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ABSTRACT. Let $(X_{\underline{\ell}})_{\underline{\ell}\in\mathbb{Z}^d}$ be a real random field (r.f.) indexed by $\mathbb{Z}^d$ with common probability distribution function F. Let $(z_k)_{k=0}^{\infty}$ be a sequence in $\mathbb{Z}^d$. The empirical process obtained by sampling the random field along (z_k) is $\sum_{k=0}^{n-1}[\mathbf{1}_{X_{z_k}\leq s} - F(s)]$.

We give conditions on (z_k) implying the Glivenko-Cantelli theorem for the empirical process sampled along (z_k) in different cases (independent, associated or weakly correlated random variables). We consider also the functional central limit theorem when the $X_{\underline{\ell}}$'s are i.i.d.

These conditions are examined when (z_k) is provided by an auxiliary stationary process. This leads to investigate local times and maximum local times for ergodic sums.

CONTENTS

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Introduction

For a sequence (X_k) of real i.i.d. random variables with common probability distribution function F, the empirical process is defined by $\sum_{k=0}^{n-1}[\mathbf{1}_{X_k\leq s}-F(s)]$. Recall two classical results.
(A) the Glivenko-Cantelli theorem:
a.s. the sequence of empirical distribution functions $F_n(s):=\frac{1}{n}\sum_{k=0}^{n-1}\mathbf{1}_{X_k\leq s}$ *converges uniformly to* F, *i.e.* $\sup_s|F_n(s)-F(s)|\to 0$*;*
(B) a functional central limit theorem (FCLT): if the random variables (r.v.'s) X_k have a common distribution F over $[0,1]$, then
the process $\frac{1}{\sqrt{n}}\sum_{k=0}^{n-1}[\mathbf{1}_{X_k\leq s}-F(s)]$ *converges weakly to a Brownian bridge in the space of cadlag functions on* $[0,1]$.

In this paper we study the extension of these results when the process is sampled along a subsequence, analogously to what is done for limit theorems in random scenery.

In the sequel, for $d\geq 1$, $(X_{\underline{\ell}})_{\underline{\ell}\in\mathbb{Z}^d}$ will be a real random field (r.f.) indexed by $\mathbb{Z}^d$ defined on a probability space $(\Omega,\mathcal{F},\mathbb{P})$ with common probability distribution function F. The expectation on $(\Omega,\mathbb{P})$ is denoted by $\mathbb{E}$. We consider in particular the case of a r.f. of i.i.d. r.v.'s or of stationary associated r.v.'s.

Let $(z_k)_{k=0}^{\infty}$ be a sequence in $\mathbb{Z}^d$. The process obtained by sampling the random field along (z_k) is $W_n(s):=\sum_{k=0}^{n-1}[\mathbf{1}_{X_{z_k}\leq s}-F(s)]$.

We will call $W_n(s)$ "empirical process sampled along (z_k)", or simply "sampled empirical process". A general question is whether the above results (A), (B) extend to the sampled empirical process $W_n(s)$, in particular when (z_k) is given by the sums of another stationary process with values in $\mathbb{Z}^d$.

In Section 1, we give conditions on (z_k) implying that (A) and (B) are still valid for the empirical process of $(X_{\underline{\ell}})$ sampled along (z_k) in different cases: independent, associated or weakly correlated random variables. The conditions are expressed in terms of the following quantities associated to the sequence (z_k) in $\mathbb{Z}^d$: local time, maximal local time and number of self-intersections (up to time n) defined, for $n\geq 1$, by

$$\begin{aligned} &N_n(\underline{\ell}):=\#\{0\leq k\leq n-1:\ z_k=\underline{\ell}\},\\ &M_n:=\max_{\underline{\ell}}N_n(\underline{\ell}),\ V_n:=\#\{0\leq j,k\leq n-1:\ z_j=z_k\}. \end{aligned}\tag{1}$$

They satisfy $\sum_{\underline{\ell}}N_n(\underline{\ell})=n$ and $n\leq V_n=\sum_{\underline{\ell}}N_n^2(\underline{\ell})\leq nM_n\leq n^2$.

In the other sections, (z_k) is given by a stationary process (or equivalently by the sequence $(S_k f(x))_{k\geq 1}$ of ergodic sums of a function f over a dynamical system).

The conditions found in Section 1 lead to study the local times, maximum number of visits, number of self-intersections for the sequence $(S_k f(x))$. Some general remarks are presented in Section 2. Then in Section 3, we consider two families of examples: random walks and some ergodic sums over a rotation.

The Glivenko-Cantelli theorem along ergodic sums (extension of (A)) is strongly related to random ergodic theorems, in particular to results in [23] and [25]. This is discussed in the last Section 4.

Finally let us mention the quenched FCLT for the 2-parameters process

$$W_n(s,t) := \sum_{k=0}^{[nt]-1} [\mathbf{1}_{X_{Z_k(x)}\leq s} - F(s)],\ (s,t)\in[0,1]^2.$$

When $(X_{\underline{\ell}})$ is a r.f. of i.i.d. r.v.'s indexed by $\mathbb{Z}^2$ and when the sampling is provided by a 2-dimension centered random walk (Z_k) with a moment of order 2, the weak convergence for a.e. x toward a Kiefer-Müller process can be shown. This will be the content of a forthcoming paper.

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1. General results on the empirical process along a sub-sequence

1.1. **Preliminaries.** In this subsection, results on the empirical process along a sub-sequence are shown for independent variables, as well for some of them for wider classes (associated, PDQ and weakly correlated random variables). We start by recalling some notions and auxiliary results.

1) Associated variables

Definition (cf. [18]): A finite set of real random variables $\mathbf{T} = (T_1, T_2, \ldots, T_n)$ is said to be *associated* if $\mathrm{Cov}[f(\mathbf{T}), g(\mathbf{T})] \geq 0$, for every coordinate-wise non-decreasing functions $f = f(x_1, ..., x_n)$ and $g = g(x_1, ..., x_n)$ for which $\mathbb{E}[f(\mathbf{T})]$, $\mathbb{E}[g(\mathbf{T})]$, $\mathbb{E}[f(\mathbf{T})\, g(\mathbf{T})]$ exist. An infinite set of random variables is associated if any finite subset of it is associated.

For the basic properties of associated r.v.'s, see [18]. We recall some of them below.

A set reduced to a single real r.v. is associated. Let us show it to illustrate this notion of association. If X is a r.v. and Y an independent copy of X, for f, g two non-decreasing

functions on $\mathbb{R}$, clearly we have $[f(X)-f(Y)][g(X)-g(Y)] \geq 0$. Expanding and taking expectation, we get: $0 \leq 2\mathbb{E}[f(X)g(X)] - 2\mathbb{E}[f(X)]\mathbb{E}[g(X)] = 2\mathrm{Cov}(f(X), g(X))$. Hence the result.

Association of random variables is preserved under taking subsets and forming unions of independent sets (of associated random variables). In particular a family of independent variables is associated.

Associated r.v.'s which are uncorrelated are jointly independent (cf. [28, Theorem 1]).

Examples of (non independent) stationary associated processes with absolutely summable series of correlations are provided by some Ising models. References to such examples of stationary $\mathbb{Z}^d$ random fields which satisfies the FKG inequalities and with absolutely summable correlations can be found in Newman's paper [27]. Notice that the FKG inequalities expresses the association property of the r.v.'s.

We will make use of the fact that if $(X_{\underline{\ell}})$ is associated, then for any sequence (z_k), (X_{z_k}) is associated.

For simplicity of notation, let us explain it on an example. Suppose we take for instance an associated family of four r.v.'s $\{X, Y, Z, V\}$. Now, if we repeat some of the r.v.'s and consider for instance $\{X, Y, X, X, Z, Y\}$, this family is associated. Indeed, let f, g be two coordinate-wise non decreasing functions on $\mathbb{R}^6$. Putting $\tilde{f}(x, y, z) = f(x, y, x, x, z, y)$, $\tilde{g}(x, y, z) = g(x, y, x, x, z, y)$, we obtain two coordinate-wise non decreasing functions on $\mathbb{R}^3$ and by association of $\{X, Y, Z, V\}$ the covariance satisfies

$$Cov(f(X, Y, X, X, Z, Y), g(X, Y, X, X, Z, Y)) = Cov(\tilde{f}(X, Y, Z), \tilde{g}(X, Y, Z)) \geq 0.$$

2) *PQD variables*

Two r.v.'s X, Y are called (cf. [24]) *positively quadrant dependent (PQD)* if,

$$\mathbb{P}(X > x, Y > y) \geq \mathbb{P}(X > x)\,\mathbb{P}(Y > y), \forall x, y \in \mathbb{R}.$$

The property is preserved by centering. Any pairwise associated r.v.'s are pairwise PQD. Pairwise independent random variables are pairwise PQD.

Two random variables X and Y are PQD if and only if for every non-decreasing functions f and g, $\mathrm{Cov}(f(X), g(Y)) \geq 0$ (whenever the covariance exists) ([18, Theorem 4.4]).

3) We will use the following results:

a) Maximal inequality of Newman and Wright [28, Inequality (12)]:

If (W_i) is a sequence of centered associated, square integrable random variables, it holds:

$$\mathbb{P}(\max_{1\leq j\leq n} |\sum_{i=1}^{j} W_i| \geq \lambda \,\|\sum_{i=1}^{n} W_i\|_2) \leq 2\mathbb{P}(|\sum_{i=1}^{n} W_i| \geq (\lambda - \sqrt{2})\,\|\sum_{i=1}^{n} W_i\|_2), \forall \lambda \geq 0. \tag{2}$$

b) Hoeffding's identity (see [2, Theorem 3.1])

Let X, Y be random variables with finite second moments. For any absolutely continuous functions f, g on $\mathbb{R}$, such that $\mathbb{E}[f^2(X) + g^2(Y)] < \infty$, it holds

$$Cov(f(X), g(Y)) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f'(x)g'(y)[\mathbb{P}(X > x, Y > y) - \mathbb{P}(X > x)\mathbb{P}(Y > y)]dxdy.$$

In particular, if X, Y are PDQ random variables, if $|f'|, |g'| \leq M$ a.e., we have

$$|Cov(f(X), g(Y))| \leq M^2 Cov(X, Y).$$

4) Uniformity in the analogues of Glivenko-Cantelli theorem will follow from the lemma:

Lemma 1.1. [7, Lemma, p 140] *Let F_n, F be a family of right continuous distribution functions on $\mathbb{R}$. Assume that, for each point x in a dense countable set $Q \subset \mathbb{R}$, we have $F_n(x) \to F(x)$. Let J be the set of jumps of F and assume that $F_n(x) - F_n(x^-) \to F(x) - F(x^-)$ for every $x \in J$. Then $F_n(x) \to F(x)$ uniformly in $\mathbb{R}$.*

A strong law of large numbers

First we state a law of large numbers for bounded r.v.'s which is valid under weak hypotheses.

In the proof, we will use a lemma, proved in [14], when φ_n is the exponential $e^{2\pi inx}$ on $[0, 1]$, and valid in the following general formulation. We are indebted to the referee for this reference [14], which allows to improve a previous version of the results of this section.

Lemma 1.2. *Let $(\varphi_n)_{n\geq 1}$ be a sequence of real or complex bounded measurable functions on a probability space $(\Omega, \mathbb{P})$ such that $\sup_n \|\varphi_n\|_\infty < \infty$. If*

$$\sum_{n\geq 1} \frac{1}{n}\int_\Omega \big(\frac{1}{n}\sum_{j=1}^{n}\varphi_j(\omega)\big)^2 d\mathbb{P} < \infty,$$

then $\dfrac{1}{n}\displaystyle\sum_{j=1}^{n}\varphi_j(\omega) \to 0$, for $\mathbb{P}$-a.e. ω.

Let $(U_{\underline{\ell}})_{\underline{\ell}\in\mathbb{Z}^d}$ be a r.f. indexed by $\mathbb{Z}^d$ of square integrable centered r.v's on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Let $(z_k)_{k\geq 0}$ be a sequence in $\mathbb{Z}^d$, $d \geq 1$, with numbers of self-intersections $V_n, n \geq 1$. The partial sums along (z_k) are denoted by $S_n := \displaystyle\sum_{k=0}^{n-1} U_{z_k}$.

By the Cauchy-Schwarz inequality, we have

$$\sum_{\underline{r}\in\mathbb{Z}^d} N_n(\underline{r}+\underline{\ell})N_n(\underline{r})\,|\langle U_{\underline{r}+\underline{\ell}}, U_{\underline{r}}\rangle| \;\leq\; \sup_{\underline{r}\in\mathbb{Z}^d}|\langle U_{\underline{r}+\underline{\ell}}, U_{\underline{r}}\rangle| \sum_{\underline{r}\in\mathbb{Z}^d} N_n^2(\underline{r}), \tag{3}$$

hence for a stationary r.f. with summable correlation:

$$\|S_n\|_2^2 = \|\sum_{i=0}^{n-1} U_{z_i}\|_2^2 \leq V_n \sum_{\underline{\ell}\in\mathbb{Z}^d} |\langle U_{\underline{\ell}}, U_{\underline{0}}\rangle|. \tag{4}$$

Proposition 1.3. *If* $\sup_{\underline{\ell}\in\mathbb{Z}^d} \|U_{\underline{\ell}}\|_\infty < +\infty$, *then* $\dfrac{S_n(\omega)}{n} \to 0$ *for* $\mathbb{P}$*-a.e* ω *under the following Conditions (A) or (B):*

Conditions (A)

$$C_0 := \sum_{\underline{\ell}\in\mathbb{Z}^d} \sup_{r\in\mathbb{Z}^d} |\langle U_{\underline{r}+\underline{\ell}}, U_{\underline{r}}\rangle| < +\infty, \tag{5}$$

and

$$\sum_{n\geq 1} \frac{V_n}{n^3} < \infty. \tag{6}$$

(The above condition is satisfied if $V_n \leq C_1 \frac{n^2}{(\log n)^\beta}$, *with* $\beta > 1$.*)*

Conditions (B) The r.f. (U_ℓ) *is stationary with* $\sum_{\underline{\ell}\in\mathbb{Z}^d} |\langle U_{\underline{\ell}}, U_{\underline{0}}\rangle|^\zeta < \infty$, *for some* $\zeta \in [1,2]$, *and*

$$\sum_{n\geq 1} \frac{V_n^{\frac{1}{\zeta}}}{n^{1+\frac{2}{\zeta}}} < +\infty. \tag{7}$$

(The above condition is satisfied if $V_n \leq C_1 \frac{n^2}{(\log n)^\beta}$, *with* $\beta > \zeta$.*)*

Proof. 1) Suppose Conditions (A) are satisfied. By (3) and (5) it holds:

$$\|\sum_{i=0}^{n-1} U_{z_i}\|_2^2 = \sum_{\underline{\ell}} \sum_{\underline{r}} N_n(\underline{r}+\underline{\ell}) N_n(\underline{r}) \langle U_{\underline{r}+\underline{\ell}}, U_{\underline{r}}\rangle \leq V_n \sum_{\underline{\ell}} \sup_{\underline{r}} |\langle U_{\underline{r}+\underline{\ell}}, U_{\underline{r}}\rangle| = C_0 V_n; \tag{8}$$

whence:

$$\sum_{n\geq 1} \frac{1}{n} \|\frac{1}{n} \sum_{i=0}^{n-1} U_{z_i}\|_2^2 \leq C_0 \sum_{n\geq 1} \frac{V_n}{n^3}.$$

The result follows from Lemma 1.2 if $\sum_{n\geq 1} \dfrac{V_n}{n^3} < \infty$.

2) Assume now Conditions (B). The r.f. is stationary. As $\sum_{\underline{\ell}\in\mathbb{Z}^d} |\langle U_{\underline{\ell}}, U_{\underline{0}}\rangle|^\zeta < \infty$ for some ζ such that $1 \leq \zeta \leq 2$, we have $\sum_{\underline{\ell}\in\mathbb{Z}^d} |\langle U_{\underline{\ell}}, U_{\underline{0}}\rangle|^2 < \infty$. (We can assume $\zeta \neq 1$, since this case is included in (A).)

Therefore the r.f. has a spectral measure ν_φ absolutely continuous with respect to the Lebesgue measure λ on $\mathbb{T}^d$ with a density $\rho \in L^2(\lambda)$ and Fourier coefficients $\int_{\mathbb{T}^d} e^{2\pi i\langle \underline{\ell}, \underline{t}\rangle} \rho(\underline{t})\, d\underline{t} =$

$\langle U_{\underline{0}}, U_{\underline{\ell}}\rangle$. Moreover the density ρ belongs to $L^{\zeta'}(\lambda)$ where ζ, ζ' are conjugate exponents (see: [32], p. 102, or [21] Th. 31.22), and it satisfies:

$$\|\rho\|_{\zeta'} \leq (\sum_{\underline{\ell}\in\mathbb{Z}^d} |\langle U_{\underline{\ell}}, U_0\rangle|^{\zeta})^{1/\zeta}.$$

If (K_n) is a sequence of constants, we have $\int_{\rho>K_n} \rho\, d\underline{t} \leq K_n^{-\zeta'+1}\|\rho\|_{\zeta'}^{\zeta'}$. It follows:

$$\frac{1}{n^2}\int_{\mathbb{T}^d} |\sum_{j=0}^{n-1} e^{2\pi i\langle z_j,\underline{t}\rangle}|^2\, d\nu_\varphi(\underline{t}) \leq K_n\frac{V_n}{n^2} + \int_{\rho>K_n} \rho\, d\underline{t} \leq K_n\frac{V_n}{n^2} + K_n^{-\zeta'+1}\|\rho\|_{\zeta'}^{\zeta'}.$$

Taking K_n such that : $K_n\frac{V_n}{n^2} = K_n^{-\zeta'+1}\|\rho\|_{\zeta'}^{\zeta'}$, that is: $K_n = \|\rho\|_{\frac{\zeta}{\zeta-1}}(\frac{n^2}{V_n})^{\frac{\zeta-1}{\zeta}}$, it gives the bound:

$$\frac{1}{n^2}\int_{\mathbb{T}^d} |\sum_{j=0}^{n-1} e^{2\pi i\langle z_j,\underline{t}\rangle}|^2\, d\nu_\varphi(\underline{t}) \leq 2\|\rho\|_{\frac{\zeta}{\zeta-1}}(\frac{V_n}{n^2})^{\frac{1}{\zeta}};$$

whence,

$$\sum_{n\geq 1}\frac{1}{n}\|\frac{1}{n}\sum_{i=0}^{n-1} U_{z_i}\|_2^2 \leq 2\|\rho\|_{\frac{\zeta}{\zeta-1}}\sum_{n\geq 1}\frac{1}{n}(\frac{V_n}{n^2})^{\frac{1}{\zeta}}.$$

Again we can apply Proposition 1.2, if

$$\sum_{n\geq 1}\frac{1}{n}(\frac{V_n}{n^2})^{\frac{1}{\zeta}} = \sum_{n\geq 1}\frac{V_n^{\frac{1}{\zeta}}}{n^{1+\frac{2}{\zeta}}} < +\infty. \quad \square$$

Remarks 1.4. 1) Let us give an example of a non stationary r.f. $(U_{\underline{\ell}})$ which satisfies (5) of the previous proposition.

We take $(U_{\underline{\ell}} = V_{\underline{\ell}}W_{\underline{\ell}}, \underline{\ell} \in \mathbb{Z}^d)$, where $(V_{\underline{\ell}})$ and $(W_{\underline{\ell}})$ are two r.f.'s independent from each other, with $(V_{\underline{\ell}})$ centered stationary and such that $\sum_{\underline{\ell}\in\mathbb{Z}^d}|\langle V_{\underline{\ell}}, V_{\underline{0}}\rangle| < \infty$, and $(W_{\underline{\ell}})$ satisfying $\sup_{\underline{\ell},\underline{p}}|\langle W_{\underline{\ell}+\underline{p}}, W_{\underline{\ell}}\rangle| < \infty$.

The r.f. $(W_{\underline{\ell}})$ can be viewed as a (multiplicative) noise (which can be non stationary) independent from the r.f. $(U_{\underline{\ell}})$. Clearly (5) is satisfied.

2) For a stationary r.f. $(U_{\underline{\ell}})$ with a bounded spectral density (but with a series of correlations which may be not absolutely summable), then like in 1) the condition $\beta > 1$ is sufficient for the conclusion of the theorem.

3) The questions of the existence of the asymptotic variance and of it's positivity will be discussed in Subsection 1.3 and 3.1 when (z_k) is given by a random walk, in particular in the transient case.

Now, we give a pointwise bound for the sampled sums, first for i.i.d. r.v.'s, then for a stationary random field $(U_{\underline{\ell}})_{\underline{\ell}\in\mathbb{Z}^d}$ of associated r.v.'s.

We denote by σ_n^2 the variance: $\sigma_n := \|\sum_{i=0}^{n-1} U_{z_i}\|_2^2$.

