New refined model for curved linear anisotropic rods with circular cross section

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Dedicaced to the memory of Professor Hui Hui Dai

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Abstract

An asymptotic reduction method is introduced to construct a curved rod theory for a general anisotropic linearized elastic material. For the sake of simplicity, the cross section is assumed to be circular. The starting point is Taylor expansions about the mean-line in curvilinear coordinates, and the goal is to eliminate the two spatial variables in the cross section in a pointwise manner in order to obtain a closed system for the displacement coefficients. We achieve this by using a Fourier series for the lateral traction condition together with the use of cylindrical coordinates in the cross section and by considering exact tridimensional equilibrium equation. We get a closed differential system of ten vector unknowns, and after a reduction process we obtain a differential system of the vector of the mean line displacement and twist angle. Six boundary conditions at each edge are obtained from the edge term in the tridimensional virtual work principle, and a unidimensional virtual work principle is also deduced from the weak forms of the rod equations. **Through one example, we show that our theory gives more accurate results than the ones of both classical Euler-Bernoulli rod theory and Timoshenko rod theory**. **The displacement field is computed for two types of material symmetry : isotropy and transverse isotropy.**

Keywords

curved rod theory, reduction method, anisotropic linearized elasticity, rod variational formulation, Fourier series, **classical rod theories.**

1. Introduction

Rods are very important engineering structures. For straight or curved rods, the dimension of the cross section is much smaller than the third one, its length. Due to this relative smallness, one may model the behaviours of these thin structures by one-dimensional rod theory through certain dimensional reduction processes. The most popular approach is to introduce some kinematic assumptions such that the displacement has certain particular forms. Classical rod theories, such as Euler-Bernoulli rod theory, Timoshenko rod theory and Reddy's third-order rod theory (Reddy, 2006) are obtained in this way. Due to its simplicity, Carrera and Giunta (2010) and Carrera and Petrolo (2012) use various forms of this approach. But if one substitutes the assumed displacement form into the equilibrium equation and traction condition, some contradictions will arise. Another approach is to expand the displacement in terms of a series of basis functions and to truncate the potential energy Pruchnicki and Dai (2019). It is interesting to note that this method gives an error estimation between the exact three dimensional plate problem and the approximate bidimensional one (Schneider and Kienzler 2020). Only for isotropic material, the number of main unknowns can be reduced by using the pseudo-reduction method (Schneider et al., 2014) and then a plate (symmetric transverse loading) model with three (due to considered symmetry) boundary conditions (for transverse shear, twisting moment and bending moment) can be constructed.

Another theory is introduced by Ciarlet and Destuynder (1979) for plate through use of asymptotic expansions in variational formulation. This theory was adapted by Bermudez and Viano (1984) for rods; for further works on rods in this direction see Trabucho and Viano (1996). Another approach is through the use of asymptotic expansion in the minimization of potential

energy. The method was first introduced by Pantz (2000) for homogeneous plates, and by Meunier (2008) and Marigo and Meunier (2006) for rods and extended by Pruchnicki(2009); (2011) for heterogeneous plates. This problem is also addressed through the use of a rigorous Gamma convergence tool, which generates one-dimensional models when the small transverse dimension goes to zero (Giovanna Mora and Müller, 2003). Different scalings lead to a hierarchy of different models, but those approaches do not yield a single one dimensional model which contains extension, bending and torsion effects together. Also, a priori scalings adopted, assumes that the loads depend on the thickness but in reality applied loads are external and thus are not linked to the geometry of the thin structure. Moreover it is difficult to apply this method to obtain error estimates.

To overcome these drawbacks, Dai and Song (2014); Song and Dai (2016 a,b); Wang et al. (2019); Yu et al. (2020) introduced for plate a novel method of dimension reduction deduced from tridimensional equilibrium equation and boundary condition on the upper and lower face of the plate. This approach was used in Chen et al. (2018) to obtain a plane-stress rod model for a linearized isotropic elastic material with pointwise error estimates. In this way Pruchnicki (2018) obtains a unidimensional model for a rod with rectangular cross section.

Obtention of a rod theory is more complicated than plate or shell ones. For rod with circular cross section, we present a one dimensional model as already done in a previous work for straight rod (Chen et al. (2020)).

The paper is arranged as follows. We first give specific used notation in section 2. Then in section 3, we define the mean line of the rod (Subsection 3.1) and for the rod the curvilinear coordinate and covariant and contravariant basis (Subsection 3.2). Then, we start the derivation process. In section 4, we give the three-dimensional equilibrium equations together with boundary conditions in both rectangular and curvilinear coordinates. In section 5, we derive a system of differential equations for the rod. In order to do, we begin with Taylor expansions in curvilinear coordinates for displacement, strain and Cauchy stress, together with the relations among their coefficients. Next the Fourier series is used to represent the given traction on the lateral surface, and the expansion for the Cauchy stress is represented in terms of polar coordinates. Then, we carefully truncate the traction conditions on the lateral boundary in order to keep bending and twisting terms. After considering three dimensional equilibrium equation in a pointwise manner, a calculation leads to a closed ten equations with ten unknown vectors. In section 6 a number of refinements based on truncation of equations at the same order of magnitude, leads to asymptotically-consistent four scalar rod equations with four unknowns (the three components of the mean line displacement and the averaged twist angle). In section 7, rod boundary conditions are derived from the edge term of the tridimensional virtual work principle and then the rod virtual work principle is deduced from the weak forms of the rod equations. **In the appendix, we give an example for a straight rod with two clamped ends and loaded by a uniform line force at the top of lateral surface along the negative y_1-axis. We compare the obtained system of differential equations with Euler-Bernoulli rod theory and Timoshenko rod theory and we see that our theory gives refined results. Finally, the displacement field is computed with Mathematica software for two types of material symmetry: isotropy and transverse isotropy.** In section 8, we conclude and give perspective.

2. Notation

Boldface letters represent vector-valued functions, tensors or spaces. Let $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ denotes an

orthonormal basis of the space $\mathbb{R}^3$. $(\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3)$ is the local orthonormal covariant basis. Also, we adopt Einstein convention summation for repeated indices. The Euclidean scalar product, vector product and tensorial product of vectors $\mathbf{a}, \mathbf{b}$ are denoted by $\mathbf{a}.\mathbf{b}$, $\mathbf{a} \wedge \mathbf{b}$ and $\mathbf{a} \otimes \mathbf{b}$ respectively. The double dot product of two tensors is **A**:**B**. The latin index take value 1, 2 and 3. Greek index take value 1, 2. B_{ij} (respectively $\widetilde{B}_{ij}$) is the component of a second order tensor **B** with respect to $\mathbf{e}_i \otimes \mathbf{e}_j$ (respectively $\mathbf{a}_i \otimes \mathbf{a}_j$), this notation also holds for a general *k*-th order tensor.

3. The curved rod

3.1. The mean line $\widetilde{\Gamma}_c$ of the rod

We begin by defining a general (not necessary plane) curve $\widetilde{\Gamma}_c$ parametrized by its curvilinear measures s in the reference configuration and its running point in this configuration will be denoted by **OG** = **r**(*s*), where **O** is the chosen origin. If *L* is the length of this curve in the reference configuration, we have $s \in [0, L]$. The orthonormal Frenet frame at a point G is denoted by $(\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3)$:

$$\mathbf{a}_3 = r_{,s}, \tag{1}$$

$$\mathbf{a}_1 = \frac{\mathbf{a}_{3,s}}{k},$$

$$\mathbf{a}_2 = \mathbf{a}_3 \wedge \mathbf{a}_1.$$

where $\mathbf{a}_1$ is the unit vector tangent to $\widetilde{\Gamma}_c$ and $k(s) = |\mathbf{a}_{3,s}(s)|$ is the curvature at point G.

The variation of $(\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3)$ is described the Frenet-Serret formulae :

$$\mathbf{a}_{1,s} = -k\,\mathbf{a}_3 + \tau\,\mathbf{a}_2, \tag{2}$$

$$\mathbf{a}_{2,s} = -\tau\,\mathbf{a}_1, \tag{3}$$

$$\mathbf{a}_{3,s} = k\,\mathbf{a}_1. \tag{4}$$

where $\tau(s) = |\mathbf{a}_{2,s}(s)|$ denotes the torsion of the curve $\widetilde{\Gamma}_c$ at point G.

3.2. Geometric and kinematic formulae for rod

In a neighbourhood of $\widetilde{\Gamma}_c$, the geometric points will be defined by the position vector **X** depending on the curvilinear coordinates $\mathbf{y} = (y_1, y_2, y_3)$ (Sanchez-Hubert and Sanchez-Palencia 1999):

$$\mathbf{X}(\mathbf{y}) = \mathbf{r}(y_3 = s) + y_1\,\mathbf{a}_1(y_3 = s) + y_2\,\mathbf{a}_2(y_3 = s), \tag{5}$$

The mapping **X** is injective for sufficiently small y_1, y_2.

For the sake of simplicity, we assume that the cross section Σ (curvilinear parametrization) of the rod is circular with radius R_0 :

$$\Sigma = \{\, \mathbf{y}' = (y_1, y_2),\ y_1^2 + y_2^2 < R_0^2 \,\}.$$

Indeed the rod is defined by the set of points such that:

$\widetilde{\mathbf{B}} = \mathbf{X}(\mathbf{B}) = \{\mathbf{P}, \mathbf{OP}(\mathbf{y}) = \mathbf{X}(\mathbf{y})$ such that $\mathbf{y}' \in \Sigma$ and $y_3 \in]0, L[\}$,

in which $B = \Sigma \times]0, L[$.

Next we record a number of formulae pertaining to the geometry of the curved rod and to the kinematics of the material in a three-dimensional neighbourhood of the mean line.

At each point of the rod, we define the covariant basis $\mathbf{g}_i = \mathbf{X}_{,i}$ as well as the contravariant basis $\mathbf{g}^i$ ($\mathbf{g}_i . \mathbf{g}^j = \delta_i^j$, where δ_i^j are the Kronecker symbols).

From formulae (4)-(5), we easily obtain:

$\mathbf{g}_1 = \mathbf{a}_1$, (6)

$\mathbf{g}_2 = \mathbf{a}_2$, (7)

$\mathbf{g}_3 = (1 - k y_1)\mathbf{a}_3 - \tau y_2 \mathbf{a}_1 + \tau y_1 \mathbf{a}_2$, (8)

$\mathbf{g}^1 = \mathbf{a}_1 + \dfrac{\tau y_2}{1 - k y_1}\mathbf{a}_3$, (9)

$\mathbf{g}^2 = \mathbf{a}_2 - \dfrac{\tau y_1}{1 - k y_1}\mathbf{a}_3$. (10)

$\mathbf{g}^3 = \dfrac{1}{1 - k y_1}\mathbf{a}_3$, (11)

We note that $\mathbf{g}_i(0,0,s) = \mathbf{g}^i(0,0,s) = \mathbf{a}_i$.

The determinant of the metric tensor of components $g_{ij} = \mathbf{g}_i . \mathbf{g}_j$ is:

$g = (1 - k y_1)^2$. (12)

From (12), we see that the volume element of the rod is given in terms of curvilinear coordinates by:

$d\widetilde{\mathbf{B}} = d\mathbf{X} = dX_1\, dX_2\, dX_3 = \sqrt{g}\, d\mathbf{y} = (1 - k y_1) d\mathbf{y}$. (13)

Also we can observe that the boundary $\partial\widetilde{\mathbf{B}}$ of the rod $\widetilde{\mathbf{B}}$ is composed of the cross sections at $s = 0$ denoted by $\widetilde{\Sigma}_0$, $s = L$ denoted by $\widetilde{\Sigma}_L$, and the lateral boundary $\widetilde{\Gamma}_{lat}$ (the boundaries of all cross sections $\widetilde{\Sigma}$=X(Σ)) .

4. The three-dimensional formulation

It is supposed that the rod is composed of a general linearized anisotropic elastic material, for which the constitutive relation in cartesian coordinate is given by :

$\tilde{\boldsymbol{\sigma}}(\tilde{\mathbf{u}}(\mathbf{X})) = \widetilde{\mathbf{E}} : \widetilde{\mathrm{grad}}(\tilde{\mathbf{u}})$, (14)

in which $\widetilde{\boldsymbol{\sigma}}$ is the Cauchy stress tensor, $\widetilde{\mathbf{E}}$ is the fourth-order elastic modulus tensor and $\tilde{\boldsymbol{\varepsilon}}$ is the linearized Green-Lagrange strain tensor and $\tilde{\mathbf{u}}$ is the displacement vector in cartesian coordinates.

The modulus tensor $\widetilde{\mathbf{E}}$ in cartesian coordinates ($\mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k \otimes \mathbf{e}_l$, *i, j, k* and *l* =1, 2, 3) possesses the symmetry relations :

$\widetilde{E}_{ijkl} = \widetilde{E}_{jikl} = \widetilde{E}_{ijlk} = \widetilde{E}_{klij}$,

thus there are at most twenty one free components in $\widetilde{\mathbf{E}}$.

It is well-known that the three-dimensional field equations together with boundary conditions are :

$$\mathrm{Div}\,\tilde{\boldsymbol{\sigma}} + \tilde{\mathbf{f}} = 0, \quad \text{in} \quad \widetilde{\mathbf{B}}, \tag{15}$$

$$\tilde{\boldsymbol{\sigma}}.\tilde{\mathbf{n}}_{lat} = \tilde{\bar{\mathbf{T}}}, \quad \text{on} \quad \widetilde{\Gamma}_{lat}, \tag{16}$$

$$\tilde{\boldsymbol{\sigma}}.\tilde{\mathbf{n}} = \tilde{\bar{\mathbf{t}}}, \quad \text{on} \quad \widetilde{\Sigma}_L, \tag{17}$$

$$\tilde{\mathbf{u}} = \tilde{\bar{\mathbf{u}}}, \quad \text{on} \quad \widetilde{\Sigma}_0, \tag{18}$$

in which $Div\,\tilde{\boldsymbol{\sigma}} = \tilde{\sigma}_{ij,j}\,\mathbf{e}_i$, $\tilde{\mathbf{n}}_{lat}$ (respectively $\tilde{\mathbf{n}}$) is the unit outward normal vector on the boundary $\widetilde{\Gamma}_{lat}$ (respectively $\widetilde{\Sigma}_L$), $\tilde{\bar{\mathbf{T}}}$ (respectively $\tilde{\bar{\mathbf{t}}}$) is the prescribed traction on the lateral boundary $\widetilde{\Gamma}_{lat}$ (respectively $\widetilde{\Sigma}_L$), at the edges $\widetilde{\Sigma}_0$,the displacement $\tilde{\bar{\mathbf{u}}}$ is imposed and finally $\tilde{\mathbf{f}}$ is the volumic force.

