



HAL
open science

Degeneration of families of projective hypersurfaces and Hodge conjecture

Johann Bouali

► **To cite this version:**

Johann Bouali. Degeneration of families of projective hypersurfaces and Hodge conjecture. 2024. hal-04377032v7

HAL Id: hal-04377032

<https://hal.science/hal-04377032v7>

Preprint submitted on 27 Mar 2024 (v7), last revised 23 Apr 2024 (v8)

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Degeneration of families of projective hypersurfaces and Hodge conjecture

Johann Bouali

March 27, 2024

Abstract

We prove by induction on dimension the Hodge conjecture for smooth complex projective varieties. Let X be a smooth complex projective variety. Then X is birational to a possibly singular projective hypersurface, hence to a smooth projective variety E_0 which is a component of a normal crossing divisor $E = \cup_{i=0}^r E_i \subset Y$ which is the singular fiber of a pencil $f : Y \rightarrow \mathbb{A}^1$ of smooth projective hypersurfaces. Using the smooth hypersurface case ([5] theorem 1), the nearby cycle functor on mixed Hodge module with rational de Rham factor and the induction hypothesis, we prove that a Hodge class of E_0 is absolute Hodge, more precisely the locus of Hodge classes inside the algebraic vector bundle given the De Rham cohomology of the rational deformation of E_0 is defined over \mathbb{Q} . By [4] theorem 4, we get the Hodge conjecture for E_0 . By the induction hypothesis we also have the Hodge conjecture for X .

Contents

1	Introduction	1
2	Preliminaries and notations	3
2.1	Notations	3
2.2	Birational projective varieties and Hodge conjecture	5
2.3	A family of smooth projective hypersurfaces associated to a smooth projective variety	6
2.4	The nearby cycle functor and the specialization map	6
3	Hodge conjecture for smooth projective varieties	8

1 Introduction

The main result of this article is an inductive proof of the Hodge conjecture using the results of [4] and [5]. More precisely, assuming the Hodge conjecture for smooth projective varieties of dimension less or equal to $d - 1$, the property of satisfying the Hodge conjecture for connected smooth projective varieties of dimension d is a birational invariant (proposition 1). On the other hand, if X is a connected smooth projective variety of dimension d , then X is birational to a possibly singular hypersurface $Y_{0_X} \subset \mathbb{P}_{\mathbb{C}}^{d+1}$ of degree $l_X := \deg(\mathbb{C}(X)/\mathbb{C}(x_1, \dots, x_d))$. Consider

$$f : \mathcal{Y}' = V(f) \subset \mathbb{P}_{\mathbb{C}}^{d-1} \times \mathbb{P}(l_X) \rightarrow \mathbb{P}(l_X)$$

the universal family of projective hypersurfaces of degree l_X , and $0_X \in \mathbb{P}(l_X)$ the point corresponding to Y_{0_X} . Denote $\Delta \subset \mathbb{P}(l_X)$ the discriminant locus parametrizing the hypersurfaces which are singular. Let $\bar{S} := 0_X^{\mathbb{Q}} \subset \mathbb{P}(l_X)$ the \mathbb{Q} -Zariski closure of 0_X inside $\mathbb{P}(l_X)$. If Y_{0_X} is smooth, then X satisfy the hodge conjecture since Y_{0_X} satisfy the Hodge conjecture by [5] and by the induction hypothesis since X

is birational to Y_{0_X} . Hence, we may assume that Y_{0_X} is singular, that is $0_X \in \Delta$. Then, $\bar{S} \subset \Delta$ since $\Delta \subset \mathbb{P}(l_X)$ is defined over \mathbb{Q} . Let $\bar{T} \subset \mathbb{P}(l_X)$ an irreducible closed subvariety of dimension $\dim(\bar{S}) + 1$ defined over \mathbb{Q} such that $\bar{S} \subset \bar{T}$ and $T := \bar{T} \cap (\mathbb{P}(l_X) \setminus \Delta) \neq \emptyset$. Since there always exists subvarieties of $\mathbb{P}(l_X) \setminus \Delta$ of arbitrary dimension defined over \mathbb{Q} which are not weakly special (take complete intersections which intersect Δ transversally), we may assume that $T \subset \mathbb{P}(l_X) \setminus \Delta$ is not contained in a weakly special subvariety. We have then $\bar{T} \cap \Delta = \bar{S} \cup \bar{S}'$ where $\bar{S}' \subset \Delta$ is a divisor. We then consider the family of algebraic varieties defined over \mathbb{Q}

$$\tilde{f}_X : \mathcal{Y}_{\bar{T}^o} \xrightarrow{\epsilon} \mathcal{Y}'_{T^o} \xrightarrow{f \times_{\mathbb{P}(l_X)} \bar{T}^o} \bar{T}^o$$