Before the next proposition, observe that if the r.v.'s are pairwise orthogonal, in particular independent, and if $\|U_{\underline{\ell}}\|_2 = 1$, then $\sigma_n^2 = V_n$ and $0 < \sigma_n^2 - \sigma_{n-1}^2 \leq 1 + 2N_{n-1}(z_{n-1}) \leq 2V_n^{\frac{1}{2}}$, so that $\frac{\sigma_n^2 - \sigma_{n-1}^2}{\sigma_n^2} \to 0$.

For associated r.v.'s, there is an analogous result, which will be used in Proposition 1.6:

Lemma 1.5. *For associated r.v.'s, under the conditions*

$$c := \inf_{\underline{\ell} \in \mathbb{Z}^d} \langle U_{\underline{\ell}}, U_{\underline{\ell}} \rangle > 0, \ C_1 := \sup_{\underline{r} \in \mathbb{Z}^d} \sum_{\underline{\ell} \in \mathbb{Z}^d} |\langle U_{\underline{\ell}}, U_{\underline{r}} \rangle|^2 < +\infty, \tag{9}$$

it holds

$$\frac{\sigma_n^2 - \sigma_{n-1}^2}{\sigma_n^2} \to 0. \tag{10}$$

Proof. Since the r.v.'s are associated, we have by Cauchy-Schwarz and (9):

$$\begin{aligned} 0 \leq \sigma_n^2 - \sigma_{n-1}^2 &= \langle U_{z_{n-1}}, U_{z_{n-1}} \rangle + 2 \sum_{i=0}^{n-2} \langle U_{z_i}, U_{z_{n-1}} \rangle \leq 2 \sum_{\underline{\ell}} N_n(\underline{\ell}) \, \langle U_{\underline{\ell}}, U_{z_{n-1}} \rangle \\ &\leq 2 \, (\sum_{\underline{\ell}} N_n(\underline{\ell})^2)^{\frac{1}{2}} \, (\sum_{\underline{\ell}} \langle U_{\underline{\ell}}, U_{z_{n-1}} \rangle^2)^{\frac{1}{2}} \leq 2 C_1^{\frac{1}{2}} \, V_n^{\frac{1}{2}}. \end{aligned} \tag{11}$$

Moreover the expansion of $\sigma_n^2 = \| \sum_{i=1}^n U_{z_i} \|_2^2$ yields a sum of non negative terms (by association) and the lower bound (by the first condition in (9)):

$$\sigma_n^2 = \| \textstyle\sum_{\underline{\ell}} N_n(\underline{\ell}) \, U_{\underline{\ell}} \|_2^2 = \sum_{\underline{\ell}, \underline{r}} N_n(\underline{\ell}) N_n(\underline{r}) \langle U_{\underline{\ell}}, U_{\underline{r}} \rangle \geq \sum_{\underline{\ell}} N_n(\underline{\ell})^2 \langle U_{\underline{\ell}}, U_{\underline{\ell}} \rangle \geq c \, V_n. \tag{12}$$

This shows (10), since by (11): $\frac{\sigma_n^2 - \sigma_{n-1}^2}{\sigma_n^2} \leq 2 \frac{C_1^{\frac{1}{2}}}{c} \frac{V_n^{\frac{1}{2}}}{V_n} \to 0.$ □

Remarks: 1) For a stationary r.f., (9) reduces to $\|U_0\|_2 \neq 0$ and $\sum_{\underline{\ell} \in \mathbb{Z}^d} |\langle U_{\underline{\ell}}, U_{\underline{0}} \rangle|^2 < +\infty$.

2) If we assume, instead of the second inequality in (9), that for some $2 \leq q < \infty$: $C_1 := \sup_r \sum_{\underline{\ell}} |\langle U_{\underline{\ell}}, U_r \rangle|^q < +\infty$, then with $p = \frac{q}{q-1}$ and since $N_n(\underline{\ell})^p \leq N_n(\underline{\ell})^2$ we obtain the same conclusion: $\frac{\sigma_n^2 - \sigma_{n-1}^2}{\sigma_n^2} \leq 2 \frac{C_1^{\frac{1}{q}}}{c} \frac{V_n^{\frac{1}{p}}}{V_n} \to 0.$

Proposition 1.6. *1) Suppose that the r.v.'s $U_{\underline{\ell}}, \underline{\ell} \in \mathbb{Z}^d$, are i.i.d., centered, such that $\|U_{\underline{0}}\|_\infty \leq K$ for a constant K and $\|U_0\|_2 = 1$. Then it holds*

$$\limsup_n \frac{|S_n|}{\sqrt{V_n} \, (2 \log \log n)^{\frac{1}{2}}} \leq K, \ \mathbb{P}\text{-}a.e. \tag{13}$$

If $V_n = o(n^2 \, (\log \log n)^{-1})$, then $\lim_n \frac{S_n}{n} = 0$, $\mathbb{P}$-a.e.

2) Suppose that the random field $(U_{\underline{\ell}})$ is stationary centered associated with $\|U_0\|_2 = 1$. Suppose that, for some $2 \leq q < \infty$, $\sum_{\underline{\ell}} |\langle U_{\underline{\ell}}, U_{\underline{0}} \rangle|^q < +\infty$.

Then for all $\varepsilon > 0$, it holds:

$$\limsup_n \frac{|S_n|}{\sigma_n (\log \sigma_n)^{\frac{1}{2}+\varepsilon}} \leq 1, \ \mathbb{P}\text{-}a.e. \tag{14}$$

If $q = 1$, then

$$|S_n| = O(\sqrt{V_n}\,(\log n)^{\frac{1}{2}+\varepsilon}), \ \mathbb{P}\text{-}a.e. \tag{15}$$

If in addition, $V_n \leq Cn^2 (\log n)^{-(1+\eta)}$ for some constants $C, \eta > 0$, then $\lim_n \frac{S_n}{n} = 0$, $\mathbb{P}$-a.e.

Proof. A) In both cases we may apply Lemma 1.5 (see also Remark 2 above). For $q = 1$, by (4): $\sigma_n \leq (\sum_{\underline{p}} \langle U_{\underline{p}}, U_{\underline{0}} \rangle)^{\frac{1}{2}} \sqrt{V_n}$ and trivially $\sigma_n \leq n$, hence (15) follows from (14). By association, σ_n is non-decreasing and tends to infinity (see (12)). We note that in case 1) $\sigma_n = \sqrt{V_n}$.

Let $\rho > 1$. We can construct a strictly increasing sequence of integers (n_k) (depending on ρ), such that $\rho^k < \sigma_{n_k} \leq \rho^{k+1}$. To prove the existence of the sequence (n_k) after a certain rank, let n_k such that $\sigma_{n_k - 1} < \rho^k \leq \sigma_{n_k}$ (which implies $n_k - 1 < \rho^{2k}$).

Claim: $\sigma_{n_k} \leq \rho^{k+1}$, for k big.

Indeed, suppose otherwise that $\sigma_{n_k} > \rho^{k+1}$.

Then we have $(\rho^2 - 1)\rho^{2k} = \rho^{2k+2} - \rho^{2k} \leq \sigma^2_{n_k} - \sigma^2_{n_k - 1}$. As $\sigma^2_{n_k - 1} < \rho^{2k}$, it follows

$$\rho^2 - 1 = \frac{\rho^{2k+2} - \rho^{2k}}{\rho^{2k}} \leq \frac{\sigma^2_{n_k} - \sigma^2_{n_k - 1}}{\sigma^2_{n_k - 1}} \leq \frac{\sigma^2_{n_k}}{\sigma^2_{n_k - 1}} \frac{\sigma^2_{n_k} - \sigma^2_{n_k - 1}}{\sigma^2_{n_k}} \to 0,$$

since by Lemma 1.5 $\frac{\sigma^2_{n_k} - \sigma^2_{n_k - 1}}{\sigma^2_{n_k}} \to 0$ (hence also $\frac{\sigma^2_{n_k}}{\sigma^2_{n_k - 1}} \to 1$). It gives a contradiction and shows the claim.

By the construction of (n_k) we have

$$\rho^k < \sigma_{n_k} \leq \rho^{k+1} < \sigma_{n_{k+1}} \leq \rho^{k+2}. \tag{16}$$

Moreover, we have $\sigma_{n_{k+1}} / \sigma_{n_k} \leq \rho^2$ and $n_k \geq \rho^k$ (since $\sigma_n \leq n$).

Assume for a while the existence of a non decreasing sequence of positive numbers (λ_n) such that

$$\lambda_{n_k} > \sqrt{2}, \ \limsup_k \lambda_{n_{k+1}} / \lambda_{n_k} \leq 1,$$
$$\sum_k \mathbb{P}\big(\big| \textstyle\sum_{i=0}^{n_k - 1} U_{z_i} \big| \geq (\lambda_{n_k} - \sqrt{2}) \, \| \sum_{i=0}^{n_k - 1} U_{z_i} \|_2 \big) < \infty. \tag{17}$$

By the previous inequalities and by Newman-Wright's inequality (2) for the sequence of centered associated random variables [1] $(W_i) = (U_{z_i})$, we have

$$\sum_k \mathbb{P}\Big(\max_{0 \leq j \leq n_k - 1} \big| \sum_{i=0}^{j} U_{z_i} \big| \geq \lambda_{n_k} \| \sum_{i=0}^{n_k - 1} U_{z_i} \|_2 \Big) \leq 2 \sum_k \mathbb{P}\Big(\big| \sum_{j=0}^{n_k - 1} U_{z_j} \big| \geq (\lambda_{n_k} - \sqrt{2}) \| \sum_{j=0}^{n_k - 1} U_{z_j} \|_2 \Big) < +\infty.$$

[1] as it is a subset of a set of associated r.v.'s

By the Borel-Cantelli lemma, it follows:

$$\limsup_k \frac{\max_{0\le j\le n_{k+1}-1}\big|\sum_{i=0}^{j} U_{z_i}\big|}{\lambda_{n_{k+1}}\,\sigma_{n_{k+1}}} \le 1, \mathbb{P}\text{-a.e.}$$

Hence $\mathbb{P}$-a.e.

$$\limsup_k \frac{\max_{0\le j< n_{k+1}-1}|\sum_{i=0}^{j} U_{z_i}|}{\lambda_{n_k}\sigma_{n_k}} \le \limsup_k\big(\frac{\lambda_{n_{k+1}}}{\lambda_{n_k}}\frac{\sigma_{n_{k+1}}}{\sigma_{n_k}}\big) \le \rho^2. \tag{18}$$

Observe that, if $|S_i| > \rho^2\lambda_i\sigma_i$, for some $i \in [n_k, n_{k+1}[$, then $\max_{0\le j<n_{k+1}}|S_j| > \rho^2\lambda_{n_k}\sigma_{n_k}$. This shows:

$$\{|S_n| > \rho^2\lambda_n\sigma_n, \text{ i.o.}\} \subset \{\max_{0\le j<n_{k+1}}|S_j| > \rho^2\lambda_{n_k}\sigma_{n_k}, \text{ i.o.}\}.$$

By this inclusion and (18) it follows: $\limsup_n \frac{|\sum_{i=0}^{n-1} U_{z_i}|}{\lambda_n\sigma_n} \le \rho^2$, $\mathbb{P}$-a.e.

Taking $\rho = \rho_n$ with $\rho_n \downarrow 1$, we obtain

$$\limsup_n \frac{|\sum_{i=0}^{n-1} U_{z_i}|}{\lambda_n\sigma_n} \le 1, \ \mathbb{P}\text{-a.e.} \tag{19}$$

B) Choice of a sequence (λ_n) such that (17) is satisfied.

Case 1)

Suppose that the U_k's are i.i.d. r.v.'s. Recall that, by Hoeffding's inequality for differences of martingale ([20]), if $(W_j, j \in J)$ is any finite family of centered bounded independent random variables on $(\Omega, \mathbb{P})$, it holds:

$$\mathbb{P}(|\sum_{j\in J} W_j| > \varepsilon) \le 2\exp(-\frac{1}{2}\frac{\varepsilon^2}{\sum_{j\in J}\|W_j\|_\infty^2}),\ \forall\varepsilon > 0. \tag{20}$$

We apply it to the family $(N_n(\underline{\ell})U_{\underline{\ell}},\ \underline{\ell} \in \mathbb{Z}^d)$. From the hypotheses, we have:

$$\sum_{\underline{\ell}} \|N_n(\underline{\ell})U_{\underline{\ell}}\|_\infty^2 \le K^2\sum_{\underline{\ell}} N_n^2(\underline{\ell}) = K^2V_n.$$

With $\varepsilon = (\lambda - \sqrt{2})\sqrt{V_n}$, (20) implies (note that here $\sigma_n^2 = V_n$):

$$\mathbb{P}\big(\big|\sum_{\ell} N_n(\ell)\,U_{\underline{\ell}}\big| \ge (\lambda-\sqrt{2})\sqrt{V_n}\big)$$
$$\le 2\exp\big(-\frac{1}{2}(\lambda-\sqrt{2})^2\frac{V_n}{K^2V_n}\big) = 2\exp\big(-\frac{1}{2K^2}(\lambda-\sqrt{2})^2\big).$$

Let c, δ be such that $c > \delta > K^2$. In the previous inequality, we take

$$\lambda = \lambda_n = (2c\log\log n)^{\frac{1}{2}}.$$

Let $k(c,\delta)$ be such that $\lambda_{n_k} > \sqrt{2}$ and $c(1 - \frac{2}{\sqrt{c \log\log n_k}}) \geq \delta$, for $k \geq k(c,\delta)$. As $n_k \geq \rho^k$, we have:

$$\sum_{k=k(c,\delta)}^{\infty} \mathbb{P}\big(\big|\sum_{i=1}^{n_k-1} U_{z_i}\big| \geq (\lambda_{n_k} - \sqrt{2}) \,\|\sum_{i=1}^{n_k-1} U_{z_i}\|_2\big) \leq 2 \sum_{k=k(c,\delta)}^{\infty} \exp\big(-\frac{1}{2K^2}(\lambda_{n_k} - \sqrt{2})^2\big)$$

$$\leq \frac{2}{\exp K^2} \sum_{k=k(c,\delta)}^{\infty} \exp\big(-\frac{c}{K^2}\log\log n_k)(1 - \frac{2}{\sqrt{c\log\log n_k}})\big)$$

$$\leq \frac{2}{\exp K^2} \sum_{k=k(c,\delta)}^{\infty} \frac{1}{(k\log\rho)^{\frac{\delta}{K^2}}} < \infty.$$

Now we can apply (19). It follows:

$$\limsup_n \frac{|\sum_{i=0}^{n-1} U_{z_i}|}{\sqrt{2c(\log\log n)V_n}} \leq 1, \mathbb{P}\text{-a.e.}$$

Taking $c = c_r > \delta_r > K$ with c_r and $\delta_r \downarrow K^2$, we get (13).

Case 2)

For general associated r.v.'s, we use simply that $\mathbb{P}\big(\big|\sum_{i=0}^{n-1} U_{z_i}\big| \geq \lambda \,\|\sum_{i=0}^{n-1} U_{z_i}\|_2\big) \leq \frac{1}{\lambda^2}$.

We take $\lambda_n = (\log\sigma_n)^{\frac{1}{2}+\varepsilon}$, with $\varepsilon > 0$. By (16) we have $\lambda_{n_k} \geq (k\log\rho)^{\frac{1}{2}+\varepsilon}$, and therefore, for a constant C_1: $\sum_k \frac{1}{\lambda^2_{n_k}} \leq C_1 \sum_k k^{-(1+2\varepsilon)} < +\infty$; hence condition (17).

Moreover we have $k\log\rho \leq \log\sigma_{n_k} \leq \log\sigma_{n_{k+1}} \leq (k+2)\log\rho$; hence

$$\frac{\lambda_{n_{k+1}}}{\lambda_{n_k}} = \big(\frac{\log\sigma_{n_{k+1}}}{\log\sigma_{n_k)}}\big)^{\frac{1}{2}+\varepsilon} \leq (1 + \frac{2}{k})^{\frac{1}{2}+\varepsilon} \to 1.$$

By (19), this proves (14). □

1.2. **A Glivenko-Cantelli type theorem.**

Empirical process

Let us consider a random field of r.v.'s $(X_{\underline{\ell}}, \underline{\ell} \in \mathbb{Z}^d)$ on $(\Omega, \mathcal{F}, \mathbb{P})$ with common distribution function F. Let $(z_k) \subset \mathbb{Z}^d$ be a sequence with self-intersections (V_n).

Notation. We say that $(X_{\underline{\ell}}, \underline{\ell} \in \mathbb{Z}^d)$ satisfies a Glivenko-Cantelli theorem along a sequence (z_k) in $\mathbb{Z}^d$ if

$$\limsup_n \sup_s |\frac{1}{n}\sum_{k=1}^{n} \mathbf{1}_{(-\infty,s]}(X_{z_k}(\omega)) - F(s)| = 0, \text{ for } \mathbb{P}\text{-a.e.}\omega.$$

We show now a Glivenko-Cantelli theorem along a sequence (z_k) under various hypotheses on (z_k) and on $(X_{\underline{\ell}})$ (mixing, i.i.d., associated or PQD).

Le $(X_{\underline{\ell}}, \underline{\ell} \in \mathbb{Z}^d)$ be a r.f. Denoting by $\sigma(X_{\underline{\ell}})$ the σ-algebra generated by the random variable $X_{\underline{\ell}}$, we define coefficients of mixing by

$$\gamma(\underline{\ell}) := \sup_{r\in\mathbb{Z}^d} \sup_{A\in\sigma(X_{\underline{r}}),\, B\in\sigma(X_{\underline{\ell}+\underline{r}})} |\mathbb{P}(A\cap B) - \mathbb{P}(A)\mathbb{P}(B)|. \tag{21}$$

Observe that for every $s, t \in \mathbb{R}$ it holds:

$$\sup_{\underline{r}\in\mathbb{Z}^d} |\langle 1_{X_{\underline{r}}\le s} - \mathbb{P}(X_{\underline{r}} \le s), 1_{X_{\underline{\ell}+\underline{r}}\le t} - \mathbb{P}(X_{\underline{\ell}+\underline{r}} \le t)\rangle| \le \gamma(\underline{\ell}), \forall \underline{\ell} \in \mathbb{Z}^d. \tag{22}$$

By (22) and Proposition 1.3, we get:

Theorem 1.7. *Let (z_k) be such that $V_n \le C_1 \dfrac{n^2}{(\log n)^\beta}$, for constants $C_1 > 0, \beta$.*
If $\sum_{\underline{\ell}\in\mathbb{Z}^d} \gamma(\underline{\ell}) < +\infty$ and $\beta > 1$,
or if the r.f. is stationary and $\sum_{\underline{\ell}\in\mathbb{Z}^d} \gamma(\underline{\ell})^\zeta < +\infty$, for some $\zeta \in [1, 2]$ and $\beta > \zeta$,
then $(X_{\underline{\ell}}, \underline{\ell} \in \mathbb{Z}^d)$ satisfies a Glivenko-Cantelli theorem along (z_k).

In the next theorem, we use Proposition 1.6 for the i.i.d. and the associated cases.

Theorem 1.8. *a) If $(X_{\underline{\ell}})_{\underline{\ell}\in\mathbb{Z}^d}$ is a r.f. of i.i.d. r.v.'s, then under the condition $V_n = o(n^2 (\log\log n)^{-1})$ it satisfies a Glivenko-Cantelli theorem along (z_k).*

b) If $(X_{\underline{\ell}})_{\underline{\ell}\in\mathbb{Z}^d}$ is a strictly stationary r.f. of associated r.v.'s such that $\sum_{\underline{\ell}} \langle X_{\underline{\ell}}, X_{\underline{0}}\rangle$ converges, then, under the condition $V_n = O(n^2 \log^{-(1+\eta)} n)$ for some $\eta > 0$, for a.e. ω, we have for each continuity point s of F:

$$\lim_n \frac{1}{n} \sum_{k=0}^{n-1} \mathbf{1}_{(-\infty,s]}(X_{z_k}(\omega)) = F(s). \tag{23}$$

If F is continuous, the convergence is uniform in s.

Proof. a) Denote by $F_n(s)(\omega)$ the means $\frac{1}{n}\sum_{k=0}^{n-1} \mathbf{1}_{(-\infty,s]}(X_{z_k}(\omega))$. Let Q be a dense countable set of continuity points of F.

For every $s \in Q$, by the assumption on V_n and Proposition 1.6, there is a null set $N(s)$ such that, for a sequence ε_n tending to 0, for every $\omega \notin N(s)$,

$$|F_n(s)(\omega) - F(s)| \le \varepsilon_n (V_n \log\log n)^{-\frac{1}{2}} |\sum_{k=0}^{n-1} \big(\mathbf{1}_{(-\infty,s]}(X_{z_k}) - F(s)\big)| \to 0.$$

Then $F_n(s)(\omega) \to F(s)$ for every ω outside the null set $N := \cup_{s\in Q} N(s)$ and for $s \in Q$.