Now we want to work in curvilinear coordinates **y,** we introduce the components of the displacement field as follows:

$$\tilde{\mathbf{u}}(\mathbf{X}) = \tilde{u}_i(\mathbf{X})\mathbf{e}_i = u_i(\mathbf{y})\mathbf{a}_i = \mathbf{u}(\mathbf{y}). \tag{19}$$

From formula (19), we see that (14) can be written as follows :

$$\boldsymbol{\sigma}(\mathbf{u}(\mathbf{y})) = \tilde{\boldsymbol{\sigma}}(\tilde{\mathbf{u}}(\mathbf{X})) = \widetilde{\mathbf{E}} : \mathrm{grad}(\mathbf{u}(\mathbf{y})), \tag{20}$$

in which by using chain rule and formulas (2)-(4), we see that

$$\mathrm{grad}(\mathbf{u}) = \mathbf{u}_{,l}(\mathbf{y}) \otimes \mathbf{g}^l = \mathrm{grad}(\mathbf{u})_{ij}\,\mathbf{a}_i \otimes \mathbf{a}_j, \tag{21}$$

with $u_i = \mathbf{u}.\mathbf{a}_i$, $g_l^m = \mathbf{g}^m.\mathbf{a}_l$, $\mathrm{grad}(\mathbf{u})_{ij} = \left(u_{i,\boldsymbol{\alpha}}\, g_j^{\boldsymbol{\alpha}} + u_{i|3}\, g_j^3\right)$, $u_{1|3} = u_{1,3} - \tau u_2 + k u_3$,

$u_{2|3} = u_{2,3} + \tau u_1$, $u_{3|3} = u_{3,3} - k u_1$ and $\sigma_{ij} = E_{ijkl}\,\mathrm{grad}(\mathbf{u})_{kl}\,\mathbf{a}_i \otimes \mathbf{a}_j$.

From formulae (20)-(21), we see that :

$$\begin{aligned}\boldsymbol{\sigma}(\mathbf{u}(\mathbf{y})) = \sigma_{ij}(\mathbf{u}(\mathbf{y}))\mathbf{a}_i \otimes \mathbf{a}_j &= \widetilde{E}_{ijkl}\,\mathrm{grad}(\mathbf{u})_{rs}(\mathbf{a}_k)_r(\mathbf{a}_l)_s\,\mathbf{e}_i \otimes \mathbf{e}_j, \\ &= \widetilde{E}_{ijkl}\,\mathrm{grad}(\mathbf{u})_{rs}(\mathbf{a}_i)_o(\mathbf{a}_j)_p(\mathbf{a}_k)_r(\mathbf{a}_l)_s\,\mathbf{a}_o \otimes \mathbf{a}_p, \\ &= E_{ijkl}\mathrm{grad}(\mathbf{u})_{kl}\,\mathbf{a}_i \otimes \mathbf{a}_j, \end{aligned} \tag{22}$$

where $E_{oprs} = \widetilde{E}_{ijkl}\,(\mathbf{a}_i)_o(\mathbf{a}_j)_p(\mathbf{a}_k)_r(\mathbf{a}_l)_s$.

Then we can write the three-dimensional field equations together with boundary conditions (17) in curvilinear coordinates :

$$\begin{aligned} &\mathrm{div}\,\boldsymbol{\sigma} + \mathbf{f} = 0, && \text{in} && \mathbf{B}, \\ &\boldsymbol{\sigma}.\mathbf{n}_{lat} = \bar{\mathbf{T}}, && \text{on} && \Gamma_{lat} = [0, L] \times \partial\Sigma, \\ &\boldsymbol{\sigma}.\mathbf{a}_3 = \bar{\mathbf{t}}, && \text{on} && \Sigma_L, \\ &\mathbf{u} = \bar{\mathbf{u}}, && \text{on} && \Sigma_0, \end{aligned} \tag{23}$$

in which $\operatorname{div}\boldsymbol{\sigma}(\mathbf{y}) = \dfrac{\partial \boldsymbol{\sigma}(\mathbf{y})}{\partial y_m}.g^m = \boldsymbol{\sigma}_{ij\|j}\,\mathbf{a}_i,$

with $\boldsymbol{\sigma}_{ij\|j} = \sigma_{ij,\alpha}\, g_j^{\alpha} + \left(\sigma_{ij,3} + \sigma_{1j}\left(\tau\,\delta_{i2} - k\,\delta_{i3}\right) - \tau\,\delta_{i1}\sigma_{2j} + k\delta_{i1}\sigma_{3j}\right) g_j^3 +$

$\left(-k\,g_3^3 + \tau g_2^3\right)\sigma_{i1} + \left(-\tau\sigma_{i2} + k\sigma_{i3}\right) g_1^3,$

$\mathbf{f}(\mathbf{y}) = \tilde{\mathbf{f}}\left(\mathbf{X}(\mathbf{y})\right), \mathbf{n}_{lat}(\mathbf{y}) = \tilde{\mathbf{n}}_{lat}\left(\mathbf{X}(\mathbf{y})\right), \overline{\mathbf{T}}(\mathbf{y}) = \tilde{\overline{\mathbf{T}}}\left(\mathbf{X}(\mathbf{y})\right), \overline{\mathbf{t}}(\mathbf{y}) = \tilde{\overline{\mathbf{t}}}\left(\mathbf{X}(\mathbf{y})\right)$ and $\overline{\mathbf{u}}(\mathbf{y}) = \tilde{\overline{\mathbf{u}}}\left(\mathbf{X}(\mathbf{y})\right).$

5. The one-dimensional rod equations

Based on the tridimensional formulation given in section 4, we intend to establish a rod theory which eliminates the cross-section variables $\mathbf{y}'$. This will be achieved through Taylor series expansions in terms of the curvilinear coordinates $\mathbf{y}'$, while on the lateral surface the polar coordinates will be used. We shall assume sufficient smoothness such that the displacement field $\mathbf{u}(\mathbf{y})$ defined by (19) has the following expansion:

$$\mathbf{u}(\mathbf{y}) = \sum_{n=0}^{5}\sum_{k=0}^{n} \frac{\mathbf{u}^{(k,\,n-k)}}{k!(n-k)!}\left(y_3\right) y_1^k\, y_2^{n-k} + O\left(R_0^6\right), \tag{24}$$

$$\text{in which } \mathbf{u}^{(k,\,n-k)}\left(y_3\right) = \frac{\partial^n \mathbf{u}}{\partial y_1^k\, \partial y_2^{n-k}}(0,0,y_3). \tag{25}$$

This approach provides under regularity assumption, the order of the remainder in every equations. This is the least order expansion such that rod equations are of leading order correctness.

If we assume that $k\,y_1 < k\,R_0 << 1$, we can write the Taylor-Young expansion of the contravariant basis:

$$\mathbf{g}^1(\mathbf{y}) = \mathbf{a}_1\left(y_3\right) + \sum_{n=1}^{4} \mathbf{g}^{1(n)}\left(y_3\right) y_1^{n-1} y_2 + O\left(R_0^5\right), \tag{26}$$

$$\mathbf{g}^2(\mathbf{y}) = \mathbf{a}_2\left(y_3\right) + \sum_{n=1}^{4} \mathbf{g}^{2(n)}\left(y_3\right) y_1^{n} + O\left(R_0^5\right), \tag{27}$$

$$\mathbf{g}^3(\mathbf{y}) = \mathbf{a}_3\left(y_3\right) + \sum_{n=1}^{4} \mathbf{g}^{3(n)}\left(y_3\right) y_1^{n} + O\left(R_0^5\right), \tag{28}$$

in which

$$\mathbf{g}^{1(n)} = \tau k^{n-1}\,\mathbf{a}_3,\ \mathbf{g}^{2(n)} = -\tau k^{n-1}\,\mathbf{a}_3,\ \mathbf{g}^{3(n)} = k^n\,\mathbf{a}_3, \text{ for } 1 \le n \le 4. \tag{29}$$

From (21), (24)-(28), we obtain the expansion of the gradient of the displacement field (24):

$$\operatorname{grad}\left(\mathbf{u}(\mathbf{y})\right) = \sum_{n=0}^{4}\sum_{k=0}^{n} \frac{\operatorname{grad}(\mathbf{u})^{(k,\,n-k)}}{k!(n-k)!}\left(y_3\right) y_1^k\, y_2^{n-k} + O\left(R_0^5\right), \tag{30}$$

in which $\operatorname{grad}(\mathbf{u})^{(k,\,n-k)} = \operatorname{grad}(\mathbf{u})_{ij}^{(k,\,n-k)}\,\mathbf{a}_i \otimes \mathbf{a}_j,$

where $\operatorname{grad}(\mathbf{u})_{ij}^{(k,\,n-k)} = u_i^{(k+1,\,n-k)}\delta_{1j} + \sum_{p=1}^{k+1} u_i^{(k+1-p,\,n-k-1)}\, g_j^{1(p)} + u_i^{(k,\,n-k+1)}\,\delta_{2j} + \sum_{p=1}^{k} u_i^{(k-p,\,n-k+1)}\, g_j^{2(p)} +$

$$u_{i|3}^{(k,n-k)}\delta_{3j}+\sum_{p=1}^{k}u_{i|3}^{(k-p,n-k)}g_j^{3(p)},$$

with $u_{1|3}^{(l,m)}=u_{1,3}^{(l,m)}-\tau u_2^{(l,m)}+k u_3^{(l,m)}$, $u_{2|3}^{(l,m)}=u_{2,3}^{(l,m)}+\tau u_1^{(l,m)}$, $u_{3|3}^{(l,m)}=u_{3,3}^{(l,m)}-k u_1^{(l,m)}$, for $l\geq 0$ and $m\geq 0$.

The expansion of the stress tensor is obtainable from (26) and (30) :

$$\boldsymbol{\sigma}=\sum_{n=0}^{4}\sum_{k=0}^{n}\frac{y_1^k y_2^{n-k}}{k!(n-k)!}\boldsymbol{\sigma}^{(k,n-k)}+O\left(R_0^5\right), \tag{31}$$

where $\boldsymbol{\sigma}^{(k,n-k)}=\widetilde{\mathbf{E}}:\mathrm{grad}(\mathbf{u})^{(k,n-k)}=\widetilde{E}_{ijkl}\,\mathrm{grad}(\mathbf{u})_{rs}^{(k,n-k))}(\mathbf{a}_k)_r(\mathbf{a}_l)_s\,\mathbf{e}_i\otimes\mathbf{e}_j,$

$$=E_{ijlm}\,\mathrm{grad}(\mathbf{u})_{lm}^{(k,n-k)}\mathbf{a}_i\otimes\mathbf{a}_j, \tag{32}$$

We point out that the stress coefficients (32) for n = 4 is not needed for the end results so in the above equation we take $0\leq n\leq 3$. An observation is that $\boldsymbol{\sigma}^{(k,n-k)}$ depends linearly on higher-order coefficients $u_i^{(k+1,n-k)}$, $u_i^{(k-p,n-k+1)}$ for p = 0, k and on lower order coefficients $u_i^{(k+1-p,n-k-1)}$ for p = 1,

k +1. For this reason the dimension reduction procedure works as in the similar approach for plate and shell theories.

To satisfy the lateral traction condition in a pointwise manner, we use Fourier series expansion for the applied traction $\overline{\mathbf{T}}(\mathbf{y})=\hat{\overline{\mathbf{T}}}(R_0\cos\theta_a, R_0\sin\theta_a, y_3)$ (polar coordinate is used on $\partial\Sigma$: $y_1=R_0\cos\theta_a$, $y_2=R_0\sin\theta_a$ **and** $\theta_a=\left(\widehat{Gy_1},\overrightarrow{GM}\right)$ **is the azimuthal angle on** y_1y_2**-plane**), then we assume that $\hat{\overline{\mathbf{T}}}$ is a sufficiently regular function of θ_a to ensure that the Fourier series expansion is convergent :

$$\hat{\overline{\mathbf{T}}}(\theta_a)=\hat{\overline{\mathbf{T}}}_0+\sum_{p=1}^{\infty}\hat{\overline{\mathbf{T}}}_{cp}\cos p\theta_a+\hat{\overline{\mathbf{T}}}_{sp}\sin p\theta_a, \tag{33}$$

where $\hat{\overline{\mathbf{T}}}_0=\frac{1}{2\pi}\int_{-\pi}^{\pi}\hat{\overline{\mathbf{T}}}(\theta_a)d\theta_a$, $\hat{\overline{\mathbf{T}}}_{cp}=\frac{1}{\pi}\int_{-\pi}^{\pi}\hat{\overline{\mathbf{T}}}(\theta_a)\cos(p\theta_a)d\theta_a$ and

$\hat{\overline{\mathbf{T}}}_{sp}=\frac{1}{\pi}\int_{-\pi}^{\pi}\hat{\overline{\mathbf{T}}}(\theta_a)\sin(p\theta_a)d\theta_a$.