where

- $\tilde{f}_X : \mathcal{Y} \xrightarrow{\epsilon} \mathcal{Y}'_{\bar{T}} \xrightarrow{f \times_{\mathbb{P}(l_X)} \bar{T}} \bar{T}$ with $\epsilon : (\mathcal{Y}, \mathcal{E}') \rightarrow (\mathcal{Y}'_{\bar{T}}, \mathcal{Y}_{\bar{S}} \cup \mathcal{Y}_{\bar{S}'})$ is a desingularization over \mathbb{Q} , in particular $\mathcal{E}' = \mathcal{E} \cup \mathcal{E}''$ with $\mathcal{E} = \cup_{i=0}^r \mathcal{E}_i := \tilde{f}_X^{-1}(\bar{S}) \subset \mathcal{Y}$ which is a normal crossing divisor, in particular $\mathcal{Y}_T = \mathcal{Y}'_T$,
- $S \subset \bar{S}$ is the open subset such that S is smooth and $\tilde{f}_X \times_{\bar{T}} S : \mathcal{E}_I := \cap_{i \in I} \mathcal{E}_i \rightarrow S$ is smooth projective for each $I \subset [0, \dots, r]$, $\bar{T}^o \subset \bar{T}$ is an open subset such that $\bar{T}^o \cap \Delta = S$ and $T^o := \bar{T}^o \cap T$ is smooth,
- $t \in \mathbb{S}(\mathbb{C})$ is the point such that $\mathcal{Y}'_{\mathbb{C}, t} = Y_{0_X}$ and $\epsilon_t : E_0 := \mathcal{E}_{0, \mathbb{C}, t} \rightarrow Y_{0_X}$ is surjective, in particular E_0 is birational to Y_{0_X} , hence E_0 is birational to X .

Assuming the Hodge conjecture for smooth projective varieties of dimension less or equal to $d - 1$, we then prove the Hodge conjecture for the E_i , more specifically for E_0 as follows : Let $p \in \mathbb{Z}$. By the hard Lefschetz theorem on $H^*(E_0)$, we may assume that $2p \leq d$. We then have then a morphism of mixed Hodge module over S (in fact of geometric variation of mixed Hodge structures over S) given by the specialization map

$$Sp_{\mathcal{E}/\tilde{f}_X} : E_{Hdg}^{2p}(\mathcal{E}/S) \rightarrow H^{2p} \psi_S E_{Hdg}(\mathcal{Y}_{T^o}/T^o),$$

whose de Rham part are morphism of filtered vector bundle over S with Gauss-Manin connexion defined over \mathbb{Q} , and whose kernel of the part supported on \mathcal{E}_0 is supported a closed subset $\mathcal{F} \subset \mathcal{E}_0$ with $\text{codim}(\mathcal{F}_{t'}, \mathcal{E}_{0, t'}) = 1$ for all $t' \in S(\mathbb{C})$. The fact that the kernel is supported on a closed subset of codimension one on fibers follows from the fact that $H^{2p-1}(Y_s, \mathbb{Q}) = 0 = H_{Y_0}^{2p-1}(Y)$, by the weak Lefschetz theorem since Y_s and Y_0 are projective hypersurfaces ($2p \leq d$). Consider the exact sequence of variation of mixed Hodge structure over S

$$0 \rightarrow W_{2p-2} E_{Hdg}^{2p}(\mathcal{E}/S) \rightarrow E_{Hdg}^{2p}(\mathcal{E}/S) \xrightarrow{(i^*)_{0 \leq i \leq r} := q} \oplus_{i=0}^r E_{Hdg}^{2p}(\mathcal{E}_i/S) \rightarrow 0.$$

It induces the exact sequence of presheaves on $S_{\mathbb{C}}^n$,

$$J(W_{2p-2} E_{Hdg}^{2p}(\mathcal{E}_{\mathbb{C}}/S_{\mathbb{C}})) := \text{Ext}^1(\mathbb{Q}_S^{Hdg}, W_{2p-2} E_{Hdg}^{2p}(\mathcal{E}_{\mathbb{C}}/S_{\mathbb{C}})) \xrightarrow{\iota} \text{Ext}^1(\mathbb{Q}_S^{Hdg}, E_{Hdg}^{2p}(\mathcal{E}_{\mathbb{C}}/S_{\mathbb{C}}))$$

$$HL^{2p,p}(\mathcal{E}_{\mathbb{C}}/S_{\mathbb{C}}) \xrightarrow{q} \oplus_{i=0}^r HL^{2p,p}(\mathcal{E}_{i, \mathbb{C}}/S_{\mathbb{C}}) \xrightarrow{\epsilon}$$

