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The Direction-Theory of Parallels

Geometry and Philosophy in the Age of Kant

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Abstract. The direction-theory of parallels was a mathematical theory that gained enormous importance and popularity for about a century, from the 1770s to the 1870s. It was conceived for the purpose of proving the famous Parallel Postulate and establishing the foundations of Euclidean geometry. The development of the geometric theory was intertwined with that of mathematical epistemology. Proponents of the theory discussed at length such topics as the analyticity of mathematics, the role of intuition in geometry, mathematical constructivism, and the relationship between geometry and the structure of space. In the first few decades of its life, the direction-theory of parallels became the most important benchmark on which to test Kant's philosophy, and Kantians and anti-Kantians alike wrote articles and books on it. The direction-theory was later generally accepted by the leading post-Kantian philosophers of the nineteenth century. It was finally subjected to fatal criticism by Lewis Carroll and Gottlob Frege.

[Enter:] The phantasm of *Herr Niemand*, carrying a pile of phantom-books, the works of Euclid's Modern Rivals, phantastically bound.
(Lewis Carroll)

§1. *Euclid's Rivals on the Theory of Parallels*

Back from Wonderland, Lewis Carroll published (under his worldly name of Charles Dodgson) a treatise on the foundations of geometry written in “dramatic form” and “lighter style”. *Euclid and His Modern Rivals* appeared in 1879, and lambasted recent treatments of elementary geometry in England, while extolling the logical virtues of the original *Elements* by Euclid. An entire act of this tragedy of geometry is dedicated to the modern theories of parallel lines, which are expounded by the German Herr Niemand (Mr. Nobody) and rebutted by the Greek (and infernal) judge Minos—as a champion of Euclid. The greatest part of this act deals with the so-called “direction-theory” of parallels.¹

¹ C.L. Dodgson, *Euclid and his Modern Rivals*, London, MacMillan 1879. The theory of parallels is discussed in Act Two, the direction-theory being dealt with in pp. 70-131 of the work.

The direction-theory of parallels was, like Herr Niemand, German. It was first conceived in 1778, with the stated aim of reforming Euclid's theory of parallels and proving the Parallel Postulate. This was a famous challenge in geometry, and since Antiquity mathematicians had been busy proposing new demonstrations of the postulate and at disproving other mathematicians' proofs.² The direction-theory fared much better for a long time and acquired great momentum at the turn of the nineteenth century. The theory began to be taught in schools, was endorsed by the most important philosophers, and was generally accepted by the mathematical community. It soon crossed Germany's borders and spread all over Europe. In 1870, the British *Association for the Improvement of Geometrical Teaching* recommended that students in England should learn geometry from modern textbooks rather than from Euclid's *Elements*—as was also the long-held custom in Germany.³ Many of these new English manuals embraced the direction-theory of parallels as a more suitable approach for students than Euclid's.

The discovery of non-Euclidean geometry by Lobachevsky and Bolyai did not slow down this booming phenomenon. The importance of these pioneering works was not recognized for several decades, and it was only in the course of the 1870s that non-Euclidean geometry began to gain wider acceptance—when Klein and Poincaré legitimized it even outside the field of foundational studies.⁴ At the end of the decade, Dodgson's witty drama disposed of this German theory belatedly imported into Britain and defeated by modern mathematics. The direction-theory, that had flourished for 101 years in more than as many books, finally stepped through the looking-glass.⁵

The gist of the direction-theory is easy to convey. Whereas Euclid had defined parallel lines as straight lines that do not meet, the direction-theorists defined them as *straight lines having the same direction*. Sameness of direction is commonly understood as to be a transitive relation: if A has the same direction as B , and B the same direction as C , then A has the same direction as C . This apparently harmless assumption, applied to parallel lines, entails however the transitivity of parallelism—something that fails in non-Euclidean geometry. As a consequence, Euclid's modern rivals surreptitiously introduced an assumption equivalent into the Parallel Postulate in the definition of parallelism, and triumphantly derived the former from the latter. Minos did not need too much effort to expose the blatant *petitio principii* of these proofs, and Dodgson's book could expound the criticism in painful detail.

Five years after Dodgson's rebuttal of the direction-theory, Gottlob Frege produced a logical analysis of its shortcomings in his *Grundlagen der Arithmetik*. Frege identified the main mistake in

² For an outstanding presentation of the history of non-Euclidean geometry, see J. Gray, *Ideas of Space: Euclidean, Non-Euclidean, and Relativistic*, Clarendon Press, Oxford 1989.

³ An exhaustive presentation of the English debate on the teaching of Euclid is found in A. Moktefi, *Geometry: The Euclid Debate*, in R. Flood, A.C. Rice, R. Wilson (eds.), *Mathematics in Victorian Britain*, Oxford, OUP 2011, pp. 320-36. For its connection with non-Euclidean geometry, see J. Gray, *János Bolyai, Non-Euclidean Geometry and the Nature of Space*, Cambridge MA, Burndy 2004, pp. 95 ff.

⁴ On the acceptance of non-Euclidean geometry in the second half of the nineteenth century, see J.-D. Voelke, *Renaissance de la géométrie non euclidienne entre 1860 et 1900*, Bern, Lang 2005, and K. Volkert, *Das Undenkbare denken. Die Rezeption der nichteuclidischen Geometrie im deutschsprachigen Raum (1860-1900)*, Berlin, Springer 2013.

⁵ An important historical discussion of various attempts to prove the Parallel Postulate through arguments from direction, composed just when the era of such attempts was drawing to a close, is to be found in H. Schotten, *Inhalt und Methode des planimetrischen Unterrichts*, Leipzig, Teubner 1890-1893. For further discussions on the *attardés* who were still hoping to prove the Postulate through the direction-theory at the end of the nineteenth century, see J.-C. Pont, *L'aventure des parallèles. Histoire de la géométrie non euclidienne: précurseurs et attardés*, Berne, Lang 1986.

the theory in an incorrect epistemological assumption. According to him, geometry must begin with concrete objects, such as straight lines and circles. Abstract notions, such as direction, cannot be assumed beforehand and employed to define these basic geometric objects. Quite the opposite: abstract notions may only be defined by looking at the relations obtaining between concrete objects. *If* such a relation is an equivalence relation – and, therefore, if it is transitive – *then* it partitions the set of objects into equivalence classes to which an abstract notion may be attached. The transitivity of relations among concrete objects is a prerequisite for abstracting notions in the first place.

The trouble is, that this is to reverse the true order of things. For surely everything geometrical must be given originally in intuition. But now I ask whether anyone has an intuition of the direction of a straight line. Of a straight line, certainly; but do we distinguish in our intuition between this straight line and something else, its direction? That is hardly plausible. The concept of direction is only discovered at all as a result of a process of intellectual activity which takes its start from intuition. On the other hand, we do have a representation of parallel straight lines.⁶

According to Frege, the only possible procedure to abstract the notion of direction is the following. We consider the set of (concrete) straight lines in a plane and Euclid's relation of parallelism (i.e. non-incidence). Since the Parallel Postulate is true, this relation is transitive and in fact an equivalence relation. Therefore, we may partition the set of straight lines into mutually exclusive equivalence classes of non-incident lines. Each of these classes defines a direction as an abstract notion. By means of this procedure, we transform the sentence "line *a* is parallel (non-incident) to line *b*", which is about concrete objects given in intuition, into the statement that "the direction of line *a* is identical to the direction of line *b*", revolving on abstract, non-intuitive notions. From here, we get to the concept of direction in general.⁷

Following Frege's analysis, one cannot even formulate the notion of direction without presupposing the Euclidean transitivity of parallelism and therefore the truth of the Parallel Postulate. The mathematical mistake of the direction-theorists was rooted in a faulty epistemology, which upturned the priority between intuition and concept. Frege's analysis attempted to reach much deeper than Dodgson's, and pointed a finger at the philosophical assumptions of the direction-theorists. In doing so, he exposed his own not inconsiderable share of philosophical commitments.⁸

It is remarkable that neither Dodgson nor Frege rebutted the direction-theory by exploiting the consistency of non-Euclidean geometry. A decade after his book on Euclid, Dodgson himself

⁶ G. Frege, *Grundlagen der Arithmetik*, Breslau, Koebner 1884, § 64, p. 75. Transl. by J.L. Austin in G. Frege, *The Foundations of Arithmetic*, New York, Harper 1960 (2nd ed.).

⁷ This last passage is further belabored by Frege in great detail, but it does not concern us here. Note that Lobachevsky and Bolyai were able to define the direction of asymptotic parallel lines, that is to say, of one special kind of parallels in hyperbolic geometry that have the transitive property.

⁸ Recently P. Mancosu, *Grundlagen, Section 64: Frege's Discussion of Definitions by Abstraction in Historical Context*, "History and Philosophy of Logic", 36 (2015), pp. 62-89, has advanced the conjecture that Frege came to know the direction-theory particularly from his reading of O. Schölmilch, *Grundzüge einer wissenschaftlichen Darstellung der Geometrie des Masses*, Eisenach, Baercke 1849. Among Frege's sources, Karl Georg Christian von Staudt (1798-1867) correctly assumed the Parallel Postulate as an axiom (in the form of Lorenz, see below), and then defined direction starting from (transitive, Euclidean) parallelism: see K.G.C. von Staudt, *Geometrie der Lage*, Nürnberg, Bauer 1847, §§ 35-36, pp. 14-15. Frege had already endorsed a similar procedure in his doctoral dissertation from 1873. See G. Frege, *Über eine geometrische Darstellung der imaginären Gebilde in der Ebene*, in *Kleine Schriften*, ed. I Angelelli, Hildesheim, Olms 1967, pp. 3-4 and 49.

was still attempting to prove the Parallel Postulate and show the impossibility of non-Euclidean geometry.⁹ Frege, committed to making use of intuition in geometry, explicitly rejected non-Euclidean geometry as a delusional, un-scientific theory engendered by an ill-conceived philosophy of mathematics.¹⁰ Dodgson's and Frege's criticisms to the direction-theory only concerned the possibility of defining parallel lines through the notion of direction and to prove the Parallel Postulate from such a definition. Dodgson took the Parallel Postulate as a statement provable by different means, Frege as an indemonstrable axiom given by pure intuition, but both of them firmly believed in its unconditional truth.