Since $\mathbf{n}_{lat}=\cos(\theta_a)\mathbf{a}_1+\sin(\theta_a)\mathbf{a}_2$, we obtain :

$$\begin{aligned}\boldsymbol{\sigma}.\mathbf{n}_{lat}=&\sum_{n=0}^{3}\sum_{k=0}^{n}\frac{R_0^n}{k!(n-k)!}(\cos\theta_a)^{k+1}(\sin\theta_a)^{n-k}\boldsymbol{\sigma}^{(k,n-k)}.\mathbf{a}_1\\&+\sum_{n=0}^{3}\sum_{k=0}^{n}\frac{R_0^n}{k!(n-k)!}(\cos\theta_a)^{k}(\sin\theta_a)^{n-k+1}\boldsymbol{\sigma}^{(k,n-k)}.\mathbf{a}_2+\\&O(R_0^4\cos\theta_a,R_0^4\sin\theta_a,R_0^4\cos 3\theta_a,R_0^4\sin 3\theta_a)+\\&O(R_0^5,R_0^5\cos 2\theta_a,R_0^5\sin 2\theta_a,R_0^5\cos 4\theta_a,R_0^5\sin 4\theta_a).\end{aligned} \tag{34}$$

Then, from equations $(23)_2$, (33) and (34), by setting equal the coefficients of $\cos(m\theta_a)$, $\sin(m\theta_a)$for $1\leq m\leq$ 3 on both sides, one can obtain seven equations (in vector form) for the stress coefficients. However, since equation (34) is not exact, for the coefficients of $\cos(m\theta_a)$(0

$\leq m \leq 1$) and $\sin(\theta_a)$, all terms are kept, while for other coefficients only leading-order terms are kept (Chen et al. (2020)). So we obtain :

$$\frac{2}{R_0}\hat{\hat{\mathbf{T}}}_0 = \boldsymbol{\sigma}^{(1,0)}.\mathbf{a}_1 + \boldsymbol{\sigma}^{(0,1)}.\mathbf{a}_2 + \frac{R_0^2}{8}\left(\boldsymbol{\sigma}^{(1,2)}.\mathbf{a}_1 + \boldsymbol{\sigma}^{(2,1)}.\mathbf{a}_2 + \boldsymbol{\sigma}^{(3,0)}.\mathbf{a}_1 + \boldsymbol{\sigma}^{(0,3)}.\mathbf{a}_2\right) + O\left(R_0^4\right),$$

$$\hat{\hat{\mathbf{T}}}_{c1} = \boldsymbol{\sigma}^{(0,0)}.\mathbf{a}_1 + \frac{R_0^2}{4}\boldsymbol{\sigma}^{(1,1)}.\mathbf{a}_2 + \frac{R_0^2}{8}\left(\boldsymbol{\sigma}^{(0,2)} + 3\boldsymbol{\sigma}^{(2,0)}\right).\mathbf{a}_1 + O\left(R_0^4\right),$$

$$\hat{\hat{\mathbf{T}}}_{s1} = \boldsymbol{\sigma}^{(0,0)}.\mathbf{a}_2 + \frac{R_0^2}{4}\boldsymbol{\sigma}^{(1,1)}.\mathbf{a}_1 + \frac{R_0^2}{8}\left(3\boldsymbol{\sigma}^{(0,2)} + \boldsymbol{\sigma}^{(2,0)}\right).\mathbf{a}_2 + O\left(R_0^4\right), \tag{35}$$

$$\frac{2}{R_0}\hat{\hat{\mathbf{T}}}_{c2} = \boldsymbol{\sigma}^{(1,0)}.\mathbf{a}_1 - \boldsymbol{\sigma}^{(0,1)}.\mathbf{a}_2 + O\left(R_0^2\right),$$

$$\frac{2}{R_0}\hat{\hat{\mathbf{T}}}_{s2} = \boldsymbol{\sigma}^{(0,1)}.\mathbf{a}_1 + \boldsymbol{\sigma}^{(1,0)}.\mathbf{a}_2 + O\left(R_0^2\right),$$

$$\frac{4}{R_0^2}\hat{\hat{\mathbf{T}}}_{c3} = -\boldsymbol{\sigma}^{(1,1)}.\mathbf{a}_2 + \frac{1}{2}\left(\boldsymbol{\sigma}^{(2,0)} - \boldsymbol{\sigma}^{(0,2)}\right).\mathbf{a}_1 + O\left(R_0^2\right),$$

$$\frac{4}{R_0^2}\hat{\hat{\mathbf{T}}}_{s3} = \boldsymbol{\sigma}^{(1,1)}.\mathbf{a}_1 + \frac{1}{2}\left(\boldsymbol{\sigma}^{(2,0)} - \boldsymbol{\sigma}^{(0,2)}\right).\mathbf{a}_2 + O\left(R_0^2\right).$$

By using (32), we see that the fifteen unknown coefficients of the expansion of the displacement field (24) intervenes in the seven equations (35). Next considering equilibrium equations, it is possible to obtain a closed system of ten equations with ten unknown coefficients of the displacement field (24).

By using expansion of stress tensor (31) and expansions (26)-(28) for the contravariant basis vectors, equilibrium equation $(23)_1$ truncated at order R_0^3 involves six vectorial equation:

$$\mathrm{div}^{(k,n-k)}\boldsymbol{\sigma}\left(y_3\right) + \mathbf{f}^{(k,n-k)}\left(y_3\right) = 0, \qquad \text{for } n = 0, 1, 2 \text{ and } k = 0, n, \tag{36}$$

in which the volumic force $\mathbf{f}$ is assumed to have expansion of the type (24) of which the vector coefficient are $\mathbf{f}^{(k,n-k)}$ and

$$\mathrm{div}^{(0,0)}\boldsymbol{\sigma}\left(y_3\right) = \boldsymbol{\sigma}^{(1,0)}\left(y_3\right).\mathbf{a}_1\left(y_3\right) + \boldsymbol{\sigma}^{(0,1)}\left(y_3\right).\mathbf{a}_2\left(y_3\right) + \boldsymbol{\sigma}_{,3}^{(0,0)}\left(y_3\right).\mathbf{a}_3\left(y_3\right),$$

$$\mathrm{div}^{(1,0)}\boldsymbol{\sigma}\left(y_3\right) = \boldsymbol{\sigma}^{(2,0)}\left(y_3\right).\mathbf{a}_1\left(y_3\right) + \boldsymbol{\sigma}^{(0,1)}\left(y_3\right).\mathbf{g}^{2(1)}\left(y_3\right) + \boldsymbol{\sigma}^{(1,1)}\left(y_3\right).\mathbf{a}_2\left(y_3\right) + \boldsymbol{\sigma}_{,3}^{(0,0)}\left(y_3\right).\mathbf{g}^{3(1)}\left(y_3\right) + \boldsymbol{\sigma}_{,3}^{(1,0)}\left(y_3\right).\mathbf{a}_3\left(y_3\right),$$

$$\mathrm{div}^{(0,1)}\boldsymbol{\sigma}\left(y_3\right) = \boldsymbol{\sigma}^{(1,1)}\left(y_3\right).\mathbf{a}_1\left(y_3\right) + \boldsymbol{\sigma}^{(0,2)}\left(y_3\right).\mathbf{a}_2\left(y_3\right) + \boldsymbol{\sigma}_{,3}^{(0,1)}\left(y_3\right).\mathbf{a}_3\left(y_3\right),$$

$$\mathrm{div}^{(2,0)}\boldsymbol{\sigma}\left(y_3\right) = \boldsymbol{\sigma}^{(3,0)}\left(y_3\right).\mathbf{a}_1\left(y_3\right) + 2\boldsymbol{\sigma}^{(0,1)}\left(y_3\right).\mathbf{g}^{2(2)}\left(y_3\right) + 2\boldsymbol{\sigma}^{(1,1)}\left(y_3\right).\mathbf{g}^{2(1)}\left(y_3\right) + \boldsymbol{\sigma}^{(2,1)}\left(y_3\right).\mathbf{a}_2\left(y_3\right) + 2\boldsymbol{\sigma}_{,3}^{(0,0)}\left(y_3\right).\mathbf{g}^{3(2)}\left(y_3\right) + 2\boldsymbol{\sigma}_{,3}^{(1,0)}\left(y_3\right).\mathbf{g}^{3(1)}\left(y_3\right) + \boldsymbol{\sigma}_{,3}^{(2,0)}\left(y_3\right).\mathbf{a}_3\left(y_3\right),$$

$$\mathrm{div}^{(1,1)}\boldsymbol{\sigma}\left(y_3\right) = \boldsymbol{\sigma}^{(1,0)}\left(y_3\right).\mathbf{g}^{1(1)}\left(y_3\right) + \boldsymbol{\sigma}^{(2,1)}\left(y_3\right).\mathbf{a}_1\left(y_3\right) + \boldsymbol{\sigma}^{(0,2)}\left(y_3\right).\mathbf{g}^{2(1)}\left(y_3\right) + \boldsymbol{\sigma}^{(1,2)}\left(y_3\right).\mathbf{g}^{2(1)}\left(y_3\right) + \boldsymbol{\sigma}^{(1,2)}\left(y_3\right).\mathbf{a}_2\left(y_3\right) + \boldsymbol{\sigma}_{,3}^{(0,1)}\left(y_3\right).\mathbf{g}^{3(1)}\left(y_3\right) + \boldsymbol{\sigma}_{,3}^{(1,1)}\left(y_3\right).\mathbf{a}_3\left(y_3\right),$$

$$\mathrm{div}^{(0,2)}\boldsymbol{\sigma}\left(y_3\right) = \boldsymbol{\sigma}^{(1,2)}\left(y_3\right).\mathbf{a}_1\left(y_3\right) + \boldsymbol{\sigma}^{(0,3)}\left(y_3\right).\mathbf{a}_2\left(y_3\right) + \boldsymbol{\sigma}_{,3}^{(0,2)}\left(y_3\right).\mathbf{a}_3\left(y_3\right),$$

in which from formulae (2)-(4) and $(22)_2$, we obtain:

$$\boldsymbol{\sigma}_{,3}^{(l,m)} = E_{ijkr}\,\mathrm{grad}\left(\mathbf{u}\right)_{kr,3}^{(l,m)}\mathbf{a}_i\otimes\mathbf{a}_j + \widetilde{E}_{ijkt}\,\mathrm{grad}\left(\mathbf{u}\right)_{rs}^{(l,m)}A_{kr}\widetilde{\left(\mathbf{a}_t\right)}_s\widetilde{\left(\mathbf{a}_i\right)}_o\widetilde{\left(\mathbf{a}_j\right)}_p\mathbf{a}_o\otimes\mathbf{a}_p +$$

$$\widetilde{E}_{ijkt}\,\mathrm{grad}\left(\mathbf{u}\right)_{rs}^{(l,m)}\widetilde{\left(\mathbf{a}_k\right)}_r A_{ts}\widetilde{\left(\mathbf{a}_i\right)}_o\widetilde{\left(\mathbf{a}_j\right)}_p\mathbf{a}_o\otimes\mathbf{a}_p. \tag{37}$$

in which $A_{kr} = \left(k\delta_{k3}-\tau\delta_{k2}\right)\widetilde{\left(\mathbf{a}_1\right)}_r + \tau\delta_{k1}\widetilde{\left(\mathbf{a}_2\right)}_r - k\delta_{k1}\widetilde{\left(\mathbf{a}_3\right)}_r$,

$$\mathrm{grad}\left(\mathbf{u}\right)_{kr,3}^{(l,m)} = u_{k,3}^{(l+1,m)}\boldsymbol{\delta}_{1r} + \sum_{p=1}^{l+1}\left(u_{k,3}^{(l+1-p,m-1)}\,g_r^{1(p)} + u_k^{(l+1-p,m-1)}\,g_{r,3}^{1(p)}\right) +$$

$$u_{k,3}^{(l,m+1)}\boldsymbol{\delta}_{2r} + \sum_{p=1}^{l}\left(u_{k,3}^{(l-p,m+1)}\,g_r^{2(p)} + u_k^{(l-p,m+1)}\,g_{r,3}^{2(p)}\right) +$$

$$u_{k\|33}^{(l,m)}\boldsymbol{\delta}_{3l'} + \sum_{p=1}^{l}\left(u_{k\|33}^{(l-p,m)}\,g_{l'}^{3(p)} + u_{k\|3}^{(l-p,m)}\,g_{l',3}^{3(p)}\right),$$

with $u_{1\|33}^{(l,m)} = u_{1,33}^{(l,m)} - \left(\tau_{,3}\,u_2^{(l,m)} + \tau u_{2,3}^{(l,m)}\right) + k_{,3}\,u_3^{(l,m)} + k\;u_{3,3}^{(l,m)}$, $u_{2\|33}^{(l,m)} = u_{2,33}^{(l,m)} + \tau_{,3}\,u_1^{(l,m)} + \tau u_{1,3}^{(l,m)}$,

$u_{3\|33}^{(l,m)} = u_{3,33}^{(l,m)} - \left(k_{,3}\,u_1^{(l,m)} + k\,u_{1,3}^{(l,m)}\right)$, for $l\geq 0$ and $m\geq 0$.

Next we modify equations (35)-(36) in order to obtain a system of ten vectors equations with ten vectors displacement field unknows. By considering (36) (for (k, n) = (0, 0), (2,0), (0,2)), we can eliminate $\boldsymbol{\sigma}^{(1,0)}.\mathbf{a}_1 + \boldsymbol{\sigma}^{(0,1)}.\mathbf{a}_2$ and $\boldsymbol{\sigma}^{(1,2)}.\mathbf{a}_1 + \boldsymbol{\sigma}^{(2,1)}.\mathbf{a}_2 + \boldsymbol{\sigma}^{(3,0)}.\mathbf{a}_1 + \boldsymbol{\sigma}^{(0,3)}.\mathbf{a}_2$ from $(35)_1$:

$$\frac{2}{R_0}\hat{\hat{\mathbf{T}}}_0 + \mathbf{f}^{(0,0)} = -\boldsymbol{\sigma}_{,3}^{(0,0)}.\mathbf{a}_3 - \frac{R_0^2}{4}\left(\boldsymbol{\sigma}^{(0,1)}.\mathbf{g}^{2(2)} + \boldsymbol{\sigma}^{(1,1)}.\mathbf{g}^{2(1)} + \boldsymbol{\sigma}_{,3}^{(0,0)}.\mathbf{g}^{3(2)} + \boldsymbol{\sigma}_{,3}^{(1,0)}.\mathbf{g}^{3(1)}\right) -$$

$$\frac{R_0^2}{8}\left(\boldsymbol{\sigma}_{,3}^{(2,0)}.\mathbf{a}_3 + \boldsymbol{\sigma}_{,3}^{(0,2)}.\mathbf{a}_3 + \mathbf{f}^{(2,0)} + \mathbf{f}^{(0,2)}\right) + O\left(R_0^4\right). \tag{38}$$

Finally equations $(35)_{2\text{-}7}$, (36) (for (k, n) = (0, 0), (1,0), (0,1))) and (38) is a system of ten vector equations with ten unknown vectors $\mathbf{u}^{(k,n)}$ (for (k, n) = (0,0), (1,0), (0,1), (2,0), (1,1), (0,2), (3,0), (2,1), (1,2), (0,3)). However, the errors in different equations are not of the same order of magnitude. Equations (38) and $(35)_{2\text{-}3}$ are truncated at $O\left(R_0^4\right)$ while equations $(35)_{4\text{-}7}$ are truncated at $O\left(R_0^2\right)$. Therefore in what follows, we shall perform a number of refinements.

6. Reduction process.

The method of section 5 presents a number of undesirable features. There are too many unknowns (ten vector unknowns). It would be better if the rod equations contains only terms with clear physical meanings (stretching, bending and twisting). This system of ten vector equations requires in total twenty four scalar boundary conditions which cannot be imposed physically. Indeed, for a force edge, one can only know six scalar boundary conditions given by the components of the torsor resultant. In this section, we reduce the number of main unknows to four which represent stretching, two bendings and twisting.