where $HL^{2p,p}(\mathcal{E}_{\mathbb{C}}/S_{\mathbb{C}}) \subset E_{DR}^{2p}(\mathcal{E}_{\mathbb{C}}/S_{\mathbb{C}})$, $HL^{2p,p}(\mathcal{E}_{i, \mathbb{C}}/S_{\mathbb{C}}) \subset E_{DR}^{2p}(\mathcal{E}_{i, \mathbb{C}}/S_{\mathbb{C}})$ are the locus of Hodge classes. Let $\lambda \in F^p H^{2p}(E_0^{an}, \mathbb{Q})$, where we recall $E_0 = \mathcal{E}_{0, \mathbb{C}, t}$. Consider then $\tilde{\lambda} \in H^{2p}(E^{an}, \mathbb{Q})$, such that $q(\tilde{\lambda}) = \lambda$, and $\tilde{\lambda} := Sp_{\mathcal{E}/\tilde{f}_X}(\tilde{\lambda}) \in Sp_{E/\tilde{f}_X}(H^{2p}(E^{an}, \mathbb{Q}))$, Since the monodromy of $R^{2p} \tilde{f}_{X*} \mathbb{Q}_{\mathcal{Y}_{T^o, \mathbb{C}}}^{an}$ is irreducible, we have

$$Sp_{\mathcal{E}/\tilde{f}_X}(\ker q) := Sp_{\mathcal{E}/\tilde{f}_X}(W_{2p-2} E_{DR}^{2p}(\mathcal{E}/S)) = 0$$

The locus of Hodge classes passing through λ

$$\mathbb{V}_S^p(\lambda) := \mathbb{V}_S(\lambda) \cap F^p E_{DR}^{2p}(\mathcal{E}_{0, \mathbb{C}}/S_{\mathbb{C}}) \subset E_{DR}^{2p}(\mathcal{E}_{0, \mathbb{C}}/S_{\mathbb{C}}),$$

inside the De Rham vector bundle of $\tilde{f}_X \circ i_{E_0} : \mathcal{E}_0 \rightarrow S$ thus satisfy

$$Sp_{\mathcal{E}/\tilde{f}_X}(q^{-1}(\mathbb{V}_S^p(\lambda))) = \mathbb{V}_S^p(Sp_{\mathcal{E}/\tilde{f}_X}(\tilde{\lambda})) \subset E_{DR, V_S}^{2p}(\mathcal{Y}_{T^o, \mathbb{C}}/T_{\mathbb{C}}^o),$$

where

- $\mathbb{V}_S(\lambda) \subset E_{DR}^{2p}(\mathcal{E}_{0, \mathbb{C}}/S_{\mathbb{C}})$, $\mathbb{V}_S(\tilde{\lambda}) \subset E_{DR}^{2p}(\mathcal{E}_{\mathbb{C}}/S_{\mathbb{C}})$, are the flat leaves, e.g. $\mathbb{V}_S(\lambda) := \pi_S(\lambda \times \tilde{S}_{\mathbb{C}}^{an})$ where $\pi_S : H^{2p}(E^{an}, \mathbb{C}) \times \tilde{S}_{\mathbb{C}}^{an} \rightarrow E_{DR}^{2p, an}(\mathcal{E}_{\mathbb{C}}/S_{\mathbb{C}})$ is the morphism induced by the universal covering $\pi_S : \tilde{S}_{\mathbb{C}}^{an} \rightarrow S_{\mathbb{C}}^{an}$,
- $q := q \otimes \mathbb{C} : E_{DR}^{2p}(\mathcal{E}_{\mathbb{C}}/S_{\mathbb{C}}) \rightarrow E_{DR}^{2p}(\mathcal{E}_{0, \mathbb{C}}/S_{\mathbb{C}})$ is the quotient map.

Now,

- if $Sp_{E/f_X}(\tilde{\lambda}) = 0$, assuming the Hodge conjecture for smooth projective varieties of dimension less or equal to $d - 1$, $\mathbb{V}_S^p(\tilde{\lambda}) \subset E_{DR}^{2p}(\mathcal{E}_{\mathbb{C}}/S_{\mathbb{C}})$ is an algebraic subvariety defined over $\bar{\mathbb{Q}}$ and its Galois conjugate are also components of the locus of Hodge classes,
- if $Sp_{E/f_X}(\tilde{\lambda}) \neq 0$, by [5] theorem 1, the Hodge conjecture holds for projective hypersurfaces, hence

$$Sp_{\mathcal{E}/\tilde{f}_X}(\mathbb{V}_S^p(\tilde{\lambda})) = \mathbb{V}_S^p(Sp_{\mathcal{E}/\tilde{f}_X}(\tilde{\lambda})) = \overline{\mathbb{V}_{T^o}^p(Sp_{E/f_X}(\tilde{\lambda}))} \cap p^{-1}(S) \subset E_{DR, V_S}^{2p}(\mathcal{Y}_{T^o, \mathbb{C}}/T_{\mathbb{C}}^o)$$

is an algebraic subvariety defined over $\bar{\mathbb{Q}}$ and its Galois conjugate are also components of the locus of Hodge classes, where the second equality follows from [6] lemma 2.11.

