-\frac{1}{2} + \frac{\epsilon}{2}} \right) \leq \exp\left(-\frac{\tilde{C}^2}{2\bar{h}^2} N^\epsilon\right).$$

Let $\epsilon_N = 2\tilde{C}N^{-\frac{1}{2} + \frac{\epsilon}{2}}$ and $\beta_N = \exp\left(-\frac{\tilde{C}^2}{2\bar{h}^2} N^\epsilon\right)$ and the conclusion follows. \square

Proof of Corollary 3.7. The result follows from the proof of Proposition 3.6 by defining $\delta = (2\bar{h}^2 \log(1/\beta)/N)^{\frac{1}{2}}$ in (D.8). \square

Appendix E. Continuity of state trajectories with respect to control input. Let $u, v \in \mathcal{U}$ and denote by x^u and x^v the associated trajectories that satisfy the SDE (5.2). For any $t \in [0, T]$, we write $\Delta b_t^{u,v} = b(x_t^u, u_t) - b(x_t^v, v_t)$ and $\Delta \sigma_t^{u,v} = \sigma(x_t^u, u_t) - \sigma(x_t^v, v_t)$. Then, for any $t \in [0, T]$ and $p \geq 2$, we have

$$\begin{aligned} \mathbb{E} \left[\sup_{0 \leq s \leq t} \|x_s^u - x_s^v\|^p \right] &= \mathbb{E} \left[\sup_{0 \leq s \leq t} \left\| \int_0^s \Delta b_r^{u,v} dr + \int_0^s \Delta \sigma_r^{u,v} dW_r \right\|^p \right] \\ &\leq C_p \left(\mathbb{E} \left[\sup_{0 \leq s \leq t} \left\| \int_0^s \Delta b_r^{u,v} dr \right\|^p \right] + \mathbb{E} \left[\sup_{0 \leq s \leq t} \left\| \int_0^s \Delta \sigma_r^{u,v} dW_r \right\|^p \right] \right) \end{aligned}$$

for some constant C_p . Then, by Hölder's inequality,

$$\mathbb{E} \left[\sup_{0 \leq s \leq t} \left\| \int_0^s \Delta b_r^{u,v} dr \right\|^p \right] \leq \mathbb{E} \left[\left(\int_0^t \|\Delta b_r^{u,v}\| dr \right)^p \right] \leq t^{p-1} \mathbb{E} \left[\int_0^t \|\Delta b_r^{u,v}\|^p dr \right]$$

and by the Burkholder-Davis-Gundy [19, Theorem 5.16] and Hölder inequalities,

$$\begin{aligned} \mathbb{E} \left[\sup_{0 \leq s \leq t} \left\| \int_0^s \Delta \sigma_r^{u,v} dW_r \right\|^p \right] &\leq C_2 \mathbb{E} \left[\left\langle \int_0^\cdot \Delta \sigma_r^{u,v} dW_r, \int_0^\cdot \Delta \sigma_r^{u,v} dW_r \right\rangle_t^{p/2} \right] \\ &= C_2 \mathbb{E} \left[\left(\int_0^t \|\Delta \sigma_r^{u,v}\|^2 dr \right)^{p/2} \right] = C_2 t^{\frac{p}{2}-1} \mathbb{E} \left[\int_0^t \|\Delta \sigma_r^{u,v}\|^p dr \right]. \end{aligned}$$

Thus, since $t \in [0, T]$, we obtain for some constants $C_{p,T}$ and $C_{p,T,K}$ that

$$\begin{aligned} \mathbb{E} \left[\sup_{0 \leq s \leq t} \|x_s^u - x_s^v\|^p \right] &\leq C_{p,T} \left(\mathbb{E} \left[\int_0^t \|\Delta b_r^{u,v}\|^p dr \right] + \mathbb{E} \left[\int_0^t \|\Delta \sigma_r^{u,v}\|^p dr \right] \right) \\ &\leq C_{p,T,K} \left(\mathbb{E} \left[\int_0^t \|x_r^u - x_r^v\|^p dr \right] + \int_0^t \|u_r - v_r\|^p dr \right) \\ &\leq C_{p,T,K} \left(\mathbb{E} \left[\int_0^t \sup_{0 \leq r' \leq r} \|x_{r'}^u - x_{r'}^v\|^p dr \right] + \int_0^t \|u_r - v_r\|^p dr \right) \end{aligned}$$

where the second inequality follows from (A5). Thus, by Grönwall's inequality,

$$\mathbb{E} \left[\sup_{0 \leq s \leq t} \|x_s^u - x_s^v\|^p \right] \leq C_{p,T,K} \left(\int_0^t \|u_r - v_r\|^p dr \right)$$