By considering linear elastic relation (32), one can see that $(35)_{4,5}$ and (36) (for $k = 0$, $n = 0$) provide 3 linear algebraic vectorial equations for 3 unknown vectors $\mathbf{u}^{(k,l)}$ (for (k, l) = (2,0), (1,1), (0,2)) and we obtain:

$$\begin{bmatrix} \mathbf{u}^{(2,0)} \\ \mathbf{u}^{(1,1)} \\ \mathbf{u}^{(0,2)} \end{bmatrix} = \mathbf{K}^{-1}\left\{\begin{bmatrix} -\mathbf{A}^{(1,3)}\mathbf{u}_{|3}^{(1,0)} \\ -\mathbf{A}^{(2,3)}\mathbf{u}_{|3}^{(1,0)} \\ -\mathbf{A}^{(3,1)}\mathbf{u}_{,3}^{(1,0)} - \mathbf{A}^{(1,3)}\mathbf{u}_{|3}^{(1,0)} \end{bmatrix} + \begin{bmatrix} \mathbf{A}^{(2,3)}\mathbf{u}_{|3}^{(0,1)} \\ -\mathbf{A}^{(1,3)}\mathbf{u}_{|3}^{(0,1)} \\ -\mathbf{A}^{(3,2)}\mathbf{u}_{,3}^{(0,1)} - \mathbf{A}^{(2,3)}\mathbf{u}_{|3}^{(0,1)} \end{bmatrix}\right\} +$$

$$\mathbf{K}^{-1}\left\{\begin{bmatrix} \frac{2}{R_0}\hat{\bar{\mathbf{T}}}_{c2} + O\left(R_0^2\right) \\ \frac{2}{R_0}\hat{\bar{\mathbf{T}}}_{s2} + O\left(R_0^2\right) \\ -\mathbf{A}^{(3,3)}\mathbf{u}_{\|33}^{(0,0)} - \mathbf{f}^{(0,0)} \end{bmatrix} - \begin{bmatrix} \mathbf{A}^{(1,3,1)} \\ \mathbf{A}^{(2,3,1)} \\ \mathbf{A}^{(3)} \end{bmatrix}\mathbf{u}_{|3}^{(0,0)} - \begin{bmatrix} \mathbf{A}^{(1,2,1)} \\ \mathbf{A}^{(2,2,1)} \\ \mathbf{A}^{(2)} \end{bmatrix}\mathbf{u}^{(0,1)} - \begin{bmatrix} 0 \\ 0 \\ \mathbf{A}^{(1)} \end{bmatrix}\mathbf{u}^{(1,0)}\right\}, \tag{39}$$

where

$$\mathbf{K} = \begin{pmatrix} \mathbf{A}^{(1,1)} & \mathbf{A}^{(1,2)} - \mathbf{A}^{(2,1)} & -\mathbf{A}^{(2,2)} \\ \mathbf{A}^{(2,1)} & \mathbf{A}^{(1,1)} + \mathbf{A}^{(2,2)} & \mathbf{A}^{(1,2)} \\ \mathbf{A}^{(1,1)} & \mathbf{A}^{(1,2)} + \mathbf{A}^{(2,1)} & \mathbf{A}^{(2,2)} \end{pmatrix},$$

$$\mathbf{A}^{(r)} = \widetilde{E}_{ijkl}\left(\left(k\boldsymbol{\delta}_{k3} - \tau\boldsymbol{\delta}_{k2}\right)\mathbf{a}_1 + \tau\boldsymbol{\delta}_{k1}\mathbf{a}_2 - k\boldsymbol{\delta}_{k1}\mathbf{a}_3\right)_{k^*}\left(\mathbf{a}_l\right)_r\left(\mathbf{a}_i\right)_{i^*}\left(\mathbf{a}_j\right)_3\mathbf{a}_{i^*} \otimes \mathbf{a}_{k^*} +$$
$$\widetilde{E}_{ijkl}\left(\mathbf{a}_k\right)_{k^*}\left(\left(k\boldsymbol{\delta}_{l3} - \tau\boldsymbol{\delta}_{l2}\right)\mathbf{a}_1 + \tau\boldsymbol{\delta}_{l1}\mathbf{a}_2 - k\boldsymbol{\delta}_{l1}\mathbf{a}_3\right)_r\left(\mathbf{a}_i\right)_{i^*}\left(\mathbf{a}_j\right)_3\mathbf{a}_{i^*} \otimes \mathbf{a}_{k^*},$$

$$\mathbf{A}^{(i,j)} = E_{oipj}\,\mathbf{a}_o \otimes \mathbf{a}_p,\ \mathbf{A}^{(i,j,k)} = E_{oipn}\,g_n^{j(k)}\,\mathbf{a}_o \otimes \mathbf{a}_p.$$

We recall that we keep $O\left(R_0^2\right)$ terms in (38) and $(35)_{2\,3}$ for two reasons: they may give bending or torsion effects and they may become leading-order terms. If the first term on the right hand side of (38) can be eliminated, then the $O\left(R_0^2\right)$ term becomes the leading one. Based on this argument, we multiply (38) by $\mathbf{a}_1$ and then add it to the derivative (with respect to y_3) of $(35)_2$ multiplied by $\mathbf{a}_3$. Then we take into account of (2), (3), (29) and (36) for $k = 0$ and $n = 1$ to obtain one scalar equation:

$$\hat{\bar{\mathbf{T}}}_{c1,3}.\mathbf{a}_3 + \frac{2}{R_0}\hat{\bar{\mathbf{T}}}_0.\mathbf{a}_1 = \mathbf{a}_3.\boldsymbol{\sigma}^{(0,0)}.\left(-k\,\mathbf{a}_3 + \tau\,\mathbf{a}_2\right) - \mathbf{f}^{(0,0)}.\mathbf{a}_1 - \frac{R_0^2}{8}\mathbf{a}_1.\left(\mathbf{f}^{(2,0)} + \mathbf{f}^{(0,2)}\right) -$$
$$\frac{R_0^2}{4}\mathbf{a}_1.\left(-\tau\,k\,\boldsymbol{\sigma}^{(0,1)}.\mathbf{a}_3 - \tau\,\boldsymbol{\sigma}^{(1,1)}.\mathbf{a}_3 + k^2\boldsymbol{\sigma}_{,3}^{(0,0)}.\,\mathbf{a}_3 + k\,\boldsymbol{\sigma}_{,3}^{(1,0)}.\,\mathbf{a}_3\right) +$$
$$\frac{R_0^2}{4}\mathbf{a}_3.\left(\boldsymbol{\sigma}_{,3}^{(1,1)}.\mathbf{a}_2 + \boldsymbol{\sigma}_{,3}^{(2,0)}\mathbf{a}_1\right) - \tau\frac{R_0^2}{4}\mathbf{a}_3.\boldsymbol{\sigma}^{(1,1)}.\mathbf{a}_1 + \frac{R_0^2}{8}\mathbf{a}_3.\left(\boldsymbol{\sigma}^{(0,2)} + 3\boldsymbol{\sigma}^{(2,0)}\right).\left(-k\,\mathbf{a}_3 + \tau\,\mathbf{a}_2\right) + O\left(R_0^4\right). \tag{40}$$

By deriving with respect to y_3 equation (36) for $k = 1$ and $n = 1$, we substitute a new expression for $\boldsymbol{\sigma}_{,3}^{(1,1)}.\mathbf{a}_2 + \boldsymbol{\sigma}_{,3}^{(2,0)}\mathbf{a}_1$ in (40) to obtain:

$$\hat{\bar{\mathbf{T}}}_{c1,3}.\mathbf{a}_3 + \frac{2}{R_0}\hat{\bar{\mathbf{T}}}_0.\mathbf{a}_1 + \mathbf{f}^{(0,0)}.\mathbf{a}_1 + \frac{R_0^2}{4}\left(\mathbf{f}_{,3}^{(1,0)}.\mathbf{a}_3 + \frac{1}{2}\left(\mathbf{f}^{(2,0)} + \mathbf{f}^{(0,2)}\right).\mathbf{a}_1\right) =$$

$$\mathbf{a}_3.\boldsymbol{\sigma}^{(0,0)}.\left(-k\,\mathbf{a}_3+\tau\,\mathbf{a}_2\right)-\frac{R_0^2}{4}\mathbf{a}_1.\left(-2\tau\,k\,\boldsymbol{\sigma}^{(0,1)}+2k^2\boldsymbol{\sigma}_{,3}^{(0,0)}+k\,\boldsymbol{\sigma}_{,3}^{(1,0)}\right).\mathbf{a}_3-$$

$$\frac{R_0^2}{4}\mathbf{a}_3.\left(-\tau\,\boldsymbol{\sigma}_{,3}^{(0,1)}.\,\mathbf{a}_3-\tau_{,3}\boldsymbol{\sigma}^{(0,1)}.\mathbf{a}_3\right)-$$

$$\frac{R_0^2}{4}\mathbf{a}_3.\left(k\boldsymbol{\sigma}_{,33}^{(0,0)}.\mathbf{a}_3+k_{,3}\boldsymbol{\sigma}_{,3}^{(0,0)}.\,\mathbf{a}_3+\boldsymbol{\sigma}_{,33}^{(1,0)}.\mathbf{a}_3+k\boldsymbol{\sigma}_{,3}^{(1,0)}.\mathbf{a}_1\right)+$$

$$\tau\frac{R_0^2}{4}\mathbf{a}_3.\boldsymbol{\sigma}^{(1,1)}.\mathbf{a}_1+\frac{R_0^2}{8}\mathbf{a}_3.\left(\boldsymbol{\sigma}^{(0,2)}+\boldsymbol{\sigma}^{(2,0)}\right).\left(-k\,\mathbf{a}_3+\tau\,\mathbf{a}_2\right)+O\left(R_0^4\right). \tag{41}$$

in which (see formulae (2)-(4) and (37)) :

$$\boldsymbol{\sigma}_{,33}^{(l,m)}=E_{ijkr}\,\mathrm{grad}\left(\mathbf{u}\right)_{kr,33}^{(l,m)}\mathbf{a}_i\otimes\mathbf{a}_j+2\widetilde{E}_{ijkt}\,\mathrm{grad}\left(\mathbf{u}\right)_{rs,3}^{(l,m)}A_{kr}\widetilde{\left(\mathbf{a}_t\right)}_s\widetilde{\left(\mathbf{a}_i\right)}_o\widetilde{\left(\mathbf{a}_j\right)}_p\mathbf{a}_o\otimes\mathbf{a}_p+$$

$$2\widetilde{E}_{ijkt}\,\mathrm{grad}\left(\mathbf{u}\right)_{rs,3}^{(l,m)}\widetilde{\left(\mathbf{a}_k\right)}_rA_{ts}\widetilde{\left(\mathbf{a}_i\right)}_o\widetilde{\left(\mathbf{a}_j\right)}_p\mathbf{a}_o\otimes\mathbf{a}_p+\widetilde{E}_{ijkt}\,\mathrm{grad}\left(\mathbf{u}\right)_{rs}^{(l,m)}B_{kr}\widetilde{\left(\mathbf{a}_t\right)}_s\widetilde{\left(\mathbf{a}_i\right)}_o\widetilde{\left(\mathbf{a}_j\right)}_p\mathbf{a}_o\otimes\mathbf{a}_p+$$

$$2\widetilde{E}_{ijkt}\,\mathrm{grad}\left(\mathbf{u}\right)_{rs}^{(l,m)}A_{kr}A_{ts}\widetilde{\left(\mathbf{a}_i\right)}_o\widetilde{\left(\mathbf{a}_j\right)}_p\mathbf{a}_o\otimes\mathbf{a}_p+$$

$$\widetilde{E}_{ijkt}\,\mathrm{grad}\left(\mathbf{u}\right)_{rs}^{(l,m)}\widetilde{\left(\mathbf{a}_k\right)}_rB_{ts}\widetilde{\left(\mathbf{a}_i\right)}_o\widetilde{\left(\mathbf{a}_j\right)}_p\mathbf{a}_o\otimes\mathbf{a}_p. \tag{42}$$

where $B_{kr}=\left(-\left(\tau^2+k^2\right)\delta_{k1}+k_{,3}\delta_{k3}-\tau_{,3}\delta_{k2}\right)\widetilde{\left(\mathbf{a}_1\right)}_r+\left(\left(k\delta_{k3}-\tau\delta_{k2}\right)\tau+\tau_{,3}\delta_{k1}\right)\widetilde{\left(\mathbf{a}_2\right)}_r-$

$$\left(k\left(k\delta_{k3}-\tau\delta_{k2}\right)+k_{,3}\delta_{k1}\right)\widetilde{\left(\mathbf{a}_3\right)}_r,$$

with $\mathrm{grad}\left(\mathbf{u}\right)_{kl,33}^{(m,n)}=u_{k,33}^{(m+1,n)}\boldsymbol{\delta}_{1l}+\sum_{p=1,n\geq1}^{m+1}\left(u_{k,33}^{(m+1-p,n-1)}\,g_l^{1(p)}+2u_{k,3}^{(m+1-p,n-1)}\,g_{l,3}^{1(p)}+u_k^{(m+1-p,n-1)}\,g_{l,33}^{1(p)}\right)+$

$$u_{k,33}^{(m,n+1)}\boldsymbol{\delta}_{2l}+\sum_{p=1}^{m}\left(u_{k,33}^{(m-p,n+1)}\,g_l^{2(p)}+2u_{k,3}^{(m-p,n+1)}\,g_{l,3}^{2(p)}+u_k^{(m-p,n+1)}\,g_{l,33}^{2(p)}\right)+$$

$$u_{k\|33}^{(m,n)}\boldsymbol{\delta}_{3l}+\sum_{p=1}^{m}\left(u_{k\|33}^{(m-p,n)}\,g_l^{3(p)}+2u_{k\|33}^{(m-p,n)}\,g_{l,3}^{3(p)}+u_{k\|33}^{(m-p,n)}\,g_{l,3}^{3(p)}\right)$$

and $u_{1\|333}^{(m,n)}=u_{1,333}^{(m,n)}-\left(\tau_{,33}\,u_2^{(m,n)}+2\tau_{,3}\,u_{2,3}^{(m,n)}+\tau\,u_{2,33}^{(m,n)}\right)+k_{,33}\,u_3^{(m,n)}+2\,k_{,3}\,u_{3,3}^{(m,n)}+k\;u_{3,33}^{(m,n)}$,

$u_{2\|333}^{(m,n)}=u_{2,333}^{(m,n)}+\tau_{,33}\,u_1^{(m,n)}+2\tau_{,3}\,u_{1,3}^{(m,n)}+\tau\,u_{1,33}^{(m,n)}$,

$u_{3\|333}^{(m,n)}=u_{3,333}^{(m,n)}-\left(k_{,33}\,u_1^{(m,n)}+2k_{,3}\,u_{1,3}^{(m,n)}+k\,u_{1,33}^{(m,n)}\right)$, for $l\geq0$ and $m\geq0$.