The cultural phenomenon of the direction-theory of parallels and its demise poses a double challenge to the historians of mathematics. On the one hand, it is astonishing that such a clearly-faulted theory was accepted and taught for a hundred years. The theory was based on a trivial mistake – assuming the truth of the postulate in the definition of parallels – and it seems that Europe must have been relinquished by all gods if no one noticed such an obvious blunder. On the other hand, the direction-theory was destroyed by the discovery of non-Euclidean geometry, and yet the most important – and very belated – criticisms of it came from two important conservative (even “countermodern”) logicians who opposed such a discovery.

The century in which the direction-theory thrived and waned marked the passage from a pre-modern epistemology to a modern – or modernist – philosophy of mathematics.¹¹ It is from this vantage point, which also people like Dodgson and Frege contributed to shape, that we may look down at the direction-theory as an inexplicable failure of logic and common sense. The direction-theorists, however, were not naïve. Their mathematics was rather based on a quite developed pre-modern epistemology. The direction-theory was elaborated in some crucial decades between the end of the eighteenth century and the beginning of the nineteenth century, in which German mathematicians and philosophers were highly engaged with epistemological questions. The direction-theorists played an important role in this debate, and they were not unaware of the challenges later raised against them by Dodgson and Frege. Only, their pre-modern answers were different from the modern ones.

In the eighteenth century, for instance, the philosopher and mathematician Christian Wolff had argued that the axioms of mathematics should follow from the definitions of the terms involved. Consequently, it was *expected* that the Parallel Postulate could be drawn from a suitable definition of parallel lines. Dodgson's complaint of a *petitio principii* would have been met with puzzlement by many mathematicians endorsing Wolff's views. According to other Leibnizians,

⁹ C. Dodgson, *A New Theory of Parallels*, London, MacMillan 1888. In the treatise, Dodgson provisionally accepted a further “quasi-axiom” in order to prove the Parallel Postulate, admitting to have failed to offer a completely unhypothetical proof of it. He also believed that reducing the proof of the Parallel Postulate to the proof of this provisional axiom may count as a step forward towards a unhypothetical demonstration of it. The 1888 book also offers a further discussion of the direction-theory of parallels in Appendix IV, § 5, and several important remarks on the Parallel Postulate in non-Archimedean planes.

¹⁰ Frege's clearest statements on this topic are in his unpublished *Über Euklidische Geometrie*. In it, he states that no one can serve two masters (Euclidean and non-Euclidean geometry), and concludes that non-Euclidean geometry is to be listed among the non-scientific disciplines having only historical interest—like astrology and mummies. See G. Frege, *Nachgelassene Schriften*, eds. H. Hermes, F. Kambartel, F. Kaulbach, Hamburg, Meiner 1983 (2nd ed.), pp. 182-84.

¹¹ The topic of modernism in mathematics is wonderfully addressed in one of Jeremy Gray's most fascinating books: J. Gray, *Plato's Ghost. The Modernist Transformation of Mathematics*, Princeton, PUP 2008. The notion of countermodernism in mathematics was famously introduced by H. Mehrtens, *Moderne Sprache, Mathematik: Eine Geschichte des Streits um die Grundlagen der Disziplin und des Subjects formaler Systeme*, Frankfurt, Suhrkamp 1990, and is discussed at length by Gray.

intuition should play no role in the foundations of geometry, and it was nonsense to claim that the abstract notion of direction should follow the concrete intuition of parallel straight lines. Quite the opposite: the concept of direction is *simpler* than, and should therefore precede, that of a straight line. Frege's objection would have appeared to these philosophers as a serious epistemological error.

These examples show that in order to understand why the direction-theory was so widely upheld for an entire century, we have to dive deeper into the philosophy of mathematics that engendered it. Similarly, in order to appreciate its demise, that did not happen for purely mathematical reasons (the discovery of non-Euclidean geometry), we should follow the development of the nineteenth-century epistemology of mathematics up to the point in which it could no longer back the assumptions of the theory. The philosophy of Kant played an important role in this debate, since the direction-theory was born in the age of Kant and was immediately drawn into the disputes on transcendentalism. Kant's shadow projected however much further into the following century, and Frege's and Dodgson's remarks on the syntheticity of geometry and the role of intuition still depended on their reading of the *Critique*.¹² In short, the history of the cultural phenomenon of the direction-theory may only be written together with a history of the philosophy of mathematics in the eighteenth and nineteenth centuries.

In the present essay, I will not attempt this much. I rather concentrate on the first period of the direction theory, roughly covering the fifty years from 1778 to the discovery of non-Euclidean geometry. In these decades the theory was still confined to Germany and the debate surrounding it was sensitive to the important transformations of the philosophical landscape. Historians have not yet explored this early history of the theory—in fact they have not even recognized it as a topic of investigation. The available narratives on the subject (including Dodgson's own history of Euclid's rivals) generally concentrate on the second half of the nineteenth century, when the direction-theory already had its current name and final shape. This may convey the false impression that the theory was originally conceived this late.

An enquiry into the sources of the German and British direction-theorists of the nineteenth century, however, discloses another story—of which the present essay offers a first sketch. Under the name of a theory of *Lage* or *situs*, the “direction-theory” (as it was later called) was born much earlier and had slowly transformed into its nineteenth-century counterpart. The appreciation of this fact is not only relevant in offering a more exact genealogy of the theory. Rather, it offers a rationale for its invention, since it can be shown that the direction-theory is deeply rooted in the logical and epistemological discussion that took place in Germany in the last three decades of the eighteenth century. Without this early history, the reasons for the success and demise of the direction-theory are destined, I claim, to remain unfathomable.

In §2 of the present essay, I detail the background of the direction-theory and its roots in Leibniz' program of an *analysis situs*. In §3, I deal with Karsten's invention (1778) of the direction-theory in the context of a broadly Wolffian epistemology. In §4, I mention the first reactions to Karsten's theory and in particular Hindenburg's own attempt and his philosophical qualms on intuition. In §5, I briefly discuss the impact of Kant's new philosophy of mathematics on the direction-theory, and the beginnings of the analyticity-syntheticity debate. In §6, I introduce Schwab's theory, that soon became the standard view of the subject. In §7, I briefly mention the reception of the direction-theory among later philosophers and its general acceptance in Germany.

¹² Frege's commitment to (broadly) Kantian views on geometry in the *Grundlagen* are well-known; see the classic paper by M. Dummett, *Frege and Kant on Geometry*, “Inquiry” 25 (1991), pp. 233-254. Dodgson quoted with approval Kant's *Critique* in order to rebut the ideas of Herr Niemand; see *Euclid and his Modern Rivals*, p. 55.

§2. Leibniz, Kästner, and the Analysis Situs

The direction-theory of parallel lines did not declare its name before its full development. The early theorists did not put any special emphasis on the notion of direction (*Richtung*) and rather defined parallels through the concept of *situs* (German: *Lage*). Over the years, the term ‘*Richtung*’ became more and more common—to the point that it gave the name to the whole theory in the 1810s.¹³

I am not aware of any important theory of parallel lines grounded on the notion of direction (or *situs*, or similar) predating the German attempts in the eighteenth century. The only exception is possibly offered by the German mathematician Nikolaus Kauffmann (Latin: Mercator, 1620-1687), who in 1678 published a reworking of Euclid’s *Elements*. In it, Mercator took the lead from Euclid’s definition of an angle as the *inclination* (κλίσις) between two lines, to define parallel lines as lines that *do not incline* the one towards the other. Following this definition, Mercator states an axiom to the effect that if two lines incline in the same way towards a third, they do not incline the one towards the other (i.e. are parallel). From this axiom he easily proves *Elements* I, 30 (the transitivity of parallelism) and from this all other standard properties of parallel lines in Euclidean geometry.¹⁴ Mercator’s attempt has many points in common with the further developments of the direction-theory but I have not been able to trace a direct filiation from these 1678 *Elementa* to the German theory of parallels presented a century later.¹⁵

By contrast, an explicit, and yet fully fabricated, filiation may be found between the direction-theories and the geometrical essays of Gottfried Wilhelm Leibniz (1646-1716). Leibniz worked for his entire life on a new mathematical theory, that he called *analysis situs*, aimed at grounding all geometry on the notion of “situation”. Leibniz did not publish any of the many essays that he wrote on the subject. In the first half of the eighteenth century, however, a few letters that he had sent to Huygens and Johann Bernoulli were published, and the German world was informed of Leibniz’ grand geometrical project—a lost science which left no traces. The imagination of several mathematicians was tickled, and in the course of the following two centuries many different mathematical endeavors (combinatorial geometry, vector calculus, projective geometry, topology) were developed under the name of Leibniz’ mysterious *analysis situs*. The theory of parallels made no exception, and a long-lasting narrative was engendered, according to which Leibniz had in fact embraced a direction-theory of parallels.¹⁶

¹³ The notion of direction was very much open to debate in the first half of the seventeenth century, and a definition of parallelism through direction was no simple matter, as it involved several commitments on the nature of space as an “affine” (rather than centered) structure. On the cosmological debate that brought the notion of direction to the core of the Copernican Revolution, see D.M. Miller, *Representing Space in the Scientific Revolution*, Cambridge, CUP 2014.

