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Generalised time-translation-invariance in simple ageing

Malte Henkel

Abstract Generalised time-translation-invariance, together with dynamical scaling, imply the entire known generic phenomenology of physical ageing.

1 Physical ageing

The physics of ageing was first studied in experiments on the mechanical relaxation in glasses [45]. Herein, the ageing process was started by rapidly quenching the glass from its molten state to a temperature below the *glass-transition temperature* where the glass solidifies. The sample response was measured and studied in dependence of the ‘waiting time’ s before a mechanical stress had been applied. Several reproducible properties, independent of the sample history or the detailed microscopic structure of the glass, were identified [53, 3]. Subsequently, it became apparent that very similar phenomenologies also arise in other situations of non-equilibrium dynamics, for example in the relaxation of magnetic systems, quenched to a low-temperature state from an initially disordered high-temperature state [7], also observed experimentally [38, 2, 50]. Characteristics of physical ageing are:

Definition: *A many-body system is said to undergo physical ageing when its relaxation dynamics from some initial state obeys the properties*

1. *it is slow (in contrast to ‘fast relaxations’ described in terms of exponentials $e^{-t/\tau}$ with a finite relaxation time τ)*
2. *dynamical scaling*
3. *broken time-translation-invariance*

Malte Henkel^{a,b}

^aLaboratoire de Physique et Chimie Théoriques (CNRS UMR 7019), Université de Lorraine Nancy, B.P. 70239, F – 54506 Vandœuvre-lès-Nancy Cedex, France

^bCentro de Física Teórica e Computacional, Universidade de Lisboa, P – 1749-016 Lisboa, Portugal

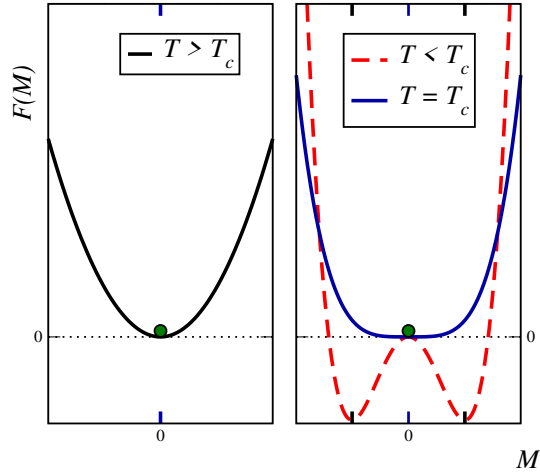


Fig. 1 Schematic free energies. The circle represents the disordered initial state.

These three conditions require several comments. First, the insistence on slow dynamics distinguishes systems in a disordered state with a single equilibrium state (when according to the general principles of thermodynamics, a fast relaxation within a finite characteristic time τ_r towards that state should occur) from systems either at a critical point or in a two-phase coexistence region. In figure 1, the equilibrium free energy $F = F(M)$ (in the language of magnetic systems, where M denotes the magnetisation) is shown.¹ Before the quench (left panel), the system is prepared in a disordered initial state and after a rapid relaxation, this state is found at the unique minimum of the free energy, with gaussian fluctuations around it. Immediately after the quench (right panel), the system's state has not yet evolved but is no longer at a stable minimum. At criticality ($T = T_c$), critical fluctuations lead to a wider form of $F(M)$ and allow for larger and non-gaussian fluctuations which also slow down any relaxation towards equilibrium. Below criticality ($T < T_c$), the system now sits at an unstable maximum of $F(M)$ and there are at least two equivalent, but distinct equilibrium states at the new minima of $F(M)$. In both cases, a rapid relaxation towards equilibrium is impossible and the system becomes spatially inhomogeneous, mathematically expressed through the relaxation time τ_r becoming formally infinite. Hence only in the situations sketched in the right panel of figure 1 ageing is at all possible. Second, the requirement of dynamical scaling asserts that the dynamics can be described in terms of a single time-dependent length scale $L = L(t)$. To be specific, we shall restrict throughout to systems where $L(t) \sim t^{1/z}$ grows algebraically for large times,² which defines the *dynamic exponent* z . Third, the breaking of time-translation-invariance implies that ageing is an intrinsically *non-equilibrium* phenomenon. The precise meaning of ‘breaking of time-translation-invariance’ will be the central topic of this work.

¹ For simplicity, we shall use the language of classical dynamics, although all considerations which follow should apply to quantum quenches as well.

² Hence glassy systems where $L(t) \sim (\ln t)^{1/\psi}$ grows logarithmically [19, 53, 3] are not considered.