Hence, $\mathbb{V}_S^p(\lambda) \subset E_{DR}^{2p}(\mathcal{E}_{0\mathbb{C}}/S_{\mathbb{C}})$ is an algebraic subvariety defined over $\bar{\mathbb{Q}}$ and its Galois conjugate are also components of the locus of Hodge classes. By [4] theorem 4, we get

$$\lambda = [Z] \in H^{2p}(E_0^{an}, \mathbb{Q}), \quad Z \in \mathcal{Z}^p(E_0).$$

We recall the key point is that since $\mathbb{V}_S^p(\lambda) \subset E_{DR}^{2p}(\mathcal{E}_{0\mathbb{C}}/S_{\mathbb{C}})$ is an algebraic subvariety defined over $\bar{\mathbb{Q}}$ and its Galois conjugate are also components of the locus of Hodge classes, we have for each $\theta \in \text{Aut}(\mathbb{C}, \mathbb{Q})$,

$$\theta(\lambda) \in \text{Gal}(\bar{\mathbb{Q}}, \mathbb{Q})(\mathbb{V}_S^p(\lambda)_{\theta}) \subset E_{DR}^{2p}(\mathcal{E}_{0\mathbb{C}, \theta}/S_{\mathbb{C}, \theta}),$$

which implies that $\theta(\lambda) \in H^{2p}(E_{0, \theta}^{an}, \mathbb{Q}) \subset H^{2p}(E_{0, \theta}^{an}, \mathbb{C})$, that is $\lambda \in H^{2p}(E_0^{an}, \mathbb{Q})$ is absolute Hodge and we apply [4] corollary 1. Since $p \in \mathbb{Z}$ and $\lambda \in F^p H^{2p}(E_0^{an}, \mathbb{Q})$ are arbitrary, this proves the Hodge conjecture for E_0 . Since X is birational to E_0 , the Hodge conjecture for E_0 and the induction hypothesis implies the Hodge conjecture for X .

I am grateful to professor F.Mokrane for help and support during this work.

2 Preliminaries and notations

2.1 Notations

- Denote by Top the category of topological spaces and RTop the category of ringed spaces.
- Denote by Cat the category of small categories and RCat the category of ringed topoi.
- For $\mathcal{S} \in \text{Cat}$ and $X \in \mathcal{S}$, we denote $\mathcal{S}/X \in \text{Cat}$ the category whose objects are $Y/X := (Y, f)$ with $Y \in \mathcal{S}$ and $f : Y \rightarrow X$ is a morphism in \mathcal{S} , and whose morphisms $\text{Hom}((Y', f'), (Y, f))$ consists of $g : Y' \rightarrow Y$ in \mathcal{S} such that $f \circ g = f'$.
- Let $(\mathcal{S}, O_{\mathcal{S}}) \in \text{RCat}$ a ringed topoi with topology τ . For $F \in C_{O_{\mathcal{S}}}(\mathcal{S})$, we denote by $k : F \rightarrow E_{\tau}(F)$ the canonical flasque resolution in $C_{O_{\mathcal{S}}}(\mathcal{S})$ (see [3]). In particular for $X \in \mathcal{S}$, $H^*(X, E_{\tau}(F)) \xrightarrow{\sim} \mathbb{H}_{\tau}^*(X, F)$.

- For $f : \mathcal{S}' \rightarrow \mathcal{S}$ a morphism with $\mathcal{S}, \mathcal{S}' \in \text{RCat}$, endowed with topology τ and τ' respectively, we denote for $F \in C_{O_{\mathcal{S}}}(\mathcal{S})$ and each $j \in \mathbb{Z}$,

$$\begin{aligned} - f^* &:= H^j \Gamma(\mathcal{S}, k \circ \text{ad}(f^*, f_*)(F)) : \mathbb{H}^j(\mathcal{S}, F) \rightarrow \mathbb{H}^j(\mathcal{S}', f^*F), \\ - f^* &:= H^j \Gamma(\mathcal{S}, k \circ \text{ad}(f^{*mod}, f_*)(F)) : \mathbb{H}^j(\mathcal{S}, F) \rightarrow \mathbb{H}^j(\mathcal{S}', f^{*mod}F), \end{aligned}$$

the canonical maps.