Similarly by Proposition 1.6, for every s in the set J of jumps of F, we have $F_n(s)(\omega) - F_n(s^-)(\omega) \to F(s) - F(s^-)$ a.e. As J is countable, this convergence holds for every $s \in J$ and $\omega \notin \tilde{N}$, where $\tilde{N}$ is a null set.

Outside the null set $N \cup \tilde{N}$, Lemma 1.1 applied with Q and J implies the result.

b) We consider now the case of a strictly stationary random field of associated r.v.'s. $(X_{\underline{\ell}})_{\underline{\ell}\in\mathbb{Z}^d}$. Recall that $\langle X_{\underline{\ell}}, X_{\underline{0}}\rangle \ge 0, \forall \underline{\ell} \in \mathbb{Z}^d$, by association.

Let s be a continuity point of the common distribution F. For every $\epsilon > 0$ there exists $\delta > 0$, such that $F(s+\delta) - F(s-\delta) \leq \epsilon$. As in [31], for $\delta > 0$ and s, define the approximated step function $h_{\delta,s}$ by $h_{\delta,s}(x) = 0$, if $x \leq s - \delta$ and $h_{\delta,s}(x) = 1 + \frac{x-s}{\delta}$ if $s - \delta \leq x \leq s$, otherwise, $h_{\delta,s}(x) = 1$. It is a non decreasing continuous function with $h'_{\delta,s}(x) = 1/\delta$ for $s - \delta < x < s$. It follows from the above Hoeffding's identity applied to this approximated step function (see [2]):

$$Cov(h_{\delta,s}(X_{\underline{\ell}}), h_{\delta,s}(X_{\underline{0}})) \leq \delta^{-2}\langle X_{\underline{\ell}}, X_{\underline{0}}\rangle,$$
$$Cov(h_{\delta,s+\delta}(X_{\underline{\ell}}), h_{\delta,s+\delta}(X_{\underline{0}})) \leq \delta^{-2}\langle X_{\underline{\ell}}, X_{\underline{0}}\rangle.$$

As we compose associated r.v.'s by non decreasing functions, $\big(h_{\delta,s}(X_{\underline{\ell}})\big)$ as well as $\big(h_{\delta,s+\delta}(X_{\underline{\ell}})\big)$ are stationary r.f.s of associated r.v.'s, and we may apply Proposition 1.6 to their centered versions (also associated). The condition simply reads, for $\tau = s, s+\delta$:

$$\sum_{\underline{\ell}} Cov(h_{\delta,\tau}(X_{\underline{\ell}}), h_{\delta,\tau}(X_{\underline{0}})) \leq \delta^{-2} \sum_{\underline{\ell}} \langle X_{\underline{\ell}}, X_{\underline{0}}\rangle < \infty.$$

We put $\overline{S}_n = \sum_{k=0}^{n-1} h_{\delta,s}(X_{z_k})$ and $\underline{S}_n = \sum_{k=0}^{n-1} h_{\delta,s+\delta}(X_{z_k})$. By $h_{\delta,s+\delta}(x) \leq \mathbf{1}_{\{x>s\}} \leq h_{\delta,s}(x)$, it holds $\underline{S}_n \leq \sum_{k=0}^{n-1} \mathbf{1}_{(s,\infty)}(X_{z_k}) \leq \overline{S}_n$. Hence by Proposition 1.6, we have almost everywhere $\frac{1}{n}\underline{S}_n \to \mathbb{E}[h_{\delta,s+\delta}(X_{\underline{0}})]$ and $\frac{1}{n}\overline{S}_n \to \mathbb{E}[h_{\delta,s}(X_{\underline{0}})]$. Since

$$\mathbb{E}[h_{\delta,s}(X_{\underline{0}})] \leq F(s) - F(s-\delta) + 1 - F(s) \leq \epsilon + 1 - F(s),$$
$$\mathbb{E}[h_{\delta,s+\delta}(X_{\underline{0}})] \geq 1 - F(s+\delta) = 1 - F(s) - (F(s+\delta) - F(s)) \geq 1 - F(s) - \epsilon,$$

we conclude

$$1 - F(s) - \epsilon \leq \liminf_n \frac{1}{n}\sum_{k=0}^{n-1} \mathbf{1}_{(s,\infty)}(X_{z_k}) \leq \limsup_n \frac{1}{n}\sum_{k=0}^{n-1} \mathbf{1}_{(s,\infty)}(X_{z_k}) \leq 1 - F(s) + \epsilon.$$

Subtracting the 1's and taking $\epsilon \to 0$, we get (23). □

PQD variables.

The result shown for associated variables can be extended to the class of PDQ variables, but with a stronger condition on the local times of the sequence (z_k).

Proposition 1.9. *Let $(U_{\underline{\ell}})$ be a stationary random field of pairwise PQD square integrable centered r.v.'s such that $\sum_{\underline{\ell}}\langle U_{\underline{\ell}}, U_{\underline{0}}\rangle$ converges. Let (z_k) be a sequence of points with maximal local times (M_n). If $\sum_{n\geq 1} \frac{M_n}{n^2} < +\infty$, then $\lim_n \frac{1}{n}\sum_{k=0}^{n-1} U_{z_k} = 0$ a.e.*

Proof. We apply the following result of [4]: let $(Y_j : j \geq 1)$ be a sequence of pairwise centered PQD r.v.'s. with finite variance. If $\sum_{j\geq 1} j^{-2}\mathrm{Cov}(Y_j, \sum_{i=1}^{j} Y_i)$ converges and $\sup_j \mathbb{E}|Y_j| < \infty$, then $n^{-1}\sum_{i=1}^{n} Y_i \to 0$ a.e.

Taking for Y_j the (still) pairwise PQD r.v.'s U_{z_j}, we get the result, since $\mathrm{Cov}(U_{z_j}, U_{z_1} + \cdots + U_{z_j}) \leq M_j \sum_{\underline{\ell}}\langle U_{\underline{0}}, U_{\underline{\ell}}\rangle$. □

Now, we consider the empirical distribution.

Theorem 1.10. *Let $(X_{\underline{\ell}})$ be a strictly stationary random field of pairwise PQD square integrable centered r.v.'s with distribution function F such that $\sum_{\underline{\ell}}\langle X_{\underline{\ell}}, X_{\underline{0}}\rangle$ converges. Let*

(z_k) be a sequence of points with maximal local times (M_n). Assume that $\sum_{n\geq 1}\frac{M_n}{n^2}<+\infty$. Then for each continuity point s of F, we have for a.e. ω: $\lim_n \frac{1}{n}\sum_{k=0}^{n-1}\mathbf{1}_{(-\infty,s]}(X_{z_k}(\omega))=F(s)$, with uniform convergence over s if F is continuous.

Proof. The r.f.s $h_{\delta,s}(X_{\underline{\ell}})$ and $h_{\delta,s+\delta}(X_{\underline{\ell}})$ are still stationary pairwise PQD. The proof is analogous to the proof of Theorem 1.8. When F is continuous, we use Lemma 1.1 to show uniformity in the convergence. □

Remark. If $M_n=O(n\,(\log n)^{-(1+\eta)})$, then $V_n=O(n^2\,(\log n)^{-(1+\eta)})$.

If $V_n\leq C\dfrac{n^2}{(\log n)^\beta}$, with $\beta>2$, then $\displaystyle\sum_{n\geq 1}\frac{M_n}{n^2}<+\infty$.

As shown in Section 3, $\sum_{n\geq 1}\frac{M_n}{n^2}$ converges when the sampling is done along random walks, but diverges in some examples of sampling along "deterministic" random walks.

1.3. **A sufficient condition for a FCLT for the sampled empirical process.**

After a Glivenko-Cantelli theorem for sampled empirical processes, we consider now the Functional Central Limit Theorem (FCLT). Let (z_k) be in $\mathbb{Z}^d, d\geq 1$, with the associated quantities $N_n(\underline{\ell})$, M_n and V_n defined by (1).

Before restricting to a r.f. of i.i.d. r.v.'s, first we examine the variance in the more general situation where the series of correlations is absolutely summable.

Kernel associated to a sequence (z_k) and variance.

Let K_n be the kernel (which is a real even function on $\mathbb{T}^d$ depending on $n\geq 0$) defined by the equivalent formulas:

$$K_n(\underline{t}) = |\sum_{k=0}^{n-1}e^{2\pi i\langle z_k,\underline{t}\rangle}|^2 = n+2\sum_{k=1}^{n-1}\sum_{j=0}^{n-k-1}\cos(2\pi\langle z_{k+j}-z_j,t\rangle) = |\sum_{\underline{\ell}\in\mathbb{Z}^d}N_n(\underline{\ell})\,e^{2\pi i\langle\underline{\ell},t\rangle}|^2$$

$$= n+2\sum_{\underline{\ell}}\Big(\sum_{k=1}^{n-1}\sum_{j=0}^{n-k-1}1_{z_{k+j}-z_j=\underline{\ell}}\Big)\cos(2\pi\langle\underline{\ell},t\rangle). \tag{24}$$

If $(X_{\underline{\ell}},\underline{\ell}\in\mathbb{Z}^d)$ is a stationary r.f. such that $\sum_{\underline{\ell}\in\mathbb{Z}^d}|\langle X_{\underline{\ell}},X_{\underline{0}}\rangle|<+\infty$, the spectral density ρ is continuous and we have:

$$\int|\sum_{k=0}^{n-1}X_{z_k}|^2d\mathbb{P}=\int_{\mathbb{T}^d}K_n(\underline{t})\,\rho(\underline{t})\,d\underline{t}\leq\|\rho\|_\infty V_n\leq(\sum_{\underline{\ell}\in\mathbb{Z}^d}|\langle X_{\underline{\ell}},X_{\underline{0}}\rangle|)V_n.$$

One can ask if there is an asymptotic variance, i.e., a limit for the normalised quantity $V_n^{-1}\displaystyle\int|\sum_{k=0}^{n-1}X_{z_k}|^2d\mathbb{P}$ which is bounded if the series of correlations is absolutely summable.

The existence of asymptotic variance is shown in [11] in the case of summation along a random walk. We will come back to the question of its positivity for transient random walks in Subsection 3.1.

Functional Central limit Theorem in the i.i.d. case

The following result gives a sufficient condition for a Functional Central limit Theorem (FCLT) along a sequence (z_k) in the i.i.d. case.

The standard Brownian bridge process $W^0(s)$ is the centered Gaussian process $W^0(s) := W(s) - sW(1)$ in $C(0,1)$, where $W(s)$ is the Wiener process. It has the properties $W^0(0) = W^0(1) = 0$ and $\mathbb{E}[W^0(s_1)W^0(s_2)] = s_1 \wedge s_2 - s_1 s_2$.

Let $(X_{\underline{k}})_{\underline{k}\in\mathbb{Z}^d}$ be i.i.d. random variables with common probability distribution F in $[0,1]$. We put $W_F^0 = W^0 \circ F$. Let $Y_n(s)$ be the random element in $D[0,1]$ defined by

$$Y_n(s) = \frac{1}{\sqrt{V_n}} \sum_{k=0}^{n-1} [\mathbf{1}_{X_{z_k}\le s} - F(s)] = \frac{1}{\sqrt{V_n}} \sum_{\underline{\ell}\in\mathbb{Z}^d} N_n(\ell)\, [\mathbf{1}_{X_{\underline{\ell}}\le s} - F(s)].$$

Theorem 1.11. *$Y_n(s) \to_{D[0,1]} W_F^0(s)$, if (z_k) satisfies the condition*

$$\lim_n \frac{M_n^2}{V_n} = 0, \tag{25}$$

Proof. The result follows from the Cramér-Wold device, if we prove convergence of the finite dimensional distributions and tightness. The variance is

$$\mathbb{E}[Y_n(s)]^2 = \frac{1}{V_n} \sum_{\underline{\ell}} N_n^2(\ell)\, \mathbb{E}[\mathbf{1}_{X_{\underline{\ell}}\le s} - F(s)]^2 = \sigma^2(s) = F(s)(1-F(s)). \tag{26}$$

1) *Finite dimensional distributions.* The convergence follows from Lindeberg's theorem for triangular arrays of independent random variables as in [3, thm 7.2]. The Lindeberg's condition for the triangular array of independent r.v.'s $\big(\frac{N_n(\ell)[\mathbf{1}_{X_{\underline{\ell}}\le s} - F(s)]}{\sqrt{V_n}}\big)_{\underline{\ell},n}$ follows from

$$\begin{aligned} & \frac{1}{V_n} \sum_{\underline{\ell}} \int_{\{N_n(\underline{\ell})|\mathbf{1}_{X_{\underline{\ell}}\le s} - F(s)| \ge \varepsilon\sqrt{V_n}\}} N_n^2(\underline{\ell})\, |\mathbf{1}_{X_{\underline{\ell}}\le s} - F(s)|^2 d\mathbb{P} \\ \le\ & \frac{1}{V_n} \sum_{\underline{\ell}} N_n^2(\underline{\ell}) \int_{\{\sup_{\underline{\ell}} N_n(\underline{\ell})|\mathbf{1}_{X_{\underline{0}}\le s} - F(s)| \ge \varepsilon\sqrt{V_n}\}} |\mathbf{1}_{X_{\underline{0}}\le s} - F(s)|^2 d\mathbb{P} \to 0, \end{aligned}$$

for every $\varepsilon > 0$, since $V_n = \sum_{\underline{\ell}} N_n^2(\underline{\ell})$ and $\dfrac{\sup_{\underline{\ell}} N_n(\underline{\ell})}{\sqrt{V_n}} \to 0$, by assumption.

For the correlation of the process taken at s_1 and s_2, it holds by independence:

$$\begin{aligned} \mathbb{E}[Y_n(s_1)Y_n(s_2)] &= \frac{1}{V_n} \sum_{\underline{\ell}_1, \underline{\ell}_2} N_n(\underline{\ell}_1) N_n(\underline{\ell}_2) \mathbb{E}[(\mathbf{1}_{X_{\underline{\ell}_1}\le s_1} - F(s_1))(\mathbf{1}_{X_{\underline{\ell}_2}\le s_2} - F(s_2))] \\ &= \frac{1}{V_n} \sum_{\ell} N_n^2(\underline{\ell})(F(s_1\wedge s_2) - F(s_1)F(s_2)) = F(s_1\wedge s_2) - F(s_1)F(s_2). \end{aligned}$$

This proves the convergence in distribution: $Y_n(s) \to W_F^0(s)$ for every s.

Let us show now the convergence of the finite dimensional distributions. Starting with the asymptotic distribution of $aY_n(s_1) + bY_n(s_2)$, by the above computation, we have

$$\mathbb{E}[(aY_n(s_1) + bY_n(s_2))^2] =$$
$$a^2F(s_1)(1 - F(s_1)) + b^2F(s_2)(1 - F(s_2)) + 2ab(F(s_1 \wedge s_2) - F(s_1)F(s_2)). \tag{27}$$

As above, it is easily seen that Lindeberg's condition is satisfied for the triangular array defined by $aY_n(s_1)+bY_n(s_2)$. It means that the asymptotic distribution of $aY_n(s_1)+bY_n(s_2)$ is centered Gaussian with variance as computed above.

Note that $\mathbb{E}[(aW^0(s_1) + bW^0(s_2))^2]$ is also given by (27) above.

Similarly, for every $s_1 \leq \cdots \leq s_r$, it holds

$$(Y_n(s_1), \cdots, Y_n(s_r)) \to_{dist} (W^0_F(s_1), \ldots, W^0_F(s_r)).$$

Tightness. First we suppose F continuous. We proceed as in Billingsley (cf. [3, Theorem 15.6]). It is enough to show that for $s \leq t \leq r$, uniformly in n,

$$\mathbb{E}[(Y_n(t) - Y_n(s))^2(Y_n(r) - Y_n(t))^2] \leq C(F(r) - F(s))^2.$$

Putting $F(u, v) := F(v) - F(u)$, $f(\underline{\ell}, u, v) := \mathbf{1}_{u<X_{\underline{\ell}}\leq v} - F(u, v)$, for $u < v$, we compute

$$\mathbb{E}[(Y_n(t) - Y_n(s))^2(Y_n(r) - Y_n(t))^2] = \frac{1}{V_n^2}\mathbb{E}\big[\big(\sum_{\underline{\ell}} N_n(\underline{\ell})f(\underline{\ell}, s, t)\big)^2\big(\sum_{\underline{\ell}} N_n(\underline{\ell})f(\underline{\ell}, t, r)\big)^2\big].$$

By expansion and independence, the above expression is sum of three types of terms:

$$\frac{1}{V_n^2}\sum_{\underline{\ell}} N_n^4(\underline{\ell})\,[A],\ \frac{1}{V_n^2}\sum_{\underline{\ell}_1,\underline{\ell}_2} N_n^2(\underline{\ell}_1)N_n^2(\underline{\ell}_2)\,[B],\ \frac{1}{V_n^2}\sum_{\underline{\ell}_1\neq\underline{\ell}_2} N_n^2(\underline{\ell}_1)N_n^2(\underline{\ell}_2)\,[C],$$

with $A = \mathbb{E}[f^2(\underline{\ell}, s, t)f^2(\underline{\ell}, t, r)], B = \mathbb{E}[f^2(\underline{\ell}_1, s, t)]\,\mathbb{E}[f^2(\underline{\ell}_2, t, r)]$,
$C = \mathbb{E}[f(\underline{\ell}_1, s, t)f(\underline{\ell}_1, t, r))]\,\mathbb{E}[f(\underline{\ell}_2, s, t)f(\underline{\ell}_2, t, r))]$.

By stationarity and since the intervals do not overlap, we have

$$\begin{aligned} A &= F(s, t)F^2(t, r) + F^2(s, t)F(t, r) - 3F^2(s, t)F^2(t, r), \\ B &= F(s, t)(1 - F(s, t)) \cdot F(t, r)(1 - F(t, r)),\ C = F^2(s, t)F^2(t, r). \end{aligned}$$

Since $0 \leq F(s, t), F(t, r), F(s, r) \leq 1$ and $F(s, t), F(t, r) \leq F(s, r)$, it follows

$$A \leq 2F^3(s, r) \leq 2F^2(s, r),\ B \leq F^2(s, r),\ C \leq F^4(s, r) \leq F^2(s, r).$$

Recall that $V_n = \sum_{\underline{\ell}} N_n^2(\underline{\ell})$. Using $\|\cdot\|_{\ell_4} \leq \|\cdot\|_{\ell_2}$ for the bound of the first term, we have for some fixed constant $C > 0$:

$$\mathbb{E}[(Y_n(t) - Y_n(s))^2(Y_n(r) - Y_n(t))^2] \leq C(F(r) - F(s))^2,\ \forall n.$$

Hence by [3, Theorem 15.6], for non decreasing continuous F, the sequence of processes $(Y_n(s))$ is tight in $D(0, 1)$. This proves that, if F is continuous, then $Y_n \to_{D(0,1)} (W^0 \circ F)$.

Now, for a general F a classical method is to use a generalized inverse. Let us describe it briefly. We consider first the uniform empirical process. Let (ζ_k) be uniformly distributed

i.i.d. r.v.'s. Denote the empirical process along (z_k) with respect to (ζ_k) by $U_n(s)$. By applying what we have just proved for a continuous distribution, $U_n(s) \to_{D(0,1)} W^0(s)$.

Now let $F^{-1}(t) := \inf\{s : t \le F(s)\}$. We get $\mathbb{P}(F^{-1}(\zeta_0) \le s) = \mathbb{P}(X_0 \le s) = F(s)$, so $Y_n(s) =_{dist.} U_n(F(s))$. As in ([3, Theorem 5.1]), we deduce the FCLT for $Y_n(s)$ with $W^0(F(s))$ as limit. □

2. **Local times for ergodic sums**

In the previous section about limit theorems for the empirical process sampled along (z_k), we found sufficient conditions on the quantities V_n and M_n associated to (z_k). When (z_k) is given by a "cocycle", $z_k = S_k f(x)$, one can ask if these conditions are satisfied.

In this section we start with some general facts and then construct counterexamples for which condition (25) is not satisfied. In the next section, we will discuss two very different examples of cocycles: first the case of random walks, then "stationary random walks" generated by a rotation.

2.1. **Auxiliary general results.**

First we introduce some notation and make general remarks.

Notation 2.1. Let $(X, \mathcal{B}, \mu)$ be a probability space and T a measure preserving transformation acting on X such that the dynamical system $(X, \mathcal{B}, \mu, T)$ is ergodic.

Let f be a measurable function on X with values in $\mathbb{Z}^d$, $d \ge 1$. Its ergodic sums generated by the iteration of T, denoted by f_k (or $S_k f$), are

$$f_k(x) := \sum_{j=0}^{k-1} f(T^j x), k \ge 1,\ f_0(x) = \underline{0}.$$

The sequence $(f_k(x), k \ge 1)$ can be viewed as a "stationary random walk" defined on $(X, \mathcal{B}, \mu)$. It will be called a "cocycle" and denoted by (μ, T, f) or simply (T, f).