The relaxation process is described via the time- and space-dependent order-parameter $\phi = \phi(t, \mathbf{r})$, conveniently coarse-grained such that a continuum description becomes feasible. Unless stated otherwise, we shall assume an initial disordered state with vanishing initial order-parameter $\langle \phi_{\text{ini}}(\mathbf{r}) \rangle = \langle \phi(0, \mathbf{r}) \rangle = 0$, where the average $\langle \cdot \rangle$ is taken over the initial configurations and thermal histories. The *two-time correlator* C and the *two-time response* R are defined as [17, 32]

$$C(t, s; \mathbf{r}) = \langle \phi(t, \mathbf{r}) \phi(s, \mathbf{0}) \rangle, \quad R(t, s; \mathbf{r}) = \left. \frac{\delta \langle \phi(t, \mathbf{r}) \rangle}{\delta h(s, \mathbf{0})} \right|_{h=0} = \langle \phi(t, \mathbf{r}) \tilde{\phi}(s, \mathbf{0}) \rangle \quad (1)$$

Because of dynamical scaling, one expects,³ for sufficiently large times $t, s \rightarrow \infty$ and $y = t/s > 1$ that

$$C(t, s; \mathbf{r}) = s^{-b} F_C \left(\frac{t}{s}; \frac{|\mathbf{r}|}{s^{1/z}} \right), \quad R(t, s; \mathbf{r}) = s^{-1-a} F_R \left(\frac{t}{s}; \frac{|\mathbf{r}|}{s^{1/z}} \right) \quad (2)$$

where a, b are ageing exponents.⁴ This kind of dynamical scaling is often referred to as *simple ageing*. Evidence for this kind of data collapse abounds in the literature, e.g. [7, 16, 27, 47]. If one restricts to auto-correlations and auto-responses with $\mathbf{r} = \mathbf{0}$, one expects asymptotically for $y \gg 1$

$$f_C(y) = F_C(y, 0) \sim y^{-\lambda_C/z}, \quad f_R(y) = F_R(y, 0) \sim y^{-\lambda_R/z} \quad (3a)$$

where λ_C, λ_R are the auto-correlation and auto-response exponents, respectively. Scale-invariance alone cannot account for such an algebraic asymptotic behaviour. Furthermore, one generally finds the exponent equality⁵

$$\lambda_C = \lambda_R = \lambda \quad (3b)$$

The universality of the values of λ_C, λ_R and of the form of the scaling functions $f_C(y), f_R(y)$ implies that they should be independent of the ‘microscopic details’ of any given model. In other words, we are looking for some as yet un-recognised dynamical symmetry in order to explain the assertions eqs. (3). In what follows, we shall generalise time-translation-invariance in such a way that it becomes applicable to ageing phenomena. The hypothesis will be presented in section 2. In section 3 we shall show that not only the examples mentioned so far, but also several other well-established facts of ageing phenomenology follow from this single symmetry.

³ This explicitly excludes multi-scaling as found e.g. in phase-separation kinetics at $T < T_c$ with conserved order-parameters [14, 9], as well as logarithmic sub-ageing [5, 18].

⁴ Spatial translation-invariance as well as spatial rotation-invariance such that $\mathbf{r} \mapsto r = |\mathbf{r}|$ will be admitted throughout for notational simplicity.

⁵ At criticality, this may be argued from the finiteness of the limit fluctuation-dissipation ratio [21] $X_\infty = \lim_{y \rightarrow \infty} (\lim_{s \rightarrow \infty} X(y, s))$ with $X(t, s) = T_C R(t, s; \mathbf{0}) / \partial_s C(t, s; \mathbf{0})$. This argument does not apply to $T < T_c$, but for short-ranged and gaussian initial correlators, (3b) still follows, either from a comparison of the scaling form of correlators and responses [6, 7] or else, if in addition $z = 2$, from local scale-invariance [41, 27].

2 Generalised time-translation-invariance

Postulate: *The Lie algebra generator X_n^{equi} of a time-space symmetry of an equilibrium system becomes a symmetry out of equilibrium by the change of representation*

$$X_n^{\text{equi}} \mapsto X_n = e^{\xi \ln t} X_n^{\text{equi}} e^{-\xi \ln t} \quad (4)$$

where ξ is a parameter whose value characterises the scaling operator ϕ on which X_n acts. When applied to a dilatation generator $X_0^{\text{equi}} = -t\partial_t - \frac{1}{z}r\partial_r - \delta$, the prescription (4) leads to a modified effective scaling dimension $\delta_{\text{eff}} = \delta - \xi$, since

$$X_0^{\text{equi}} \mapsto X_0 = -t\partial_t - \frac{1}{z}r\partial_r - (\delta - \xi) \quad (5a)$$

However, the time-translation generator $X_{-1}^{\text{equi}} = -\partial_t$ becomes

$$X_{-1}^{\text{equi}} \mapsto X_{-1} = -\partial_t + \frac{\xi}{t} \quad (5b)$$

Whenever $\xi \neq 0$, time-translation-invariance is broken, but this is achieved here through the choice of a different representation rather than changing the Lie algebra of dynamical symmetries by suppressing X_{-1} . In general, it does not seem to be possible to require the presence of other symmetries, even if extended via the prescription (4), i.e. following ideas of local scale-invariance [22, 23, 27]. Such requirements, when applied to correlators, would in general imply that $C = 0$, because of certain Bargman superselection rules [41]. At most, one might hope that certain response functions R might admit larger dynamical symmetry groups [27]. This subject however, must be postponed to a later date, since it is not the aim of this work to restrain the form of certain correlators or responses as much as possible by considering their co-variance under a maximal Lie algebra but rather to identify the minimal requirements on dynamical symmetries needed for an understanding of the general phenomenology of ageing. Generalised time-translations (5b) occurred first as a by-product of studies of local-scale-invariance for systems with dynamical exponent $z = 2$ [26]. They were advocated as a special case of much more general representations [39] in the context of holographic geometry of ageing. Physical examples arose only recently in the context of dynamical symmetries of biased systems [43, 30, 44].