for a new constant $C_{p,T,K}$ and all $t \in [0, T]$. Thus, for $p = 2$ and $t = T$,

$$\mathbb{E} \left[\sup_{0 \leq s \leq T} \|x_s^u - x_s^v\|^p \right] \leq C_{T,K} \left(\left(\int_0^T \|u_s - v_s\|^2 ds \right)^{\frac{1}{2}} \right)^{1+\beta}$$

for $\beta = 1$. Then, since \mathcal{U} can be identified with a (compact) subset of \mathbb{R}^{Sm} , we apply Kolmogorov's Lemma [19, Theorem 2.9] and obtain that there exists a modification \tilde{x}^u of x^u whose sample paths are α -Hölder continuous for every $\alpha \in (0, \frac{\beta}{p}) = (0, \frac{1}{2})$. Thus, for every $\alpha \in (0, \frac{1}{2})$, \mathbb{P} -almost-surely, there exists a finite constant $C_\alpha(\omega)$ with

$$\sup_{0 \leq t \leq T} \|\tilde{x}_t^u(\omega) - \tilde{x}_t^v(\omega)\| \leq C_\alpha(\omega) \left(\left(\int_0^T \|u_s - v_s\|^2 ds \right)^{\frac{1}{2}} \right)^\alpha.$$

This shows that up to a modification, \mathbb{P} -almost-surely, the map $x(\cdot, \omega) : \mathcal{U} \rightarrow C([0, T]; \mathbb{R}^n) : u \mapsto x_u(\cdot, \omega)$ is continuous, with the metric topology on $C([0, T]; \mathbb{R}^n)$ induced by the norm $\|x\|_\infty = \sup_{0 \leq t \leq T} \|x(t)\|$ and the metric topology on $\mathcal{U} \cong U^S \subset \mathbb{R}^{Sm}$ for $\|u\| = \left(\int_0^T u^2(s) ds \right)^{\frac{1}{2}}$.

To show that $\mathbb{E}[C_\alpha^2(\omega)] < \infty$, by adopting the notation and results introduced in the proof of [19, Theorem 2.9] (the variables (q, ϵ) in [19] and in the proof below evaluate to $(2, 1)$ in this work), it suffices to prove that $\mathbb{E}[K_\alpha^q(\omega)] < \infty$, where

$$K_\alpha(\omega) \triangleq \sup_{n \in \mathbb{N}} \left(\max_{1 \leq i \leq 2^n} 2^{\alpha n} d \left(X_{\frac{i-1}{2^n}}(\omega), X_{\frac{i}{2^n}}(\omega) \right) \right).$$

To prove that $\mathbb{E}[K_\alpha^q(\omega)] < \infty$, thanks to the monotone convergence theorem, we write

$$\begin{aligned} \mathbb{E}[K_\alpha^q] &= \mathbb{E} \left[\left(\sup_{n \in \mathbb{N}} \left(\max_{1 \leq i \leq 2^n} 2^{\alpha n} d \left(X_{\frac{i-1}{2^n}}, X_{\frac{i}{2^n}} \right) \right) \right)^q \right] \leq \mathbb{E} \left[\sum_{n \in \mathbb{N}} \left(\max_{1 \leq i \leq 2^n} 2^{\alpha n} d \left(X_{\frac{i-1}{2^n}}, X_{\frac{i}{2^n}} \right) \right)^q \right] \\ &= \sum_{n \in \mathbb{N}} \mathbb{E} \left[\left(\max_{1 \leq i \leq 2^n} 2^{\alpha n} d \left(X_{\frac{i-1}{2^n}}, X_{\frac{i}{2^n}} \right) \right)^q \right] \leq \sum_{n \in \mathbb{N}} 2^{q\alpha n} \sum_{i=1}^{2^n} \mathbb{E} \left[d \left(X_{\frac{i-1}{2^n}}, X_{\frac{i}{2^n}} \right)^q \right] \\ &\leq \sum_{n \in \mathbb{N}} 2^{q\alpha n} \sum_{i=1}^{2^n} \left(\frac{1}{2^n} \right)^{1+\epsilon} = \sum_{n \in \mathbb{N}} \left(\frac{1}{2^{\epsilon-q\alpha}} \right)^n = \frac{2^{\epsilon-q\alpha}}{2^{\epsilon-q\alpha} - 1} < \infty \end{aligned}$$

given that $\epsilon - q\alpha > 0$. In this work, $\alpha \in (0, \frac{1}{2})$, so we conclude that $\mathbb{E}[C_\alpha^2(\omega)] < \infty$.

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