¹⁴ Mercator, *Euclidis elementa geometrica novo ordine ac methodo fere demonstrata*, London, Martyn 1678. The definition of parallelism is on p. 2. The axiom is the third one on the same page, and the easy proof of *Elements* I, 30 happens as Theorem 7 on p. 7.

¹⁵ A few decades later, Edmund Scarburgh also talked about the “Tendency and Inclination towards one another” of non-parallel lines, even though he did not rely on transitivity to prove the Postulate (*The English Euclide*, Oxford, Theatrum Sheldonianum 1705). This could be a further source for the British direction-theorists of the nineteenth century.

¹⁶ For a late assessment of Leibniz’s theory of parallels as a theory of direction, see W. Killing, *Einführung in die Grundlagen der Geometrie*, Paderborn, Schöningh 1893-1898, vol. 1, p. 5.

Unbeknownst to the direction-theorists, Leibniz had indeed strived to prove the Parallel Postulate through his *analysis situs*. Among his unpublished papers, now preserved in the *Leibniz-Archiv* in Hannover, we find many different attempts at establishing Euclidean geometry.¹⁷ Several of these attempts are quite ingenious, and the notion of *situs* is employed in them in unexpected ways. None of them, however, ever attempted to define parallel lines as lines having the same situation, or to assume the transitivity of parallelism through the notion of situation (or direction). The standard references made by the direction-theorists to Leibniz' *analysis situs* were therefore entirely unwarranted.

While Leibniz' geometry had no authentic impact on the eighteenth-century theory of parallels, his epistemology exerted an important influence on the debate on the theory of direction. Several generations of German philosophers and mathematicians (and especially Christian Wolff, 1679-1754) shared Leibniz' views on philosophy of science. Leibniz' idea that all axioms of geometry could be proven starting from the definitions of the terms employed, had a pivotal role in orienting the German debate towards the search for a definition of parallel lines that could improve on Euclid's. Leibniz' ideas on the analyticity of truth brought mathematicians to disregard intuition in geometry. Leibniz' insistence that geometry is the science of space (rather than the science of the individual figures in it), and that space is a complex structure of situational relations, offered a completely new perspective on the meaning of the Parallel Postulate. In short, Leibniz remained a looming figure throughout all the discussions on the theory of parallels that took place in Germany in the crucial years 1770-1830. The great majority of the direction-theorists styled themselves as Leibnizians.

We can follow in detail how Leibniz' heritage inspired the birth of the direction-theory. In the second half of the eighteenth century, the most famous mathematician in Germany was a devoted Leibnizian: the Göttingen professor Abraham Gotthelf Kästner (1719-1800). After having attempted to prove the Parallel Postulate himself for many years, Kästner became disillusioned with obtaining a demonstration of it with standard mathematical tools. He collected a large number of treatises dealing with parallels, and turned with great hopes to Leibniz' analysis of situation.¹⁸ One of his students, Georg Simon Klügel (1739-1812), defended a dissertation in which he took advantage of his professor's library, and expounded a good number of failed attempts at proving the Postulate—including several by living mathematicians. This dissertation, the *Conatuum praecipuorum theoriam parallelarum demonstrandi recensio*, was published in 1763 and attracted considerable attention. Kästner added a note at the end of his student's book:

I hardly hope that we will ever obtain the true demonstration [of the Parallel Postulate] – of which you, bringing the light of geometry, have vanished the specters – unless we cultivate more assiduously the theory of *situs*, the analysis of which perished with Leibniz.¹⁹

¹⁷ Leibniz' papers on the theory of parallels have been published in V. De Risi, *Leibniz on the Parallel Postulate and the Foundations of Geometry*, Basel/Boston, Birkhäuser 2015.

¹⁸ See the preface to G.A. Kästner, *Anfangsgründe der Arithmetik, Geometrie, ebenen und sphärischen Trigonometrie und Perspectiv*, Göttingen, Vandenhoeck 1758, as well as §§ 27-28, pp. 13-14, on the provability of axioms from definitions. On Kästner's involvement with the Parallel Postulate, see W.S. Peters, *Das Parallelenproblem bei A.G. Kästner*, "Archive for History of Exact Sciences", 1 (1962), pp. 480-87.

¹⁹ Klügel, *Recensio*, p. 33: "Habituros nos aliquando, veram eam cuius admoto geometriae lumine spectra dissipasti demonstrationem, vix speraverim nisi diligentius exculpta doctrina situs, cuius analysis cum Leibnitio interiit".

Kästner also remarked that since all axioms of mathematics are grounded on, and provable from, the definitions of the terms, the major challenge for a theory of parallels was to find a new and appropriate definition of parallel lines.²⁰

The gauntlet had been thrown in the name of Leibniz' *analysis situs*.

§3. Karsten and the Birth of the Direction-Theory

The two main living mathematicians criticized by Klügel were Wenceslaus Johann Gustav Karsten (1732-1787), a professor at Bützow; and János András Segner (1704-1777), a Hungarian scholar who had been the first professor of mathematics at the University of Göttingen (and Kästner's predecessor in that chair) and later moved to Halle.

Segner had employed the notion of *Lage* (or *situs*) in his theory of parallels, but had made no use of it in his proof of the Postulate in his 1756 *Cursus mathematicus*.²¹ Segner's proof implicitly assumed that any straight line, passing through a point inside an angle, cuts this angle. This principle is in turn a reformulation of the transitivity of parallelism, itself equivalent to the Parallel Postulate, and in hyperbolic geometry a straight line may be entirely contained within an angle. Klügel was not able to pinpoint Segner's mistake, and concentrated on a marginal issue of no consequence for the demonstration.²² Segner may have taken note of Klügel's criticism, since in 1764 he slightly revised his proof in the German translation of the *Cursus*; but since the criticism was incorrect, the new demonstration did not fare any better.²³ The first complete explanation of Segner's *petitio principii* was offered by Johann Friedrich Lorenz (1738-1808) only in 1791.²⁴

Karsten, by contrast, was struck hard by Klügel's *Recensio*. In 1758, he had published a work in which he accepted Segner's proof of the Postulate. By 1760, however, he had also offered an alternative demonstration of it, that was loosely inspired by a standard (and faulty) proof offered

²⁰ Kästner, however, seems to have thought that the culprit here was Euclid's famously obscure definition of a straight line, rather than Euclid's definition of parallels. Accordingly, Kästner did not attempt himself to define parallel lines through the notion of direction. See for instance his statement from 1790: "Der Grund, warum man in diesem Axiome [the Parallel Postulate] nicht die Evidenz der übrigen findet, ist ... daß man von der geraden Linie nur einen klaren Begriff hat, nicht einen deutlichen" (G.A. Kästner, *Über den mathematischen Begriff des Raums*, "Philosophisches Magazin", II, 4 (1790), pp. 403-19; p. 414). Accordingly, Kästner himself did not attempt to define parallel lines through the notion of direction.

²¹ The reference to *Lage* in relation to the theory of parallels occurred already in 1747: see J.A. Segner, *Vorlesungen über die Rechenkunst und Geometrie*, Lemgo, Meyer 1747, p. 218; in this treatise there is a rather naïve proof of the Parallel Postulate. Segner's more mature proof of the postulate was expounded in his Latin treatise on mathematics: J.A. Segner, *Cursus mathematici pars I. Elementa arithmeticae, geometriae et calculi geometrici*, Halle, Renger 1756 (in § 11 of the *Geometria*, p. 144, is Segner's assumption on angles; the proof of the Parallel Postulate, depending on such an assumption, is in § 31, pp. 150-51).

²² Klügel, *Recensio*, § 11, pp. 15-16.

²³ J.A. Segner, *Anfangsgründe der Arithmetik, Geometrie, und der geometrischen Berechnungen*, Halle, Renger 1764; § 259, pp. 198-200. The preface to this work is dated 1763, and it is not obvious that Segner had read Klügel's *Recensio* at the time. Note that Segner published in 1767 a second edition of his 1747 treatise without relevant changes to the simple proof expounded there.

²⁴ Lorenz' aim in this book was to vindicate Segner by making explicit the hidden assumptions in his work. Therefore, he assumed the principle on the straight line inside an angle as an *axiom* that he thought to be self-evident and much clearer than the original postulate of Euclid. See J.F. Lorenz, *Grundriß der reinen und angewandten Mathematik, oder der erste Cursus der gesamten Mathematik*, Helmstädt, Fleckeisen 1791, where the new principle is stated in § 44 of the *Planimetrie* (pp. 102-103), and the Parallel Postulate is proven, as Prop. 10, a few pages later (§ 83, pp. 118-21).

by the Persian mathematician Nasīr al-Dīn al-Tusi in the thirteenth century, which had later become commonplace in the European literature on the subject.²⁵ Klügel criticized this latter proof with valid reasons, and Karsten realized that he needed a different demonstration.²⁶ In 1778, Karsten was called to Halle to take Segner's chair, and he took this momentous occasion to give an inaugural speech on the theory of parallels—which was published as a *Versuch einer völlig berichtigten Theorie von den Parallellinien*. In this important paper, Karsten publicly recognized that Klügel's "*bekannte Disputation*" has disproven his and Segner's proofs, and accepted Kästner's suggestion of developing Leibniz' theory of *Lage* in order to ground a novel approach to the theory of parallels.