As for straight rod (Chen et al. 2018), the term linked to bending along y_2 ($-\frac{R_0^2}{4}\mathbf{a}_3.\boldsymbol{\sigma}_{,33}^{(1,0)}.\mathbf{a}_3$ in the right hand side of (41)) may be the leading-order term when *k* and τ are sufficiently small (due to presence of term $\mathbf{a}_3.\boldsymbol{\sigma}^{(0,0)}.\left(-k\,\mathbf{a}_3+\tau\,\mathbf{a}_2\right)$).

In the same way, by taking into account of (2), (3), (29) and (36) for $k = 0$ and $n = 1$, then we multiply (38) by $\mathbf{a}_2$ and add it to the derivative (with respect to y_3) of $(35)_3$ multiplied by $\mathbf{a}_3$ to obtain one scalar equation :

$$\hat{\bar{\mathbf{T}}}_{s1,3}.\mathbf{a}_3+\frac{2}{R_0}\mathbf{a}_2.\hat{\bar{\mathbf{T}}}_0=-\mathbf{a}_2.\mathbf{f}^{(0,0)}-\boldsymbol{\tau}\,\mathbf{a}_3.\boldsymbol{\sigma}^{(0,0)}.\mathbf{a}_1+\frac{R_0^2}{4}\mathbf{a}_3.\left(\boldsymbol{\sigma}_{,3}^{(1,1)}.\mathbf{a}_1+\boldsymbol{\sigma}_{,3}^{(0,2)}.\mathbf{a}_2\right)+$$

$$\frac{R_0^2}{4}\mathbf{a}_3.\boldsymbol{\sigma}^{(1,1)}.\left(-k\,\mathbf{a}_3+2\tau\,\mathbf{a}_2\right)-\frac{R_0^2}{8}\mathbf{a}_2.\left(\mathbf{f}^{(2,0)}+\mathbf{f}^{(0,2)}\right)-\frac{R_0^2}{8}\boldsymbol{\tau}\,\mathbf{a}_3.\left(3\boldsymbol{\sigma}^{(0,2)}+\boldsymbol{\sigma}^{(2,0)}\right).\mathbf{a}_1-$$
$$\frac{R_0^2}{4}\mathbf{a}_2.\left(-\tau\,k\,\boldsymbol{\sigma}^{(0,1)}.\mathbf{a}_3+k^2\boldsymbol{\sigma}_{,3}^{(0,0)}.\,\mathbf{a}_3+k\,\boldsymbol{\sigma}_{,3}^{(1,0)}.\,\mathbf{a}_3\right)+O\left(R_0^4\right). \tag{43}$$

By deriving with respect to y_3 equation (36) for $k = 0$ and $n = 1$, we substitute a new expression for $\boldsymbol{\sigma}_{,3}^{(1,1)}.\mathbf{a}_1+\boldsymbol{\sigma}_{,3}^{(0,2)}\mathbf{a}_2$ in (43) and we obtain :

$$\hat{\hat{\mathbf{T}}}_{s1,3}.\mathbf{a}_3+\frac{2}{R_0}\mathbf{a}_2.\hat{\hat{\mathbf{T}}}_0+\mathbf{f}^{(0,0)}.\mathbf{a}_2+\frac{R_0^2}{4}\left(\mathbf{f}_{,3}^{(0,1)}.\mathbf{a}_3+\frac{1}{2}\left(\mathbf{f}^{(2,0)}+\mathbf{f}^{(0,2)}\right).\mathbf{a}_2\right)=$$
$$-\boldsymbol{\tau}\,\mathbf{a}_3.\boldsymbol{\sigma}^{(0,0)}.\mathbf{a}_1-\frac{R_0^2}{4}\mathbf{a}_3.\left(\boldsymbol{\sigma}_{,33}^{(0,1)}.\mathbf{a}_3+k\,\boldsymbol{\sigma}_{,3}^{(0,1)}.\mathbf{a}_1\right)-$$
$$\frac{R_0^2}{4}\mathbf{a}_2.\left(-\tau\,k\,\boldsymbol{\sigma}^{(0,1)}.\mathbf{a}_3+k^2\boldsymbol{\sigma}_{,3}^{(0,0)}.\,\mathbf{a}_3+k\,\boldsymbol{\sigma}_{,3}^{(1,0)}.\,\mathbf{a}_3\right)-$$
$$\frac{R_0^2}{8}\boldsymbol{\tau}\,\mathbf{a}_3.\left(\boldsymbol{\sigma}^{(0,2)}+\boldsymbol{\sigma}^{(2,0)}\right).\mathbf{a}_1+\tau\frac{R_0^2}{4}\mathbf{a}_2.\boldsymbol{\sigma}^{(1,1)}.\mathbf{a}_3+O\left(R_0^4\right). \tag{44}$$

As for straight rod (Chen et al. 2018), the term linked to bending along y_1 ($-\mathbf{a}_3.\boldsymbol{\sigma}_{,33}^{(0,1)}.\mathbf{a}_3\,R_0^2/4$ in the right hand side of (44)) may be the leading order term when the torsion curvature τ can be small ($-\boldsymbol{\tau}\,\mathbf{a}_3.\boldsymbol{\sigma}^{(0,0)}.\mathbf{a}_1$ in the right hand side of (44)) .

We show that we can simplify equations (41) and (44). Firstly, in formula (41), we observe that $k\,\mathbf{a}_1.\boldsymbol{\sigma}_{,3}^{(1,0)}.\mathbf{a}_3\,R_0^2/4$ and $k\,\mathbf{a}_3.\boldsymbol{\sigma}_{,33}^{(0,0)}.\mathbf{a}_3\,R_0^2/4$ (respectively $\tau\,\mathbf{a}_3.\boldsymbol{\sigma}_{,3}^{(0,1)}.\,\mathbf{a}_3\,R_0^2/4$) are (respectively is) smaller than $k\,\mathbf{a}_3.\boldsymbol{\sigma}^{(0,0)}.\mathbf{a}_3$ (respectively $\tau\mathbf{a}_3.\boldsymbol{\sigma}^{(0,0)}.\mathbf{a}_2$) and therefore these terms can be neglected.

Secondly, by neglecting the last lines of the right hand side of formulae (41) and (44) (for consistency similar terms will be neglected below and lead to formulas (48)-(50)) and by dividing by $R_0^2/4$, we obtain:

$$\frac{4}{R_0^2}\hat{\hat{\mathbf{T}}}_{c1,3}.\mathbf{a}_3+\frac{8}{R_0^3}\hat{\hat{\mathbf{T}}}_0.\mathbf{a}_1+\frac{4}{R_0^2}\mathbf{f}^{(0,0)}.\mathbf{a}_1+\mathbf{f}_{,3}^{(1,0)}.\mathbf{a}_3+\frac{1}{2}\left(\mathbf{f}^{(2,0)}+\mathbf{f}^{(0,2)}\right).\mathbf{a}_1=$$
$$\frac{4}{R_0^2}\mathbf{a}_3.\boldsymbol{\sigma}^{(0,0)}.\left(-k\,\mathbf{a}_3+\tau\,\mathbf{a}_2\right)-\mathbf{a}_1.\left(-2\tau\,k\,\boldsymbol{\sigma}^{(0,1)}+2k^2\boldsymbol{\sigma}_{,3}^{(0,0)}\right).\mathbf{a}_3+$$
$$\mathbf{a}_3.\left(\tau_{,3}\boldsymbol{\sigma}^{(0,1)}-k_{,3}\boldsymbol{\sigma}_{,3}^{(0,0)}-\boldsymbol{\sigma}_{,33}^{(1,0)}\right).\mathbf{a}_3+O\left(R_0^2\right), \tag{45}$$

$$\frac{4}{R_0^2}\hat{\hat{\mathbf{T}}}_{s1,3}.\mathbf{a}_3+\frac{8}{R_0^3}\mathbf{a}_2.\hat{\hat{\mathbf{T}}}_0+\mathbf{f}^{(0,0)}.\mathbf{a}_2+\mathbf{f}_{,3}^{(0,1)}.\mathbf{a}_3+\frac{1}{2}\left(\mathbf{f}^{(2,0)}+\mathbf{f}^{(0,2)}\right).\mathbf{a}_2=$$
$$-\frac{4}{R_0^2}\tau\,\mathbf{a}_3.\boldsymbol{\sigma}^{(0,0)}.\mathbf{a}_1-\mathbf{a}_3.\left(\boldsymbol{\sigma}_{,33}^{(0,1)}.\mathbf{a}_3+k\,\boldsymbol{\sigma}_{,3}^{(0,1)}.\mathbf{a}_1\right)-$$
$$\mathbf{a}_2.\left(-\tau\,k\,\boldsymbol{\sigma}^{(0,1)}.\mathbf{a}_3+k^2\boldsymbol{\sigma}_{,3}^{(0,0)}.\,\mathbf{a}_3+k\,\boldsymbol{\sigma}_{,3}^{(1,0)}.\,\mathbf{a}_3\right)+O\left(R_0^2\right). \tag{46}$$

By taking into account of (29) and (36) for (k , n) = (0, 1) and (1, 1) then we multiply $(35)_2$ by $\mathbf{a}_2$

and substract $(35)_3$ multiplied by $\mathbf{a}_1$, finally we divide by $R_0^2/4$ and obtain the scalar equation:

$$\frac{4}{R_0^2}\left(\hat{\hat{\mathbf{T}}}_{c1}.\mathbf{a}_2-\hat{\hat{\mathbf{T}}}_{s1}.\mathbf{a}_1\right)=-\mathbf{a}_2.\left(-\tau\,\boldsymbol{\sigma}^{(0,1)}.\mathbf{a}_3+k\boldsymbol{\sigma}_{,3}^{(0,0)}.\mathbf{a}_3+\boldsymbol{\sigma}_{,3}^{(1,0)}.\mathbf{a}_3+\mathbf{f}^{(1,0)}\right)+$$
$$\mathbf{a}_1.\left(\boldsymbol{\sigma}_{,3}^{(0,1)}.\mathbf{a}_3+\mathbf{f}^{(0,1)}\right)+O\left(R_0^2\right). \tag{47}$$

We observe that in (47), the torsion term $\left(-\mathbf{a}_2.\left(-\tau\,\boldsymbol{\sigma}^{(0,1)}+\boldsymbol{\sigma}_{,3}^{(1,0)}\right)+\mathbf{a}_1.\boldsymbol{\sigma}_{,3}^{(0,1)}\right).\mathbf{a}_3$ is the part of the leading order term.

By considering (39) and dropping the error $O\left(R_0^2\right)$, we see that (45)-(47) forms three scalar vector equations with nine unknows which are the components of the displacement field vectors $\mathbf{u}^{(0,0)},\mathbf{u}^{(1,0)}$ and $\mathbf{u}^{(0,1)}$. So it remains to find six scalar equations.

Equation (38) projected on $\mathbf{a}_3$ has not yet been used so we keep only the relative $O\left(R_0^2\right)$terms into the remainder to obtain the scalar equation :

$$\left(\frac{2}{R_0}\hat{\hat{\mathbf{T}}}_0+\mathbf{f}^{(0,0)}\right).\mathbf{a}_3=-\mathbf{a}_3.\boldsymbol{\sigma}_{,3}^{(0,0)}.\mathbf{a}_3+O\left(R_0^2\right). \tag{48}$$

Now we put into the remainders the $O\left(R_0^2\right)$ terms for equations $(35)_2$ multiplied by $\mathbf{a}_1$ and $\mathbf{a}_3$ respectively and also for equations $(35)_3$ multiplied by $\mathbf{a}_2$ and $\mathbf{a}_3$ respectively :

$$\hat{\hat{\mathbf{T}}}_{c1}.\mathbf{a}_1=\mathbf{a}_1.\boldsymbol{\sigma}^{(0,0)}.\mathbf{a}_1+O\left(R_0^2\right),\qquad \hat{\hat{\mathbf{T}}}_{c1}.\mathbf{a}_3=\mathbf{a}_3.\boldsymbol{\sigma}^{(0,0)}.\mathbf{a}_1+O\left(R_0^2\right), \tag{49}$$

$$\hat{\hat{\mathbf{T}}}_{s1}.\mathbf{a}_2=\mathbf{a}_2.\boldsymbol{\sigma}^{(0,0)}.\mathbf{a}_2+O\left(R_0^2\right),\qquad \hat{\hat{\mathbf{T}}}_{s1}.\mathbf{a}_3=\mathbf{a}_3.\boldsymbol{\sigma}^{(0,0)}.\mathbf{a}_2+O\left(R_0^2\right). \tag{50}$$

Finally, we add $(35)_2$ multiplied by $\mathbf{a}_2$ and substract $(35)_3$ multiplied by $\mathbf{a}_1$ then we take into account of (29) and (36) for $(k\,,\,\mathrm{n}) = (0,\,1)$ and $(1,\,1)$ and finally we put into the remainders the $O\left(R_0^2\right)$ terms to obtain the scalar equation :

$$\hat{\hat{\mathbf{T}}}_{c1}.\mathbf{a}_2+\hat{\hat{\mathbf{T}}}_{s1}.\mathbf{a}_1=2\,\mathbf{a}_1.\boldsymbol{\sigma}^{(0,0)}.\mathbf{a}_2+O\left(R_0^2\right). \tag{51}$$

From Ciarlet (1998), it is easy to see that the asymmetric part of the displacement gradient is $\boldsymbol{\omega}=\frac{1}{2}\left(u_{k,l}-u_{l,k}\right)\mathbf{g}^k\otimes\mathbf{g}^l$ and then becomes on the *mean line* $\tilde{\Gamma}_c$ $(\mathbf{y}'=0)$:

$$\boldsymbol{\omega}=\frac{1}{2}\left(u_{k,l}-u_{l,k}\right)\mathbf{a}_k\otimes\mathbf{a}_l. \tag{52}$$

By taking into account of (2)-(3), equation (52) shows that the twist angle at $(\mathbf{y}'=0)$ is :

$$\theta=\frac{1}{2}\left(u_2^{(1,0)}-u_1^{(0,1)}\right). \tag{53}$$

By inserting the rod constitutive relation (32) for $(k\,,\,\mathrm{n}) = (0,\,0)$ into (49)-(51) and by considering (53), we see that we have six linear algebraic equations from which the six components of $\mathbf{u}^{(1,0)}$ and $\mathbf{u}^{(0,1)}$ can be expressed in terms of $\mathbf{u}^{(0,0)}$ and θ:

$$\begin{bmatrix} \mathbf{u}^{(1,0)} \\ \mathbf{u}^{(0,1)} \end{bmatrix} = \mathbf{G}^{-1}\left\{ -\begin{bmatrix} \mathbf{c}^{(1,1)T} \\ \mathbf{c}^{(1,2)T} \\ \mathbf{c}^{(1,3)T} \\ \mathbf{c}^{(2,2)T} \\ \mathbf{c}^{(2,3)T} \\ 0^T \end{bmatrix} \mathbf{u}_{|3}^{(0,0)} + \begin{bmatrix} \hat{\hat{\mathbf{T}}}_{c1}.\mathbf{a}_1 \\ \frac{1}{2}\left(\hat{\hat{\mathbf{T}}}_{c1}.\mathbf{a}_2 + \hat{\hat{\mathbf{T}}}_{s1}.\mathbf{a}_1\right) \\ \hat{\hat{\mathbf{T}}}_{c1}.\mathbf{a}_3 \\ \hat{\hat{\mathbf{T}}}_{s1}.\mathbf{a}_2 \\ \hat{\hat{\mathbf{T}}}_{s1}.\mathbf{a}_3 \\ 2\theta_a \end{bmatrix} + O\left(R_0^2\right) \right\}, \tag{54}$$

in which $\mathbf{G} = \begin{pmatrix} \mathbf{a}^{(1,1)} & \mathbf{a}^{(1,2)} & \mathbf{a}^{(1,3)} & \mathbf{a}^{(2,2)} & \mathbf{a}^{(2,3)} & \mathbf{a}_2 \\ \mathbf{b}^{(1,1)} & \mathbf{b}^{(1,2)} & \mathbf{b}^{(1,3)} & \mathbf{b}^{(2,2)} & \mathbf{b}^{(2,3)} & -\mathbf{a}_1 \end{pmatrix}^T$,

with $\mathbf{a}^{(i,j)} = E_{ijo1}\mathbf{a}_0$, $\mathbf{b}^{(i,j)} = E_{ijo2}\mathbf{a}_o$, $\mathbf{c}^{(i,j)} = E_{ijo3}\mathbf{a}_o$.