The mathematical basis of the prescription (4) can be stated as follows.

Proposition 1: [39, 28] *For the constant γ and the non-constant function $g(z)$, the generators*

$$\ell_n = -z^{n+1}\partial_z - n\gamma z^n - g(z)z^n \quad (6)$$

obey the conformal algebra $[\ell_n, \ell_m] = (n-m)\ell_{n+m}$ for $n, m \in \mathbb{Z}$.

Physically, the relevance of ξ for non-equilibrium dynamics can be understood from the kinetics of the *spherical model* [15, 21, 40, 44]. This exactly solvable model can be formulated in terms of real-valued spin $S(t, \mathbf{r}) \in \mathbb{R}$ subject to the con-

straint $\sum_{\mathbf{r}} \langle S^2(t, \mathbf{r}) \rangle = \mathcal{N}$, where \mathcal{N} is the number of sites of the lattice. With the hamiltonian $\mathcal{H}[S] = -\sum_{(\mathbf{r}, \mathbf{r}')} S(t, \mathbf{r})S(t, \mathbf{r}') - \mu \sum_{\mathbf{r}} S^2(t, \mathbf{r})$ the over-damped dynamics is given by $\partial_t S(t, \mathbf{r}) = -\frac{\delta \mathcal{H}[S]}{\delta S(t, \mathbf{r})} + \mathfrak{z}(t)S(t, \mathbf{r}) + \eta(t, \mathbf{r})$, where η is a thermal white noise. Herein, the Lagrange multiplier $\mathfrak{z}(t)$ is fixed from the spherical constraint. Defining $\ln g(t) := 2 \int_0^t d\tau \mathfrak{z}(\tau)$, it can be shown that this leads to a Volterra integral equation

$$g(t) = A(t) + 2T \int_0^t d\tau f(t-\tau)g(\tau) \quad (7)$$

for the temperature T where for nearest-neighbour interactions $f(t) = e^{-4dt} I_0(4t)^d$ and I_0 is a modified Bessel function. If $d > 2$, for a fully uncorrelated initial state $A(t) = f(t)$ such that at criticality with $T = T_c(d)$ one finds that $g(t) \sim t^{\xi}$ giving $\xi = \frac{d}{2} = \min(d/4 - 1, 0)$ [21]. Hence, for $d > 4$ the model is in the mean-field universality class. On the other hand, whenever $\xi \neq 0$, fluctuation effects must be invoked for the required change of representation (4). On the other hand, if one prepares the system at $T = T_c(d)$ at critical equilibrium, the model merely undergoes *equilibrium critical dynamics*. Equilibrium initial conditions are spatially long-ranged and modify $A(t)$ such that $g(t) = \text{cste.}$ [40] and $\xi = 0$ throughout, for all $d > 2$. Therefore, *representations with $\xi \neq 0$ constructed via eq. (4) should be relevant for non-equilibrium critical dynamics* (where ageing occurs) and by extension also for phase-separation or phase-ordering kinetics, depending on whether the order-parameter is conserved or not, after a quench into the phase-coexistence region with $T < T_c$.

Our main use of dynamical symmetries will concern the scaling form of two-point functions. A scaling operator $\phi_a = \phi_a(t, \mathbf{r})$ is characterised by the two parameters (δ_a, ξ_a) . The co-variance of a *two-point function*

$$\mathcal{C} = \mathcal{C}(t, s; \mathbf{r}) = \langle \phi_1(t, \mathbf{r}) \phi_2(s, \mathbf{0}) \rangle \quad (8)$$

under generalised time-translations (5b) and dilations (5a) leads to the conditions

$$X_{-1} \mathcal{C} = \left(-\partial_t - \partial_s + \frac{\xi_1}{t} + \frac{\xi_2}{s} \right) \mathcal{C} = 0 \quad (9a)$$

$$X_0 \mathcal{C} = \left(-t \partial_t - s \partial_s - \frac{1}{z} r \partial_r - \delta_1 + \xi_1 - \delta_2 + \xi_2 \right) \mathcal{C} = 0 \quad (9b)$$

Eq. (9a) gives the meaning of ‘broken time-translation-invariance’ used in this work.

In what follows, all physical applications will taken from the following

Proposition 2: *The general solution of the co-variance conditions (9) is*

$$\mathcal{C}(t, s; \mathbf{r}) = s^{-\delta_1 - \delta_2 + \xi_1 + \xi_2} \left(\frac{t}{s} \right)^{-\xi_1} \left(\frac{t}{s} - 1 \right)^{-\delta_1 - \delta_2 + 2\xi_1} \mathcal{F} \left(\frac{r}{(t-s)^{1/z}} \right) \quad (10)$$

where \mathcal{F} is an undetermined scaling function.