Karsten proposed a new definition of parallel lines that he intended to supplant Euclid's. To this effect, Karsten introduced the notion of the *Lage* of a straight line. He claimed, with Leibniz, that a general definition of *Lage* cannot be given, as this is one of the most basic notions that we make use of in geometry: a *simple concept*, in fact, that admits of no conceptual analysis. This notwithstanding, it is possible to give an implicit definition of the *Lage* of a straight line by stating the conditions under which two straight lines have the same *Lage*. Needless to say, the relation of the sameness of *Lage* is called *parallelism*, and two lines having the same *Lage* may be called *parallel lines*. Karsten made a comparison with the simple notion of a magnitude, which cannot be explicitly defined either, but may be implicitly defined through congruence (i.e. sameness of magnitude).²⁷ We can note that Karsten's argument is not much different from Frege's introduction of the notion of direction by abstraction, even if their epistemological motivations were completely different.

The sameness of *Lage*, according to Karsten, may be rigorously constructed in geometry. This is the core of Karsten's theory, which is neither a purely analytic discussion based on definitions, nor a mere appeal to the intuition of parallel lines.

Karsten writes that the relative position (the *Lage*) of two intersecting straight lines is expressed by their angle of incidence (as in Mercator's theory, mentioned above). From this it follows that each of two straight lines which form equal angles with a common transversal has the same *Lage* with respect to this transversal. So far, so good; but then Karsten changed the relation of "having the same *Lage* with respect to a third line" into that of "having the same *Lage*" *überhaupt*. That is to say, he claimed that two straight lines which form equal angles with a transversal have the same *Lage* with respect to one another, irrespective of the specific transversal that has been used to establish this relation. Thanks to this unjustified assumption, Karsten claimed that two straight lines which have the same *Lage* with respect to a certain transversal, will also have the same *Lage* with respect to any other transversal. Given Karsten's definition, this amounts to saying that two straight lines forming equal angles with one transversal will also form equal angles with any other

²⁵ The first proof is in W.J.G. Karsten, *Praelectiones matheseos theoreticae elementaris*, Rostock, Berger 1758; Segner's assumption on angles is here accepted in § 73 (pp. 34-35), and the Parallel Postulate is proven in § 76, p. 36. The second proof is in W.J.G. Karsten, *Mathesis theoretica elementaris atque sublimior*, Rostock, Röse 1760, § 91, pp. 31-35. Al-Tusi's proof had been expounded by Clavius, Wallis, Arnauld, and others, and was well-known in the eighteenth century.

²⁶ Klügel, *Recensio*, §§ 11-12, pp. 15-17, on Karsten's first proof from 1758; and § 8, pp. 12-13, on Karsten's demonstration from 1760.

²⁷ Even though Karsten is not quoting Leibniz explicitly, he is clearly drawing on the point made at § 47 of Leibniz's *Fifth Letter* to Samuel Clarke (1716), in which Leibniz claims that one cannot define the notion of "place" but only the notion of "having the same place" (see A. Robinet, *Correspondance Leibniz-Clarke*, Paris, PUF 1957, pp. 142-45).

transversal. This is a false claim in hyperbolic geometry, and one from which one can deduce the Parallel Postulate—as Karsten himself did in the subsequent pages.²⁸

It should be remarked that Karsten himself did raise a few doubts about the soundness of his proof and was not completely satisfied with it. He claimed that the unanalyzability of the notion of *Lage* forces us to accept several principles which immediately flow from the nature of this simple notion. In this respect, Karsten introduced some latitude into the rather severe epistemology developed by Leibniz and Wolff.²⁹

Karsten restated his theory without relevant changes in the 1780 edition of his textbook of elementary mathematics, and again in 1786, in an extended and (slightly) improved version, as an essay *Von den Parallellinien* included in his monumental *Mathematische Abhandlungen*.³⁰ In the following years, Karsten's books enjoyed a large diffusion, and the direction-theory became well-known in the German world. We find traces of Karsten's theory even in technical schools and in the teaching at military academies.³¹

Karsten did not call his own theory a “direction-theory” of parallels, even though he often equated the notion of *Lage* with that of the *Richtung* (direction) of the straight lines.³² There can be no doubt, however, that in his inaugural lecture of 1778 he exploited for the first time a line of argument that in the following years transformed into the fully-fledged theory of direction later criticized by Dodgson and Frege.

²⁸ Dodgson did not refer to Karsten (nor to other earlier German authors) in the course of his books, but in his own attempt at proving the Parallel Postulate (*A New Theory*, p. 69) he was crystal clear that the notion of the sameness of direction is in fact equivalent to the property that two parallels make equal angles with any transversal whatsoever—which is in fact Karsten's unwarranted assumption. As we know today, the latter assumption is only equivalent to the Parallel Postulate if we also accept the Axiom of Archimedes.

²⁹ Karsten's epistemology had already been stated in abridged form in his 1778 essay, where he explicitly claimed that his first six propositions are to be considered as axioms (*Versuch*, pp. 14 and 16). He returned to the question, however, in §§ 1-16 of his essay *Von den Parallellinien* (*Mathematische Abhandlungen*, Halle, Renger 1786, pp. 115-30), which is a short essay in Wolffian epistemology—with a twist. In the latter essay, it is pretty clear that Karsten was not thinking about assuming the propositions on the *Lage* of straight lines as authentically unprovable axioms, but rather as statements that accept some kind of proof (*Beweis*) or at least of an explanation or an exhibition in a figure. He had no concept of different axiom systems, and believed that everything about Euclidean geometry could be justified in one way or another.

³⁰ W.J.G. Karsten, *Anfangsgründe der mathematischen Wissenschaften*, Greifswald, Röse 1780, pp. 383-414; the 1768 first edition did not include a theory of parallels. W.J.G. Karsten, *Von den Parallellinien*, in *Mathematische Abhandlungen*, pp. 113-202.

³¹ For instance, the military textbook of F. Meinert, *Lehrbuch der gesammten Kriegswissenschaften für Officiere bei der Infanterie und Kavallerie: zweiter Theile, Gemeine Geometrie*, Halle, Hemmerde 1790, briefly mentions many attempts to prove the Parallel Postulate but states that Karsten's theory is the best one, and “Die vorstehende Theorie der Parallellinien ist völlig die Karstensche [...]. Fast sollte man glauben, wenn diese noch nicht alle Schwierigkeiten gehoben hat, daß sie schwerlich durch Hülfe der Elementargeometrie gehoben werden können. Auf die ausübende Mathematik haben die bisher erregten Zweifeln gegen die völlige Nichtigkeit der euklideischen Theorie der Parallellinien keinen Einfluß” (pp. 59-60; I thank Thomas Morel for pointing out this passage to me). But even a more theoretical work (even though practically oriented) such as G.G. Schmidt, *Anfangsgründe der Mathematik zum Gebrauch auf Schulen und Universitäten*, Frankfurt a.M., Barrentrapp 1797, still followed Karsten's proof (cf. pp. 132-34).

³² This is clear already in Karsten's essay from 1778. It should be noted, however, that the notion of *Richtung* is especially used by him in the later presentation of his theory, to be found in his *Mathematische Abhandlungen* from 1786. Here he still distinguished between *Lage* and *Richtung*, in the weak sense that in lines with the same situation two different directions may be spelled out (say, toward the left or the right side); more often, however, he simply wrote “*Lage oder Richtung*” as synonyms.

§4. Hindenburg and Transitivity

After Karsten, the most important step forward in the direction-theory of parallelism was made by the Leipzig professor Carl Friedrich Hindenburg (1741-1808), better known as the leading figure of the German group of mathematicians working on combinatorics.³³ In his 1781 *Neues System der Parallellinien*, Hindenburg made good use of a few of Karsten's ideas and attempted to prove the Parallel Postulate through a different route.³⁴

Hindenburg's epistemology was also different from Karsten's. He was critical towards the Leibnizian tradition in logic, and rejected the idea that an axiom should be proven from definitions.³⁵ This notwithstanding, Hindenburg appreciated the theory of *Lage*, and agreed with Kästner and Karsten that a fully developed *analysis situs* was required to handle the theory of parallels. The latter theory, however, had to be freed from the constraints of Leibniz' epistemology and developed through constructive, ruler-and-compass constructions.

Hindenburg maintained that geometry has two parts: one, which had been developed since ancient times, dealing with magnitudes; and another, instantiated by his own studies on combinatorics, dealing with *Lage*. Karsten had failed to keep these two branches apart and committed a *μετάβασις εἰς ἄλλο γένος* of sorts by defining the *Lage* of a straight line through a reference to the *magnitude* of an angle. The geometry of *Lage*, and the theory of parallels in particular, should be developed without any recourse to the notion of quantity. Accordingly, Hindenburg put much emphasis on the transitivity of parallelism, that seemed to him a purely situational (non-metric) property.

In sum, Hindenburg's program in the theory of parallels aimed to offer a geometrical (rather than a merely logical) and non-metric demonstration of the transitivity of parallelism.

Hindenburg considers two straight lines, *a* and *b* both parallel to a third line *c*, and attempts to prove that *a* and *b* are also parallel to one another. The demonstration is articulated in two different cases, depending on the position of line *c*. In Case One, lines *a* and *b* lie on different sides of the common parallel *c*. Hindenburg disposes of this case quickly, stating that if *a* and *b* were not parallel, the straight line *c* would meet one of them in the direction of their intersection, against the hypothesis. In the more difficult Case Two, *a* and *b* are both on the same side of the common

³³ For a biographical sketch of Hindenburg and a detailed analysis of his involvement with German combinatorics (as well as his relation with Leibniz' thinking), see the recent E. Noble, *The Rise and Fall of the German Combinatorial Analysis*, Cham, Birkhäuser 2022.