By substituting the rod constitutive relation (32) for (k, n) = (0, 0), (0, 1), (1, 1) (with the use of recursive relations (39) and (54)) into equation (45)-(48) after dropping the $O\left(R_0^2\right)$, we obtain four scalar rod equations for the leading order coefficient $\mathbf{u}^{(0,0)}$ (the displacement vector on the *mean line* $\tilde{\Gamma}_c$) and the twist angle θ

$$\left(\frac{2}{R_0}\hat{\hat{\mathbf{T}}}_0 + \mathbf{f}^{(0,0)}\right).\mathbf{a}_3 = -\mathbf{a}_3.\boldsymbol{\sigma}_{,3}^{(0,0)}.\mathbf{a}_3, \tag{55}$$

$$\begin{aligned} &\frac{4}{R_0^2}\hat{\hat{\mathbf{T}}}_{c1,3}.\mathbf{a}_3 + \frac{8}{R_0^3}\hat{\hat{\mathbf{T}}}_0.\mathbf{a}_1 + \frac{4}{R_0^2}\mathbf{f}^{(0,0)}.\mathbf{a}_1 + \mathbf{f}_{,3}^{(1,0)}.\mathbf{a}_3 + \frac{1}{2}\left(\mathbf{f}^{(2,0)} + \mathbf{f}^{(0,2)}\right).\mathbf{a}_1 = \\ &\qquad \frac{4}{R_0^2}\mathbf{a}_3.\boldsymbol{\sigma}^{(0,0)}.\left(-k\,\mathbf{a}_3 + \tau\,\mathbf{a}_2\right) - \mathbf{a}_1.\left(-2\tau\,k\,\boldsymbol{\sigma}^{(0,1)} + 2k^2\boldsymbol{\sigma}_{,3}^{(0,0)}\right).\mathbf{a}_3 + \\ &\qquad \mathbf{a}_3.\left(\tau_{,3}\boldsymbol{\sigma}^{(0,1)} - k_{,3}\boldsymbol{\sigma}_{,3}^{(0,0)} - \boldsymbol{\sigma}_{,33}^{(1,0)}\right).\mathbf{a}_3, \end{aligned} \tag{56}$$

$$\begin{aligned} &\frac{4}{R_0^2}\hat{\hat{\mathbf{T}}}_{s1,3}.\mathbf{a}_3 + \frac{8}{R_0^3}\mathbf{a}_2.\hat{\hat{\mathbf{T}}}_0 + \mathbf{f}^{(0,0)}.\mathbf{a}_2 + \mathbf{f}_{,3}^{(0,1)}.\mathbf{a}_3 + \frac{1}{2}\left(\mathbf{f}^{(2,0)} + \mathbf{f}^{(0,2)}\right).\mathbf{a}_2 = \\ &\qquad -\frac{4}{R_0^2}\tau\,\mathbf{a}_3.\boldsymbol{\sigma}^{(0,0)}.\mathbf{a}_1 - \mathbf{a}_3.\left(\boldsymbol{\sigma}_{,33}^{(0,1)}.\mathbf{a}_3 + k\,\boldsymbol{\sigma}_{,3}^{(0,1)}.\mathbf{a}_1\right) - \\ &\qquad \mathbf{a}_2.\left(-\tau\,k\,\boldsymbol{\sigma}^{(0,1)}.\mathbf{a}_3 + k^2\boldsymbol{\sigma}_{,3}^{(0,0)}.\mathbf{a}_3 + k\,\boldsymbol{\sigma}_{,3}^{(1,0)}.\mathbf{a}_3\right), \end{aligned} \tag{57}$$

$$\frac{4}{R_0^2}\left(\hat{\hat{\mathbf{T}}}_{c1}.\mathbf{a}_2 - \hat{\hat{\mathbf{T}}}_{s1}.\mathbf{a}_1\right) + \mathbf{f}^{(1,0)}.\mathbf{a}_2 - \mathbf{f}^{(0,1)}.\mathbf{a}_1 = \left(\mathbf{a}_2.\left(\tau\,\boldsymbol{\sigma}^{(0,1)} - k\boldsymbol{\sigma}_{,3}^{(0,0)} - \boldsymbol{\sigma}_{,3}^{(1,0)}\right) + \mathbf{a}_1.\boldsymbol{\sigma}_{,3}^{(0,1)}\right).\mathbf{a}_3. \tag{58}$$

For the ease of variational formulation of (55)-(58), we rewrite them by using formula of product of derivative and formulae (2)-(4) (for (60), we use the fact that $k\,\mathbf{a}_3.\boldsymbol{\sigma}_{,33}^{(0,0)}.\mathbf{a}_3$ and $k\,\mathbf{a}_1.\boldsymbol{\sigma}_{,3}^{(1,0)}.\mathbf{a}_3$ are smaller than $\frac{4k}{R_0^2}\boldsymbol{\sigma}_{33}^{(0,0)}$ and can be neglected):

$$\left(\frac{2}{R_0}\hat{\hat{\mathbf{T}}}_0+\mathbf{f}^{(0,0)}\right).\mathbf{a}_3-2k\sigma_{13}^{(0,0)}=-\sigma_{33,3}^{(0,0)}, \tag{59}$$

$$\frac{4}{R_0^2}\left(\left(\hat{\hat{\mathbf{T}}}_{c1}\right)_{3,3}-k\left(\hat{\hat{\mathbf{T}}}_{c1}\right)_1\right)+\frac{8}{R_0^3}\left(\hat{\hat{\mathbf{T}}}_0\right)_1+\frac{4}{R_0^2}f_1^{(0,0)}+f_{3,3}^{(1,0)}-kf_1^{(1,0)}+\frac{1}{2}\left(f_1^{(2,0)}+f_1^{(0,2)}\right)=$$

$$\frac{4}{R_0^2}\mathbf{a}_3.\boldsymbol{\sigma}^{(0,0)}.\left(-k\mathbf{a}_3+\tau\mathbf{a}_2\right)-\left(k\ \mathbf{a}_3.\boldsymbol{\sigma}_{,3}^{(0,0)}.\mathbf{a}_3\right)_{,3}+\left(\tau\ \sigma_{33}^{(0,1)}\right)_{,3}-\left(\mathbf{a}_3.\boldsymbol{\sigma}_{,3}^{(1,0)}.\mathbf{a}_3\right)_{,3} \tag{60}$$

$$\frac{4}{R_0^2}\left(\left(\hat{\hat{\mathbf{T}}}_{s1}\right)_{3,3}-k\left(\hat{\hat{\mathbf{T}}}_{s1}\right)_1\right)+\frac{8}{R_0^3}\left(\hat{\hat{\mathbf{T}}}_0\right)_2+f_2^{(0,0)}+f_{3,3}^{(0,1)}-kf_1^{(0,1)}+\frac{1}{2}\left(f_2^{(2,0)}+f_2^{(0,2)}\right)=$$

$$-\frac{4}{R_0^2}\tau\sigma_{13}^{(0,0)}-\left(\mathbf{a}_3.\boldsymbol{\sigma}_{,3}^{(0,1)}.\mathbf{a}_3\right)_{,3}+k\mathbf{a}_3.\boldsymbol{\sigma}_{,3}^{(0,1)}.\mathbf{a}_1$$

$$-\tau k\sigma_{23}^{(0,1)}+k^2\mathbf{a}_2.\boldsymbol{\sigma}_{,3}^{(0,0)}.\mathbf{a}_3+k\mathbf{a}_2.\boldsymbol{\sigma}_{,3}^{(1,0)}.\mathbf{a}_3, \tag{61}$$

$$\frac{4}{R_0^2}\left(\left(\hat{\hat{\mathbf{T}}}_{c1}\right)_2-\left(\hat{\hat{\mathbf{T}}}_{s1}\right)_1\right)+f_2^{(1,0)}-f_1^{(0,1)}=\tau\left(\sigma_{23}^{(0,1)}-\sigma_{13}^{(1,0)}-\sigma_{23}^{(0,1)}\right)-k\mathbf{a}_2.\boldsymbol{\sigma}_{,3}^{(0,0)}.\mathbf{a}_3-$$

$$\left(\mathbf{a}_2.\boldsymbol{\sigma}^{(1,0)}.\mathbf{a}_3\right)_{,3}+\left(\mathbf{a}_1.\boldsymbol{\sigma}^{(0,1)}.\mathbf{a}_3\right)_{,3}+k\left(\sigma_{12}^{(1,0)}-\sigma_{11}^{(0,1)}+\sigma_{13}^{(0,1)}\right). \tag{62}$$

The above system consists of two second order equations for $u_3^{(0,0)}$ and θ ((59), (62)) and two fourth-order equations for $u_1^{(0,0)}, u_2^{(0,0)}$((60)-(61)). Then at each rod edge six boundary conditions are needed, which will be addressed in the next section.

7. Boundary conditions and unidimensional rod virtual work principle

Now, we shall derive the rod boundary conditions and unidimensional rod virtual work principle. By using Green's theorem and strong formulation (15)-(18), we obtain the tridimensional virtual work :

$$-\int_{\tilde{\mathbf{B}}}\mathrm{Div}\tilde{\boldsymbol{\sigma}}.\tilde{\mathbf{v}}^*\,d\tilde{\mathbf{B}}-\int_{\tilde{\mathbf{B}}}\tilde{\mathbf{f}}.\tilde{\mathbf{v}}^*\,d\tilde{\mathbf{B}}+\int_{\tilde{\Gamma}_{lat}}\left(\tilde{\boldsymbol{\sigma}}.\tilde{\mathbf{n}}_{lat}-\tilde{\tilde{\mathbf{T}}}\right).\tilde{\mathbf{v}}^*\,d\tilde{\Gamma}_{lat}+\int_{\tilde{\Sigma}_L}\left(\tilde{\boldsymbol{\sigma}}.\tilde{\mathbf{n}}-\tilde{\tilde{\mathbf{t}}}\right).\tilde{\mathbf{v}}^*\,d\tilde{\Sigma}_L=0, \tag{63}$$

in which the test function $\tilde{\mathbf{v}}^*$ vanishes on $\tilde{\Sigma}_0$.

In order to write variational formulation (63), we define the element surface of the lateral boundary of the rod $\tilde{\Gamma}_{lat}$. Let s_{lat} measure the arc length of the boundary (denoted by $\partial\tilde{\Sigma}$) of a cross section $\tilde{\Sigma}$ (for $s\in[0,L]$), and suppose that the unit tangent $\boldsymbol{\tau}_{lat}$ to $\partial\tilde{\Sigma}$ is given by:

$$\boldsymbol{\tau}_{lat}=\tau_{lat\,1}\mathbf{a}_1+\tau_{lat\,2}\mathbf{a}_2. \tag{64}$$

We assume that the cross section of the rod is uniform, and then its lateral boundary is a collection of curve $\tilde{\mathbf{C}}$ defined by position vector points $\mathbf{X}$ given by (5) such that $s\in[0,L]$ and $\mathbf{y}'$ is constant. Consequently, the oriented differential surface area induced by the parametrization (s_{lat},s) of $\tilde{\Gamma}_{lat}$ is $\tilde{\mathbf{n}}_{lat}\,d\tilde{\Gamma}_{lat}=ds_{lat}\,\boldsymbol{\tau}_{lat}\wedge ds\,\mathbf{g}_3$. Then by considering equations (8) and (64), we

obtain:

$$d\widetilde{\Gamma}_{lat} = \left\|\boldsymbol{\tau}_{lat} \wedge \mathbf{g}_3\right\| ds\, ds_{lat} = \left\|\boldsymbol{\tau}_{lat} \wedge \mathbf{g}_3\right\| d\Gamma_{lat}, \tag{65}$$

where $\left\|\boldsymbol{\tau}_{lat}(\mathbf{y}') \wedge \mathbf{g}_3(\mathbf{y})\right\| = \sqrt{1 + 2C_{lat\,1}(y_1, y_3) + C_{lat\,1}(y_1, y_3)^2 + C_{lat\,2}(\mathbf{y})}$,

with $C_{lat\,1}(y_1, y_3) = -k(y_3) y_1$, $C_{lat\,2}(\mathbf{y}) = \tau(y_3)^2 \left(\boldsymbol{\tau}_{lat\,1}(\mathbf{y}') y_1 + \boldsymbol{\tau}_{lat\,2}(\mathbf{y}') y_2\right)^2$.

By inserting formula (13) and (65) into variational formulation (64), we obtain :

$$-\int_{\mathbf{B}} (\operatorname{div}\boldsymbol{\sigma} + \mathbf{f}).\mathbf{w}^* \sqrt{g}\, d\mathbf{y} + \int_{\Gamma_{lat}} (\boldsymbol{\sigma}.\mathbf{n}_{lat} - \overline{\mathbf{T}}).\mathbf{w}^* \left\|\boldsymbol{\tau}_{lat} \wedge \mathbf{g}_3\right\| d\Gamma_{lat} + \int_{\Sigma_L} (\boldsymbol{\sigma}.\mathbf{a}_3 - \overline{\mathbf{t}}).\mathbf{w}^*\, d\mathbf{y}' = 0, \tag{66}$$

in which the test function $\mathbf{w}^*(\mathbf{y}) = \tilde{\mathbf{v}}^*(\mathbf{X}(\mathbf{y}))$ is such that $\mathbf{w}^*(\mathbf{y}) = 0$ on Σ_0 ($y_3 = 0$).