Proof: Clearly, (10) solves both eqs. (9a,9b). General theorems on linear partial differential equations [35] guarantee the uniqueness of this form, since a normalisation constant is absorbed into the undetermined function \mathcal{F} . \square

3 Phenomenological consequences

3.1 Preliminaries: the ageing exponents b, a

For magnetic systems quenched from a disordered initial state to a temperature $T \leq T_c$, the two-time correlation function should obey dynamical scaling [7]

$$C(t, s; \mathbf{r}) = \kappa^{bz} C(\kappa^z t, \kappa^z s; \kappa \mathbf{r}) = s^{-b} F_C \left(\frac{t}{s}; \frac{|\mathbf{r}|}{s^{1/z}} \right) \quad (11)$$

An important boundary condition is furnished by the equal-time correlator. If the system is quenched into the phase-coexistence region where $T < T_c$, one expects

$$C(t, t; \mathbf{0}) \sim \begin{cases} 1 & \text{discrete (e.g. Ising model)} \\ M_{\text{eq}}^2 & \text{continuous} \end{cases} \quad (12)$$

whereas for a quench onto the critical point $T = T_c$, one should find

$$C_{\text{eq}}(\mathbf{r}) = C(\infty, \infty; \mathbf{r}) \sim |\mathbf{r}|^{-(d-2+\eta)} \quad (13)$$

and where η is an equilibrium critical exponent. These are captured by the asymptotic scaling function, for $u \ll 1$

$$F_C(1; u) \sim u^{-bz} ; b = \begin{cases} 0 & \text{if } T < T_c \\ (d-2+\eta)/z & \text{if } T = T_c \end{cases} \quad (14)$$

which furnishes the value of the ageing exponent b [7, 16]. For phase-ordering, with a non-conserved order-parameter and short-ranged interactions, one has $z = 2$ [7, 8]. Similarly, for a response function one has [24, 25, 27]

$$R(t, s; \mathbf{r}) = s^{-1-a} F_R \left(\frac{t}{s}; \frac{|\mathbf{r}|}{s^{1/z}} \right) ; a = \begin{cases} 1/z & \text{if class S} \\ (d-2+\eta)/z & \text{if class L} \end{cases} \quad (15)$$

where class S are systems with short-ranged equilibrium correlators which fall off exponentially fast and class L are systems with an algebraic long-range decay $C_{\text{eq}}(\mathbf{r}) \sim |\mathbf{r}|^{-(d-2+\eta)}$ and which includes all systems at criticality $T = T_c$. Eq. (15) furnishes the value of the ageing exponent a [27].

3.2 Auto-correlation function

The two-time auto-correlator $C(t,s) = \langle \phi(t, \mathbf{r}) \phi(s, \mathbf{r}) \rangle$ is built from two copies of the magnetic order-parameter ϕ . Because of the identity of the scaling operators, we have

$$\delta = \delta_1 = \delta_2, \quad \xi = \xi_1 = \xi_2 \quad (16)$$

such that the two-point function (10) becomes, in the limit of large separations $y = t/s \gg 1$

$$C(t,s) = s^{-2\delta+2\xi} \left(\frac{t}{s}\right)^{-2\delta+\xi} \mathcal{F}_C(0) \quad (17)$$

This does indeed agree with the expectations (2,3) and in particular *proves the expected algebraic asymptotics (3a) of the scaling function $f_C(y)$* . We can identify

$$\text{if } T = T_c: \quad b = 2(\delta - \xi) = \frac{d-2+\eta}{z}, \quad \frac{\lambda_C}{z} = 2\delta - \xi = \xi + \frac{d-2+\eta}{z} \quad (18a)$$

$$\text{if } T < T_c: \quad b = 2(\delta - \xi) = 0, \quad \frac{\lambda_C}{z} = 2\delta - \xi = \xi \quad (18b)$$

where we used (14).

3.3 Auto-response function

The auto-response function $R(t,s) = \langle \phi(t, \mathbf{r}) \tilde{\phi}(s, \mathbf{r}) \rangle$ can be written as a correlator of the order-parameter ϕ with the conjugate response scaling operator $\tilde{\phi}$. Furthermore, since responses are indeed co-variant under larger algebras of local scale-transformations, notably conformal transformations which make up local scale-invariance [22, 23, 27], we now have

$$\delta = \delta_1 = \delta_2, \quad \xi = \xi_1, \quad \tilde{\xi} = \xi_2 \quad (19)$$

where the first identity is a consequence of local-scale invariance. However, the two parameters ξ and $\tilde{\xi}$ are now independent. Then, for $y = t/s \gg 1$ we have from (10)

$$R(t,s) = s^{-2\delta+\xi+\tilde{\xi}} \left(\frac{t}{s}\right)^{-2\delta+\xi} \mathcal{F}_R(0) \quad (20)$$

once more in agreement with (2,3a). We can identify, for both classes S and L

$$1 + a = 2\delta - \xi - \tilde{\xi}, \quad \frac{\lambda_R}{z} = 2\delta - \xi \quad (21)$$

The results obtained so far can be summarised as follows.

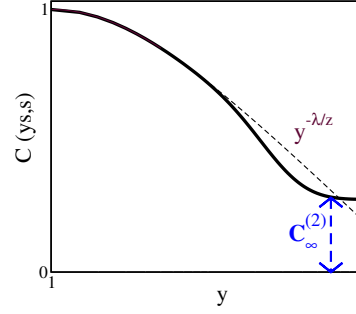


Fig. 2 Scaling of the autocorrelator $C(y, s; N^{-1})$ in a finite volume (with $b = 0$).