- For $m : A \rightarrow B$, $A, B \in C(\mathcal{A})$, \mathcal{A} an additive category, we denote $c(A) : \text{Cone}(m : A \rightarrow B) \rightarrow A[1]$ and $c(B) : B \rightarrow \text{Cone}(m : A \rightarrow B)$ the canonical maps.
- Denote by $\text{Sch} \subset \text{RTop}$ the subcategory of schemes (the morphisms are the morphisms of locally ringed spaces). We denote by $\text{PSch} \subset \text{Sch}$ the full subcategory of proper schemes. For a field k , we consider $\text{Sch}/k := \text{Sch}/\text{Spec } k$ the category of schemes over $\text{Spec } k$. The objects are $X := (X, a_X)$ with $X \in \text{Sch}$ and $a_X : X \rightarrow \text{Spec } k$ a morphism and the morphisms are the morphisms of schemes $f : X' \rightarrow X$ such that $f \circ a_{X'} = a_X$. We then denote by
 - $\text{Var}(k) = \text{Sch}^{ft}/k \subset \text{Sch}/k$ the full subcategory consisting of algebraic varieties over k , i.e. schemes of finite type over k ,
 - $\text{PVar}(k) \subset \text{QPVar}(k) \subset \text{Var}(k)$ the full subcategories consisting of quasi-projective varieties and projective varieties respectively,
 - $\text{PSmVar}(k) \subset \text{SmVar}(k) \subset \text{Var}(k)$, $\text{PSmVar}(k) := \text{PVar}(k) \cap \text{SmVar}(k)$, the full subcategories consisting of smooth varieties and smooth projective varieties respectively.
- Denote by $\text{AnSp}(\mathbb{C}) \subset \text{RTop}$ the subcategory of analytic spaces over \mathbb{C} , and by $\text{AnSm}(\mathbb{C}) \subset \text{AnSp}(\mathbb{C})$ the full subcategory of smooth analytic spaces (i.e. complex analytic manifold).
- For $X \in \text{Var}(k)$ and $X = \cup_{i \in I} X_i$ with $i_i : X_i \hookrightarrow X$ closed embeddings, we denote $X_{\bullet} \in \text{Fun}(\Delta, \text{Var}(k))$ the associated simplicial space, with for $J \subset I$, $i_{IJ} : X_I := \cap_{i \in I} X_i \hookrightarrow X_J := \cap_{i \in J} X_i$ the closed embedding.
- Let $(X, O_X) \in \text{RTop}$. We consider its De Rham complex $\Omega_X^{\bullet} := DR(X)(O_X)$.
 - Let $X \in \text{Sch}$. Considering its De Rham complex $\Omega_X^{\bullet} := DR(X)(O_X)$, we have for $j \in \mathbb{Z}$ its De Rham cohomology $H_{DR}^j(X) := \mathbb{H}^j(X, \Omega_X^{\bullet})$.
 - Let $X \in \text{Var}(k)$. Considering its De Rham complex $\Omega_X^{\bullet} := \Omega_{X/k}^{\bullet} := DR(X/k)(O_X)$, we have for $j \in \mathbb{Z}$ its De Rham cohomology $H_{DR}^j(X) := \mathbb{H}^j(X, \Omega_X^{\bullet})$. The differentials of $\Omega_X^{\bullet} := \Omega_{X/k}^{\bullet}$ are by definition k -linear, thus $H_{DR}^j(X) := \mathbb{H}^j(X, \Omega_X^{\bullet})$ has a structure of a k vector space.
 - Let $X \in \text{AnSp}(\mathbb{C})$. Considering its De Rham complex $\Omega_X^{\bullet} := DR(X)(O_X)$, we have for $j \in \mathbb{Z}$ its De Rham cohomology $H_{DR}^j(X) := \mathbb{H}^j(X, \Omega_X^{\bullet})$.
- For $Y \in \text{Var}(\mathbb{Q})$ and $X = V(I) = V(f_1, \dots, f_r) \subset Y_{\mathbb{C}}$, we consider $\mathcal{X} = V(\tilde{I}) = V(\tilde{f}_1, \dots, \tilde{f}_r) \subset Y \times \mathbb{A}_{\mathbb{Q}}^s$ the canonical rational deformation, where

$$\begin{aligned} \text{if } f &= \sum_{I \subset [1, \dots, d]^n} a_I x_1^{n_1} \cdots x_n^{n_n} \in \mathbb{C}[x_1, \dots, x_n], \\ \tilde{f} &= \sum_{I \subset [1, \dots, d]^n} a_I x_1^{n_1} \cdots x_n^{n_n} \in \mathbb{Q}[(a_I)_{I \subset [1, \dots, d]^n}, x_1, \dots, x_n]. \end{aligned}$$

- For $X \in \text{AnSp}(\mathbb{C})$, we denote $\alpha(X) : \mathbb{C}_X \hookrightarrow \Omega_X^{\bullet}$ the embedding in $C(X)$. For $X \in \text{AnSm}(\mathbb{C})$, $\alpha(X) : \mathbb{C}_X \hookrightarrow \Omega_X^{\bullet}$ is an equivalence usu local by Poincare lemma.

- We denote $\mathbb{I}^n := [0, 1]^n \in \text{Diff}(\mathbb{R})$ (with boundary). For $X \in \text{Top}$ and R a ring, we consider its singular cochain complex

$$C_{\text{sing}}^*(X, R) := (\mathbb{Z} \text{Hom}_{\text{Top}}(\mathbb{I}^*, X)^\vee) \otimes R$$

and for $l \in \mathbb{Z}$ its singular cohomology $H_{\text{sing}}^l(X, R) := H^n C_{\text{sing}}^*(X, R)$. For $f : X' \rightarrow X$ a continuous map with $X, X' \in \text{Top}$, we have the canonical map of complexes

$$f^* : C_{\text{sing}}^*(X, R) \rightarrow C_{\text{sing}}^*(X', R), \sigma \mapsto f^* \sigma := (\gamma \mapsto \sigma(f \circ \gamma)).$$