³⁴ C.F. Hindenburg, *Über die Schwürigkeit bey der Lehre von den Parallellinien. Neues System der Parallellinien*, "Leipziger Magazin zur Naturkunde, Mathematik und Oekonomie", 1781, pp. 145-68; *Anmerkungen über das neue System der Parallellinien*, *ibid.*, pp. 342-71. The first section, *Über die Schwürigkeit*, is a collection of criticisms of previous proofs of the Parallel Postulate (Hindenburg knew Klügel's *Recensio*); the second section, *Neues System*, contains Hindenburg's own proof; and the third section, *Anmerkungen*, published in a later issue of the journal (still in the same year) replies to a few criticisms that had been levelled against the *Neues System* in a review which had appeared in the "Königsbergsche Gelehrte und Politische Zeitungen".

³⁵ It is possible that Hindenburg's epistemological ideas were indebted to Johann Heinrich Lambert (1728-1777), who had sharply criticized Wolff's abuse of definitions in mathematics. Hindenburg had been a correspondent of Lambert in the latter's last year of life; their letters are to be found in Lambert, *Briefwechsel*, vol. 5.1, pp. 137 ff., and do not concern geometry. In the *System*, Hindenburg explicitly quotes both Lambert's *Briefwechsel* (whose first volume had been published by Johann III Bernoulli in the same year 1781) and the *Neues Organon*. Later on, in 1786, Hindenburg published for the first time Lambert's important *Theorie der Parallellinien*, written in 1766 and left by him in manuscript form. On Lambert, see J. Gray, L. Tilling, *Johann Heinrich Lambert, mathematician and scientist, 1728–1777*, "Historia Mathematica", 5 (1978), pp. 13-41.

parallel c , and Hindenburg builds a complex network of logical implications to bring back this case to the previous one, thus proving the theorem.

The system of logical implications underpinning Case Two was seen as the most problematic part of Hindenburg's proof. Several German mathematicians levelled objections to the logical form of the demonstration, and Karsten himself was among its fiercest opponents.³⁶ Hindenburg got caught in a difficult logical controversy. He published an essay *Noch etwas über die Parallellinien* (1786), almost entirely dedicated to the matter, and in the course of the debate he became convinced that the peculiar logical reasoning employed in Case Two was a form of *consequentia mirabilis*, a correct inferential scheme that had been employed in the theory of parallels by Gerolamo Saccheri in 1733.³⁷

This entire logical dispute eventually proved to have been in vain. Even though no mathematician recognized it for many years, the problem of Hindenburg's proof was not in the difficult Case Two, but rather in the "self-evident" Case One—that no one discussed. In it, Hindenburg tacitly assumed that the straight line c cannot be entirely contained within the angle formed by a and b (if they are not parallel), and this assumption is equivalent to the Parallel Postulate. In fact, this was just Segner's presupposition in his faulty proof from the 1750s.

While Hindenburg's proof was believed wrong for the wrong reason, his engagement in a logical dispute showed how the alleged superiority of his mathematical method over Karsten's more logical approach was an illusion. As a matter of fact, Karsten "analytic" proof of the Parallel Postulate was much more constructive and geometrical than Hindenburg's "synthetic" demonstration. The Karsten-Hindenburg dispute was the prelude to a much wider debate on the role of intuition in geometrical proofs.

§5. Kant and the Analyticity Debate

In the same year, 1781, in which Hindenburg published his essay on parallel lines, Immanuel Kant (1724-1804) printed the first edition of his *Critique of Pure Reason*. In the following ten years, the philosophical landscape of Germany was completely transformed, and the debate on the analyticity and the syntheticity of mathematics came to the forefront. Kant's statements on synthetic *a priori* judgments and space as a pure intuition engendered a number of reactions. Leibnizian philosophers restated that no recourse to intuition was needed to prove mathematical theorems, and attempted to offer purely analytic demonstrations on the theory of parallels. Geometry was taken as the benchmark of Critical philosophy, and the direction-theory as the crowning effort in the foundations of mathematics. At times, it seemed that Kant's whole philosophy had to stand or fall according to whether the direction-theory itself fell or stood.

Kant was aware of Karsten's attempts to develop a direction-theory of parallels, and in private correspondence and notes, he explicitly criticized his analytic approach to geometry. In particular, Kant complained that the notion of direction (*Richtung*) cannot be defined without the concept of a straight line, and that, therefore, one cannot define straight lines (or parallel straight lines) through

³⁶ See Karsten, *Von den Parallellinien*, §§ 33-45 (*Mathematische Abhandlungen*, pp. 145-162).

³⁷ Hindenburg's defense of his Case Two had already begun in his *Anmerkungen* from 1781, which are almost entirely devoted to this purpose, but the main efforts in this direction came in the second section of C.F. Hindenburg, *Noch etwas über die Parallellinien*, "Leipziger Magazin für reine und angewandte Mathematik", 1786, pp. 359-404. On the history of the *consequentia mirabilis* in geometry, see G. Saccheri, *Euclid Vindicated from Every Blemish*, ed. V. De Risi, Basel/Boston, Birkhäuser 2014.

direction—a remark, which is not much different from Frege’s later claim on the same subject.³⁸ Kant agreed that the notion of *Lage* and Leibniz’ famous *analysis situs* may perhaps be employed to prove the Parallel Postulate, but was skeptical on a non-metric approach to the notion of parallelism.

In the 1780s, Kant himself attempted to prove the Parallel Postulate, and seems to have arrived at the conclusion that no synthetic *a priori* proof of it can be given. This was not, however, a statement of absolute indemonstrability, and much less an opening towards non-Euclidean geometries. Kant thought, on the contrary, that a purely analytic proof of the Postulate (that he called a “philosophical proof”) could be obtained starting from a viable definition of parallel lines. This proof could be an example of Leibniz’ lost *analysis situs*, here interpreted as a metric theory, but Kant’s reflections on the topic are unfortunately too brief and obscure to offer a perspicuous mathematical meaning. In any case, given the epistemology professed in the *Critique*, and the criticisms that Kant moved to Karsten’s approach, the recourse to a philosophical proof to prove a mathematical theorem is quite an astonishing claim. Kant never published his thoughts on parallels and his tentative demonstration remained buried among his private notes.³⁹

But while Kant did not publicly express his views on the theory of parallels, this was repeatedly done by the mathematician Johann Friedrich Schultz (1739-1805), who was a friend and follower of Kant, as well as one of the most prolific authors of proofs of the Parallel Postulate. In 1784, Schultz published his first important treatise on the topic, the *Entdeckte Theorie der Parallelen*, in which he expounded a theory based on the manipulation of infinite magnitudes that Kant himself found dubious and untenable. In the same essay, however, Schultz also strongly criticized both Karsten’s and Hindenburg’s attempts on the basis of Kant’s philosophy of mathematics. He claimed that their proofs were entirely grounded on the *analysis* of the notion of *Lage*, and merely deduced from a definition what they had themselves introduced in it. By contrast, geometry should be grounded on synthetic *a priori* judgments, which are the only ones that can actually extend our knowledge.⁴⁰

³⁸ The context of Kant’s claim is a reply to the philosopher Salomon Maimon (1753-1800). In his *Versuch über die Transzendentalphilosophie* from 1790 (pp. 65-68), Maimon endorsed a broadly Leibnizian epistemology of mathematics with several Kantian nuances. He attempted to show that the proposition stating that a straight line is the shortest line between two points, which Kant had famously claimed to be synthetic and thus irreducible to the definition of a straight line (*KrV*, B16), could in fact be proven by conceptual analysis alone. To this effect, Maimon defined a straight line as a line such that every part of it has the same direction or *Lage*. Maimon had the opportunity to send the draft of his philosophical essay to Kant himself, who replied by criticizing Maimon’s definition (Kant to Herz, May 26th, 1789; in *KgS* 11, pp. 48-54). In the very same years of Dodgson’s criticisms of the direction-theory, Kant’s opinion on direction was endorsed by Helmholtz in his 1878 presidential speech at the University of Berlin on *Die Tatsachen in der Wahrnehmung*. See H. von Helmholtz, *Schriften zur Erkenntnistheorie*, eds. M. Schlick and P. Hertz, Berlin, Springer 1921, p. 182.

³⁹ Kant himself attempted to prove the Parallel Postulate, making use of the metrical notion of “equidistance” and was convinced that this was the only viable way to address the issue. In this connection, he mentioned the *Geometrie der Lage*. See the *Reflexionen* nn. 8-10 in *KgS* 14, pp. 33-51, probably dating from 1784-1790. Kant’s standard distinction between philosophical proofs and mathematical demonstrations is to be found (among other places) in *KrV*, A734-35/B762-63. On the topic, see V. De Risi, *La dimostrazione kantiana del Quinto Postulato*, in S. Bacin, A. Ferrarin, C. La Rocca, M. Ruffing (eds.), *Kant und die Philosophie in weltbürgerlicher Absicht*, Berlin, de Gruyter 2013, vol. 5, pp. 31-43; and J. Heis, *Kant on Parallel Lines. Definitions, Postulates, and Axioms*, in C. Posy and O. Rechter (eds.), *Kant’s Philosophy of Mathematics*, Cambridge, CUP 2020, pp. 157-80.