For $x \in X$, we put (cf. (1)) $N_0(x, \underline{\ell}) = 0$ and, for $n \ge 1$,

$$\begin{aligned} N_n(T, f, x, \underline{\ell}) &:= \#\{1 \le k \le n :\ f_k(x) = \underline{\ell}\} = \sum_{k=1}^{n} 1_{f_k(x)} = \underline{\ell},\ \underline{\ell} \in \mathbb{Z}^d, \\ M_n(T, f, x) &:= \max_{\underline{\ell} \in \mathbb{Z}^d} N_n(T, f, x, \underline{\ell}), \\ V_n(T, f, x) &:= \#\{1 \le j, k \le n :\ f_j(x) = f_k(x)\} = \sum_{\underline{\ell} \in \mathbb{Z}^d} N_n^2(T, f, x, \underline{\ell}) \ge M_n^2(T, f, x). \end{aligned}$$

Most of the time, we will omit T and f in the notation and write simply $N_n(x, \underline{\ell})$, $M_n(x)$, $V_n(x)$. We have $\sum_{\underline{\ell}} N_n(x, \underline{\ell}) = n$ and $n \le V_n(x) \le n\, M_n(x)$; hence

$$\frac{M_n^2(x)}{n^2} \le \frac{V_n(x)}{n^2} \le \frac{M_n(x)}{n}. \tag{28}$$

In order to apply the previous results, a question is the validity of the following conditions for a.e. x:

$$V_n(x) = o(n^2\,(\log\log n)^{-1}) \text{ or } V_n(x) \le C_1\frac{n^2}{(\log n)^\beta},\ \text{with } \beta > 1, \tag{29}$$

$$\lim_n \frac{M_n^2(x)}{V_n(x)} = 0. \tag{30}$$

For a random walk this is related to a question studied in [17] and later in [15]: How many times does the walk revisit the most frequently visited site in the first n steps?

Cylinder map. We denote by $\tilde{T}_f$ the map (sometimes called cylinder map) $(x, \underline{\ell}) \to (Tx, \underline{\ell} + f(x))$ acting on $X \times \mathbb{Z}^d$, endowed with the infinite invariant measure $\tilde{\mu}$ defined as the product of μ by the counting measure on $\mathbb{Z}^d$.

For $\varphi : X \times \mathbb{Z}^d \to \mathbb{R}$ the ergodic sums for the cylinder map are

$$\tilde{S}_n\varphi(x, \underline{\ell}) := \sum_{k=0}^{n-1} \varphi(\tilde{T}_f^k(x, \underline{\ell})) = \sum_{k=0}^{n-1} \varphi(T^k x, \underline{\ell} + f_k(x)).$$

With $\varphi_0 := \mathbf{1}_{X\times\{\underline{0}\}}$, it holds

$$\tilde{S}_n\varphi_0(x, -\underline{\ell}) = \sum_{k=0}^{n-1} \mathbf{1}_{X\times\{\underline{0}\}}(T^k x, -\underline{\ell} + f_k(x)) = \#\{0 \le k \le n-1 :\ f_k(x) = \underline{\ell}\}.$$

Therefore, $\tilde{S}_n\varphi_0(x, -\underline{\ell}) = N_n(\underline{\ell})(x)$.

For $x \in X$, let $\underline{\ell}(n, x)$ (a most visited site by $S_k(x)$ up to time n) be defined by

$$\begin{aligned} \underline{\ell}(n,x) &:= \underline{0},\ \text{if } N_n(x,\underline{0}) \ge N_n(x,\underline{\ell}),\ \text{for all } \underline{\ell} \ne 0, \\ \text{else} &:= \underline{\ell}_1,\ \text{if } \ell_1 \text{ is such that } M_n(x) = N_n(x,\underline{\ell}_1) > N_n(x,\underline{0}). \end{aligned}$$

Lemma 2.2. *The following formulas hold for the quantities defined in 2.1.*

$$V_n(x) = n + 2\sum_{k=1}^{n-1}\sum_{j=0}^{n-k-1}(1_{f_k(T^j x)=\underline{0}}), \tag{31}$$

$$V_n(x) = 2[N_{n-1}(Tx,0) + N_{n-2}(T^2x,0) + ... + N_1(T^{n-1}x,0)] + n,\ n \ge 2, \tag{32}$$

$$M_n(x) = \max[N_n(x,0), 1 + \max_{1\le k\le n-1} N_{n-k}(T^k x,0)] \le 1 + \max_{0\le k\le n-1} N_n(T^k x,0), \tag{33}$$

$$M_n(x) = M_{n-1}(Tx) + 1_{\underline{\ell}(n-1,Tx)=\underline{0}} \le M_{n-1}(Tx) + 1. \tag{34}$$

Proof. a) From $f_k(x) = f(x) + f_{k-1}(Tx)$, $k \ge 1$, it follows

$$N_n(x, \underline{\ell}) = N_{n-1}(Tx, \underline{\ell} - f(x)) + 1_{f(x)=\underline{\ell}},\ n \ge 1. \tag{35}$$

Therefore we have:

$$\sum_{\underline{\ell}\in\mathbb{Z}^d} N_n^2(x,\underline{\ell}) = \sum_{\underline{\ell}\in\mathbb{Z}^d}[N_{n-1}(Tx,\underline{\ell}-f(x)) + 1_{f(x)=\underline{\ell}}]^2 = \sum_{\underline{\ell}\in\mathbb{Z}^d}[N_{n-1}(Tx,\underline{\ell}) + 1_{\underline{\ell}=\underline{0}}]^2$$
$$= \sum_{\underline{\ell}\neq 0}[N_{n-1}(Tx,\underline{\ell})]^2 + [N_{n-1}(Tx,\underline{0})+1]^2 = \sum_{\underline{\ell}}[N_{n-1}(Tx,\underline{\ell})]^2 + 2N_{n-1}(Tx,\underline{0}) + 1.$$

Hence the relation

$$V_n(x) = V_{n-1}(Tx) + 2N_{n-1}(Tx,\underline{0}) + 1. \tag{36}$$

We have $V_1(x) = 1$ and by the previous relation we get by induction (31) and (32).

b) Let $p_n(x) \in [1,n]$ be the first visit of $S_k(x)$ to $\underline{\ell}(n,x)$ for $k = 1,...,n$. By definition $M_n(x) = N_n(x,\underline{\ell}(n,x))$.

We have $M_n(x) = N_n(x,\underline{0})$ if $\underline{\ell}(n,x) = \underline{0}$, else $M_n(x) = N_{n-p_n(x)}(T^{p_n(x)}x,\underline{0}) + 1$, by the cocycle relation $S_{p_n(x)+k}(x) - S_{p_n(x)}(x) = S_k(T^{p_n(x)}x)$. This implies:

$$M_n(x) \leq \max[N_n(x,\underline{0}), N_{n-p_n(x)}(T^{p_n(x)}x,\underline{0})+1] \leq \max[N_n(x,\underline{0}), \max_{1\leq k\leq n} N_{n-k}(T^k x,\underline{0})+1].$$

It follows (noticing that $N_0(x,\underline{0}) = 0$):

$$M_n(x) \leq 1 + \max_{0\leq k\leq n-1} N_{n-k}(T^k x,\underline{0}) \leq 1 + \max_{0\leq k\leq n-1} N_n(T^k x,\underline{0}). \tag{37}$$

This shows (33).

c) Observe also the following relation: by (35) we have:

$$\begin{aligned} M_n(x) &= \sup_{\underline{\ell}}[N_{n-1}(Tx,\underline{\ell}-f(x)) + 1_{\underline{\ell}-f(x)=\underline{0}}] = \sup_{\underline{\ell}}[N_{n-1}(Tx,\underline{\ell}) + 1_{\underline{\ell}=\underline{0}}] \\ &= \max\,[\sup_{\underline{\ell}\neq\underline{0}} N_{n-1}(Tx,\underline{\ell}),\ N_{n-1}(Tx,\underline{0})+1]. \end{aligned}$$

If $\underline{\ell}(n-1,Tx) = \underline{0}$, then $N_{n-1}(Tx,\underline{0}) \geq \sup_{\underline{\ell}\neq\underline{0}} N_{n-1}(Tx,\underline{\ell})$. If $\underline{\ell}(n-1,Tx) \neq \underline{0}$, then $N_{n-1}(Tx,\underline{0}) < \sup_{\underline{\ell}\neq\underline{0}} N_{n-1}(Tx,\underline{\ell})$. This shows (34). □

Remark 2.3. By (33), if K_n is a uniform bound over x of $N_n(x,\underline{0})$, then $M_n(x) \leq K_n$. Likewise, if $N_n(x,\underline{0}) \leq K_n$, for a.e. x, then $M_n(x) \leq K_n$, for a.e. x.

Now we show that the set of $x \in X$ such that $\lim_n \frac{M_n^2(x)}{V_n(x)} = 0$ has measure 0 or 1.

Lemma 2.4. *It holds:* $\lim_n [\dfrac{M_n^2(x)}{V_n(x)} - \dfrac{M_n^2(Tx)}{V_n(Tx)}] = 0.$

If T is ergodic, there is a constant $\gamma \in [0,1]$ such that $\limsup_n \dfrac{M_n^2(x)}{V_n(x)} = \gamma$ *for a.e. x.*

Proof. We use (36) and (34). Putting $\varepsilon = 1_{\underline{\ell}(n-1,Tx)=\underline{0}}$, we have:

$$\begin{aligned}
&|\frac{M_n^2(x)}{V_n(x)} - \frac{M_{n-1}^2(Tx)}{V_{n-1}(Tx)}| = |\frac{M_{n-1}^2(Tx) + \varepsilon(2M_{n-1}(Tx)+1)}{V_{n-1}(Tx) + 2N_{n-1}(Tx,\underline{0})+1} - \frac{M_{n-1}^2(Tx)}{V_{n-1}(Tx)}| \\
&= |\frac{\varepsilon(2M_{n-1}(Tx)+1)}{V_n(x)} - \frac{(2N_{n-1}(Tx,\underline{0})+1)}{V_n(x)}\,\frac{M_{n-1}^2(Tx)}{V_{n-1}(Tx)}| \\
&\leq \frac{2M_{n-1}(Tx)+1}{V_n(x)} + \frac{2N_{n-1}(Tx,\underline{0})+1}{V_n(x)} \leq \frac{4M_{n-1}(Tx)}{V_n(x)} + \frac{2}{V_n(x)} \leq \frac{4}{\sqrt{n}} + \frac{2}{V_n(x)}.
\end{aligned}$$

For the last inequality we use that either $M_n(x) \geq \sqrt{n}$, hence $\frac{M_n(x)}{V_n(x)} \leq \frac{1}{M_n(x)} \leq \frac{1}{\sqrt{n}}$, or $M_n(x) < \sqrt{n}$, hence $\frac{M_n(x)}{V_n(x)} \leq \frac{\sqrt{n}}{n} = \frac{1}{\sqrt{n}}$. Therefore,

$$|\frac{M_n^2(x)}{V_n(x)} - \frac{M_{n-1}^2(Tx)}{V_{n-1}(Tx)}| \to 0. \tag{38}$$

Observe now that

$$M_n(x) = M_{n-1}(x) + \varepsilon_n, \text{ where } \varepsilon_n = 0 \text{ or } = 1,$$

and $\varepsilon_n = 1$ if and only if there is $\underline{\ell}_n$ such that

$$M_{n-1}(x) = \#\{1 \leq k \leq n-1 : \ f_k(x) = \underline{\ell}_n\} \text{ and } f_n(x) = \underline{\ell}_n.$$

We have

$$M_n^2(x) = M_{n-1}^2(x) + c_n, \text{ with } c_n = \varepsilon_n(1 + 2M_{n-1}(x))$$

and $N_n(x,\underline{\ell}) = N_{n-1}(x,\underline{\ell}) + \varepsilon'_n(\underline{\ell})$, with $\varepsilon'_n(\underline{\ell}) = 1_{f_n(x)=\underline{\ell}}$ and $\sum_{\underline{\ell}\in\mathbb{Z}^d} \varepsilon'_n(\underline{\ell}) = 1$.

Therefore,

$$\begin{aligned}
&V_n(x) = \sum_{\underline{\ell}\in\mathbb{Z}^d} (N_{n-1}(x,\underline{\ell}) + \varepsilon'_n(\underline{\ell}))^2 = V_{n-1}(x) + 2\sum_{\underline{\ell}\in\mathbb{Z}^d} \varepsilon'_n(\underline{\ell}) N_n(x,\underline{\ell})) + 1, \\
&0 \leq V_n(x) = V_{n-1}(x) + d_n, \text{ with } d_n \leq 2M_n(x) + 1.
\end{aligned}$$

$$\begin{aligned}
&|\frac{M_n^2(x)}{V_n(x)} - \frac{M_{n-1}^2(x)}{V_{n-1}(x)}| = |\frac{M_{n-1}^2(x) + c_n}{V_{n-1}(x) + d_n} - \frac{M_{n-1}^2(x)}{V_{n-1}(x)}| \leq \frac{c_n}{V_n(x)} + \frac{d_n}{V_n(x)}\,\frac{M_{n-1}^2(x)}{V_{n-1}(x)} \\
&\leq \frac{\varepsilon_n(1+2M_{n-1}(x)}{V_n(x)} + \frac{2M_n(x)+1}{V_n(x)}\,\frac{M_{n-1}^2(x)}{V_{n-1}(x)} \leq \frac{2}{V_n(x)} + \frac{4M_n(x)}{V_n(x)} \leq \frac{2}{V_n(x)} + \frac{4}{\sqrt{n}} \to 0.
\end{aligned}$$

From (38) and the convergence above, it follows $\lim_n [\frac{M_n^2(x)}{V_n(x)} - \frac{M_n^2(Tx)}{V_n(Tx)}] = 0$. By ergodicity of T, this shows the lemma □

Recurrence and transience, application of the subadditive ergodic theorem.

It can be shown that a cocycle (μ, T, f) (over an ergodic dynamical system) is either *recurrent* or *transient*. For f with values in $\mathbb{Z}^d$, it means that either $S_k f(x) = 0$ infinitely often for a.e. x, or $S_k f(x) = 0$ finitely often for a.e. x. In the latter case, we have $\lim_k |S_k f(x)| = +\infty$, μ-a.e.

Let $\mathcal{R}_n(x) = \{\underline{\ell} \in \mathbb{Z}^d : \ f_k(x) = \underline{\ell} \text{ for some } k \leq n\}$ be the "range" of the cocycle, i.e., the set of points visited by $f_k(x)$ up to time n.

The sequence $(\mathrm{Card}(\mathcal{R}_n(x)), n \geq 1)$ satisfies the conditions of Kingman's ergodic subadditive theorem. Therefore, $\frac{1}{n}\mathrm{Card}(\mathcal{R}_n(x))$ converges a.e. and in L^1. The limit is a constant since the system (X, μ, T) is ergodic. We refer to [16], where the limit is explicited:

$$\frac{\mathrm{Card}(\mathcal{R}_n(x))}{n} \to \mu(\cap_{n\geq 1}\{S_n f \neq \underline{0}\}), \mu\text{-a.e. and in } L^1. \tag{39}$$

It follows that the limit is 0 in the recurrent case and > 0 in the transient case. By the lemma below, this implies for a.e. x:

$$\frac{V_n(x)}{n} \to +\infty \text{ in the recurrent case.} \tag{40}$$

To show (40) we use the following inequality valid for a general sequence (z_k):

Lemma 2.5. *If $\mathcal{A}$ is a non empty subset in $\mathbb{Z}^d$, we have:*

$$V_n \geq \frac{\left(\sum_{k=0}^{n-1} 1_{z_k \in \mathcal{A}}\right)^2}{\mathrm{Card}(\mathcal{A})}. \tag{41}$$

Proof. Cauchy-Schwarz inequality implies:

$$\sum_{k=0}^{n-1} 1_{z_k \in \mathcal{A}} = \sum_{\underline{\ell}\in\mathcal{A}} \sum_{k=0}^{n-1} 1_{z_k = \underline{\ell}} \leq (\sum_{\underline{\ell}\in A}(\sum_{k=0}^{n-1} 1_{z_k=\underline{\ell}})^2)^{\frac{1}{2}} \, (\mathrm{Card}(\mathcal{A}))^{\frac{1}{2}} \leq V_n^{\frac{1}{2}} \, (\mathrm{Card}(\mathcal{A}))^{\frac{1}{2}}. \quad \square$$

If $z_k = S_k f(x)$, this shows (40). Indeed by taking $\mathcal{A} = \mathcal{R}_n(x)$ we get

$$V_n(x) \geq \frac{n^2}{\mathrm{Card}(\mathcal{R}_n(x))}. \tag{42}$$

Like the cardinal of the range, $(M_n(x), n \geq 1)$ and $(V_n^{\frac{1}{2}}(x), n \geq 1)$ satisfy the conditions of the ergodic subadditive theorem. Indeed we have:

$$M_{n+p}(x) = \sup_{\underline{\ell}} N_{n+p}(x, \underline{\ell}) \leq \sup_{\underline{\ell}} N_n(x, \underline{\ell}) + \sup_{\underline{\ell}} N_p(T^n x, \underline{\ell} - S_n f(x)) = M_n(x) + M_p(T^n x).$$

$$\begin{aligned}
V_{n+p}(x) &= \sum_{\underline{\ell}} N_{n+p}^2(x, \underline{\ell}) = \sum_{\underline{\ell}} (N_n(x, \underline{\ell}) + N_p(T^n x, \underline{\ell} - S_n f(x))^2 \\
&\leq \sum_{\underline{\ell}} N_n^2(x, \underline{\ell}) + \sum_{\underline{\ell}} N_p^2(T^n x, \underline{\ell} - S_n f(x)) + 2(\sum_{\underline{\ell}} N_n^2(x, \underline{\ell}))^{\frac{1}{2}} \, (\sum_{\underline{\ell}} N_p^2(T^n x, \underline{\ell} - S_n f(x)))^{\frac{1}{2}} \\
&= \sum_{\underline{\ell}} N_n^2(x, \underline{\ell}) + \sum_{\underline{\ell}} N_p^2(T^n x, \underline{\ell}) + 2(\sum_{\underline{\ell}} N_n^2(x, \underline{\ell}))^{\frac{1}{2}} \, (\sum_{\underline{\ell}} N_p^2(T^n x, \underline{\ell}))^{\frac{1}{2}} \\
&= (V_n^{\frac{1}{2}}(x) + V_p^{\frac{1}{2}}(T^n x))^2.
\end{aligned}$$

As $0 \leq M_n(x) \leq n$, $0 \leq V_n(x) \leq n^2$ and since the dynamical system is assumed to be ergodic, this implies that for two constants $\lambda, \beta \in [0, 1]$

$$\lim \frac{M_n(x)}{n} = \lambda, \ \lim_n \frac{V_n(x)}{n^2} = \beta, \mu\text{-a.e.} \tag{43}$$

By (28), we have $\lambda^2 \leq \beta \leq \lambda$.

We will see later that $\beta > 0$ (and therefore $\lambda > 0$), if and only if the cocycle (T, f) is a coboundary.

Observe also that $(V_n(x), n \geq 1)$ is positive superadditive in the sense that: $V_n(x) + V_m(T^n x) \leq V_{n+m}(x)$, for every $n, m \geq 0$. Therefore, by a variant of Kingman's theorem (for a short proof see [26]), it holds for a.e. x:

$$\lim \frac{V_n(x)}{n} = \nu, \text{ for a constant } \nu \in [1, +\infty].$$

By (32), for each fixed integer $p \geq 1$, we have, for $n \geq p + 1$,

$$\frac{V_n(x)}{n} \geq \frac{2}{n}[N_p(Tx, 0) + N_p(T^2 x, 0) + ... + N_p(T^{n-p} x, 0)] + 1.$$

Taking the limit when $n \to +\infty$, we get by the ergodic theorem, for each $p \geq 1$,

$$\nu = \lim_n \frac{V_n(x)}{n} \geq 2 \int N_p(., \underline{0}) d\mu + 1.$$

Let Γ be the limit of the increasing sequence of positive functions $(N_p(., \underline{0}), p \geq 1)$. From the above formula, by taking the limit when $p \to +\infty$, we obtain:

$$\nu = \lim_n \frac{V_n(x)}{n} \geq 2 \int N_p(., \underline{0}) d\mu + 1 \to_{p \to +\infty} 2 \int \Gamma \, d\mu + 1.$$

The function Γ is integrable if $\lim_n \frac{V_n(x)}{n}$ is finite.