To obtain the rod boundary conditions from (66), we need to specialize $\boldsymbol{\sigma}$ and $\mathbf{v}$ to the one dimensional case by using Taylor expansions for them. Due to equilibrium equation (36), the first term of (66) is automatically satisfied up to $O(R_0^5)$. Due to (35), the second term of (66) is automatically satisfied up to $O(R_0^4)$. Then only the last edge integral is left and boundary conditions will be derived from it.

If we take $\mathbf{v}$ to be arbitrary, from (66) we deduce the tridimensional boundary condition on Σ_L which cannot be satisfied for a rod theory obviously. Therefore, we need to specialize the test displacement field $\mathbf{w}^*$ at $y_3 = L$ and the criterion is that the applied force and torque should generate the virtual work. According to this criterion, we take:

$$\mathbf{w}^* = v_i^* \mathbf{a}_i - \left(y_1 v_{1,3}^* + y_2 v_{2,3}^*\right)\mathbf{a}_3 + \left(-y_2 \mathbf{a}_1 + y_1 \mathbf{a}_2\right)\boldsymbol{\theta}^*. \tag{67}$$

Then expansion of the stress tensor (31) and the specially chosen test displacement $\mathbf{w}^*$ are taken into the third integral of (66), as this term vanishes, we have :

$$L_k v_k^* - \frac{\boldsymbol{\pi} R_0^4}{4}\left(\boldsymbol{\sigma}_{33}^{(1,0)} v_{1,3}^* + \boldsymbol{\sigma}_{33}^{(0,1)} v_{2,3}^*\right) + \frac{\boldsymbol{\pi} R_0^4}{4}\left(\boldsymbol{\sigma}_{32}^{(1,0)} - \boldsymbol{\sigma}_{31}^{(0,1)}(\mathbf{a}_1)_j\right)\boldsymbol{\theta}^* =$$

$$\hat{t}_i v_i^* + \hat{m}_2 v_{1,3}^* - \hat{m}_1 v_{2,3}^* + \hat{m}_3 \boldsymbol{\theta}^* + O(R_0^6). \tag{68}$$

where

$$L_{\boldsymbol{\alpha}} = \boldsymbol{\pi} R_0^2 \left(\boldsymbol{\sigma}_{\boldsymbol{\alpha}3}^{(0,0)} + \frac{R_0^2\left(\boldsymbol{\sigma}_{\boldsymbol{\alpha}3}^{(2,0)} + \boldsymbol{\sigma}_{\boldsymbol{\alpha}3}^{(0,2)}\right)}{8}\right), L_3 = \boldsymbol{\pi} R_0^2 \boldsymbol{\sigma}_{33}^{(0,0)},$$

$$\hat{t}_i = \int_{\Sigma_L} \overline{t}_i\, d\mathbf{y}', \overline{t}_i = \overline{\mathbf{t}}.\mathbf{a}_i, \hat{m}_1 = \int_{\Sigma_L} y_2 \overline{t}_3\, d\mathbf{y}', \hat{m}_2 = -\int_{\Sigma_L} y_1 \overline{t}_3\, d\mathbf{y}', \hat{m}_3 = \int_{\Sigma_L} \left(y_1 \overline{t}_2 - y_2 \overline{t}_1\right) d\mathbf{y}'.$$

For L_3, we neglect $\boldsymbol{\pi} \dfrac{R_0^4\left(\boldsymbol{\sigma}_{33}^{(2,0)} + \boldsymbol{\sigma}_{33}^{(0,2)}\right)}{8}$ as already done for establishing equation (48) .

Now we show that L_k in (68) can be simplified. Firstly for L_1, from $(35)_2$, we see that :

$$\mathbf{a}_3.\boldsymbol{\sigma}^{(0,0)}.\mathbf{a}_1 = \hat{\overline{\mathbf{T}}}_{c1}.\mathbf{a}_3 - \frac{R_0^2}{4}\mathbf{a}_3.\boldsymbol{\sigma}^{(1,1)}.\mathbf{a}_2 - \frac{R_0^2}{8}\mathbf{a}_3.\left(\boldsymbol{\sigma}^{(0,2)} + 3\boldsymbol{\sigma}^{(2,0)}\right).\mathbf{a}_1 + O(R_0^4). \tag{69}$$

Then applying relation (69) to (68) to replace $\mathbf{a}_3.\boldsymbol{\sigma}^{(0,0)}.\mathbf{a}_1$, together with the use of the equilibrium equation (36) for (*k, n*) = (1,1) leads to :

$$L_1 = \pi R_0^2 \hat{\hat{\mathbf{T}}}_{c1}.\mathbf{a}_3 + \frac{\pi R_0^4}{4}\left(\mathbf{a}_3.\left(\boldsymbol{\sigma}_{,3}^{(1,0)} + k\boldsymbol{\sigma}_{,3}^{(0,0)} - \tau\,\boldsymbol{\sigma}^{(0,1)}\right).\mathbf{a}_3 + \mathbf{f}^{(1,0)}.\mathbf{a}_3\right). \tag{70}$$

Secondly for L_2, from $(35)_3$, we see that :

$$\mathbf{a}_3.\boldsymbol{\sigma}^{(0,0)}.\mathbf{a}_2 = \hat{\hat{\mathbf{T}}}_{s1}.\mathbf{a}_3 - \frac{R_0^2}{4}\mathbf{a}_3.\boldsymbol{\sigma}^{(1,1)}.\mathbf{a}_1 - \frac{R_0^2}{8}\mathbf{a}_3.\left(3\boldsymbol{\sigma}^{(0,2)} + \boldsymbol{\sigma}^{(2,0)}\right).\mathbf{a}_2 + O\left(R_0^4\right). \tag{71}$$

Then applying relation (71) to (68) to replace $\mathbf{a}_3.\boldsymbol{\sigma}^{(0,0)}.\mathbf{a}_2$, together with the use of the equilibrium equation (36) for $(k, n) = (0,1)$ leads to:

$$L_2 = \pi R_0^2 \hat{\hat{\mathbf{T}}}_{s1}.\mathbf{a}_3 + \frac{\pi R_0^4}{4}\left(\mathbf{a}_3.\boldsymbol{\sigma}_{,3}^{(0,1)} + \mathbf{f}^{(0,1)}\right).\mathbf{a}_3. \tag{72}$$

Finally, by taking L_1 and L_2 back to (68) and equating the coefficients of $\mathbf{v}^*$, $v_{1,3}^*$, $v_{2,3}^*$ and θ^* on both sides of (68), we get the rod boundary conditions at the traction edge at $y_3 = L$:

$$\mathbf{L} = \hat{\mathbf{t}} \tag{73}$$

$$\frac{\pi R_0^4}{4}\mathbf{a}_3.\boldsymbol{\sigma}^{(0,1)}.\mathbf{a}_3 = \hat{m}_1, \tag{74}$$

$$-\frac{\pi R_0^4}{4}\mathbf{a}_3.\boldsymbol{\sigma}^{(1,0)}.\mathbf{a}_3 = \hat{m}_2, \tag{75}$$

$$\frac{\pi R_0^4}{4}\mathbf{a}_3.\left(\boldsymbol{\sigma}^{(1,0)}.\mathbf{a}_2 - \boldsymbol{\sigma}^{(0,1)}.\mathbf{a}_1\right) = \hat{m}_3, \tag{76}$$

where the vector force resultant $\hat{\mathbf{t}}$ and the torque resultant $\hat{\mathbf{m}} = \hat{m}_i\,\mathbf{a}_i$ are supposed to be prescribed.

In the variational formulation (68), we see that the virtual vectors $\mathbf{v}^{*(0,0)}$ and the virtual scalar functions $v_{1,3}^{*(0,0)}, v_{2,3}^{*(0,0)}$ and θ^* vanish at $y_3 = 0$ (which is center of gravity of Σ_0). Thus the boundary conditions at $y_3 = 0$, are :

$$\mathbf{u}^{(0,0)} = \overline{\mathbf{u}}^{(0,0)},\ u_{1,3}^{(0,0)} = \overline{\alpha}_1, u_{2,3}^{(0,0)} = \overline{\alpha}_2, \theta = \overline{\theta}. \tag{77}$$

As a consequence, the vector displacement $\overline{\mathbf{u}}^{(0,0)}$, the two rotation angles $\overline{\alpha}_1$, $\overline{\alpha}_2$ and the twist angle $\overline{\theta}$ are prescribed values for respectively the vector displacement $\mathbf{u}^{(0,0)}$ and the scalar functions $u_{1,3}^{(0,0)}$, $u_{2,3}^{(0,0)}$ and θ.

We derive the unidimensional rod virtual work principle. Firstly, we multiply each components of the virtual test vectors v_i^* (for i = 3, 1, 2) and the virtual twist angle $\boldsymbol{\theta}^*$ to equations (59)-(62) respectively and we integrate each side of these equalities between 0 and L. Secondly, we integrate by parts and we take into account of Neumann type boundary conditions (73)-(76) at $y_3 = L$ (the virtual scalar functions $\mathbf{v}^*$, $v_{1,3}^*$, $v_{2,3}^*$ and θ^* vanish at $y_3 = 0$):

$$\pi R_0^2\left(\int_0^L \sigma_{33}^{(0,0)} v_{3,3}^*\, dy_3 + 2\int_0^L k\,\sigma_{13}^{(0,0)} v_3^*\, dy_3\right) = \pi R_0^2 \int_0^L \left(\frac{2}{R_0}\left(\hat{\hat{\mathbf{T}}}_0\right)_3 + f_3^{(0,0)}\right) v_3^*\, dy_3 + \hat{t}_3 v_{3|_{y_3=L}}^*,$$

$$\frac{\pi R_0^4}{4}\int_0^L A\,v_1^* + B\,v_{1,3}^* - \sigma_{33}^{(1,0)} v_{1,33}^*\,dy_3 = \hat{t}_1 v_{1|y_3=L}^* + \hat{m}_2 v_{1,3|y_3=L}^*,$$

$$\frac{\pi R_0^4}{4}\int_0^L C\,v_2^* + D\,v_{2,3}^* - \sigma_{33}^{(0,1)} v_{2,33}^*\,dy_3 = \hat{t}_2 v_{2|y_3=L}^* - \hat{m}_1 v_{2,3|y_3=L}^*, \tag{78}$$

$$\frac{\pi R_0^4}{4}\int_0^L E\,\theta^* + \left(\sigma_{23}^{(1,0)} - \sigma_{13}^{(0,1)}\right)\theta_{,3}^*\,dy_3 = \hat{m}_3\,\theta^*_{|y_3=L},$$

with $A = \frac{4k}{R_0^2}\left(\hat{\hat{\mathbf{T}}}_{c1}\right)_1 - \frac{4}{R_0^2} f_1^{(0,0)} + k f_1^{(1,0)} - \frac{1}{2}\left(f_1^{(2,0)} + f_1^{(0,2)}\right) - \frac{8}{R_0^3}\left(\hat{\hat{\mathbf{T}}}_0\right)_1 - \frac{4k}{R_0^2}\sigma_{33}^{(0,0)} + \frac{4\tau}{R_0^2}\sigma_{23}^{(0,0)},$

$B = -2k\sigma_{13}^{(1,0)} - \tau\,\sigma_{33}^{(0,1)} + k\;\mathbf{a}_3.\boldsymbol{\sigma}_{,3}^{(0,0)}.\mathbf{a}_3 + \frac{4}{R_0^2}\left(\hat{\hat{\mathbf{T}}}_{c1}\right)_3 + f_3^{(1,0)},$

$C = \frac{4k}{R_0^2}\left(\hat{\hat{\mathbf{T}}}_{s1}\right)_1 - \frac{8}{R_0^3}\left(\hat{\hat{\mathbf{T}}}_0\right)_2 - f_2^{(0,0)} - \frac{1}{2}\left(f_2^{(2,0)} + f_2^{(0,2)}\right) + k f_1^{(0,1)} - \frac{4}{R_0^2}\tau\sigma_{13}^{(0,0)} + k\,\mathbf{a}_3.\boldsymbol{\sigma}_{,3}^{(0,1)}.\mathbf{a}_1 -$

$\tau k\sigma_{23}^{(0,1)} + k^2\mathbf{a}_2.\boldsymbol{\sigma}_{,3}^{(0,0)}.\mathbf{a}_3 + k\,\mathbf{a}_2.\boldsymbol{\sigma}_{,3}^{(1,0)}.\mathbf{a}_3,$

$D = \frac{4}{R_0^2}\left(\hat{\hat{\mathbf{T}}}_{s1}\right)_3 + f_3^{(0,1)} - 2k\sigma_{13}^{(0,1)}$ and

$E = \frac{4}{R_0^2}\left(\left(\hat{\hat{\mathbf{T}}}_{s1}\right)_1 - \left(\hat{\hat{\mathbf{T}}}_{c1}\right)_2\right) + f_1^{(0,1)} - f_2^{(1,0)} + \tau\left(\sigma_{23}^{(0,1)} - \sigma_{13}^{(1,0)} - \sigma_{23}^{(0,1)}\right) - k\,\mathbf{a}_2.\boldsymbol{\sigma}_{,3}^{(0,0)}.\mathbf{a}_3 -$

$k\,\mathbf{a}_2.\boldsymbol{\sigma}_{,3}^{(0,0)}.\mathbf{a}_3 + k\left(\sigma_{12}^{(1,0)} - \sigma_{11}^{(0,1)} + \sigma_{13}^{(0,1)}\right).$

It is easy to see that all the edge terms on the right hand side represent the virtual work done by the force resultant and torque resultant. This weak formulation provides a framework for implementing finite-element schemes.

8. Conclusion and discussion

The main purpose of this work is to provide a new asymptotic reduction method for constructing a consistent curved rod theory for linearized anisotropic elastic material. The starting point of our derivation is a Taylor–Young expansion of the displacement field. Then we consider the corresponding expansion of the deformation gradient and the stress tensor and make some development which are needed for the success of our procedure. More precisely we summarize the main idea. To write lateral boundary condition we use polar coordinates in cross section together with Fourier series expansion. This leads to seven equations with fifteen unknown coefficients displacement field. Equilibrium equations gives supplementary relations between stress coefficients and we show that it is possible to obtain a closed system of ten equations with ten unknows. The linear dependence with respect to three second order displacement coefficients is used to eliminate them. Elaborated calculations furnish bending and torsion terms and also lead to asymptotically-consistent closed three vector equations with three unknwons. By introducing the twist angle as a new unknown, it is possible to eliminate the two first-order displacement coefficients and we obtain four scalar equations with the three components of the mean line displacement and the twist angle as unknows. From the boundary term of the tridimensional variational formulation, we derive six physical and natural boundary conditions at each edge.