⁴⁰ See for instance the appreciation of Kant’s mathematical epistemology in J. Schultz, *Entdeckte Theorie der Parallelen, nebst einer Untersuchung über den Ursprung ihrer bisherigen Schwierigkeit*, Königsberg, Kanter 1784, pp. 27-28, and Schultz’ criticisms of Karsten and Hindenburg in pp. 31-65. Discussing the analyticity of Karsten’s proof,

Karsten, who honestly believed that any genuine geometrical theory must be analytic, did not reply to Schultz' criticisms. He did state, however, that he could not accept Schultz' demonstration, since arguments employing the infinite can only provide probable conclusions. Any strict mathematical proof should be analytic and finitistic.⁴¹

Hindenburg, on the other hand, was outraged by Schultz' allegations, which were not very different from those he himself had raised against Karsten. This time, he defended Karsten against Schultz, claiming that Karsten's new principles on the theory of *Lage* were to be understood as true synthetic axioms rather than analytic consequences of the definition of *Lage*. Karsten had not proven these principles, Hindenburg added, but he himself had done so in a geometric way: thus synthetically proving the Parallel Postulate and securing the direction-theory. Hindenburg also retorted Schultz' accusations, and stated that a theory of parallels like the one Schultz was advocating – employing the notion of infinite magnitudes – was wholly philosophical and un-mathematical. *This* was an analytic theory if there ever was one.⁴²

Finally, Hindenburg dragged Kästner into the dispute, by publishing a letter of his in which Kästner criticized a proof of the Parallel Postulate advanced by the Swiss mathematician Louis Bertrand (1731-1812), that was very similar to Schultz'. Pushed by the events, Kästner took up the pen himself and, as a good Leibnizian, strongly criticized Schultz' theory of parallels as a monstrosity and restated, against Schultz' master Kant, that geometry is wholly analytic.⁴³

Kant was unhappy to have been drawn into the fight. He had avowed Schultz' theory of parallels and did not want to be attacked by a famous mathematician such as Kästner. In 1790, he wrote a dense and deep reply to him, that he transmitted to Schultz with the request of publishing it in his name. Schultz did publish Kant's text, but could not help adding a conclusion with a further endorsement of the theory of infinite magnitudes.

Kant was extremely disappointed by Schultz' insubordination, as he feared a further rejoinder by Kästner. When it did not arrive, and he and Kästner exchanged polite private letters, Kant hoped – for a moment – to have settled the matter.⁴⁴

He was greatly mistaken.

Schultz remarked “daß es aber überhaupt schlechterdings unmöglich sey, die Lehre von den Parallellinien *durch bloße Analysirung des Begriffs ihrer Lage* festzustellen” (p. 41, my emphasis).

⁴¹ Karsten's treatment of infinity was intended as an answer to a *Preisaufrage* of the Berlin Academy (probably suggested by Lagrange), which had asked, in 1784, for papers dealing with the mathematics of the infinite. Karsten also explicitly discussed Schultz' attempt in §§ 52-76 of his essay *Von den Parallellinien* in the *Mathematische Abhandlungen* (pp. 168-202).

⁴² Hindenburg's reply can be read in Hindenburg, *Noch etwas*, pp. 392-97; Hindenburg stated that Schultz's proof was merely philosophical rather than mathematical on p. 368 of the same essay.

⁴³ Bertrand's proof is in L. Bertrand, *Développement nouveau de la partie élémentaire des mathématiques*, Genève, 1778, vol. 2, p. 20. For Kästner's letter on Bertrand, see above Hindenburg's *Noch etwas*. Kästner's views on infinity had already been expounded in his *De vera infiniti notione*, in G.A. Kästner, *Dissertationes mathematicae et physicae*, Oldenburg, Richter 1771, pp. 35-38. Kästner's anti-Kantian essays are: *Was heißt in Euclids Geometrie möglich?*, *Über den mathematischen Begriff des Raums*, *Über die geometrischen Axiome*, “Philosophisches Magazin”, 2.4 (1790), pp. 391-430. Kant is never explicitly mentioned in them, but Schultz is.

⁴⁴ Kant's reply to Kästner is now to be found in KgS 10, pp. 410-23. Kant complained to Schultz, in his letters from August 2nd and 16th, 1790, in KgS 11, pp. 184 and 200-201. The happy ending of Kant's controversy with Kästner is witnessed to by a very respectful exchange in the same months (Kant to Kästner, August 5th, 1790, in KgS 11, p. 186; Kästner to Kant, October 2nd, 1790, KgS 11, pp. 213-15).

§6. Schwab and his Critics

Schultz' reply had awakened the most relentless adversary of Kant's philosophy of mathematics and the most prominent proponent of the direction-theory of parallels.

Johann Christoph Schwab (1743-1821) was one of the founders of the *Philosophisches Magazin*, a journal that soon became the anti-Kantian organ in Germany. In a 1791 issue of the journal, Schwab published a paper against Schultz, and offered an analytic proof of the fact that one side of a triangle is shorter than the sum of the other two. The real target of the paper was Kant, who however decided not to reply. Other Kantians took up the fight, and Schwab's proof was later rebutted by August Wilhelm Rehberg (1757-1836). Rehberg, who was an important correspondent for Kant in mathematical matters, exposed several hidden assumptions in Schwab's proof, and restated the position that, without an appeal to an *a priori* intuition, one cannot hope to prove such basic geometrical statements. Schwab counter-replied to Rehberg's criticisms, and wrote further papers against Schultz.⁴⁵

A long debate arose in Germany concerning Kant's philosophy of mathematics, the role of intuition in geometry, and the analyticity of proofs. The philosophical positions of the protagonists had many nuances, which cannot be recounted in full here, and sometimes the terms of discussion were uncertain. For instance, Schwab claimed against Rehberg that a single analytic proof of a geometrical theorem was enough to ruin Kantianism, whereas Rehberg maintained that showing the non-analyticity of a single proof was sufficient to smite "Leibnizians" down. As a result, they played with *ad hoc* examples of analytic and synthetic proofs, without being able to settle the matter or find common ground for decision.

A decade later, Schwab decided that the time was ripe for a final assessment, and began writing books presenting his views on the analyticity of mathematics. He set himself no less an aim than to resuscitate Leibniz' *analysis situs* and prove the Parallel Postulate.

Schwab's first work on the topic is a *Tentamen novae parallelarum theoriae*, published in 1801, in which he presented his epistemological views and a new theory of parallels. This was later supplemented by a longer essay in French, the *Essai sur la situation* from 1808, which expounded Schwab's more general views about Leibniz's project of an *analysis situs*. And finally by a *Commentatio* on the First Book of the *Elements* (1814), in which Schwab attempted to reform elementary geometry and prove all the axioms in Euclid.⁴⁶

⁴⁵ Schwab's first paper is *Über die geometrischen Beweise, aus Gelegenheit einer Stelle in der Allgemeinen Litteratur-Zeitung, "Philosophisches Magazin"*, 3.4 (1791), pp. 397-407. See above for Maimon's attempt at the same result. Rehberg's reply to Schwab is the paper *Über die Natur der geometrischen Evidenz, "Philosophisches Magazin"*, 4.4 (1792), pp. 447-60. Schwab counter-replied to Rehberg in the same issue of the "Philosophisches Magazin" (pp. 461-69), but the discussion also continued outside the journal for many years. A further attack of Schwab against Schultz, for instance, is to be found in his *Über das Unendliche des Herrn Hofpredigers Schultz, "Philosophisches Archiv"*, 1.3 (1792), pp. 70-75. On the Schwab-Rehberg discussion, see J. Webb, *Immanuel Kant and the Greater Glory of Geometry*, in *Naturalistic Epistemology*, eds. D. Nails, A. Shimony, Dordrecht, Reidel 1987, pp. 17-70.

⁴⁶ See J.C. Schwab, *Tentamen novae parallelarum theoriae notione situs fundatae*, Stuttgart, Erhard 1801; *Essai sur la situation, pour servir de supplément aux principes de la géométrie*, Stuttgart, Cotta 1808; *Commentatio in primum elementorum Euclidis librum*, Stuttgart, Steinkopf 1814. Schwab was a prolific writer, and published many more books on philosophy, often discussing Kant's views (sometimes critically, and sometimes agreeing with him); in a few of them he restated his geometrical examples and the theory of parallels. Nonetheless, he did not write those eight volumes on the theory of parallels that De Morgan found somewhere attributed to him: "Eight volumes on the theory of parallels? If there be such a work, I trust I and it never meet, though ever so far produced" (A. De Morgan, *A Budget of Paradoxes*, ed. D.E. Smith, Chicago, Open Court 1915 (first ed. 1872), vol. 1, p. 230).

Schwab followed Hindenburg's idea that a theory of *situs* (i.e. *Lage*) should complement the geometry of magnitudes, and that the whole of mathematics should be grounded on the two independent pillars of quantity and situation. These are simple notions, as Karsten had said, and cannot be defined. Schwab did not provide a definition of them by abstraction, and instead conceded to his adversaries that they can only be given in intuition. This does not mean, however, that geometry is based upon intuition, and much less upon synthetic *a priori* judgments.⁴⁷ Schwab rather endorsed an epistemology of eidetic abstraction, and stated that we are able to develop, out of the intuitions of *situs* and magnitude, a purely intellectual science which needs no further reference to the senses. Geometry is thus sensible in its origins, but nonetheless purely intellectual in its developments, and should rely solely upon sound logical reasoning without any recourse to any intuitive or diagrammatic support. Intuition does not provide us with any propositional content, and all axioms of geometry are provable from definitions and logic alone.