Using again (32), we get: $\frac{V_n(x)}{n} \leq \frac{2}{n} \sum_{k=1}^{n-1} \Gamma(T^k x, \underline{0}) + 1$. Taking the limit when $n \to +\infty$, it follows:

$$\lim_n \frac{V_n(x)}{n} \leq 2 \int \Gamma \, d\mu + 1.$$

So we have shown:

Proposition 2.6. *In the transient case, it holds, for a.e. x:*

$$\lim_n \frac{V_n(x)}{n} = 1 + 2 \int \Gamma \, d\mu = 1 + 2 \sum_{k=1}^{\infty} \mu(f_k = \underline{0}) \in [1, \infty].$$

In the recurrent case there is also equality with $\lim_n \frac{V_n(x)}{n} = 1 + 2 \sum_{k=1}^{\infty} \mu(f_k = \underline{0}) = +\infty$.

Remark also that there are transient cocycles such that $\sum_{k=1}^{\infty} \mu(f_k = \underline{0}) = +\infty$. This will be shown in Subsection 2.3.

Case of a coboundary

Proposition 2.7. *If f is a coboundary, then for two constants $\beta > 0, \gamma > 0$, for a.e. x, a) $\lim_n \frac{V_n(x)}{n^2} = \beta$, b) $\lim_n \frac{M_n(x)}{n} = \gamma$, c) $\lim_n \frac{M_n^2(x)}{V_n(x)} = \frac{\gamma^2}{\beta} > 0$.*

Proof. Already we know by Kingman's theorem that $\beta = \lim_n \frac{1}{n^2} V_n(x)$ and $\gamma = \lim_n \frac{M_n(x)}{n}$ exist for a.e. x. When f is coboundary, the proof below explicits β and shows $\beta > 0$.

Suppose that f is coboundary, $f = T\Phi - \Phi$. Since f has values in $\mathbb{Z}^d$ and T is ergodic, for all component Φ_j of Φ, $e^{2\pi i \Phi_j}$ is a constant. It follows that Φ has also its values in $\mathbb{Z}^d$ up to an additive constant and we can assume that Φ has values in $\mathbb{Z}^d$. Therefore we have

$$N_n(x, \underline{0}) = \sum_{k=1}^n 1_{f_k(x)=0} = \sum_{k=1}^n 1_{\Phi(x)=\Phi(T^k x)} = \sum_{\underline{\ell}} 1_{(\Phi=\underline{\ell})}(x) \sum_{k=1}^n 1_{(\Phi=\underline{\ell})}(T^k x),$$

$$\begin{aligned} V_n(f,x) &= \sum_{\underline{\ell}\in\mathbb{Z}^d} N_n^2(x,\underline{\ell}) = \sum_{\underline{\ell}\in\mathbb{Z}^d} \#\{1 \le k \le n : \ \Phi(T^k x) - \Phi(x) = \underline{\ell}\}^2 \\ &= \sum_{\underline{\ell}\in\mathbb{Z}^d} \#\{1 \le k \le n : \ \Phi(T^k x) = \underline{\ell}\}^2 = \sum_{\underline{\ell}\in\mathbb{Z}^d} \big(\sum_{1\le k\le n} 1_{\Phi(T^k x)=\underline{\ell}}\big)^2, \end{aligned}$$

For $R \ge 1$, let A_R denote the set $\cup_{\underline{\ell}:\|\underline{\ell}\|\le R}(\Phi = \underline{\ell})$. For each $\underline{\ell}$, by Birkhoff's theorem we have, for a.e. x, $\lim_n \frac{1}{n} \sum_{0\le k\le n-1} 1_{\Phi(T^k x)=\underline{\ell}} = \mu(\Phi = \underline{\ell})$. Therefore it holds

$$\frac{1}{n^2} \sum_{\underline{\ell}\in A_R} \big(\sum_{1\le k\le n} 1_{\Phi(T^k x)=\underline{\ell}}\big)^2 = \sum_{\underline{\ell}\in A_R} \big(\frac{1}{n} \sum_{1\le k\le n} 1_{\Phi(T^k x)=\underline{\ell}}\big)^2 \to \sum_{\underline{\ell}\in A_R} (\mu(\Phi = \underline{\ell}))^2.$$

This implies, for every $R \ge 1$,

$$\liminf_n \frac{1}{n^2} \sum_{\underline{\ell}\in\mathbb{Z}^d} \big(\sum_{1\le k\le n} 1_{\Phi(T^k x)=\underline{\ell}}\big)^2 \ge \lim_n \frac{1}{n^2} \sum_{\underline{\ell}\in A_R} \big(\sum_{1\le k\le n} 1_{\Phi(T^k x)=\underline{\ell}}\big)^2 = \sum_{\underline{\ell}\in A_R} (\mu(\Phi = \underline{\ell}))^2.$$

It follows: $\liminf_n \frac{1}{n^2} \sum_{\underline{\ell}\in\mathbb{Z}^d} \big(\sum_{1\le k\le n} 1_{\Phi(T^k x)=\underline{\ell}}\big)^2 \ge \sum_{\underline{\ell}\in\mathbb{Z}^d} \mu(\Phi = \underline{\ell})^2$. For the complementary of A_R, it holds:

$$\sum_{\underline{\ell}:\|\underline{\ell}\|>R} \big(\sum_{1\le k\le n} 1_{\Phi(T^k x)=\underline{\ell}}\big)^2 = \sum_{1\le j,k\le n} \sum_{\underline{\ell}:\|\underline{\ell}\|>R} 1_{\Phi(T^j x)=\underline{\ell}}\, 1_{\Phi(T^k x)=\underline{\ell}}$$

$$\le \sum_{1\le j,k\le n} \big(\sum_{\underline{\ell}:\|\underline{\ell}\|>R} 1_{\Phi(T^j x)=\underline{\ell}}\big) \big(\sum_{\underline{\ell}:\|\underline{\ell}\|>R} 1_{\Phi(T^k x)=\underline{\ell}}\big) \le \sum_{1\le j,k\le n} 1_{A_R^c}(T^j x) 1_{A_R^c}(T^k x) = \big(\sum_{0\le k<n} 1_{A_R^c}(T^k x)\big)^2.$$

It follows for the upper bound:

$$\begin{aligned} &\limsup_n \frac{1}{n^2} \sum_{\underline{\ell}\in\mathbb{Z}^d} \big(\sum_{1\le k\le n} 1_{\Phi(T^k x)=\underline{\ell}}\big)^2 \le \lim_n \sum_{\underline{\ell}\in A_R} \big(\frac{1}{n} \sum_{1\le k\le n} 1_{\Phi(T^k x)=\underline{\ell}}\big)^2 + \lim_n \big(\frac{1}{n} \sum_{1\le k\le n} 1_{A_R^c}(T^k x)\big)^2 \\ &= \sum_{\underline{\ell}\in A_R} (\mu(\Phi = \underline{\ell}))^2 + \mu(A_R^c)^2 \underset{R\to\infty}{\to} \sum_{\underline{\ell}\in\mathbb{Z}^d} \mu(\Phi = \underline{\ell})^2. \end{aligned}$$

This shows a) with $\beta = \sum_{\underline{\ell}\in\mathbb{Z}^d} \mu(\Phi = \underline{\ell})^2 > 0$.

The point b) follows from the inequality $\gamma \geq \beta > 0$ and c) is clear. □

Proposition 2.8. *The constant $\beta = \lim_n \frac{V_n(x)}{n^2}$ (for a.e. x) is > 0 if and only if the cocycle (T, f) is a coboundary.*

Proof. The case of a coboundary follows from Proposition 2.7. We know already the existence of the limit by the subadditive ergodic theorem, but the proof below shows again its existence and its nullity when f is not a coboundary.

Suppose now that f is not a coboundary. From (31), we can write

$$\begin{aligned}\frac{V_n(x)}{n^2} &= \frac{1}{n} + \frac{2}{n}\sum_{k=1}^{n-1}\frac{1}{n}\sum_{j=0}^{n-k-1}(1_{f_k(T^jx)=\underline{0}}) \\ &\leq \frac{1}{n} + \frac{2}{n}\sum_{k=1}^{n-1}\frac{1}{n}\sum_{j=0}^{n-1}(1_{f_k(T^jx)=\underline{0}}) = \frac{1}{n} + \frac{2}{n}\sum_{j=1}^{n-1}\frac{N_n(T^jx,0)}{n}.\end{aligned}$$

We will show that $\lim_n \frac{1}{n}\sum_{j=0}^{n-1}\frac{N_n(T^jx,0)}{n} = 0$ a.e.

By the ergodic theorem of Dunford and Schwarz (in the space of infinite measure $X\times\mathbb{Z}$) applied to $\tilde{T}_f$ and $\phi_0 = \mathbf{1}_{X\times\{0\}}$, which is bounded and in $L^p(X\times\mathbb{Z})$, for every $p \geq 1$, we get a function $\tilde{\phi}_0(x)$ which is $\tilde{T}_f$-invariant and in $L^1(X\times\mathbb{Z})$ and

$$\lim_n \frac{N_n(x,0)}{n} = \tilde{\phi}_0(x), \text{ a.s.}$$

As f is not a coboundary, $\tilde{\phi}_0$ is zero a.e. (cf. for instance [12].)

Observe that $\|\sup_{n\geq L}\frac{N_n(x,0)}{n}\|_2 \to 0$, as L goes to $+\infty$. Indeed, for every $0 < \varepsilon \leq 1$, letting $A_{\varepsilon,L} := \{x : \sup_{n\geq L}\frac{N_n(x,0)}{n} > \varepsilon\}$, we have $\mu(A_{\varepsilon,L}) \to 0$, when $L \to +\infty$. Since $\frac{N_n(x,0)}{n} \leq 1$, it follows, for L big enough:

$$\int \big(\sup_{n\geq L}(\frac{N_n(x,0)}{n})\big)^2 d\mu \leq \varepsilon^2 + \mu(A_{\varepsilon,L}) \leq 2\varepsilon.$$

We put $\Lambda_n(x) := \sup_{s\geq n}\frac{N_s(x,0)}{s}$. By the previous observation, we have $\lim_n \|\Lambda_n\|_2 = 0$.

Let us consider the following maximal function for the action of T:

$$\tilde{\Lambda}_n(x) = \sup_{1\leq r<\infty}\frac{1}{r}\sum_{j=0}^{r-1}\Lambda_n(T^jx) = \sup_{1\leq r<\infty}\frac{1}{r}\sum_{j=0}^{r-1}\sup_{s\geq n}\frac{N_s(T^jx,0)}{s}. \tag{44}$$

From a classical maximal inequality (see, e.g. [19, Corollary 2.2.1]), we have $\|\tilde{\Lambda}_n\|_2 \leq 2\|\Lambda_n\|_2 \to 0$.

Observe also that, from the definition of $\tilde{\Lambda}_n$ in (44), the following inequalities hold:

$$\tilde{\Lambda}_n(x) \geq \sup_{r,s\geq n} \frac{1}{r}\sum_{j=0}^{r-1} \frac{N_s(T^j x,0)}{s} \geq \frac{1}{n}\sum_{j=0}^{n-1} \frac{N_n(T^j x,0)}{n}.$$

The sequence $\sup_{r,s\geq n} \frac{1}{r}\sum_{j=0}^{r-1} \frac{N_s(T^j x,0)}{s}$ is non negative and decreasing. Since $\|\tilde{\Lambda}_n\|_2 \to 0$, the L_2-norm of its limit in (X,μ) is zero. The result follows. □

Remark 2.9. (see also section 4 and [25])

Let $(U_{\underline{\ell}})_{\underline{\ell}\in\mathbb{Z}^d}$ be a r.f. of square integrable r.v.'s on a probability space $(\Omega,\mathcal{F},\mathbb{P})$ stationary in the weak sense and such that $\sum_{\underline{\ell}} |\langle U_{\underline{\ell}}, U_{\underline{0}}\rangle| < +\infty$. By (4) and Proposition 2.8, if f is not a coboundary, it holds

$$\frac{1}{n^2}\,\|\sum_{k=0}^{n-1} U_{f_k(x)}\|_2^2 \leq C\,\frac{V_n(x)}{n^2} \to 0, \text{ for } \mu\text{-a.e. } x.$$

Another result of norm convergence whose proof is like the proof of Proposition 1.3 is the following. Suppose that the r.f. is stationary. Let φ be an observable on the dynamical system $(\Omega,\mathbb{P},\theta)$ with a spectral measure ν_φ. We have:

$$\int_\Omega |\sum_{j=0}^{n-1} \varphi\circ\theta^{z_j}|^2\, d\mathbb{P} = \int_{\mathbb{T}^1} |\sum_{j=0}^{n-1} e^{2\pi i z_j t}|^2\, d\nu_\varphi(t).$$

Assume that ν_φ is absolutely continuous with respect to the Lebesgue measure on the torus, and let $\rho \in L^1(d\underline{t})$ such that $d\nu_\varphi(\underline{t}) = \rho(\underline{t})d\underline{t}$. For $\varepsilon > 0$ there is M such that $\int_{\rho>M} \rho\, d\underline{t} < \varepsilon$. We have

$$\frac{1}{n^2}\int_{\mathbb{T}^d} |\sum_{j=0}^{n-1} e^{2\pi i\langle z_j,\underline{t}\rangle}|^2\, d\nu_\varphi(\underline{t}) \;\leq\; \frac{M}{n^2}\int_{\mathbb{T}^d} |\sum_{j=0}^{n-1} e^{2\pi i\langle z_j,\underline{t}\rangle}|^2\, d\underline{t} + \int_{\rho>M} \rho\, d\underline{t} \leq M\frac{V_n}{n^2} + \varepsilon.$$

This shows that $\frac{V_n}{n^2} \to 0$ implies $\frac{1}{n^2}\int_\Omega |\sum_{j=0}^{n-1} \varphi\circ\theta^{z_j}|^2\, d\mathbb{P} \to 0$. This is satisfied by every $\varphi \in L^2(\mathbb{P})$, if the dynamical system has a Lebesgue spectrum.

In particular, taking $z_k = f_k(x)$, by Proposition 2.8, if f is not a coboundary, it holds

$$\frac{1}{n^2}\int_\Omega |\sum_{j=0}^{n-1} \varphi(\theta^{f_j(x)}\omega)|^2\, d\mathbb{P}(\omega) \to 0, \text{ for a.e. } x.$$

2.2. **Non centered cocycles.**

In an ergodic dynamical system (X,μ,T), if $f : X \to \mathbb{R}$ is an integrable function with $\mu(f) > 0$, by the ergodic theorem for the ergodic sums $S_n^T f(x) = \sum_{k=0}^{n-1} f(T^k x)$, it holds for a.e. x: $\lim_n \frac{1}{n} S_n f(x) > 0$ and therefore $\lim_n S_n^T f(x) = +\infty$. If f has values in $\mathbb{Z}$, as

the process $S_n^T f(x)$ visits finitely often each site, one can think there is a chance that the following condition is satisfied:

$$\lim_n \frac{M_n^2(T,f,x)}{V_n(T,f,x)} = 0. \tag{45}$$

A case where (45) is satisfied is the following: let X be a topological compact space, $T : X \to X$ a continuous map, which is uniquely ergodic with μ as unique invariant measure. Let $f : X \to \mathbb{Z}$ be an integrable function such that $\mu(f) \neq 0$. Assume f to be Riemann-integrable (i.e. such that, for every $\varepsilon > 0$, there are two continuous functions ψ_0, ψ_1 with $\psi_0 \leq f \leq \psi_1$ and $\mu(\psi_1 - \psi_0) \leq \varepsilon$).

Then, the ergodic means of f converge uniformly, and this implies the existence of N such that $\frac{1}{n}|S_n^T f(x)| \geq \frac{1}{2}|\mu(f)| > 0$ for $n \geq N$ and every x. It follows that the number of visits of $S_n^T f(x)$ to 0 is $\leq N$, for every x. By remark 2.3, $M_n(x) \leq N$, for every x, and a fortiori (45) is satisfied.

Nevertheless, we will see below that (45) *may fail in non uniform cases: there are dynamical systems and sets B of positive measure such that, for $f = 1_B$,*

$$\limsup_n \frac{M_n^2(T,f,x)}{V_n(T,f,x)} = 1. \tag{46}$$

2.3. **Counterexamples.**

1) We construct counterexamples, a transient one and a recurrent one with a function f of null integral, such that (46) is satisfied.

To construct these counterexamples, we start by considering a general ergodic dynamical system (X, μ, T) and a measurable set $B \subset X$ of positive measure. Let T_B be the induced map on B, $R(x) = R^B(x) = \inf\{k \geq 1 : T^k x \in B\}$ the first return time of x in B and $R_n(x) = R_n^B(x) := \sum_{k=0}^{n-1} R(T_B^k x)$ the n-th return time of x in B.

We take $x \in B$. If f is a function such that $f = 0$ outside B, the position of the sums up to time $R_{n-1}(x)$ are the positions of the ergodic sums $S_n^{T_B} f$ for the induced map up to time n, that is:

$$\{f(x), f(x) + f(T_B x), ..., f(x) + f(T_B x) + ... + f(T_B^{n-1} x)\}.$$

For a site ℓ, the number of visits up to time $R_{n-1}(x)$ of the ergodic sums for T is

$$N_{R_{n-1}(x)}(x, \ell) = \sum_{k=0}^{n-1} R^B(T_B^k x)\, 1_{S_k^{T_B} f(x) = \ell}$$

and therefore

$$V_{R_{n-1}(x)}(T, x) = \sum_\ell [\sum_{k=0}^{n-1} R^B(T_B^k x)\, 1_{S_k^{T_B} f(x) = \ell}]^2. \tag{47}$$

Case $f = 1_B$. Clearly $\sum_{k=0}^{n-1} f(T_B^k x) = n$. For the map T, the ergodic sums of f are incremented by 1 when and only when the iterates $T^j x$ visit the set B. Otherwise, they stay fixed. The times of visits in B, for $x \in B$, are $0, R(x), R(x) + R(T_B x), ...$ We have:

$$\text{for } x \in B, \quad \sum_{j=0}^{R_{n-1}(x)+t} f(T^j x) = n, \text{ for } t = 0, ..., R_n(x) - R_{n-1}(x) - 1.$$

For $N_n(T, x, \ell) = N_n(T, f, x, \ell)$, it holds:

$$\begin{aligned} N_n(T, x, \ell) &= 0, \text{ if } n < R_\ell(x), \\ &= t, \text{ if } n = R_\ell(x) + t, \text{ with } 0 \le t < R_{\ell+1}(x) - R_\ell(x), \\ &= R_{\ell+1}(x) - R_\ell(x) = R(T_B^\ell x), \text{ if } n \ge R_{\ell+1}(x). \end{aligned}$$

For $L \ge 1$, we have for the time preceding the L-th return to the basis for $f = 1_B$:

$$M_{R_L(x)-1}(T, f, x) = \max_{\ell \le L} R(T_B^\ell x), \; V_{R_L(x)-1}(T, f, x) = \sum_{\ell \le L} R^2(T_B^\ell x). \tag{48}$$

Special map

In order to compute an explicit example, it is easier to start from a given map S and to construct a special map T over this map.

Let $\varphi : X \to \mathbb{N}$ be integrable and ≥ 1. The (discrete time) special map $T = T_\varphi$ is defined

$$\text{on } \tilde{X} := \{(x, k), \; x \in X, k = 0, ..., \varphi(x) - 1\} \subset X \times \mathbb{R},$$
$$\text{by } T(x, k) := (x, k + 1), \text{ if } 0 \le k < \varphi(x) - 1, \; := (Sx, 0), \text{ if } k = \varphi(x) - 1.$$

Let $\tilde{\mu}$ be the probability measure defined on $\tilde{X}$ by $\tilde{\mu}(A \times \{k\}) = \mu(\varphi)^{-1} \mu(A)$, for $k \ge 0$ and $A \subset \{x : k \le \varphi(x) - 1\}$. It is T_φ-invariant. The space X can be identified with the subset $B = \{(x, 0), x \in X\}$ of $\tilde{X}$ with normalized measure. The set B is the basis and $\varphi - 1$ the roof function of the special map T_φ.

As for the map S we will take an ergodic rotation, the special map T_φ will be also ergodic for the measure $\tilde{\mu}$ on $\tilde{X}$.

Observe that the recurrence time $R(x) = R^B(x)$ for the special map in the basis B is $\varphi(x)$ and the n-th return time of x in B is $R_n(x) = R_n^B(x) = \sum_{k=0}^{n-1} \varphi(S^k x)$.

For S, let us take a rotation $S = S_\alpha$ on $X = \mathbb{T}/\mathbb{Z}$ by $\alpha \bmod 1$, where α is irrational. We denote by p_n and q_n the numerator and denominator of the convergents of α (see Subsection 3.2 for the notation). We will construct a measure preserving transformation T (the special map over S_α with roof function φ) such that the cocycle generated by $f = 1_B$ in the system $(\tilde{X}, \tilde{\mu}, T)$ satisfies $\limsup_n \dfrac{M_n^2(T, f, x)}{V_n(T, f, x)} = 1$.