Proposition 3: For systems obeying generalised time-translation-invariance and dilation-invariance, the two-time auto-correlator $C(t, s) = s^{-b} f_C(t/s)$ and two-time auto-response $R(t, s) = s^{-1-a} f_R(t/s)$ have the properties:

1. For large arguments $y \gg 1$, both scaling functions are algebraic

$$f_C(y) \sim y^{-\lambda_C/z} \quad , \quad f_R(y) \sim y^{-\lambda_R/z} \quad (22)$$

2. Comparison of (18) and (21) implies the exponent equality $\lambda_C = \lambda_R$.

Hence both properties (3) find their natural explanation as a consequence of generalised time-translation-invariance (9a), combined with dynamical scaling.

3.4 Finite-size effects

If the ageing system is placed in a volume of linear size N , then the shape of the auto-correlator $C(t, s; N^{-1})$ is modified in a characteristic way which is schematised in figure 2, here for the case of phase-ordering kinetics at $T < T_c$, where also $b = 0$. For reasonably large, but not too large, values of y , the autocorrelator's behaviour is well approximated by the infinite-size behaviour $C_{\infty}(y, s) = C(y, s; 0) \sim y^{-\lambda/z}$ (dashed line in figure 2). However, if y is increased further, the data first decay more rapidly and then saturate at a y -independent plateau

$$C_{\infty}^{(2)} = \lim_{y \rightarrow \infty} C\left(y, s; \frac{1}{N}\right) \quad (23)$$

Can one understand this behaviour in terms of a scaling description based on (4)? How does the plateau height $C_{\infty}^{(2)}$ depend on either N or the waiting time s ? Finite-size effects are also studied experimentally [51, 52].

According to the theory of finite-size scaling [20, 4, 46], in a finite volume the inverse size $1/N$ becomes a further relevant variable. Therefore, the two-time auto-correlator (with spatial separation $\mathbf{r} = \mathbf{0}$) should obey the two co-variance conditions

$$X_{-1}C = \left(-\partial_t - \partial_s + \frac{\xi}{t} + \frac{\xi}{s} \right) C = 0 \quad (24a)$$

$$X_0C = \left(-t\partial_t - s\partial_s - \frac{1}{z} \frac{1}{N} \partial_{1/N} - 2(\delta - \xi) \right) C = 0 \quad (24b)$$

which have the unique solution (re-using eqs. (18) and (3b), of course) for $t \gg s$

$$C \left(t, s; \frac{1}{N} \right) = s^{-b} \left(\frac{t}{s} \right)^{-\lambda/z} \mathcal{F}_C \left(\frac{N}{t^{1/z}} \right) \quad (25)$$

The interpolation between systems of infinite and finite volume, respectively, is controlled by the finite-size scaling function $\mathcal{F}_C(u)$, of the single argument $u = Nt^{-1/z}$. Clearly, for large arguments $\mathcal{F}_C(\infty) = \text{cste.}$, such that the infinite-size behaviour (3) is recovered. On the other hand, for $u \ll 1$, one should have $\mathcal{F}_C(u) \sim u^{-\lambda}$ such that the plateau is indeed y -independent. This implies for the plateau height $C_\infty^{(2)} \sim s^{\lambda/z-b} N^{-\lambda}$. The requirement (24a) is essential to reduce the scaling function \mathcal{F}_C to a function of a single variable on which the traditional finite-size scaling arguments [20, 4] can be applied. Of course, one might as well assemble heuristic arguments for the scaling $C_\infty^{(2)} \sim N^{-\lambda}$ for s fixed, see e.g. [29].

Proposition 4: *For quenches to $T \leq T_c$, the plateau height $C_\infty^{(2)}$ of the auto-correlator in a finite volume of linear size N obeys the scalings*

$$\text{if the waiting time } s \text{ is fixed: } C_\infty^{(2)} \sim N^{-\lambda} \quad (26a)$$

$$\text{if the system size } N \text{ is fixed: } C_\infty^{(2)} \sim s^{\lambda/z-b} \quad (26b)$$

where the values (18) must be used for the exponent b .

For a test of (26) in the spherical model at $T < T_c$, see [29]. Tests in the 2D Glauber-Ising model are forthcoming [49]. In practise, the determination of λ from simulations and/or experiments may become difficult through the need to study sufficiently large waiting times and also $y = t/s$ large enough such that the asymptotics $f_C(y) \sim y^{-\lambda/z}$ can be observed. This requires to create data for a sufficiently wide window. Eq. (26) rather suggests to let the auto-correlator relax to the plateau and extract λ from the scaling of $C_\infty^{(2)}$ which may be the better computational tool [49]. Since in experiments [2], N is likely fixed from the sample preparation techniques, eq. (26b) might be an useful alternative for the determination of λ/z .

Analogous results can be easily derived for instantaneous responses $R(t, s; \mathbf{r})$, or their time-integrated counterparts $\chi = \int du R(t, u)$, for systems in a finite volume.