Schwab supplemented this broadly Leibnizian and Aristotelian picture of science with a number of mathematical developments. He defined a straight line as a line "all the points of which have the same situation" (whatever this may mean), and parallel lines as straight lines having the same reciprocal situation.⁴⁸ Finally, Schwab claimed that the statement that two things identical to a third are also identical to one another (i.e. the transitivity of identity) is a general logical principle that enjoys no less validity than does the Principle of Contradiction itself. From its application to the simple notions of magnitude and *situs*, there immediately follows Euclid's first Common Notion ("if two things are equal in magnitude to a third, they are equal in magnitude to one another") and the transitivity of situation ("two things that have the same situation with respect to the same thing, have the same situation with respect to one another"). The latter principle, applied to the definition of parallel lines, entails the transitivity of parallelism and thus (as Schwab correctly proved) the Parallel Postulate itself.⁴⁹

Schwab's "demonstration" collected a few aspects of the previous studies on the direction-theory, and offered a purely philosophical (and mathematically trivial) proof of the Parallel Postulate embedded in a rich, if eclectic, epistemology. For this very reason, Schwab's construal of the theory of parallels represented an important standpoint in the German debate. His epistemological views were clearly stated and defended and he was considered for many years to be the Leibnizian champion against the new wave of Kantianism in mathematics.

The historical relevance of Schwab's proposal is also due to the criticisms that his theory elicited among mathematicians close to Gauss.

Karl Felix Seyffer (1762-1822), an important astronomer in Göttingen who was one of Gauss' teachers, reviewed Schwab's essay only to conclude that the only correct attitude toward the Parallel

⁴⁷ On Kant, see for instance Schwab's *Essai*, pp. 28-29; *Commentatio*, §§ 5-6 (pp. 8-10).

⁴⁸ Schwab gave no definition of a straight line in the *Tentamen*, and the latter was only added in the *Essai*: "Une ligne droite est donc celle dont tous les points ont la même situation" (p. 20). His definition of parallel lines in the *Tentamen*, "Duae lineae rectae in eodem plano jacentes sunt parallelae inter se, si eundem situm habent ad se invicem" (p. 1), is, in turn, identical to that in the *Essai*: "Deux droites sont parallèles, lorsqu'elles ont la même situation entr'elles, ou lorsque la situation de l'une est identique avec celle de l'autre" (§ 28, p. 26).

⁴⁹ In the *Tentamen*, Schwab simply stated the transitivity of parallelism as an axiom ("Axioma: Si duae rectae in plano eundem situm habent ad se invicem; habent etiam eundem situm ad rectam tertiam", p. 3), followed by a discussion about the grounds of its validity. In the following works, however, and especially in the *Commentatio*, it is very clear that every axiom has to be proven from purely logical principles. A Greek source of Schwab's proof may be Proclus, who mentioned in passing that some relations have the property of transitivity; in particular, "similarity" is considered by Proclus to be transitive in general, and parallelism is just a "similarity of position" (ὁμοιότης θέσεως); thus, parallelism is transitive and the Parallel Postulate true (Proclus, *In primum Euclidis* 373).

Postulate was to accept it as an unprovable principle, since there is no salvation outside the Church of Euclid (“*nulla salus extra Euclidem*”).⁵⁰

Ferdinand Carl Schweikart (1780-1857) is particularly remembered nowadays since he was among the first to propose, in a private letter from 1818, the consistency of an “astral geometry” (*astralische Größenlehre*) in which the Parallel Postulate is false. He also realized that there are in fact as many different astral geometries as the real values of a *Constante* which, with quite some interpretative generosity, may be related to Gaussian negative curvature.⁵¹ Schweikart’s early treatise on parallel lines (1807), however, was very much concerned with Schwab’s theory of direction, which he criticized harshly and at great length. On that occasion, Schweikart indulged in some Kantian or post-Kantian reflections on the need of mathematical intuition for any finite mind (*endlicher Geist*).⁵²

Finally, Carl Friedrich Gauss (1777-1855) himself wrote a review of Schwab’s *Commentatio* which proved in the end to be his sole (very limited) public expression of his thoughts on the Parallel Postulate. Against Schwab, Gauss claimed that definitions and logical principles

are able to accomplish nothing by themselves, and that they put forth only sterile blossoms unless the fertilizing living intuition (*Anschauung*) of the object itself prevails everywhere.⁵³

This statement appeared to many scholars to be an endorsement of that very Kantianism which Schwab was attempting to wreck. Gauss’ appeal to intuition was in all probability addressed toward an *empirical* acquaintance with the structure of space rather than an *a priori* intuition, and elsewhere he decidedly opposed Kant’s views on mathematics, advocating rather an empiricist stance on the matter. Still, in the absence of further evidence, Gauss’ criticisms of Schwab’s anti-Kantianism appeared to foster Kant’s transcendental epistemology. In later times, when his views on non-Euclidean geometries were finally disclosed to the public, a few neo-Kantian philosophers were able to claim – thanks to this review of Schwab – that Gauss had admitted the compatibility between non-Euclidean space and *a priori* intuition.⁵⁴

⁵⁰ Seyffer’s review of Schwab’s *Tentamen* appeared in the *Göttingische Anzeigen von gelehrtten Sachen*, 39 (1801), pp. 377-89.

⁵¹ Schweikart’s note was attached to Gerling’s letter to Gauss from January 25th, 1819, and may now be found in Gauss’s *Werke* (vol. 8, pp. 179-81), along with Gauss’s first reaction to it (vol. 8, pp. 181-82).

⁵² Schweikart’s criticisms of Schwab are to be found in F.C. Schweikart, *Die Theorie der Parallellinien nebst dem Vorschlage ihrer Verbannung aus der Geometrie*, Jena, Gabler 1807, which also refers to Kant.

⁵³ Gauss’s review of Schwab’s *Commentatio* was published in 1816 in the *Göttingische gelehrte Anzeigen*, and can now be read in C.F. Gauss, *Werke*, Hildesheim, Göttingen/Leipzig 1863-1917, reprint Olms 1975-1987, vol. 4, pp. 364-68. The quoted sentence is translated into English in W. Ewald, *From Kant to Hilbert: a source book in the foundations of mathematics*, Oxford, Clarendon Press 1996, vol. 1, p. 300. See Gauss’s (much later) letter to Farkas Bolyai from March 6th, 1832, on the fact that the new geometrical discoveries made by János Bolyai (and Gauss himself) show that Kant was definitely wrong in believing that space is only a form of our intuition (“...der klarste Beweis, dass Kant Unrecht hatte zu behaupten, der Raum sei *nur* Form unserer Anschauung”; Gauss, *Werke*, vol. 8, pp. 220-224; also in F. Schmidt, P. Stäckel (eds.), *Briefwechsel zwischen Carl Friedrich Gauss und Wolfgang Bolyai*, Leipzig, Teubner 1899, pp. 108-113). On Gauss’ role (or lack thereof) in the discovery of non-Euclidean geometry, see J. Gray, *Gauss and non-Euclidean Geometry*, in A. Prékopa, E. Molnar (eds.), *Non-Euclidean Geometries: János Bolyai memorial volume*, New York, Springer 2003, pp. 61-80.

⁵⁴ The first attempts to claim that the Kantian conception of geometry is compatible with non-Euclidean geometries are probably those by L. Nelson, *Bemerkungen über die nicht-Euklidische Geometrie und den Ursprung der*

§7. Post-Kantian Philosophy and the Direction-Theory

These few dissonant voices notwithstanding, Schwab acted as a pied-piper to the whole scholarly world, and in the subsequent decades the direction-theory of parallels became mainstream in Germany. Like all pieces of standard science, the direction-theory was provided with a history and a name. In 1807, Johann Ephraim Scheibel (1736-1809) published a history of the theory of parallels from Euclid to his own time, which was crowned by the modern theory of *Lage*. He traced the origin of the notion back to Euclid himself, who had exploited the notion of position ($\theta\acute{\epsilon}\sigma\iota\varsigma$) in his book on *Data*.⁵⁵ In 1816, Carl Christian Hermann Vermehren (1792-1858) wrote an essay on the direction-theory in which the notion of *Richtung* supplanted that of *Lage* (or *situs*).⁵⁶ It is probably at this time that the theory began to be widely known with its current name.

In the meantime, the mathematician Karl Christian Langsdorf (1757-1834) updated Wolff's textbook for students, the *Anfangsgründe aller mathematischen Wissenschaften*, adding to it the theory of direction.⁵⁷ Among the other important and early essays on the theory, we can mention at least those by Andreas Jacobi (1801-1875) from 1824 and by Joseph Knar (1800-1864) from 1827 and 1828. The success of the theory was so extensive that even the Kantian philosopher Carl Sigmund Ouvrier (1751-1819) wrote an essay on parallel lines and the faculties of the mind, in which he accepted pure intuition as the basis of mathematics and also a theory of direction as the foundation of the theory of straight and parallel lines.⁵⁸

mathematischen Gewißheit, "Abhandlungen der Fries'schen Schule", 1-3 (1905-1906), now in Nelson, *Gesammelte Schriften*, Hamburg, Meiner 1974, vol. 3, pp. 9-52; cf. in the same years also W. Meinecke, *Die Bedeutung der Nicht-Euklidischen Geometrie in ihrem Verhältnis zu Kants Theorie der mathematischen Erkenntnis*, "Kant-Studien", 11 (1906), pp. 209-32; and later P. Natorp, *Die logischen Grundlagen der exacten Wissenschaften*, Leipzig, Teubner 1910.

⁵⁵ See J.E. Scheibel, *Zwey mathematische Abhandlungen: I. Vertheidigung der Theorie der Parallellinien nach dem Euclides*, Breslau, Korn 1807. Scheibel's work includes a long discussion of Kästner's three essays on space and geometry which had been published against Kant and Schultz.

⁵⁶ C.C.H. Vermehren, *Versuch die Lehre von der parallelen und convergenten Linien aus einfachen Begriffen vollständig herzuleiten*, Güstrow, Ebert 1816. The discussion on direction is to be found on pp. 19-26, while on p. 21 Vermehren assumed an axiom regarding the transitivity of sameness of direction.