We use the next lemma with $p = p_n, q = q_n$.

Lemma 2.10. *Let $p, q \ge 1$, $(p, q) = 1$, be such that $|\alpha - p/q| < 1/q^2$. For every x, there is a value $0 \le i < q$ such that $x + i\alpha \bmod 1 \in [0, 2/q]$.*

More generally, for every interval I of length $2/q$, for every x, there is a value $0 \leq i < q$ such that $x + i\alpha \bmod 1 \in I$.

Proof. It is well known that there is exactly one value of $j\alpha \bmod 1$, for $0 \leq j < q$, in each interval $[\frac{\ell}{q}, \frac{\ell+1}{q}[$, $\ell = 0, ..., q-1$. Let us recall a proof. For $j = 0$, $j\alpha \in [0, 1/q[$. The map $j \to \ell_j = jp \bmod q$, which is injective, is a permutation of the set $\{1, ..., q-1\}$ onto itself. We have $\alpha = p/q + \delta$, with $|\delta| < 1/q^2$.

Assuming $\delta > 0$, it follows: $j\alpha \bmod 1 \in [\frac{\ell_j}{q}, \frac{\ell_j}{q} + \frac{j}{q^2}] \subset [\frac{\ell_j}{q}, \frac{\ell_j+1}{q}[$, for $j = 1, ..., q-1$. The case $\delta < 0$ is treated the same way.

Now let us prove the first point. Let x be in $[0, 1[$. There is $i_0 \in \{0, ..., q-1\}$ such that $x = \frac{i_0}{q} + \theta$, with $0 \leq \theta < 1/q$. By the claim, there is $i \in [0, q[$ such that $i\alpha \bmod 1 \in [\frac{q-i_0}{q}, \frac{q-i_0+1}{q}]$. Hence $x + i\alpha \bmod 1 \in [\theta, \frac{1}{q} + \theta] \subset [0, \frac{2}{q}]$. □

Let (λ_n) be an increasing sequence of positive integers which will be subjected below to growth conditions. First we assume that it satisfies the condition:

$$q_{\lambda_{n+1}} \geq 3q_{\lambda_n}, \forall n \geq 1. \tag{49}$$

Denote by J_n the interval $J_n = [\dfrac{3}{q_{\lambda_{n+1}}}, \dfrac{3}{q_{\lambda_n}}]$. For the roof function, we take, with $\varepsilon_n = \frac{1}{n^2}$,

$$\varphi = 1 + \sum_{n \geq 1} \lfloor \varepsilon_n q_{\lambda_n} \rfloor 1_{J_n}.$$

The function φ is integrable: $\int \varphi d\mu \leq 1 + 3\sum_n \varepsilon_n$. Observe also that, by (49), the length of J_n is $> 2/q_{\lambda_n}$ and that $(\varepsilon_n q_{\lambda_n})$ is not decreasing for $n \geq 2$.

Let x be in the basis. By construction, the orbit of x under the iteration of T_φ is that of the rotation S_α until it enters the set B^c, complementary of B, at some time. Then it stays in this set, until it reaches the roof and comes down to the basis. Then the dynamic is that of the rotation, until again $S_\alpha^j x$ falls in the set $\varphi > 1$ and so on.

Let $W_n(x)$ be the first visit of $S^j x$ in J_n. By lemma 2.10, we have $W_n(x) \leq q_{\lambda_n}$.

Now we choose f to get a transient counterexample and a recurrent one.

Counterexample in the transient case.

We take $f = 1$ on the basis and 0 outside.

The sequence (λ_n) is taken such that

$$q_{\lambda_n} \geq n\,(q_{\lambda_{n-1}})^2, n \geq 1. \tag{50}$$

By (48), we obtain (recall that now T_B, the induced map in the basis B, is the rotation $S = S_\alpha$ and $R(T_B^j x) = \varphi(S_\alpha^j x)$):

$$M_{R_{W_n(x)}(x)-1}(T, x) = \max_{j \leq W_n(x)} \varphi(S^j x), \tag{51}$$

$$V_{R_{W_n(x)}(x)-1}(T, x) = \sum_{j \leq W_n(x)} \varphi^2(S^j x). \tag{52}$$

In the above formula, $\varphi(S^j x)$ is either 1 or (for some $k \leq n-1$) $1+\lfloor \varepsilon_k q_{\lambda_k} \rfloor \leq 1+\varepsilon_{n-1} q_{\lambda_{n-1}}$, excepted for the last term which is $1+\lfloor \varepsilon_n q_{\lambda_n} \rfloor$.

The maximum in (51) (given by the first visit to J_n) is $1+\lfloor \varepsilon_n q_{\lambda_n} \rfloor \geq \varepsilon_n q_{\lambda_n}$. As we have seen, this first visit for the iterates $S^j x$ occurs at a time $\leq q_{\lambda_n}$. It follows by (50):

$$\begin{aligned}\frac{V_{R_{W_n(x)}(x)-1}(T,x)}{M^2_{R_{W_n(x)}(x)-1}(T,x)} &\leq q_{\lambda_n}\frac{(\varepsilon_{n-1}\, q_{\lambda_{n-1}})^2}{(\varepsilon_n\, q_{\lambda_n}-1)^2}+1 \leq (\frac{\varepsilon_{n-1}}{\varepsilon_n})^2\,\frac{(q_{\lambda_{n-1}})^2}{q_{\lambda_n}}\frac{1}{(1-(\varepsilon_n\, q_{\lambda_n})^{-1})^2}+1\\ &\leq 2\,(\frac{n}{n-1})^2\,\frac{(q_{\lambda_{n-1}})^2}{q_{\lambda_n}}+1 \leq \frac{4}{n}+1,\ \text{for } n \text{ big enough.}\end{aligned}$$

This shows: $\limsup_n \dfrac{M_n^2(T,f,x)}{V_n(T,f,x)} = 1$. The result is proved for x in the basis B, but is satisfied for a.e. $x \in \tilde{X}$, since $\limsup_n \dfrac{M_n^2(T,f,x)}{V_n(T,f,x)}$ is a.e. constant by ergodicity of the special map and Lemma 2.4.

Remark that $S_k f(x) \to +\infty$ for every point x.The sequence $(N_n(x,0))$ is bounded for every x, but not uniformly in x.

Counterexample in the recurrent case.

In order to obtain a recurrent counterexample, we now use a special cocycle over a rotation by α (with α an irrational number with bounded partial quotients (bpq)) studied later (see Subsection 3.2).

Let f defined on the basis by $f(x) = 1_{[0,\frac{1}{2}[}(x) - 1_{[\frac{1}{2},1[}(x)$ and 0 outside, and $S_k f(x) = \sum_{i=0}^{k-1} f(x+i\alpha \bmod 1)$. By (47), we have

$$\begin{aligned}V_{R_{n-1}(x)}(T,f,x) &= \sum_\ell [\sum_{k=0}^{n-1} \varphi(x+k\alpha)\, 1_{S_k f(x)=\ell}]^2\\ &= \sum_\ell [\sum_{k=0}^{n-1} (1+\sum_j \varepsilon_j q_{\lambda_j} 1_{J_j}(x+k\alpha))\, 1_{S_k f(x)=\ell}]^2.\end{aligned}$$

Observe that for a constant C, $1+\sum_{j<n} \varepsilon_j q_{\lambda_j} 1_{J_j}(x+k\alpha)) \leq C q_{\lambda_{n-1}}$. Using the bound (60) obtained later for the special function f and α bpq, this implies:

$$\begin{aligned}V_{R_{W_{n-1}(x)}}(T,f,x) &\leq \sum_\ell [\sum_{k=0}^{q_{\lambda_n}} (1+\sum_{j<n} \varepsilon_j q_{\lambda_j} 1_{J_j}(x+k\alpha))\, 1_{S_k f(x)=\ell}]^2\\ &\leq C^2 \sum_\ell [\sum_{k=0}^{q_{\lambda_n}} q_{\lambda_{n-1}}\, 1_{S_k f(x)=\ell}]^2\\ &\leq C^2 q_{\lambda_{n-1}}^2 \sum_\ell [\sum_{k=0}^{q_{\lambda_n}} 1_{S_k f(x)=\ell}]^2 \leq C^2 q_{\lambda_{n-1}}^2\, q_{\lambda_n}^2/\sqrt{\log q_{\lambda_n}}.\end{aligned}$$

Put $L_n = S_{W_n(x)}f(x)$ for the site visited by the cocycle when $S^j x$ enters J_n. We have

$$M_{R_{W_n(x)}}(T,f,x) \geq N_{R_{W_n(x)}}(T,f,x,L_n(x)) = \varepsilon_n q_{\lambda_n}.$$

Hence:

$$0 \leq \frac{V_{R_{W_n(x)}}(T,f,x)}{M^2_{R_{W_n(x)}}(T,f,x)} - 1 \quad \leq \quad C^2 \frac{q^2_{\lambda_{n-1}}\, q^2_{\lambda_n}}{\sqrt{\log q_{\lambda_n}}} \frac{1}{\varepsilon_n^2 q^2_{\lambda_n}} = C^2 \frac{n^4 q^2_{\lambda_{n-1}}}{\sqrt{\log q_{\lambda_n}}}.$$

Now, we choose a growth condition on (λ_n) (stronger than (50)), such that the above bound tends to 0.

This shows the result for x in the basis, hence on the whole space using again Lemma 2.4.

2) *A transient example where* $\lim_n \frac{V_n(x)}{n} = +\infty$.

We know that $\lim \frac{V_n}{n} = \beta \in [1, +\infty]$ and $\beta = \sum_{k\geq 1} \mu(f_k = \underline{0})$.

We would like to construct a transient cocycle such that $\beta = +\infty$. The example will be given simply by taking a dynamical system (X, μ, T) and $f = 1_D$ for some subset D of positive measure of X. With such a cocycle, we have $\{S_k f = 0\} = \bigcap_{r=0}^{k-1} T^{-r} D^c$. Therefore the question is to construct a set $A = D^c$ of measure < 1 such that

$$\sum_{k\geq 1} \mu(\bigcap_{r=0}^{k-1} T^{-r} A) = +\infty.$$

Like in the previous construction of counterexemples, we consider a special flow (with discrete time) T_φ defined over a dynamical system (X, μ, S). We use the same notation for the special flow as above. The ceiling function φ is the following:

Let $\varepsilon_k = \frac{1}{k^2}$, $k \geq 1$, and let $(B_k, k \geq 1)$ be a sequence of disjoints subsets of X with measure $\mu(B_k) = \frac{\varepsilon_k}{k}$. We take $\varphi = \sum_k k\, 1_{B_k}$. The function φ is integrable, with $\int \varphi\, d\mu < \infty$.

The space X, the basis of the special flow, is identified to a subset of the space $\tilde{X}$. Denote by B_k^j the subset $B_k \times \{j\}$, $j < k$, of $\tilde{X}$, and by $\tilde{B}_k$ the union $\cup_{0\leq j<k} B_k^j$.

We put $A = \bigcup_k \tilde{B}_k$. By definition of the special flow, for $0 \leq j < k$, we have

$$T_\varphi^r B_k^j = B_k^{j+r} \subset \tilde{B}_k, \text{ if } 0 \leq r < k - j.$$

It follows:

$$\begin{aligned} \tilde{\mu}(\bigcap_{r=0}^{n-1} T_\varphi^r A) \quad &\geq \quad \sum_{k>n} \tilde{\mu}(\bigcap_{r=0}^{n-1} T_\varphi^r(\cup_{j=0}^{k-r-1} B_k^j)) = \sum_{k>n} \tilde{\mu}(\bigcap_{r=0}^{n-1}(\cup_{j=r}^{k-1} B_k^j) \\ &\geq \quad \sum_{k>n} \tilde{\mu}(\cup_{j=n}^{k-1} B_k^j) = \sum_{k>n} \varepsilon_k \frac{k-n}{k}. \end{aligned}$$

This implies:

$$\sum_{n\geq 1} \tilde{\mu}(\bigcap_{r=0}^{n-1} T_\varphi^r A) \geq \sum_{n\geq 1}\sum_{k>n} \varepsilon_k \frac{k-n}{k} = \sum_{n\geq 1}\sum_{k>n} \frac{1}{k^2}(1 - \frac{n}{k}) \geq \sum_{n\geq 1} \frac{1}{2} \sum_{k>2n} \frac{1}{k^2} \geq \frac{1}{4} \sum_{n\geq 1} \frac{1}{n} = +\infty.$$

3. **Examples**

In general, for a dynamical system (X, μ, T) and a cocycle (T, f), it seems difficult to get a precise estimate of the quantities $N_n(x, \underline{\ell}), M_n(x), V_n(x)$. In this section we present two types of cocycles for which this is possible, first in the classical case of random walks, then when they are generated by step functions over rotations.

3.1. **Random walks.**

1-dimensional cocycle satisfying the LIL.

We start by a remark on the the law of iterated logarithm (LIL). Suppose that (T, f) is a 1-dimensional cocycle which satisfies the LIL. Then for a constant $c_1 > 0$, for a.e. x, the inequality $|f_n(x)| > c_1 (n \ \ln\ln\ n)^{\frac{1}{2}}$ is satisfied only for finitely many values of n. This implies that, for a.e. x, there is $N(x)$ such that $|f_n(x)| \leq (c_1\, n \ \ln\ln\ n)^{\frac{1}{2}}$, for $n \geq N(x)$; so that, for $N(x) \leq k < n$, $|f_k(x)| \leq (c_1\, k \ \ln\ln\ k)^{\frac{1}{2}} \leq (c_1\, n \ \ln\ln\ n)^{\frac{1}{2}}$.

Therefore we have $\mathrm{Card}(\mathcal{R}_n(x)) \leq c_2(x)\,(n \ \ln\ln\ n)^{\frac{1}{2}}$, with an a.e. finite constant $c_2(x)$.

In dimension 1, by (42), we get that for a.e. x there is $c(x) > 0$ such that

$$V_n(x) \geq C(x)\, n^{\frac{3}{2}}\, (\ln\ln\ n)^{-\frac{1}{2}}.$$

The case where a LIL is valid includes the case of a 1-dimensional r.w. centered with finite variance, but also the class of cocycles for which a martingale method can be used.

Random walks.

Now we consider sequences given by a random walk. For random walks in $\mathbb{Z}^d$, the quantities $V_n(x), M_n(x)$ have been studied in many papers since the 50's. $M_n(x)$ is called "maximal multiplicity of points on a random walk" by Erdös and Taylor [17]. Below, we give a brief survey of several results for r.w.s. First we recall some definitions.

Let $(\zeta_i)_{i\geq 0}$ be a sequence of i.i.d. random vectors on a probability space $(X,\, \mu)$ with values in $\mathbb{Z}^d$ and common probability distribution ν. The associated *random walk* (r.w.) $Z = (Z_n)$ in $\mathbb{Z}^d$ starting from $\underline{0}$ is defined by $Z_0 := \underline{0}$,

$$Z_n := \zeta_0 + ... + \zeta_{n-1}, n \geq 1.$$

A r.w. can be seen as a special case of cocycle. Indeed, the r.v.'s ζ_i can be viewed as the coordinate maps on $(X,\, \mu)$ obtained as $(\mathbb{Z}^d)^{\mathbb{Z}}$ equipped with the product measure $\nu^{\otimes\mathbb{Z}}$ and with the shift T acting on the coordinates. We have $\zeta_i = \zeta_0 \circ T^i$ and the cocycle relation $Z_{n+n'} = Z_n + Z_{n'} \circ T^n, \forall n, n' \geq 0$.

Let $\mathcal{S} := \{\underline{\ell} \in \mathbb{Z}^d : \mathbb{P}(\zeta_0 = \underline{\ell}) > 0\}$ be the support of ν and L the sub-lattice of $\mathbb{Z}^d$ generated by $\mathcal{S}$. Let D be the sub-lattice of $\mathbb{Z}^d$ generated by $\{\underline{\ell} - \underline{\ell}', \underline{\ell}, \underline{\ell}' \in \mathcal{S}\}$.

For simplicity (and without loss of generality) in what follows we will assume that the random walk Z is *aperiodic* ($L = \mathbb{Z}^d$). We exclude also the "deterministic" case (i.e.,

when $\mathbb{P}(\zeta_0 = \underline{\ell}) = 1$ for some $\underline{\ell} \in \mathbb{Z}^d$) in dimension 1 (the deterministic case in higher dimension is excluded by aperiodicity).

Notice that all the pointwise limits or bounds mentioned now for random walks are *a.s.* statements. These bounds will show that conditions (30), (29) are satisfied by $V_n(x), M_n(x)$ a.s. for random walks under some assumptions.

Recurrence/transience.

Recall that a r.w. $Z = (Z_n)$ is recurrent if $\sum_{n=1}^{\infty} \mu(Z_n = \underline{0}) = +\infty$ and otherwise transient.

Recurrence occurs if and only if $\mu(Z_n = \underline{0}$ infinitely often$) = 1$, and transience if and only if $\mu(Z_n = \underline{0}$ infinitely often$) = 0$ (cf. [8], [9]).

For an aperiodic r.w. Z in dimension d with a moment of order 2 (for $d = 1$, a moment of order 1 suffices), for $d = 1, 2$, Z is recurrent if and only if it is centered. For $d \geq 3$, it is always transient.

Asymptotic variance for the sampling of a stationary centered r.f.

Let $(X_{\underline{\ell}}, \underline{\ell} \in \mathbb{Z}^d)$ be a stationary centered r.f. with summable correlation and spectral density ρ. We have

$$\frac{1}{n} \| \sum_{k=1}^{n-1} X_{Z_k(x)} \|_2^2 = \int_{\mathbb{T}^d} \frac{1}{n} | \sum_{k=0}^{n-1} e^{2\pi i \langle Z_k(x), \underline{t} \rangle} |^2 \, \rho(\underline{t}) \, d\underline{t} = \int_{\mathbb{T}^d} \frac{1}{n} K_n(x, \underline{t}) \, \rho(\underline{t}) \, d\underline{t},$$

where, using (24) with $z_k = Z_k(x)$ and $Z_k(x) - Z_j(x) = Z_k(T^j x)$, $\frac{1}{n} K_n$ reads

$$\frac{1}{n} K_n(x, \underline{t}) \quad = 1 + 2 \sum_{\underline{\ell}} \Big(\sum_{k=1}^{n-1} \frac{1}{n} \sum_{j=0}^{n-k-1} 1_{Z_k(T^j x) = \underline{\ell}} \Big) \, e^{2\pi i \langle \underline{\ell}, t \rangle}. \tag{53}$$

As already recalled, the existence of the asymptotic variance

$$\lim_n V_n(x)^{-1} \int | \sum_{k=0}^{n-1} X_{Z_k(x)} |^2 d\mu$$

has been shown in [11]. An interesting situation is that of the sums along a transient (non deterministic) r.w., where the asymptotic variance is always > 0. Below we will recall briefly a proof of this result.

By Proposition 2.6, $\lim_n \frac{V_n(x)}{n} = 1 + 2 \sum_{k=1}^{\infty} \mu(f_k = \underline{0})$, so that for a transient r.w. $\frac{V_n(x)}{n}$ has a finite limit.

Variance in the non deterministic transient case.

Now we prove the positivity of the asymptotic variance.

Let $\Psi(\underline{t}) = \mathbb{E}[e^{2\pi i \langle \zeta_0, \underline{t} \rangle}]$, $\underline{t} \in \mathbb{T}^d$. Observe that $\Psi(\underline{t}) \neq 1$ for $\underline{t} \neq \underline{0}$ in $\mathbb{T}^d$, when the r.w. is aperiodic and $|\Psi(\underline{t})| < 1$, for $\underline{t} \notin \Gamma_1$, where Γ_1 is the closed subgroup $\{\underline{t} \in \mathbb{T}^d : e^{2\pi i \langle \underline{r}, \underline{t} \rangle} =$

$1, \forall \underline{r} \in D\}$. We put, for $\underline{t} \in \mathbb{T}^d \backslash \{\underline{0}\}$ and $0 \leq \lambda < 1$,

$$\Phi(\underline{t}) := \frac{1-|\Psi(\underline{t})|^2}{|1-\Psi(\underline{t})|^2} = \Re e[\frac{1+\Psi(\underline{t})}{1-\Psi(\underline{t})}],$$

$$\Phi_\lambda(\underline{t}) := \frac{1-\lambda^2|\Psi(\underline{t})|^2}{|1-\lambda\Psi(\underline{t})|^2} = -1 + 2\sum_{k=0}^{\infty} \lambda^k \Re e(\Psi(\underline{t})^k) = -1 + 2\sum_{k=0}^{\infty} \lambda^k \mu(Z_k = \underline{\ell})\, \cos(2\pi\langle \underline{\ell}, \underline{t}\rangle),$$

where the last relation follows from $\Re e(\Psi(\underline{t})^k) = \Re e(\mathbb{E}[e^{2\pi i\langle Z_k, \underline{t}\rangle}]) = \sum_{\underline{\ell}} \mu(Z_k = \underline{\ell})\, \cos(2\pi\langle \underline{\ell}, \underline{t}\rangle)$.