Finally, the characteristic length scale $L(t)$ may for example be extracted as a second moment of the single-time correlator $C(t; \mathbf{r})$, using (25) as follows

$$L^2(t) = \frac{\int_V d\mathbf{r} \mathbf{r}^2 C(t; \mathbf{r}; N^{-1})}{\int_V d\mathbf{r} C(t; \mathbf{r}; N^{-1})} = t^{2/z} \mathcal{F}_L \left(\frac{N}{t^{1/z}} \right) \sim \begin{cases} t^{2/z} & \text{volume infinite} \\ N^2 & \text{volume finite} \end{cases} \quad (27)$$

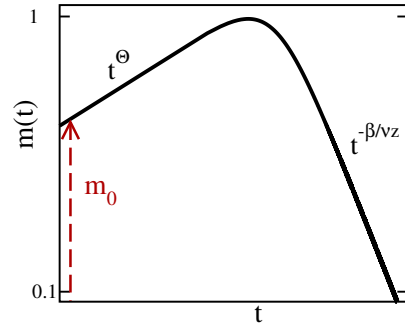


Fig. 3 Time-dependence of the non-equilibrium magnetisation $m(t)$.

which gives a method to measure the dynamic exponent z . This finite-size crossover has been routinely observed in numerical simulations of lattice systems [10, 11, 12, 13, 31].

3.5 Magnetic initial states

For a quench onto criticality $T = T_c$, the magnetisation $m(t)$ has a non-trivial time-evolution [33] sketched in figure 3. If the system is prepared with an initial magnetisation $m(0) = m_0 \neq 0$ but is otherwise uncorrelated, there exists a short-time critical regime where $m(t) \sim t^\Theta$ where the *slip exponent* Θ is independent⁶ of the equilibrium critical exponents and z . Its value can be found from deep field-theoretic considerations based on the renormalisation group and short-time operator product expansions [33, 34]. This leads to the celebrated JANSSEN-SCHAUB-SCHMITTMANN (JSS) scaling relation [33]

$$\Theta = \frac{d - \lambda}{z} \quad (28)$$

In this formulation, the identity (3b) between auto-correlation and auto-response exponents was used. After passing through a maximum, the magnetisation crosses over into a regime of critical decay, according to $m(t) \sim t^{-\beta/(vz)}$, where β and v are standard equilibrium critical exponents [1, 47]. Can one understand this behaviour and the existence of two distinct scaling regimes in figure 3, in terms of a scaling description based on (4) ?

We shall first relate the behaviour of $m(t)$ to a response function. In order to see how this comes about, recall from Janssen-de Dominicis non-equilibrium field-theory [17, 32, 34, 47] the calculation of averages of an observable A

$$\langle A \rangle = \int \mathcal{D}\phi \mathcal{D}\tilde{\phi} A[\phi] e^{-\mathcal{J}[\phi, \tilde{\phi}]} \quad (29)$$

⁶ In mean-field theories, $\Theta = 0$. Both $\Theta > 0$ and $\Theta < 0$ is seen in models, see [27].

where the action $\mathcal{J}[\phi, \tilde{\phi}] = \mathcal{J}_b[\phi, \tilde{\phi}] + \mathcal{J}_{\text{ini}}[\tilde{\phi}]$ is decomposed into a bulk term and an initial term

$$\mathcal{J}_b[\phi, \tilde{\phi}] = \int d\mathbf{r} d\mathbf{t} \left(\tilde{\phi} (\partial_t - \Delta_{\mathbf{r}} - V'[\phi]) \phi - T \tilde{\phi}^2 \right) \quad (30a)$$

$$\mathcal{J}_{\text{ini}}[\tilde{\phi}] = \int d\mathbf{r} \left(\frac{1}{2\tau_0} \tilde{\phi}^2(0, \mathbf{r}) - m_0 \tilde{\phi}(0, \mathbf{r}) \right) \quad (30b)$$

(with the usual re-scalings) which describes in the bulk a thermal white noise of temperature T , and an initially magnetised state of average magnetisation m_0 together with gaussian fluctuations of width τ_0 . Herein, ‘initial time’ means of course a time-scale τ_{mic} at the beginning of the scaling regime. The bulk action at temperature $T = 0$, $\mathcal{J}_0[\phi, \tilde{\phi}] = \lim_{T \rightarrow 0} \mathcal{J}_b[\phi, \tilde{\phi}]$ is called the *deterministic action*. Deterministic averages $\langle \cdot \rangle_0$ are obtained analogously to (29) where the action \mathcal{J} is replaced by its deterministic part \mathcal{J}_0 . Either from causality considerations [33, 34] or local-scale-invariance [41] one obtains the Bargman superselection rules

$$\left\langle \overbrace{\phi \cdots \phi}^{n \text{ times}} \overbrace{\tilde{\phi} \cdots \tilde{\phi}}^{m \text{ times}} \right\rangle_0 \sim \delta_{n,m} \quad (31)$$

for the deterministic averages. This means that only observables built from an equal number of order-parameters ϕ and conjugate response operators $\tilde{\phi}$ can have non-vanishing averages.

Averages should now be computed from (29). They can be reduced to deterministic averages by formally expanding in powers of T , $\frac{1}{\tau_0}$ and m_0 to all orders. It follows that the time-dependent magnetisation can be obtained as

$$m(t) = \langle \phi(t, \mathbf{0}) \rangle = m_0 \int_{\mathbb{R}^d} d\mathbf{r} \langle \phi(t, \mathbf{0}) \tilde{\phi}(0, \mathbf{r}) \rangle_0 = m_0 \int_{\mathbb{R}^d} d\mathbf{r} R(t, 0; \mathbf{r}) = m_0 \hat{R}(t, 0; \mathbf{0}) \quad (32)$$

This follows from (29) and the form of the action $\mathcal{J}[\phi, \tilde{\phi}]$ detailed in (30). Namely, the only term admissible, because of the Bargman rule (31), is the one linear in m_0 as can be seen from (30b). By definition, this is a response function and in the last step, we introduced the spatial Fourier transform

$$\hat{R}(t, s; \mathbf{q}) = \int_{\mathbb{R}^d} d\mathbf{r} e^{-i\mathbf{q} \cdot \mathbf{r}} R(t, s; \mathbf{r}) \quad (33)$$

Eq. (32) is the starting point of our scaling analysis. Again, the formally vanishing waiting time $s = 0$ in (32) should be physically interpreted as a microscopically small time $s_{\text{mic}} \ll t$ at the beginning of the scaling regime.