In particular, we get by functoriality the complex

$$C_{X, R \text{sing}}^* \in C_R(X), (U \subset X) \mapsto C_{\text{sing}}^*(U, R)$$

We recall that

- For $X \in \text{Top}$ locally contractible, e.g. $X \in \text{CW}$, and R a ring, the inclusion in $C_R(X)$ $c_X : R_X \rightarrow C_{X, R \text{sing}}^*$ is by definition an equivalence top local and that we get by the small chain theorem, for all $l \in \mathbb{Z}$, an isomorphism $H^l c_X : H^l(X, R_X) \xrightarrow{\sim} H_{\text{sing}}^l(X, R)$.
- For $X \in \text{Diff}(\mathbb{R})$, the restriction map

$$r_X : \mathbb{Z} \text{Hom}_{\text{Diff}(\mathbb{R})}(\mathbb{I}^*, X)^\vee \rightarrow C_{\text{sing}}^*(X, R), w \mapsto w : (\phi \mapsto w(\phi))$$

is a quasi-isomorphism by Whitney approximation theorem.

- Let $S \in \text{AnSm}(\mathbb{C})$ and $L \in \text{Shv}(S)$ a local system. Consider $E := L \otimes \mathcal{O}_S \in \text{Vect}_{\mathcal{D}}(S)$ the corresponding holomorphic vector bundle with integrable connection ∇ . Let $\pi_S : \tilde{S} \rightarrow S$ be the universal covering. Consider the canonical fiber

$$L_0 := \Gamma(\tilde{S}, \pi_S^* L) = \Gamma(\tilde{S}, \pi_S^{* \text{mod}} E)^\nabla.$$

For $\lambda \in L_0$, we will consider the flat leaf

$$\mathbb{V}_S(\lambda) := \pi_S(\lambda \times \tilde{S}) \subset E, \pi_S : L_0 \times \tilde{S} \rightarrow E, \pi_S(\nu, z) := (\nu(z), \pi_S(z)).$$

Note that for $t \in S$, $\mathbb{V}_S(\lambda) \cap p_S^{-1}(t) = \pi_1(S, t)(\lambda)$ is the orbit of λ under the monodromy action, where $p_S : E \rightarrow S$ is the projection.

2.2 Birational projective varieties and Hodge conjecture

In this subsection, we recall that assuming the Hodge conjecture for smooth projective varieties of dimension less or equal to $d - 1$, the property of satisfying the Hodge conjecture for connected smooth projective varieties of dimension d is a birational invariant :

Proposition 1. (i) *Let $X \in \text{PSmVar}(\mathbb{C})$ connected of dimension d . Let $\epsilon : \tilde{X}_Z \rightarrow X$ be the blow up of X along a smooth closed subvariety $Z \subset X$. Assume the Hodge conjecture hold for smooth complex projective varieties of dimension less or equal to $d - 1$. Then the Hodge conjecture hold for X if and only if it hold for \tilde{X}_Z .*

(ii) *Let $X, X' \in \text{PSmVar}(\mathbb{C})$ connected of dimension d . Assume the Hodge conjecture hold for smooth complex projective varieties of dimension less or equal to $d - 1$. If X is birational to X' , then the Hodge conjecture hold for X if and only if it hold for X'*

Proof. (i): Follows from the fact that

$$(\epsilon^*, \oplus_{l=1}^c i_{E^*}((-).h^{l-1})) : H^k(X, \mathbb{Q}) \oplus \oplus_{l=1}^c H^{k-2l}(E, \mathbb{Q}) \rightarrow H^k(\tilde{X}_Z, \mathbb{Q})$$

is an isomorphism of Hodge structures for each $k \in \mathbb{Z}$, where $c = \text{codim}(Z, X)$, $i_E : E \hookrightarrow \tilde{X}_Z$ is the closed embedding, $\epsilon|_E : E \rightarrow Z$ being a projective vector bundle.

(ii): Follows from (i) since if $\pi : X^o \rightarrow X'$ is a birational map, $X^o \subset X$ being an open subset, then X is connected to X' by a sequence of blow up of smooth projective varieties with smooth center by [1]. \square

2.3 A family of smooth projective hypersurfaces associated to a smooth projective variety

In this subsection, we recall that given a smooth complex projective variety X , there exists a family of smooth complex projective hypersurfaces which degenerates into a normal crossing divisor with one irreducible component birational to X .