⁵⁷ Langsdorf edited the geometrical section of Wolff's *Anfangsgründe* in 1797 (Marburg, Akad. Buchhandlung), that is, before Schwab's contribution to the theory of parallels. Already in this book, Langsdorf mentioned the new theories by Karsten and Hindenburg, which he did not see as fundamentally opposing Wolff's original views. He later wrote his own *Anfangsgründe der reinen elementar- und höheren Mathematik aus Revision der bisherigen Principien gegründet*, Erlangen, Palm 1802. In this book, Langsdorf started with a definition of the *Richtung* of a straight line (*Anfangsgründe der Geometrie*, § 14, pp. 129-30) so as to be able to give a definition of parallel lines (§ 15, pp. 131-32), which, albeit grounded upon it, recurred to the notion of equidistance; a little further on in the text, the Parallel Postulate is proven (Theorem 14, pp. 173-76). In this work, written a year after Schwab's first essay on parallels, the latter's influence is manifest.

⁵⁸ A. Jacobi, *De undecimo Euclidis axioma iudicium*, Jena, Croeck 1824. The main theorem is on p. 58: "Per punctum quoddam una tantum recta unius ejusdemque certae directionis duci potest", which is "proven" without much discussion. Andreas Jacobi was the brother of the more famous mathematician Karl Friedrich Jacobi. J. Knar, *Über die Theorie der Parallellinien*, "Zeitschrift für Physik und Mathematik", 3 (1827), pp. 414-39; *Berichtigung meiner Ansicht über die Theorie der Parallellinien*, "Zeitschrift für Physik und Mathematik", 4 (1828), pp. 427-36. The first essay also discusses several earlier attempts and quotes Karsten as the initiator of the theory that Knar himself advocated and tried to perfect. C.S. Ouvrier, *Theorie der Parallelen als Ankündigung eines neuen Versuchs über das Erkenntnißvermögen*, Leipzig, Schiegg 1808.

We will not follow in any detail the diffusion of the direction-theory in German textbooks on geometry, which was as pervasive as it is mathematically uninteresting. Rather, I would like to point out the lesser-known fact that the most important philosophers in Germany plainly endorsed Schwab's theory.

In 1805, Johann Gottlieb Fichte (1762-1814) rejected Kant's views about the givenness of spatial intuition and claimed that space itself is generated by the thinking subject. He stated that the first determination of space is given by tracing an infinite straight line which is the archetypal diameter of space and the beginning of any further construction of finite magnitudes and figures. This line determines a "primal direction" (*Ur-Richtung*) in space, and therefore defines a first sheaf of parallel lines as those straight lines sharing the same original direction. As a consequence, the Parallel Postulate is firmly established upon the basis of the original construction of space. The direction-theory of parallels appears to be the metaphysical foundation of space itself.⁵⁹

Jakob Friedrich Fries (1773-1843) offered his own "synthetic" proof of the Parallel Postulate in 1822. This proof is far more mathematical than Fichte's but is still entirely based upon the notion of the direction of a straight line. Fries defined the straight line as a line the parts of which have the same *Richtung*, then complemented this definition by three *Axiome der Richtung*, and finally proved from all these assumptions that the interior angle sum of a triangle is equal to two right angles. The Parallel Postulate is an easy consequence of the latter fact.⁶⁰ Fries' philosophical influence on mathematicians was remarkable: Gauss appreciated his work, and Schlömilch (Frege's source on the direction-theory) was a student of his. No one, however, seems to have followed his quite naïve proof of the postulate.

Johann Friedrich Herbart (1776-1841) conceived of space as a structure produced by a system of monads, and for this reason he is sometimes credited for a philosophy of space that could possibly accommodate non-Euclidean manifolds. Historians generally regard him as the main philosophical source for Riemann's thinking, and speculate on his role in establishing the possibility of a plurality of geometries. Yet, in 1829 Herbart also claimed that the Parallel Postulate can be analytically proven – without any recourse to pure or empirical intuition – from the very notion of *Richtung*.⁶¹

Georg Wilhelm Friedrich Hegel (1770-1831), in the posthumous second edition of the *Wissenschaft der Logik* (1832), discussed at length the epistemology of mathematics and stated (in perfect agreement with the previous tradition) that any axiom can be proven starting from the definitions of the terms. The Parallel Postulate, in particular, may be proven from the definition of direction, or better of the sameness thereof, *die Gleichheit der Richtung*, even though Euclid and the

⁵⁹ Fichte's discussion of the theory of parallels was not published during his lifetime, and is to be found among the lecture notes of his from 1805 called the "*Erlangen Logik*", in which he also mentions Schultz's proof, judging it to be unpersuasive. See J.G. Fichte, *Gesamtausgabe*, ed. R. Lauther *et al.*, Cannstatt, Frommann 1962-, vol. II 9, pp. 124-37. An English translation of this text, with a commentary, is to be found in D.W. Wood, "*Mathesis of the Mind*". *A Study of Fichte's Wissenschaftslehre and Geometry*, Amsterdam, Rodopi 2012.

⁶⁰ Fries's discussion on the Parallel Postulate is to be found in his *Die mathematische Naturphilosophie*, Heidelberg, Mohr 1822, §§ 66-69 (pp. 355-380; the axioms of direction are on p. 369, the definition of a straight line on p. 376, the proof of the interior angle sum on p. 379). On Fries and geometry, see F. Gregory, *Neo-Kantian Foundations of Geometry in the German Romantic Period*, "Historia Mathematica", 10 (1983), pp. 184-201.

⁶¹ Herbart's statement on the Parallel Postulate is to be found in § 257 of the second volume of his major book on metaphysics: J.F. Herbart, *Allgemeine Metaphysik*, Königsberg, Unzer 1828-1829, vol. 2, pp. 240-42. The extent of the influence of Herbart's philosophical views on Riemann's foundational mathematical works continues to be a subject of great debate. For an informed assessment, see E. Scholz, *Herbart's Influence on Bernhard Riemann*, "Historia Mathematica", 9 (1982), pp. 413-40. On Herbart's theory of intelligible space, see E. Banks, *Kant, Herbart and Riemann*, "Kant-Studien", 96 (2005), pp. 208-34.

other ancient writers had done well in assuming it without proof given the *plastische Charakter* of the beautiful endeavour of Greek science, which did not waste time on such trifles.⁶²

None of these philosophers ever doubted of the truth of the Parallel Postulate. The influence of their views on the generations that followed was enormous, and it may explain the general acceptance of the direction-theory even outside the mathematical community. Still many decades later, in 1879, the same year in which Dodgson published his rebuke of Euclid's modern rivals, the illustrious philosopher Hermann Lotze (1817-1881) wrote a book in which he rejected non-Euclidean geometry on the basis of the direction-theory of parallels.⁶³

The direction-theory of parallels was one of the most prominent phenomena in the history of the philosophy of mathematics between the eighteenth and the nineteenth centuries. We regard it nowadays as a huge misunderstanding, a conceptual dead end, and a reactionary force in the development of mathematics towards non-Euclidean geometry. Yet, the debate it engendered revolved around the most important topics in the epistemology of mathematics – abstraction, intuition, infinity, analyticity – and shaped the history of thought for a century. These magmatic reflections on the nature of mathematics were the philosophical background in which Lobachevsky, Bolyai and Gauss first conceived non-Euclidean geometry as an alternative theory of parallels. These same reflections, continuing into the following generations, produced the rebuttals by Dodgson and Frege, who – no longer pre-modern and yet not modern, and even possibly countermodern – seriously engaged the epistemology of a waning age. Modernism itself, eventually, came out of this debate.

The philosophical interest of the century of the direction-theory, with its supporters and its opponents, lies precisely in its being a passageway to modernity.

⁶² In his young years, Hegel studied a proof of *Elements* I, 29 (a proposition equivalent to the Parallel Postulate) through the definition of parallel lines as equidistant straight lines—in a broadly Wolffian perspective. These private notes have now been published in J. Hoffmeister, *Dokumente zu Hegels Entwicklung*, Stuttgart, Frommann 1936, pp. 288-300. Hegel's first public mention of the Parallel Postulate was in the third volume of the *Wissenschaft der Logik* (1816), where he claimed (in the section dedicated to the notion of a "Theorem") that all propositions of mathematics can be derived from their definitions, and the Parallel Postulate no less than any other. In the second edition of the first volume (1832; first remark to the notion of "Number"), however, he provided a much longer discussion of the analytic status of mathematics, criticizing Kant for his claim that the latter science should rather be understood to be synthetic and grounded in intuition. Here, in particular, Hegel concentrated on the definition of a straight line, attempting to show that its property of being the shortest line between two points can in fact be proven by pure logic and without any appeal to intuition (Schwab's example). The reference to Greek science, the equality of directions and the Parallel Postulate follows (cf. G.W.F. Hegel, *Gesammelte Werke*, vol. 21, ed. F. Hogemann, W. Jaeschke, Hamburg, Meiner 1984, p. 200).

⁶³ Lotze strenuously opposed non-Euclidean geometry on philosophical grounds and criticized the views of both Helmholtz and Riemann simply restating Karsten's old (1778) argument on the transitivity of direction in parallelism—as nothing had happened in this field in the previous century (H. Lotze, *System der Philosophie*, Leipzig, Hirzel 1874-1879, vol. 2, § 131, pp. 247-49). The notions of *Richtung* and parallelism were already at work in Lotze's first attempt in philosophy, even though at the time he had not dared to propose a proof of the Parallel Postulate: see H. Lotze, *Metaphysik*, Leipzig, Weidmann 1841, pp. 184 ff.