In the physical situation at hand, the initial magnetisation m_0 should be considered as a further dimensionful scaling variable, with scaling dimension x_0 . The co-variance conditions of the global response function $\hat{R} = \hat{R}(t, s; \mathbf{0}; m_0)$ then read

$$X_{-1}\hat{R} = \left(-\partial_t - \partial_s + \frac{\xi}{t} + \frac{\tilde{\xi}}{s} \right) \hat{R} = 0 \quad (34a)$$

$$X_0\hat{R} = \left(-t\partial_t - s\partial_s - \frac{x_0}{z}m_0\partial_{m_0} - 2\delta + \frac{d}{z} + \xi + \tilde{\xi} \right) \hat{R} = 0 \quad (34b)$$

In analogy to what was done before, we have for $t/s \gg 1$ the asymptotics

$$\hat{R}(t, s; \mathbf{0}; m_0) = s^{-2\delta + d/z + \xi + \tilde{\xi}} \left(\frac{t}{s} \right)^{-2\delta + d/z + \xi} \mathcal{F}_R \left(m_0 t^{-x_0/z} \right) \quad (35)$$

with the scaling function $\mathcal{F}_R(u)$ of the single argument $u = m_0 t^{-x_0/z}$. Since we need the response with respect to an ‘initial’ perturbation, we must now send $s \rightarrow s_{\text{mic}}$ and then can absorb it into the scaling function. Combination with (32) gives for the sought time-dependent magnetisation

$$m(t) = m_0 t^\Theta \mathcal{F}_R \left(m_0 t^{-x_0/z} \right), \quad \Theta = \frac{d}{z} + \xi - 2\delta = \frac{d - \lambda}{z} \quad (36)$$

Herein, we identified the slip exponent and then observe that we indeed reproduce the JSS scaling relation (28). The scaling function \mathcal{F}_R interpolates between the two regimes of non-equilibrium critical scaling, through the assumed properties (i) $\mathcal{F}_R(0) = \text{cste.}$ which reproduces the short-time scaling regime and (ii) $\mathcal{F}_R(u) \sim u^{-1}$ for $u \gg 1$ which makes the late-time scaling regime independent of m_0 and which also fixes x_0 .

Proposition 5: *A critical system, with initial magnetisation m_0 but otherwise uncorrelated, has the time-dependent magnetisation*

$$m(t) = m_0 t^\Theta \mathcal{F}_R \left(m_0 t^{\Theta + \beta/(vz)} \right) \quad (37)$$

where Θ is given by (28) and β, v are standard equilibrium critical exponents.

Hence the qualitative behaviour of figure 3 is reproduced. Given the generic relationship (32), the derivation of the JSS scaling relation (28) reduces to a simple Fourier transform. Since the scaling function stems from a response function, it is conceivable that larger dynamical symmetries such as local scale-invariance might be brought to furnish further details on its behaviour. We hope to return to this elsewhere.

This result should only be applicable to critical quenches. Any attempt to prepare an initially magnetised state should lead for $T < T_c$ to a rapid relaxation towards one of the several distinct equilibrium states in the phase-coexistence region such that the system will leave the regions in parameter space where dynamical scaling holds.

3.6 Global correlators

While all observables discussed so far have been *local* correlators or responses, we now consider global quantities. For example, in an Ising model with spins $\sigma_n = \pm 1$ at each site n of a lattice $\Lambda \subset \mathbb{Z}^d$, the *global* spin-spin correlator is defined as

$$Q(t, s) := \frac{1}{|\Lambda|} \left\langle \sum_{n, m \in \Lambda} \sigma_n(t) \sigma_m(s) \right\rangle \quad (38)$$

where $|\Lambda|$ is the number of sites of the lattice. Analogous definitions apply to other spin models. What can be said on the long-time behaviour of such observables ?

Proposition 6: *For a magnetic system quenched to $T \leq T_c$, the global spin-spin correlator (38) obeys for large observation times t*

$$Q(t, 0) \sim t^\Theta \quad (39)$$

where the slip exponent Θ is given by (28).