Proposition 2. *Let $X \in \text{PSmVar}(\mathbb{C})$ connected of dimension d . Then there exists a family of smooth projective hypersurfaces $f_X : Y \rightarrow \mathbb{A}_{\mathbb{C}}^1$ with*

- $Y \in \text{SmVar}(\mathbb{C})$, f_X flat projective,
- $Y_s := f_X^{-1}(s) \subset \mathbb{P}_{\mathbb{C}}^{d+1}$ for $s \in \mathbb{A}_{\mathbb{C}}^1 \setminus 0$ are smooth projective hypersurfaces,
- $E := f_X^{-1}(0) = \cup_{i=0}^r E_i = \epsilon^{-1}(Y_0) \subset Y$ is a normal crossing divisor, where $Y_0 \subset \mathbb{P}_{\mathbb{C}}^{d+1}$ is a possibly singular hypersurface birational to X , $\epsilon : (Y, E) \rightarrow (Y', Y_0)$ a desingularization, and E_0 is birational to X ,

Proof. See [2] lemma 1.5.8 . It use the fact that X is birational to a possibly singular projective hypersurface Y_0 and consider the desingularization of a generic pencil $f'_X : Y' = V(f) \subset \mathbb{P}_{\mathbb{C}}^{d+1} \times \mathbb{A}_{\mathbb{C}}^1 \rightarrow \mathbb{A}_{\mathbb{C}}^1$ passing through Y_0 and $f_X = f'_X \circ \epsilon$. \square

2.4 The nearby cycle functor and the specialization map

Recall from [3] that for $k \subset \mathbb{C}$ a subfield and $S \in \text{Var}(k)$ quasi-projective, and $l : S \hookrightarrow \tilde{S}$ a closed embedding with $\tilde{S} \in \text{SmVar}(k)$, we have the full subcategory

$$\iota : MHM_{k, gm}(S) \hookrightarrow \text{PSh}_{\mathcal{D}fil, S}(\tilde{S}) \times_I P(S_{\mathbb{C}}^{an})$$

consisting of geometric mixed Hodge module whose De Rham part is defined over k , where, using the fact that the V -filtration is defined over k , $\text{PSh}_{\mathcal{D}(1,0)fil, S}(\tilde{S}) \times_I P_{fil}(S_{\mathbb{C}}^{an})$ is the category

- whose objects are $((M, F, W), (K, W), \alpha)$ where (M, F, W) is a filtered $D_{\tilde{S}}$ module supported on S , where $l : S \hookrightarrow \tilde{S}$ is a closed embedding and $\tilde{S} \in \text{SmVar}(k)$, (K, W) is a filtered Perverse sheaf on $S_{\mathbb{C}}^{an}$, and $\alpha : l_*(K, W) \otimes \mathbb{C} \rightarrow DR(\tilde{S})((M, W)^{an})$ is an isomorphism in $D_{fil}(\tilde{S}_{\mathbb{C}}^{an})$.
- whose morphism between $((M, F, W), (K, W), \alpha)$ and $((M', F, W), (K', W), \alpha')$, are couples of morphisms $(\phi : (M, F, W) \rightarrow (M', F, W), \psi : (K, W) \rightarrow (K', W))$ such that $\phi \circ \alpha = \alpha' \circ \psi$.

Then, by [3], we get for $S \in \text{Var}(k)$ quasi-projective and $l : S \hookrightarrow \tilde{S}$ is a closed embedding with $\tilde{S} \in \text{SmVar}(k)$ an embedding

$$\iota : D(MHM_{k, gm}(S)) \hookrightarrow D_{\mathcal{D}fil, S}(\tilde{S}) \times_I D(S_{\mathbb{C}}^{an})$$

consisting of mixed Hodge module whose De Rham part is defined over k , where $D_{\mathcal{D}(1,0)fil, S}(\tilde{S}) \times_I D_{fil, c}(S_{\mathbb{C}}^{an})$ is the category

- whose objects are $((M, F, W), (K, W), \alpha)$ where (M, F, W) is a complex of filtered $D_{\tilde{S}}$ module supported on S , (K, W) is a filtered complex of presheaves on $S_{\mathbb{C}}^{an}$ whose cohomology are constructible sheaves, and $\alpha : l_*(K, W) \otimes \mathbb{C} \rightarrow DR(\tilde{S})((M, W)^{an})$ is an isomorphism in $D_{fil}(\tilde{S}_{\mathbb{C}}^{an})$.
- whose morphism are given in [3],

We have then the six functors formalism :

$$\begin{aligned} D(MHM_{k, gm}(-)) : \text{Var}(k) &\mapsto \text{TriCat}, \quad S \mapsto D(MHM_{k, gm}(S)), \quad (f : T \rightarrow S) \mapsto \\ (f^*, f_*) : D(MHM_{k, gm}(S)) &\rightarrow D(MHM_{k, gm}(T)), \quad (f!, f^!) : D(MHM_{k, gm}(T)) \rightarrow D(MHM_{k, gm}(S)) \end{aligned}$$

and for $D \subset S$ a (Cartier) divisor the nearby cycle functor

$$\begin{aligned} \psi_D &: D(MHM_{k,gm}(S)) \rightarrow D(MHM_{k,gm}(D)), \\ \psi_D((M, F, W), (K, W), \alpha) &:= (\psi_D(M, F, W), \psi_D(K, W), \psi_D(\alpha)). \end{aligned}$$