We put $\Phi(\underline{0}) = 0$.The function Φ is even, non-negative and $\Phi(\underline{t}) = 0$ only on Γ_1, which is $\neq \mathbb{T}^d$ when the r.w. is non deterministic (if the r.w. is deterministic, $\mu(\zeta_0 = \underline{\ell}) = 1$ for some $\underline{\ell} \in \mathbb{Z}^d$ and this implies $|\Psi(\underline{t})| \equiv 1$, but this case is excluded). Therefore Φ is $\neq 0$ a.e. for the Lebesgue measure on $\mathbb{T}^d$.

The proof of the following proposition uses the method of [29], section II.8.

Proposition 3.1. *Let $Z = (Z_n)$ be a transient aperiodic random walk in $\mathbb{Z}^d$. There is a non-negative constant M such that the Fourier coefficients of $\frac{1}{n}K_n$ converges to those of $\Phi + M\delta_{\underline{0}}$ and $\lim_n \int \frac{1}{n} K_n\, \rho\, d\underline{t} > 0$.*

Proof. We use that, if (Z_n) is a transient, for all $\underline{\ell} \in \mathbb{Z}^d$, we have $\sum_{k=1}^{\infty} \mu(Z_k = \underline{\ell}) < +\infty$. Therefore, the series $I(\underline{\ell}) := -1_{\underline{\ell}=\underline{0}} + \sum_{k=0}^{\infty} [\mu(Z_k = \underline{\ell}) + \mu(Z_k = -\underline{\ell})]$ converges and by (53), the even functions $\frac{1}{n}K_n(x, .)$ satisfy:

$$\int_{\mathbb{T}^d} \frac{1}{n} K_n(x, .)\, \cos 2\pi\langle \underline{\ell}, .\rangle\, d\underline{t} \quad = -1_{\underline{\ell}=\underline{0}} + \sum_{k=0}^{n-1} \frac{1}{n} \sum_{j=0}^{n-k-1} [1_{Z_k(T^j x)=\underline{\ell}} + 1_{Z_k(T^j x)=-\underline{\ell}}] \underset{n\to\infty}{\to} I(\underline{\ell}).$$

Note that above the sum over k is written starting from 0. By letting n tend to infinity in the relation

$$\begin{aligned} & -1_{\underline{\ell}=\underline{0}} + \sum_{k=0}^{\infty} \lambda^k [\mu(Z_k = \underline{\ell}) + \mu(Z_k = -\underline{\ell})] \\ = \quad & \int_{\mathbb{T}^d} \cos 2\pi\langle \underline{\ell}, .\rangle\, [-1 + 2\Re e(\frac{1}{1-\lambda\Psi(.)})]\, d\underline{t} \;= \int_{\mathbb{T}^d} \cos 2\pi\langle \underline{\ell}, \underline{t}\rangle\, \Phi_\lambda(.)\, d\underline{t}, \end{aligned}$$

we get since the left sum tends to $I(\underline{\ell})$:

$$I(\underline{\ell}) = \lim_{\lambda\uparrow 1} \int_{\mathbb{T}^d} \cos 2\pi\langle \underline{\ell}, \underline{t}\rangle\, \Phi_\lambda(\underline{t})\, d\underline{t}.$$

Taking $\underline{\ell} = \underline{0}$ in the previous formula, it follows from Fatou's lemma:

$$I(\underline{0}) = 1 + 2\sum_{k=1}^{\infty} \mu(Z_k = \underline{0}) = \lim_{\lambda\uparrow 1} \int_{\mathbb{T}^d} \Phi_\lambda(\underline{t})\, d\underline{t} \geq \int_{\mathbb{T}^d} \lim_{\lambda\uparrow 1} \Phi_\lambda(\underline{t})\, d\underline{t} = \int_{\mathbb{T}^d} \Phi(\underline{t})\, d\underline{t}.$$

This shows the integrability of Φ on $\mathbb{T}^d$ and we can write with a constant $M \geq 0$

$$I(\underline{0}) = \lim_{\lambda\uparrow 1} \int_{\mathbb{T}^d} \Phi_\lambda(\underline{t})\, d\underline{t} = \int_{\mathbb{T}^d} \lim_{\lambda\uparrow 1} \Phi_\lambda(\underline{t})\, d\underline{t} + M = \int_{\mathbb{T}^d} \Phi(\underline{t})\, d\underline{t} + M.$$

Let U_η be the ball of radius $\eta > 0$ centered at $\underline{0}$. By aperiodicity of the r.w., $\Psi(\underline{t}) \neq 1$ for $\underline{t}$ in U_η^c, the complementary in $\mathbb{T}^d$ of U_η, This implies $\sup_{\underline{t}\in U_\eta^c} \sup_{\lambda<1} \Phi_\lambda(\underline{t}) < +\infty$.

Therefore, we get: $\lim\limits_{\lambda\uparrow 1} \int_{U_\eta^c} \cos 2\pi\langle \underline{\ell}, \underline{t}\rangle\, \Phi_\lambda(\underline{t})\, d\underline{t} = \int_{U_\eta^c} \cos 2\pi\langle \underline{\ell}, \underline{t}\rangle\, \Phi(\underline{t})\, d\underline{t}$, hence:

$$I(\underline{\ell}) = \int_{U_\eta^c} \cos 2\pi\langle \underline{\ell}, \underline{t}\rangle\, \Phi(\underline{t})\, d\underline{t} + \lim_{\lambda\uparrow 1} \int_{U_\eta} \cos 2\pi\langle \underline{\ell}, \underline{t}\rangle\, \Phi_\lambda(\underline{t})\, d\underline{t},\ \forall \eta > 0,$$

which can be be written:

$$-\int_{U_\eta} \cos 2\pi\langle \underline{\ell}, .\rangle\, \Phi\, d\underline{t} = I(\underline{\ell}) - \int_{\mathbb{T}^d} \cos 2\pi\langle \underline{\ell}, .\rangle\, \Phi\, d\underline{t} - \lim_{\lambda\uparrow 1} \int_{U_\eta} \cos 2\pi\langle \underline{\ell}, .\rangle\, \Phi_\lambda\, d\underline{t}. \tag{54}$$

Let $\varepsilon > 0$. By positivity of Φ_λ, we have, for $\eta(\varepsilon)$ small enough:

$$(1-\varepsilon) \int_{U_{\eta(\varepsilon)}} \Phi_\lambda\, d\underline{t} \leq \int_{U_{\eta(\varepsilon)}} \cos 2\pi\langle \underline{\ell}, .\rangle\, \Phi_\lambda\, d\underline{t} \leq (1+\varepsilon) \int_{U_{\eta(\varepsilon)}} \Phi_\lambda\, d\underline{t};$$

By subtracting $\int_{U_{\eta(\varepsilon)}} \cos 2\pi\langle \underline{\ell}, \underline{t}\rangle\, \Phi(\underline{t})\, d\underline{t}$ in the previous inequalities and (54), we get:

$$\begin{aligned}
&(1-\varepsilon) \int_{U_{\eta(\varepsilon)}} \Phi_\lambda\, d\underline{t} - \int_{U_{\eta(\varepsilon)}} \cos 2\pi\langle \underline{\ell}, .\rangle\, \Phi\, d\underline{t} \\
&\leq I(\underline{\ell}) - \int_{\mathbb{T}^d} \cos 2\pi\langle \underline{\ell}, .\rangle\, \Phi\, d\underline{t} - \lim_{\lambda\uparrow 1} \int_{U_{\eta(\varepsilon)}} \cos 2\pi\langle \underline{\ell}, .\rangle\, \Phi_\lambda\, d\underline{t} + \int_{U_{\eta(\varepsilon)}} \cos 2\pi\langle \underline{\ell}, .\rangle\, \Phi_\lambda\, d\underline{t} \\
&\leq (1+\varepsilon) \int_{U_{\eta(\varepsilon)}} \Phi_\lambda\, d\underline{t} - \int_{U_{\eta(\varepsilon)}} \cos 2\pi\langle \underline{\ell}, .\rangle\, \Phi\, d\underline{t};
\end{aligned}$$

As we can chose λ such that

$$\left| - \lim_{\lambda\uparrow 1} \int_{U_{\eta(\varepsilon)}} \cos 2\pi\langle \underline{\ell}, .\rangle\, \Phi_\lambda\, d\underline{t} + \int_{U_{\eta(\varepsilon)}} \cos 2\pi\langle \underline{\ell}, .\rangle\, \Phi_\lambda\, d\underline{t} \right| \leq \varepsilon,$$

we obtain:

$$\begin{aligned}
&-\varepsilon + (1-\varepsilon) \int_{U_{\eta(\varepsilon)}} \Phi_\lambda\, d\underline{t} - \int_{U_{\eta(\varepsilon)}} \cos 2\pi\langle \underline{\ell}, .\rangle\, \Phi\, d\underline{t} \\
&\leq I(\underline{\ell}) - \int_{\mathbb{T}^d} \cos 2\pi\langle \underline{\ell}, .\rangle\, \Phi\, d\underline{t} \leq \varepsilon + (1+\varepsilon) \int_{U_{\eta(\varepsilon)}} \Phi_\lambda\, d\underline{t} - \int_{U_{\eta(\varepsilon)}} \cos 2\pi\langle \underline{\ell}, .\rangle\, \Phi\, d\underline{t}
\end{aligned}$$

For ε small enough, $\int_{U_{\eta(\varepsilon)}} \cos 2\pi\langle \underline{\ell}, .\rangle\, \Phi\, d\underline{t}$ can be made arbitrary small, as well as $\varepsilon \sup_{\lambda<1} \int_{U_\eta} \Phi_\lambda\, d\underline{t}$, since Φ is integrable and $\sup_{\lambda<1} \int_{\mathbb{T}^d} \Phi_\lambda\, d\underline{t} < \infty$.

This shows that $I(\underline{\ell}) - \int_{\mathbb{T}^d} \cos 2\pi\langle \underline{\ell}, .\rangle\, \Phi\, d\underline{t} - \int_{U_{\eta(\varepsilon)}} \Phi_\lambda\, d\underline{t}$ can be made arbitrarily small for $\varepsilon > 0$ small and λ close to 1. The same is true for $\underline{\ell} = 0$ and also for the difference

$$\begin{aligned}
&[I(\underline{\ell}) - \textstyle\int_{\mathbb{T}^d} \cos 2\pi\langle \underline{\ell}, .\rangle\, \Phi\, d\underline{t} - \int_{U_{\eta(\varepsilon)}} \Phi_\lambda\, d\underline{t}] - [I(\underline{0}) - \int_{\mathbb{T}^d} \Phi\, d\underline{t} - \int_{U_{\eta(\varepsilon)}} \Phi_\lambda\, d\underline{t}] \\
&= [I(\underline{\ell}) - \textstyle\int_{\mathbb{T}^d} \cos 2\pi\langle \underline{\ell}, .\rangle\, \Phi\, d\underline{t}] - [I(\underline{0}) - \int_{\mathbb{T}^d} \Phi\, d\underline{t}] = [I(\underline{\ell}) - \int_{\mathbb{T}^d} \cos 2\pi\langle \underline{\ell}, .\rangle\, \Phi\, d\underline{t}] - M].
\end{aligned}$$

Therefore $I(\underline{\ell}) = \int_{\mathbb{T}^d} \cos 2\pi\langle \underline{\ell}, \underline{t}\rangle\, \Phi(\underline{t})\, d\underline{t} + M$ for all $\underline{\ell}$ and the Fourier coefficients of $\frac{1}{n} K_n$ converges to those of $\Phi + M\delta_{\underline{0}}$.

As the non-negative sequence $(\frac{1}{n}K_n)$ is bounded in L^1-norm and the density ρ is continuous, this proves $\int \frac{1}{n}K_n\rho\, d\underline{t} \to \int \Phi\rho\, d\underline{t} + M\rho(\underline{0})$. Moreover, the limit is > 0 since both Φ and ρ are not 0 a.e.

It is shown in [29] that $M = 0$ for $d > 1$. □

Behaviour of $M_n(x)$.

In the transient case ($d \geq 3$) (at least for a simple r.w.), Erdös and Taylor (1960) proved that for a constant $\gamma > 0$ depending on the dimension,

$$\lim_n \frac{M_n(x)}{\log n} = \gamma.$$

Recurrent case

In dimension 1, H. Kesten has shown that $\limsup_n \frac{M_n}{\sqrt{n\ \ln\ln n}} = \sqrt{2}/\sigma$. Therefore in dimension 1, we have the following lower and upper bounds for V_n:

$$C_1(x)\, n^{\frac{3}{2}}\, (\ln\ln\ n)^{-\frac{1}{2}} \leq V_n(x) \leq C_2(x)\, n^{\frac{3}{2}} (\ln\ln n)^{\frac{1}{2}}.$$

Dimension $d = 2$.

There is a deterministic rate (law of large numbers): for a constant C_0.

$$\frac{\int V_n\, d\mu}{n\log n} \to C_0 \text{ and } \frac{V_n(x)}{n\log n} \to C_0, \text{ for a.e. } x.$$

For a planar simple random walk, Erdös and Taylor [17] proved:

$$\limsup_n \frac{M_n(x)}{(\log n)^2} \leq \frac{1}{\pi}. \tag{55}$$

The result has been extended by Dembo, Peres, Rosen and Zeitouni [15], who proved for an aperiodic centered random walk on $\mathbb{Z}^2$ with moments of all orders:

$$\lim_n \frac{M_n(x)}{(\log n)^2} = \frac{1}{2\pi\, \det(A)^{\frac{1}{2}}},$$

where A is the covariance matrix associated to the random walk.

As shown in the proof in [15], it suffices to suppose that the 2-dimensional r.w. is aperiodic. Moreover, the proof for the upper bound is based on the local limit theorem which uses only the existence of the moment of order 2. Therefore, assuming the existence of the moment of order 2, the upper bound (55) holds.

It follows in this case: there exist $C(x)$ a.e finite such that:

$$\frac{M_n^2(x)}{V_n(x)} \leq C(x)\frac{(\log n)^3}{n}.$$

3.2. **Step functions over rotations.**

Now we take $X = \mathbb{T}^r$, $r \geq 1$ endowed with μ, the uniform measure and we consider cocycles over rotations. When they are centered, such cocycles are strongly recurrent and therefore the associated quantities V_n and M_n are big. The difficult part is to bound them from above. We will give an example where an upper bound can be obtained.

Let T_α be the rotation by an irrational α. For $f : X \to \mathbb{Z}^d$, recall that the cylinder map (cf. Subsection 2.1) is $\tilde{T}_{f,\alpha} = \tilde{T}_\alpha : X \times \mathbb{Z}^d \to X \times \mathbb{Z}^d$ defined by $\tilde{T}_\alpha(x, \underline{\ell}) = (x+\alpha, \underline{\ell}+f(x))$.

Non centered step cocycles over a rotation.

Let f be a non centered function with a finite number of values values in $\mathbb{Z}^d$. Suppose that f is Riemann integrable, which amounts to assume that, for the uniform measure of the torus, the measure of the set of discontinuity points of f is zero.

Then by a remark in Subsection 2.2, $M_n(x)$ is bounded uniformly in x and n. Therefore, for $V_n(x)$, the bounds $n \leq V_n(x) \leq Cn$ are satisfied.

Centered step cocycles over a 1-dimensional rotation.

The interesting situation is that of centered functions. We will consider the case $r = 1$ and when the irrational number α has *bounded partial quotients*.

Recall that an irrational α with continued fraction expansion $[0; a_1, a_2, ..., a_n, ...]$ is said to have bounded partial quotients (bpq) if $\sup_n a_n < +\infty$. The set of bpq numbers has Lebesgue measure zero and Hausdorff dimension 1.

In the sequel of this subsection, α will be an irrational bpq number (for instance a quadratic irrational) and f a centered function with values in $\mathbb{Z}$ and bounded variation.

By Denjoy-Koksma inequality, there is a logarithmic bound for the cocycle (T_α, f): $|f_n(x)| \leq C \ln n$, for a constant C.

The cocycle is strongly recurrent to 0 (and this is true for $d \geq 1$ if f centered has values in $\mathbb{Z}^d$, when its components have bounded variation). This makes the corresponding maximum $M_n(x)$ big. Nevertheless, we will see that condition (25) is satisfied, at least for a special example.

Lower bound.

Lower bound for V_n and variance, case $d = 1$.

For a general sequence (z_k), we can obtain a lower bound for V_n by an elementary method when there is an upper bound for the variance defined below.

Lemma 3.2. *Defining the mean m_n and the variance σ_n^2 by*

$$m_n = \frac{1}{n}\sum_{k=1}^{n} z_k, \;\; \sigma_n^2 = \frac{1}{n}\sum_{k=1}^{n}(z_k - m_n)^2,$$

we have

$$V_n \geq \frac{1}{9}\frac{n^2}{\sigma_n},\ \textit{if}\ \sigma_n > 1. \tag{56}$$

Proof. Suppose that $\sigma_n > 0$. For $\lambda > 1$, let $\Delta_\lambda := [-\lambda\sigma_n + m_n,\ \lambda\sigma_n + m_n] \bigcap \mathbb{Z}$. We have:

$$\sigma_n^2 \geq \frac{1}{n}\sum_{k=0}^{n-1}(z_k - m_n)^2 1_{z_k \in \Delta_\lambda^c} \geq \frac{1}{n}\sum_{k=0}^{n-1}(1_{z_k \in \Delta_\lambda^c})\,\lambda^2\sigma_n^2.$$

Therefore: $\sum_{k=0}^{n-1} 1_{z_k \in \Delta_\lambda} \geq n(1-\lambda^{-2})$. As $\mathrm{Card}(\Delta_\lambda) \leq 2\lambda\sigma_n + 1$. It follows by (41):

$$V_n \geq \frac{(1-\lambda^{-2})^2}{2\lambda\sigma_n + 1}\, n^2.$$

For $\lambda = 2$ we get: $V_n \geq \ \frac{\frac{9}{16}}{4\sigma_n+1}\, n^2 \geq \frac{9}{80}\,\frac{n^2}{\sigma_n},$ if $\sigma_n > 1$; hence (56). □

If z_k is given by ergodic sums, i.e., $z_k = f_k(x)$, let

$$m_n(x) := \frac{1}{n}\sum_{k=1}^{n} f_k(x),\ \sigma_n^2(x) = \frac{1}{n}\sum_{k=1}^{n}(f_k(x) - m_n(x))^2.$$

By [5, Proposition 13], for α bpq and f with bounded variation, it holds $\sigma_n^2(x) \leq C\ln n$. Using (56) and $V_n(x) \leq nM_n(x)$, this gives a lower bound for $V_n(x)$ and $M_n(x)$:

$$V_n(x) \geq c\,\frac{n^2}{\sqrt{\ln n}},\ M_n(x) \geq c\,\frac{n}{\sqrt{\ln n}}. \tag{57}$$

Below we will get an estimate from above in the following example.

Example 3.3. $f = \mathbf{1}_{[0,\frac{1}{2})} - \mathbf{1}_{[\frac{1}{2},1)}$ and α bpq.

Upper bound for the example (3.3).

For f as above and α bpq, we have by [1], for some constant $C_1 > 0$,

$$\|N_n(\cdot,0)\|_{ess\ sup} = \|\tilde{S}_n(\mathbf{1}_{\mathbb{T}^1\times\{0\}})(\cdot,0)\|_\infty \leq \frac{C_1 n}{\sqrt{\log n}}. \tag{58}$$

Remark that the bound (58) is obtained in [1] as the limit of $\|N_n(\cdot,0)\|_p$, the L^p-norm of $N_n(\cdot,0)$, as p goes to ∞. Therefore the bound is written for the norm $\|.\|_{ess\ sup}$,

The function $N_n(x,0)$ is locally constant outside the discontinuity points $x = -j\alpha \bmod 1$, $x = \frac{1}{2} - j\alpha \bmod 1$, $0 \leq j < n$. Therefore, if $\mathcal{D}$ denotes the countable set of discontinuity points of the sums $f_k, k \geq 1$, the bound in (58) holds for every $x \notin \mathcal{D}$.

Below, x is any point $\notin \mathcal{D}$.

Using (33), it follows:

$$M_n(x) \leq C_1\frac{n}{\sqrt{\log n}}. \tag{59}$$

By (59) and since $V_n(x) \leq n\, M_n(x)$, we obtain

$$V_n(x) \leq C_1 \frac{n^2}{\sqrt{\log n}}. \tag{60}$$

From (57), (59) and (60), it follows: $V_n(x) \asymp n^2/\sqrt{\log n}$ and $M_n(x) \asymp n/\sqrt{\log n}$, where $a_n \asymp b_n$ for two sequences (a_n) and (b_n) means $c\, a_n \leq b_n \leq C\, a_n, \forall n \geq 1$, with two positive constants c, C.