Proof: At criticality, the result has been known since a long time [48]. It is derived here for all $T \leq T_c$ as a further example of generalised time-translation-invariance (5b). Taking into account spatial translation-invariance, in the continuum limit the discrete sums over lattice sites are replaced by integrations over a spatial domain $V \subset \mathbb{R}^d$. Sending the integration volume $|V|$ to infinity, one has

$$Q(t, s) = \lim_{|V| \rightarrow \infty} \frac{1}{|V|} \int_{V \times V} \mathbf{dr} \mathbf{dr}' \langle \phi(t, \mathbf{r}) \phi(s, \mathbf{r}') \rangle = \int_{\mathbb{R}^d} \mathbf{dr} C(t, s; \mathbf{r}) = \widehat{C}(t, s; \mathbf{0})$$

where the definition (33) of the Fourier transform was used. Recall the scaling forms (11,17) and carry out the spatial integration. This gives for the $t \gg s$ asymptotics

$$\widehat{C}(t, s; \mathbf{0}) = s^{-b+\lambda/z} t^{-\lambda/z+d/z} \underbrace{\int_{\mathbb{R}^d} \mathbf{du} \mathcal{F}_C(|\mathbf{u}|)}_{= \text{cste.}} \sim \begin{cases} s^{-\Theta+(2-\eta)/z} t^\Theta & \text{if } T = T_c \\ s^{-\Theta+d/z} t^\Theta & \text{if } T < T_c \end{cases} \quad (40)$$

where the scaling forms (2,17) and the identifications (18,36) were used, which relate the slip exponent Θ with the autocorrelation exponent λ . We point out that this establishes the validity of the identity (28) beyond the realm of non-equilibrium critical dynamics for which it was originally derived [33]. Finally, the mathematical limit $s \rightarrow 0$ corresponds physically to the limit $s \rightarrow s_{\text{mic}}$, where s_{mic} is a microscopic time-scale for the onset of the dynamic scaling regime. Hence the dependence on s_{mic} can be absorbed into the scaling function \mathcal{F}_C . Then $Q(t, 0) \sim Q(t, s_{\text{mic}}) \sim t^\Theta$ as asserted. \square

For non-equilibrium critical dynamics, eq. (39) has been routinely used to measure the exponents Θ (or, via (28), equivalently the autocorrelation exponent λ , if z is known) [48, 1, 42]. The derivation given here makes it clear that eq. (39) along

with the scaling relation (28) furnishes an alternative method to find λ from simulations for quenches into the low-temperature phase $T < T_c$. We are not aware of any numerical work along this line.

For estimating the dynamical exponent z , computing $L(t)$ from a second moment is tedious. An often-used alternative relies on the scaling of the single-time correlator, e.g. for phase-ordering where $b = 0$ one may look for pairs $(t, L(t))$ by solving numerically a scaling equation such as $C(t, \mathbf{r}) = F_C(|\mathbf{r}|/L(t)) = 0.5$ [36, 37]. Yet a further alternative studies the short-time dynamics of the squared magnetisation $\langle m^2(t) \rangle$ [31]. Repeating the above arguments, notably eq. (40), it is easily seen that

$$\langle m^2(t) \rangle \sim \widehat{C}(t, t; \mathbf{0}) \sim t^{d/z-b} \sim \begin{cases} t^{d/z} & \text{if } T < T_c \\ t^{(2-\eta)/z} & \text{if } T = T_c \end{cases} \quad (41)$$

and in agreement with earlier heuristic arguments [48, 42, 31]. The numerical requirements of this technique are somewhat smaller than in the other methods mentioned above. In recent applications to phase-ordering kinetics in the $2D$ Ising model it was concluded that as a tool for numerical computation the accuracy of (41) is at least equal to the one of other techniques [31].

Proposition 7: *In magnetic systems quenched to $T \leq T_c$, the time-dependent averaged squared magnetisation scales as*

$$\langle m^2(t) \rangle \sim t^{d/z-b} = \begin{cases} t^{d/z} & \text{if } T < T_c \quad [31] \\ t^{(2-\eta)/z} & \text{if } T = T_c \quad [48] \end{cases} \quad (42)$$

where η is a standard equilibrium critical exponent.

The extension of Propositions 6 and 7 to systems in a finite volume is obvious.

4 Conclusions

The phenomenology of simple physical ageing is based on several expectations which are met in almost all systems and models under study. However, these auxiliary assumptions have always been presented as being independent of each other. The point of this work is to provide an unifying perspective such that a single hypothesis is sufficient from which the entire generic phenomenology of simple ageing can be derived. As stated in eq. (4), non-equilibrium dynamics distinguishes itself from equilibrium dynamics merely through a different choice of Lie algebra representation. We have considered here an intertwining operator which only depends on time such that merely the time-translation and dilation generators change their form. In Propositions 3 to 7 we showed how the entire known phenomenology of the simple ageing at $T \leq T_c$ of local and global correlators and responses can be

reproduced: in spatially infinite or finite systems or in systems with or without an initial magnetisation. Simple proofs of the JSS scaling relation (28) were given.

Exponent values are not predicted.

Herein, it is essential that time-translation-invariance is broken ‘softly’ through time-dependent terms in its Lie algebra generator X_{-1} , rather than leaving it out altogether, as it was usually done up to now. In this way, the observables contain at most a scaling function of a single variable, which allows the application of standard phenomenological scaling arguments.

Complementary hypothesis needed to carry out this programme include:

1. simple ageing with an algebraically growing length scale $L(t) \sim t^{1/z}$ was admitted throughout.
2. all initial correlations were assumed short-ranged. Otherwise, results such as $\lambda_C = \lambda_R$ need no longer hold true [7, 27].
3. attention was restricted to the sole intertwining operator $W = e^{-\xi \ln t}$, see eq. (4).

Future work should relax or further clarify at least some of these assumptions.

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