Then we formulate the unidimensional rod virtual work principle. **By considering a specific example, comparison with classical Euler-Bernoulli rod theory and Timoshenko rod theory shows that our theory gives refined results.** The method developed here appears to be general, and we think that it would be applicable to other rod problems : dynamic, double symmetric cross section, geometric and/or material non linearity, incompressible material, multilayered rod.

Appendix: Two case studies for respectively isotropic and transversely isotropic material.

For the sake of simplicity, we consider a rod with constant curvature $k(s)=k$ and without torsion curvature of the mean line $\widetilde{\Gamma}_c$ $\boldsymbol{\tau}=0$ and the volumic force $\mathbf{f}$ vanishes. The rod equilibrium equations (59)-(62) become :

$$
\begin{aligned}
&-\sigma_{33,3}^{(0,0)}+2k\sigma_{13}^{(0,0)}=\frac{2}{R_0}\left(\hat{\hat{\mathbf{T}}}_0\right)_3,\\
&-\frac{4k}{R_0^2}\mathbf{a}_3.\boldsymbol{\sigma}^{(0,0)}.\mathbf{a}_3-k\left(\mathbf{a}_3.\boldsymbol{\sigma}_{,3}^{(0,0)}.\mathbf{a}_3\right)_{,3}-\left(\mathbf{a}_3.\boldsymbol{\sigma}_{,3}^{(1,0)}.\mathbf{a}_3\right)_{,3}=\frac{4}{R_0^2}\left(\left(\hat{\hat{\mathbf{T}}}_{c1}\right)_{3,3}-k\left(\hat{\hat{\mathbf{T}}}_{c1}\right)_1\right)+\frac{8}{R_0^3}\left(\hat{\hat{\mathbf{T}}}_0\right)_1,\\
&-\left(\mathbf{a}_3.\boldsymbol{\sigma}_{,3}^{(0,1)}.\mathbf{a}_3\right)_{,3}+k\,\mathbf{a}_3.\boldsymbol{\sigma}_{,3}^{(0,1)}.\mathbf{a}_1+k^2\mathbf{a}_2.\boldsymbol{\sigma}_{,3}^{(0,0)}.\mathbf{a}_3+k\,\mathbf{a}_2.\boldsymbol{\sigma}_{,3}^{(1,0)}.\mathbf{a}_3\\
&\qquad\qquad=\frac{4}{R_0^2}\left(\left(\hat{\hat{\mathbf{T}}}_{s1}\right)_{3,3}-k\left(\hat{\hat{\mathbf{T}}}_{s1}\right)_1\right)+\frac{8}{R_0^3}\left(\hat{\hat{\mathbf{T}}}_0\right)_2,\\
&-k\,\mathbf{a}_2.\boldsymbol{\sigma}_{,3}^{(0,0)}.\mathbf{a}_3-\left(\mathbf{a}_2.\boldsymbol{\sigma}^{(1,0)}.\mathbf{a}_3\right)_{,3}+\left(\mathbf{a}_1.\boldsymbol{\sigma}^{(0,1)}.\mathbf{a}_3\right)_{,3}+k\left(\sigma_{12}^{(1,0)}-\sigma_{11}^{(0,1)}+\sigma_{13}^{(0,1)}\right)\\
&\qquad\qquad=\frac{4}{R_0^2}\left(\left(\hat{\hat{\mathbf{T}}}_{c1}\right)_2-\left(\hat{\hat{\mathbf{T}}}_{s1}\right)_1\right).
\end{aligned}
\tag{A1}
$$

One can see the parameter *k* brings about the coupling between the stress components. When *k* = 0, equations (A.1) degenerate to equations (27) in the paper Chen et al. 2018, which are :

$$
\begin{aligned}
&-\sigma_{33,3}^{(0,0)}=\frac{2}{R_0}\left(\hat{\hat{\mathbf{T}}}_0\right)_3,\\
&-\sigma_{33,33}^{(1,0)}=\frac{4}{R_0^2}\left[\left(\hat{\hat{\mathbf{T}}}_{c1}\right)_{3,3}+\frac{2}{R_0^3}\left(\hat{\hat{\mathbf{T}}}_0\right)_1\right],\\
&-\sigma_{33,33}^{(0,1)}=\frac{4}{R_0^2}\left[\left(\hat{\hat{\mathbf{T}}}_{s1}\right)_{3,3}+\frac{2}{R_0^3}\left(\hat{\hat{\mathbf{T}}}_0\right)_2\right],
\end{aligned}
$$

(A2)

$$
\sigma_{13,3}^{(0,1)}-\sigma_{23,3}^{(1,0)}=\frac{4}{R_0^2}\left(\left(\hat{\hat{\mathbf{T}}}_{c1}\right)_2-\left(\hat{\hat{\mathbf{T}}}_{s1}\right)_1\right).
$$

Further, to compare with the classical beam theories (Euler-Bernoulli beam theory and Timoshenko rod theory), consider the isotropic material with Young's modulus *E* and Poisson's ratio ν, and a line force loading at the top of lateral surface along the negative y_1-axis, its value per unit length is represented as $q(y_3)$. Then the rod equations from equations (A.2) are obtained by applying the recursive relations (39) and (54) as well as the constitutive relations (32), which are :

$$
\begin{aligned}
&E A u_{3,33}^{(0,0)} = \frac{l\nu}{2\pi} q_{,3}, \\
&-E I u_{1,3333}^{(0,0)} = q + O(R_0^2), \\
&E I u_{2,3333}^{(0,0)} = 0, \\
&G I \theta_{,33} = 0,
\end{aligned} \tag{A.3}
$$

where $G = \dfrac{E}{2(1+\nu)}$ is the shear modulus, $I = \dfrac{\pi R_0^4}{4}$ is the planar second moment of the cross-section, $l = 2\pi R_0$ and $A = \pi R_0^2$ are respectively the circumference and area of the cross section. One can see our theory actually recover the results of the classical rod theories.

We compare (A.3) with two classical rod theories Euler-Bernoulli rod theory and Timoshenko rod theory. From (A.3), one can see for deflection $u_1^{(0,0)}$ to the leading order, our theory recover the result of both classical theories. On the other hand, for displacement $u_3^{(0,0)}$, Euler-Bernoulli rod theory assumes it to be zero and Timoshenko's theory assumes it to be zero to the leading order, and our results show that it depends on the derivative of the loading $q(y_3)$. Both classical theories assume $u_2^{(0,0)}$ to be zero while in our rod theory, it depends on the external loadings at two ends. The two classical theories do not give the equation for θ explicitly. We conclude that, compared with the two classical rod theories, our rod theory is refined and capable of capturing the rod deformation with higher level accuracy and is also applicable in a much wider range of loadings.

In the following, we give two examples for isotropic material and transversely isotropic material. First we give the loading conditions and the material parameters. Consider a straight rod subjected to a uniform line force ($q(y_3)$ is a constant) along the negative y_1-axis with two clamped ends.

The line force is represented as $\hat{\hat{\mathbf{T}}} = -\dfrac{q\delta(\theta_a)}{R_0}\mathbf{a}_1$ where δ is the Kronecker delta function. Since $\delta(\theta_a) = \dfrac{1}{2\pi} + \dfrac{1}{\pi}\sum_{m=1}^{\infty} \cos m\theta_a$, taking it into $\hat{\hat{\mathbf{T}}}$, we have the Fourier coefficients of $\hat{\hat{\mathbf{T}}}$ as

$$
\hat{\hat{\mathbf{T}}}_0 = -\frac{q}{2\pi R_0}\mathbf{a}_1, \hat{\hat{\mathbf{T}}}_{cm} = -\frac{q}{\pi R_0}\mathbf{a}_1, \hat{\hat{\mathbf{T}}}_{sm} = 0, \ m=1, 2, 3\ldots\ldots \tag{A.4}
$$

And the boundary conditions from formula (77) are :

$$
\mathbf{u}^{(0,0)} = 0,\ u_{1,3}^{(0,0)} = 0, u_{2,3}^{(0,0)} = 0, \theta = 0 \ \text{ at } \ y_3 = 0, L. \tag{A.5}
$$

For the materials, we consider the isotropic material with parameter E and ν (Young's modulus and Poisson's ratio), and the transversely isotropic material with five parameters E_1 , E_3, ν_{21}, ν_{31} and G_{13}. These parameters are interpreted as $E_j = \dfrac{\text{normal stress}\,\sigma_{jj}}{\text{normal strain}\,\varepsilon_{jj}\,\text{caused by}\,\sigma_{jj}}$

$$\nu_{ij} = -\frac{\text{normal strain } \boldsymbol{\varepsilon}_{jj} \text{ caused by } \sigma_{ii}}{\text{normal strain } \boldsymbol{\varepsilon}_{ii} \text{ caused by } \sigma_{ii}} \text{ and } G_{ij} = \frac{\text{shearing stress } \sigma_{ij}}{\text{shearing strain } 2\varepsilon_{ij} \text{ caused by } \sigma_{ij}}.$$

For the isotropic material, from equation (A.3) (specifying $q(y_3) = q$) and equations (A.5), the displacement components scaled by the quantity $q/E\ (L/R_0)^4$ are obtained as (denoted by $\overline{u}_i$)

$$\begin{aligned}
\overline{u}_1 &= -\frac{1}{6\pi}\overline{s}^2(\overline{s}-1)^2 + \frac{\varepsilon^2\nu}{6\pi}\left(-1+6\overline{s}-6\overline{s}^2\right)\left(\overline{y}_1^2-\overline{y}_2^2\right) \\
&\quad + \frac{\varepsilon^4\left(\nu^2-1\right)}{\pi}\overline{y}_1 + \frac{\varepsilon^4}{2\pi}\left(-1+\nu+2\nu^2\right)\overline{y}_1^2 + \frac{\varepsilon^4}{2\pi}\left(3+\nu-2\nu^2\right)\overline{y}_2^2 \\
&\quad + \frac{\varepsilon^4\left(-1+3\nu+4\nu^2\right)}{6\pi}\overline{y}_1^3 - \frac{\varepsilon^4}{2\pi}(1+\nu)(-5+4\nu)\overline{y}_1\overline{y}_2^2, \\
\overline{u}_2 &= \frac{\varepsilon^2\nu}{3\pi}\left(-1+6\overline{s}-6\overline{s}^2\right)\overline{y}_1\overline{y}_2 + \frac{\varepsilon^4\nu(1+\nu)}{\pi}\overline{y}_2 + \frac{\varepsilon^4\left(-1+\nu+2\nu^2\right)}{\pi}\overline{y}_1\overline{y}_2 \\
&\quad + \frac{\varepsilon^4\nu}{2\pi}\left(-3+\nu+4\nu^2\right)\overline{y}_1^2\overline{y}_2 - \frac{\varepsilon^4}{6\pi}(1+\nu)(1+4\nu)\overline{y}_2^3, \qquad \text{(A.6)} \\
\overline{u}_3 &= \frac{\varepsilon\overline{s}}{3\pi}\left(1-3\overline{s}+2\overline{s}^2\right)\overline{y}_1 + \frac{\varepsilon^3}{2\pi}(1-2\overline{s})\overline{y}_1^3 + \frac{\varepsilon^3}{2\pi}(1-2\overline{s})\overline{y}_1\overline{y}_2^2.
\end{aligned}$$

where $\overline{y}_i = y_i / R_0$, $(i = 1, 2)$, $\overline{s} = s/L$ and $\varepsilon = R_0 / L$. The twist angle is $\theta = 0$.

For the transversely isotropic material, we have the rod equations similar to equations (A.3), and the differences are E, G in equations (A.3) being replaced by E_3 and G_{13} respectively. The solutions for the displacement components scaled by $q/E\ (L/R_0)^4$ are

$$\begin{aligned}
\overline{u}_1 &= -\frac{1}{6\pi}\overline{s}^2(\overline{s}-1)^2 + \frac{\varepsilon^2\nu_{31}}{6\pi}\left(-1+6\overline{s}-6\overline{s}^2\right)\left(\overline{y}_1^2-\overline{y}_2^2\right) \\
&\quad + \frac{\varepsilon^4}{\pi}\left[-\frac{E_3}{E_1}+\nu_{31}^2\right]\overline{y}_1 + \frac{\varepsilon^4}{2\pi}\left[\frac{E_3}{E_1}\left(-1+\nu_{21}\right)+2\nu_{31}^2\right]\overline{y}_1^2 + \frac{\varepsilon^4}{2\pi}\left[\frac{E_3}{E_1}\left(3+\nu_{21}\right)-2\nu_{31}^2\right]\overline{y}_2^2 \\
&\quad + \frac{\varepsilon^4}{6\pi}\left[\frac{E_3}{E_1}\left(-1+3\nu_{21}\right)+4\nu_{31}^2\right]\overline{y}_1^3 + \frac{\varepsilon^4}{2\pi}\left[\frac{E_3}{E_1}\left(5+\nu_{21}\right)-4\nu_{31}^2\right]\overline{y}_1\overline{y}_2^2. \\
\overline{u}_2 &= \frac{\varepsilon^2\nu_{31}}{3\pi}\left(-1+6\overline{s}-6\overline{s}^2\right)\overline{y}_1\overline{y}_2 + \frac{\varepsilon^4}{\pi}\left[\frac{E_3}{E_1}\nu_{21}+\nu_{31}^2\right]\overline{y}_2 + \frac{\varepsilon^4}{\pi}\left[\frac{E_3}{E_1}\left(-1+\nu_{21}\right)+2\nu_{31}^2\right]\overline{y}_1\overline{y}_2 \\
&\quad + \frac{\varepsilon^4}{2\pi}\left[\frac{E_3}{E_1}\left(-3+\nu_{21}\right)+4\nu_{31}^2\right]\overline{y}_1^2\overline{y}_2 - \frac{1}{6\pi}\varepsilon^4\left[\frac{E_3}{E_1}\left(5+\nu_{21}\right)+4\nu_{31}^2\right]\overline{y}_2^3. \qquad \text{(A.7)} \\
\overline{u}_3 &= \frac{1}{3\pi}\varepsilon\overline{s}(-1+\overline{s})(-1+2\overline{s})\overline{y}_1 + \frac{1}{4\pi}\varepsilon^3(1-2\overline{s})\left[\frac{E_3}{G_{13}}-2\nu_{31}\right]\overline{y}_1\left(\overline{y}_1^2+\overline{y}_2^2\right).
\end{aligned}$$

and the twist angle is still $\theta = 0$.

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