Therefore we get in this special example 3.3:

$$\frac{M_n^2(x)}{V_n(x)} \leq (\frac{C_1 n}{\sqrt{\log n}})^2 / \frac{cn^2}{\sqrt{\log n}} = \frac{C_1^2}{c}\, \frac{1}{\sqrt{\log n}} \to 0. \tag{61}$$

Condition (25) of Theorem 1.11 is satisfied in this example, as well as the condition of Theorem 1.8 a), hence a Glivenko-Cantelli theorem along $(S_n f(x))$ for i.i.d. r.v.'s.

But the sufficient conditions for the Glivenko-Cantelli theorems 1.7, 1.8 b), 1.10 are not satisfied by this cocycle and more generally, in view of the lower bound (57), by a cocycle defined by step functions over a bpq irrational rotation.

4. About limit theorems along ergodic sums

4.1. Glivenko-Cantelli theorem along ergodic sums.

The Glivenko-Cantelli theorem recalled in the introduction is a (pointwise) law of large numbers uniform over a set of functions (here the indicators of intervals). When the r.v.'s X_k are i.i.d., the proof is an easy consequence of the strong law of large numbers applied to the sequence of i.i.d. bounded r.v.'s $(1_{X_k \leq s})$. Using Birkhoff's ergodic theorem, the Glivenko-Cantelli theorem has been extended to the setting of a strictly stationary sequence (X_k) of random variables. More precisely, formulated in terms of dynamical systems, the following holds:

Let $(Y, \mathcal{A}, \nu)$ be a probability space and S an ergodic measure preserving transformation on Y. For any measurable function $\varphi : Y \to \mathbb{R}$, let us consider the strictly stationary sequence (X_k) defined by $X_k = \varphi \circ S^k, k \geq 0$. Then the sequence of empirical distribution functions satisfies: for ν a.e. $y \in Y, \sup_s |\frac{1}{n} \sum_{k=0}^{n-1} 1_{X_k(y) \leq s} - F(s)| \to 0$, where $F(s) = \nu(\varphi \leq s)$.

Observe that the result is an application of Birkhoff's theorem and Lemma 1.1 recalled in Section 1. Its extension to the non ergodic case has been formulated by Tucker [30], the distribution function $F(s)$ being replaced by the conditional distribution function $\mathbb{E}(1_{\varphi \leq s} | \mathcal{J})$, where $\mathcal{J}$ is the σ-algebra of S-invariant sets. In others words, we have:

$$\text{for } \nu \text{ a.e. } y \in Y, \lim_{n \to \infty} \sup_s |\frac{1}{n} \sum_{k=0}^{n-1} 1_{\varphi(S^k y) \leq s} - \mathbb{E}(1_{\varphi \leq s} | \mathcal{J})(y)| = 0.$$

The above formula relies on the ergodic decomposition which can be used in the proof.

In the previous framework, for a process, a Glivenko-Cantelli like theorem sampled along a sequence generated by a dynamical system can be obtained as follows:

As in Subsection 2, let T be an ergodic measure preserving transformation on a probability space $(X, \mathcal{B}, \mu)$ and f a measurable function on X with values in $\mathbb{Z}^d$, $d \geq 1$.

Let us take a second system $(\Omega, \mathbb{P}, \theta)$, where $\theta = (\theta^{\underline{\ell}})_{\underline{\ell} \in \mathbb{Z}^d}$ is a $\mathbb{Z}^d$-action preserving $\mathbb{P}$.

The skew product associated to the cocycle (T, f) and θ is the map: $T_{\theta,f} : (x, \omega) \to (Tx, \theta^{f(x)}\omega)$ from $X \times \Omega$ to itself. By iteration we get:

$$T^k_{\theta,f}(x, \omega) = (T^k x, \theta^{f_k(x)}\omega).$$

For example, as $\mathbb{Z}^d$-action, we can take a $\mathbb{Z}^d$-Bernoulli shift $(\Omega, \mathbb{P}, (\theta^{\underline{\ell}})_{\underline{\ell} \in \mathbb{Z}^d})$, with $\mathbb{P}$ a product measure and θ the shift on the coordinates. If X_0 is the first coordinate map, then $(X_{\underline{\ell}}) = (X_0 \circ \theta^{\underline{\ell}})$ is a family of i.i.d. r.v.'s indexed by $\mathbb{Z}^d$.

In general, let $\mathcal{I}_{\theta,f}$ denote the conditional expectation with respect to the σ-algebra of $T_{\theta,f}$-invariant sets. The ergodic theorem for $T_{\theta,f}$ shows that, for $\psi \in L^1(\mu \times \mathbb{P})$,

$$\lim_n \frac{1}{n} \sum_{k=0}^{n-1} \psi(T^k x, \theta^{f_k(x)}\omega) = \mathcal{I}_{\theta,f}(\psi)(x, \omega), \text{ for } \mu \times \mathbb{P}\text{-a.e.}(x, \omega). \tag{62}$$

If φ is a measurable function on Ω, putting $\psi_s(x, \omega) = \mathbf{1}_{I_s}(\varphi(\omega))$, where I_s is the half-line $]-\infty, s]$, we have

$$\psi_s(T^k_{\theta,f}(x, \omega)) = \mathbf{1}_{I_s}(\varphi(\theta^{f_k(x)}\omega)).$$

By the quoted Tucker's result, the convergence in (62) for each ψ_s, $s \in \mathbb{R}$, can be strengthened into a uniform convergence with respect to s:

$$\text{for } \mu \times \mathbb{P}\text{-a.e } (x, \omega),\ \tfrac{1}{n} \sup_s |\textstyle\sum_{k=0}^{n-1} \mathbf{1}_{I_s}(\varphi(\theta^{f_k(x)}\omega)) - \mathcal{I}(\psi_s)(x, \omega)| \to 0.$$

Therefore, by the Fubini theorem, there is a "sampled" version of the Glivenko-Cantelli theorem for the empirical process of a stationary sequence:

Proposition 4.1. *For μ-a.e x, we have*

$$|\sup_s \tfrac{1}{n} \textstyle\sum_{k=0}^{n-1} \mathbf{1}_{I_s}(\varphi(\theta^{f_k(x)}\omega)) - \mathcal{I}(\psi_s)(x, \omega)| \to 0, \text{ for } \mathbb{P}\text{-a.e } \omega.$$

When $T_{\theta,f}$ is ergodic, if $\psi \in L^1(\mu \times \mathbb{P})$, we have $\mathcal{I}_{\theta,f}(\psi)(x, \omega) = \int \psi \, d\mu \, d\mathbb{P}$, for $\mu \times \mathbb{P}$-a.e. (x, ω), and the centering $\mathcal{I}(\psi_s)(x, \omega)$ is given by the distribution function $F(s) = \mu(\varphi \leq s)$. In this case, for a.e. x, a Glivenko-Cantelli theorem with the usual centering holds for the empirical process sampled along the sequence (z_n) given by $z_n = S_n f(x)$ (with a set of ω's of $\mathbb{P}$-measure 1 depending on x).

The lemma below shows, as it is known, that ergodicity of the cylinder map $\tilde{T}_f$ implies ergodicity of the skew map $T_{\theta,f}$. Let us sketch a proof.

Lemma 4.2. *Suppose that the cocycle (T, f) is recurrent and the map $\tilde{T}_f$ ergodic. If the action of $\mathbb{Z}^d$ by θ on $(\Omega, \mathbb{P})$ is ergodic, then $T_{\theta,f}$ is ergodic on $(X \times \Omega, \mu \times \mathbb{P})$.*

Proof. : Let Φ be a $T_{\theta,f}$ invariant measurable function on $X \times \Omega$:

$$\Phi(Tx, \theta^{f(x)}\omega) = \Phi(x, \omega), \text{ for a.e. } (x, \omega).$$

For a.e. x, there is a set Ω_x^0 of full $\mathbb{P}$-measure in Ω such that $\Phi(Tx, \theta^{f(x)}\omega) = \Phi(x, \omega)$, for all $\omega \in \Omega_x^0$. As $\mathbb{Z}^d$ is countable, for a.e. x, there is a set Ω_x of full measure such that

$$\Phi(Tx, \theta^{f(x)}\theta^{\underline{\ell}}\omega) = \Phi(x, \theta^{\underline{\ell}}\omega), \text{ for all } \omega \in \Omega_x.$$

Let $\omega \in \Omega_x$. The function $\varphi_\omega(x, \underline{\ell}) := \Phi(x, \theta^{\underline{\ell}}\omega)$ on $X \times \mathbb{Z}^d$ is measurable, $\tilde{T}_f$-invariant:

$$\begin{aligned} \varphi_\omega(\tilde{T}_f(x, \underline{\ell})) &= \varphi_\omega(Tx, \underline{\ell} + f(x)) = \Phi(Tx, \theta^{\underline{\ell}+f(x)}\omega) \\ &= \Phi(Tx, \theta^{f(x)}\theta^{\underline{\ell}}\omega) = \Phi(x, \theta^{\underline{\ell}}\omega) = \varphi_\omega(x, \underline{\ell}). \end{aligned}$$

It follows from the ergodicity of $\tilde{T}_f$ that there is a constant c_ω such that $\varphi_\omega(x, \underline{\ell}) = c_\omega$ for a.e. x. Therefore Φ coincides a.e. with a function ψ on Ω which is θ-invariant, hence a constant by the assumption of ergodicity of the action of $\mathbb{Z}^d$ on Ω. □

With Fubini's argument, we get a Glivenko-Cantelli theorem for a.e. x, if we can show that the skew map $T_{\theta,f}$ is ergodic.

There are many examples of cylinder maps $\tilde{T}_f$ which are shown to be ergodic in the literature and so providing examples via Lemma 4.2. For instance, we can take for T an irrational rotation and $f = \mathbf{1}_{[0,\frac{1}{2})} - \mathbf{1}_{[\frac{1}{2},1)}$. The cocycle (T, f) is ergodic and the above version of Glivenko-Cantelli theorem applies for any stationary sequence (X_k) (with a conditional distribution if the stationary sequence is not ergodic). See also examples for which the skew map is ergodic in [25].

4.2. **Discussion: universal sequences.**

The weakness in the approach of the previous subsection for a sampled Glivenko-Cantelli theorem along ergodic sums $(S_k f(x), k \geq 0)$ is that it yields a set of x's of μ-measure 1 depending on the dynamical system $(\Omega, \mathbb{P}, \theta)$ and on φ. One can try to reinforce the statement by introducing a notion of "universal property".

In this direction, the LLN for sums sampled along ergodic sums is closely related in the following way to the random ergodic theorems which have been studied in several papers.

First, let us call "universally good" a sequence (z_k) such that, for every dynamical system $(\Omega, \mathbb{P}, \theta)$, for every $\varphi \in L^1(\mathbb{P})$, the sequence $\frac{1}{n}\sum_{k=0}^{n-1} \varphi \circ \theta^{z_k}$ converges $\mathbb{P}$-a.e.

We say that (T, f) a "(pointwise) good averaging cocycle" (or a universally representative sampling scheme) if, for μ-a.e. x, the sequence $(S_k f(x))$ is universally good, i.e., for every dynamical system $(\Omega, \mathbb{P}, \theta)$, for every $\varphi \in L^1(\mathbb{P})$, $\frac{1}{n}\sum_{k=0}^{n-1} \varphi \circ \theta^{S_k f(x)}$ converges $\mathbb{P}$-a.e.

The definition of a "mean good averaging cocycle" is similar, changing the above convergence into convergence in $L^2(\mathbb{P})$-norm, for every φ in $L^2(\mathbb{P})$.

A question which has been studied is to find mean or pointwise good averaging cocycles. In the first direction, examples and counterexamples of mean good averaging 1-dimensional cocycles are studied in [25],

For pointwise convergence, there are 1-dimensional examples given by cocycles with a drift. In [23], the following result is shown: the cocycle defined by a random walk with a moment of order 2 is a pointwise good averaging cocycle if and only if it is not centered. Moreover it is shown that any ergodic integrable integer-valued stochastic process with nonzero mean is universally representative for bounded stationary processes. The proofs are based on the recurrence time theorem ([6]).

Notice that a related, but different, notion can be introduced by restricting the dynamical system $(\Omega, \mathbb{P}, \theta)$ to belong to a given class $\mathcal{C}$ of dynamical systems.

Let us call "pointwise good for a class $\mathcal{C}$ of dynamical systems", a sequence (z_k) such that, for every dynamical system $(\Omega, \mathbb{P}, \theta)$ in the class $\mathcal{C}$, for every $\varphi \in L^1(\mathbb{P})$, $\lim_n \frac{1}{n} \sum_{k=0}^{n-1} \varphi \circ \theta^{z_k} = \int \varphi \, d\mathbb{P}$, $\mathbb{P}$-a.e. There is a similar property for the mean convergence.

This can be also expressed for a class of random fields satisfying a condition on the decay of correlations.

For example, by Remark 2.9, every cocycle with values in $\mathbb{Z}^d$ which is not a coboundary is a mean good averaging cocycle for the stationary r.f.s on $\mathbb{Z}^d$ such that $\sum_{\underline{\ell}} |\langle U_{\underline{\ell}}, U_{\underline{0}} \rangle| < +\infty$.

If (z_k) is pointwise universally good for a class $\mathcal{C}$, clearly we get the Glivenko-Cantelli property for any dynamical system $(\Omega, \mathbb{P}, \theta)$ in $\mathcal{C}$ and every measurable function φ, i.e.:

$$\sup_s |\tfrac{1}{n} \textstyle\sum_{k=0}^{n-1} \mathbf{1}_{I_s}(\varphi(\theta^{z_k}\omega)) - \mathbb{P}(\varphi \leq s)| \to 0, \text{ for } \mathbb{P}\text{-a.e } \omega. \tag{63}$$

As we see, there are two different approaches of the notion of universal sequences for a law of large numbers: either we ask for a LLN along such a sequence for every dynamical system $(\Omega, \mathbb{P}, \theta)$ and all functions in $L^1(\mathbb{P})$ or we fix a class of dynamical systems, or a class of functions in $L^1(\mathbb{P})$. In the latter case, the condition on the sequence (z_k) may be expressed in a quantitative way. To finish, let us give a known example (cf. [13]) and recall briefly the proof.

Proposition 4.3. *Let (z_k) be a strictly increasing sequence of positive integers. If the sequence satisfies: for a finite constant C, $z_k \leq Ck, \forall k \geq 1$, then (z_k) is a pointwise good averaging sequence for the class $\mathcal{C}$ of dynamical systems $(\Omega, \mathbb{P}, \theta)$ with Lebesgue spectrum.*

Proof. There is a dense set of functions $\varphi \in L^1(\mathbb{P})$ such that

$$\frac{1}{n} \sum_{k=0}^{n-1} \varphi(\theta^{z_k}\omega) \text{ converges } \mathbb{P}\text{-a.e.} \tag{64}$$

Indeed, by Rajchman's SLLN for random variables in L^2 which are orthogonal and with a common bound for their second moment (cf. [7]), (64) is satisfied by $\varphi \in L^2(\mathbb{P})$ such that $\langle \varphi, \varphi \circ \theta^k \rangle = 0, \forall k$. The Lebesgue spectrum property implies that such functions span a dense linear space in $L^2(\mathbb{P})$, hence in $L^1(\mathbb{P})$.

Moreover, in view of the assumption on (z_k), by the ergodic maximal lemma, the space of functions φ such that (64) holds is closed. Therefore (64) is satisfied by every $\varphi \in L^1(\mathbb{P})$. □

References

[1] Aaronson, J., Bromberg, M. and Nakada, H.: Discrepancy skew products and affine random walks, Israel J. Math. 221 (2017), no. 2, 973-1010.

[2] Ambrose L.: Functional generalizations of Hoeffding's covariance lemma and a formula for Kendall's tau, Statistics and Probability Letters, 122 (2017), 218-226.

[3] Billingsley P.: *Convergence of probability measures*, 1rst ed. John Wiley & Sons, Inc., 1968.

[4] Birkel, T.: A note on the strong law of large numbers for positively dependent random variables, Statist. Probab. Lett. 7 (1988), no. 1, 17-20.

[5] Borda, B.: On the distribution of Sudler products and Birkhoff sums for the irrational rotation, Arkiv 2021.

[6] Bourgain J., Furstenberg H., Katznelson Y. and Ornstein D.: Appendix on return-time sequences. Publ. Math. IHES, (69) 42-45, 1989.

[7] Chung, K.L.: *A course in probability theory*, 3rd ed. (2001), Acad. Press, Inc., San Diego, CA.

[8] Chung, K.L., Fuchs, W.H.J.: *On the distribution of values of sums of random variables*, Mem. Amer. Math. Soc. 6 (1951), 157-168.

[9] Chung, K.L., Ornstein, D.: *On the recurrence of sums of random variables*, Bull. Amer. Math. Soc. 68 (1962), 30-32.

[10] Cohen, G., Conze, J.-P.: On the quenched functional CLT in random sceneries, to appear in Studia Matematica.

[11] Cohen, G., Conze, J.-P.: CLT for random walks of commuting endomorphisms on compact abelian groups, J. Theoret. Probab. 30 (2017), no. 1, 143-195.

[12] Conze, J.-P.: Remarques sur les transformations cylindriques et les équations fonctionnelles, Pub. sém. math. info. de Rennes, 1976, fasc. 2, http://www.numdam.org/actas/PSMIR/

[13] Conze, J.-P.: Convergence des moyennes ergodiques pour des sous-suites, Bull. Soc. Math. Fr., Mémoire 35, (1973), p. 7-15.

[14] Davenport, H.; Erdös, P.; LeVeque, W.J.: On Weyl's criterion for uniform distribution, Michigan Math. J. 10 (1963), 311-314.

[15] Dembo, A.; Peres, Y.; Rosen, J.; Zeitouni, O.: Thick points for planar Brownian motion and the Erdos-Taylor conjecture on random walk. Acta Math. 186 (2001), no. 2, 239-270.

[16] Derriennic, Y.: Quelques applications du théorème ergodique sous-additif, Astérisque, tome 74 (1980), p. 183-201.

[17] Erdös, P.; Taylor, S.J.: Some problems concerning the structure of random walk paths. Acta Math. Acad. Sci. Hungar. 11 (1960), 137–162.

[18] Esary, J.D.; Proschan, F.; Walkup, D.W.: Association of random variables, with applications, Ann. Math. Statist. 38 (1967), 1466-1474.

[19] Garsia, A.M.: Topics in almost everywhere convergence, Lectures in Advanced Mathematics, No. 4 Markham Publishing Co., Chicago, Ill. (1970).

[20] Hoeffding W.: Probability Inequalities for Sums of Bounded Random Variables, Journal of the AMS vol. 58 (1963), no. 301, 13-30.

[21] Hewitt, E.; Ross, K.A.: *Abstract harmonic analysis*, Vol. II. Die Grundlehren der mathematischen Wissenschaften, Band 152 Springer-Verlag, New York-Berlin 1970.

[22] Kesten, H.: An iterated logarithm law for local time. Duke Math. J. 32 (1965), 447-456.

[23] Lacey, M.; Petersen, K.; Wierdl, M.; Rudolph, D.: Random ergodic theorems with universally representative sequences., Ann. I. H. Poincaré Prob. Stat. 30 (1994), no. 3, 353-395.

[24] Lehmann, E.L.: Some concepts of dependence, Ann. Math. Statist. 37 (1963), 1137-1153.

[25] Lemańczyk, M.; Lesigne, E.; Parreau, F.; Volný, D.; Wierdl, M.: Random ergodic theorems and real cocycles, Israel J. Math. 130 (2002), 285-321.

[26] Neveu, J.: Courte démonstration du théorème ergodique sur-additif, Ann. Inst. Henri Poincaré, Vol. XIX, no 1, 1983, p. 87-90.

[27] Newman, C.M.: Normal fluctuations and the FKG inequalities, Comm. Math. Phys. 74 (1980), no. 2, 119-128.

[28] Newman, C.M. and Wright, A.L.: An invariance principle for certain dependent sequences, Ann. Probab. 9 (1981), no. 4, 671-675.

[29] Spitzer, F.: *Principles of random walk*, The University Series in Higher Mathematics D. Van Nostrand Co., Inc., Princeton, N.J.-Toronto-London (1964). doi: 10.1007/978-1-4757-4229-9

[30] Tucker, H.: A generalization of the Glivenko-Cantelli theorem, Annals of Math. Stat., vol. 20, no 3, p. 828-830 (1959).

[31] Yu Hao: A Glivenko-Cantelli lemma and weak convergence for empirical processes of associated sequences, Probab. Theory Related Fields 95 (1993), no. 3, 357-370.

[32] Zygmund, A.: *Trigonometric series*, Second edition, Cambridge University Press, London-New York 1968.

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