For $T \in \text{Var}(k)$ and $\epsilon_\bullet : T_\bullet \rightarrow T$ a desingularization with $T_\bullet \in \text{Fun}(\Delta, \text{SmVar}(k))$, we have

$$\mathbb{Z}_T^{Hdg} := a_T^{*Hdg} \mathbb{Z}^{Hdg} = (\epsilon_{\bullet*} E_{zar}(O_{T_\bullet}, F, W), \epsilon_{\bullet*} E_{usu}(\mathbb{Z}_{T_\bullet^{an}}, W), \alpha(T_\bullet))$$

For $f : X \rightarrow S$ a projective morphism with $X, S \in \text{SmVar}(k)$, we denote

$$E_{Hdg}(X/S) := f_* \mathbb{Z}_X^{Hdg} := \left(\int f(O_X, F), Rf_* \mathbb{Z}_{X_\mathbb{C}^{an}}, f_* \alpha(X) \right) \in D(MHM_{k,gm}(S)).$$

For $f : X \hookrightarrow \mathbb{P}^N \times S \xrightarrow{p} S$ a projective morphism with $X, S \in \text{Var}(k)$, S smooth, we denote

$$E_{Hdg}(X/S) := f_* \mathbb{Z}_X^{Hdg} := \left(\int_p \Gamma_X^{Hdg}(O_{\mathbb{P}^N \times S}, F), Rf_* \mathbb{Z}_{X_\mathbb{C}^{an}}, f_* \alpha(X) \right) \in D(MHM_{k,gm}(S)).$$

We then have the specialization map :

Definition-Proposition 1. *Let k be field of characteristic zero.*

(i) *Let $f' : X' \rightarrow T$ be a projective morphism with $X', T \in \text{SmVar}(k)$. Assume that*

- $f^\circ := f \otimes_T T : X^\circ \rightarrow T^\circ$ is smooth, $T^\circ \subset T$ is an open subset,
- $X'_S := f'^{-1}(S) \subset X'$, $S := T \setminus T^\circ$. We have by definition $X^\circ = X' \setminus X'_S$.

We have then, see [3], the distinguished triangles in $D(MHM_{k,gm}(X_S))$,

$$\begin{aligned} \mathbb{Z}_{X'_S}^{Hdg} &:= i_{X'_S}^* \mathbb{Z}_{X'}^{Hdg} \xrightarrow{sp_{X'_S/X'}} \psi_{X'_S} \mathbb{Z}_{X^\circ}^{Hdg} := (\psi_{X'_S}(O_{X'}, F), \psi_{X'_S} \mathbb{Z}_{X^\circ}, \psi_{X'_S} \alpha(X')) \\ &\xrightarrow{c(\psi_{X'_S} \mathbb{Z}_{X^\circ}^{Hdg})} \phi_{X'_S} \mathbb{Z}_{X^\circ}^{Hdg} := \text{Cone}(sp_{X'/X_S}) \xrightarrow{c(\mathbb{Z}_{X'_S}^{Hdg})} \mathbb{Z}_{X'_S}^{Hdg}[1]. \end{aligned}$$

Applying the functor $(f' \circ i_{X'_S})_*$, we obtain the generalized distinguished triangle in $D(MHM_{k,gm}(S))$

$$\begin{aligned} E_{Hdg}(X'_S/S) &\xrightarrow{sp_{X'_S/f} := (f' \circ i_{X'_S})_* sp_{X'_S/X'}} \\ (f' \circ i_{X'_S})_* (\psi_{X'_S} \mathbb{Z}_{X^\circ}^{Hdg}) &= \psi_0 E_{Hdg}(X^\circ/T^\circ) \xrightarrow{(f' \circ i_{X'_S})_* c(\Phi_{X_0} \mathbb{Z}_{X^\circ}^{Hdg})} \\ (f' \circ i_{X'_S})_* (\phi_{X'_S} \mathbb{Z}_{X^\circ}^{Hdg}) &= \phi_S E_{Hdg}(X^\circ/T^\circ) \xrightarrow{(f' \circ i_{X'_S})_* c(\mathbb{Z}_{X'}^{Hdg})} E_{Hdg}(X'_S/S)[1] \end{aligned}$$

(i)' *If for each $t \in T^\circ(\mathbb{C})$ we have $H^{k-1}(X'_t, \mathbb{Q}) = 0$ and for each $t' \in S(\mathbb{C})$ we have $H_{X'_{t'}}^{k-1}(X', \mathbb{Q}) = 0$, then the induced map on cohomology*

$$sp_{X'_S/f} : E_{Hdg}^k(X'_S/S) \rightarrow H^k \psi_S E_{Hdg}(X^\circ/T^\circ)$$

is injective.

(ii) *Let $f : X \rightarrow T$ be a projective morphism with $X, T \in \text{SmVar}(k)$. Assume that*

- $f^\circ := f \otimes_T T^\circ : X^\circ \rightarrow T^\circ$ is smooth, $T^\circ \subset T$ an open subset
- $S := T \setminus T^\circ$, $E := f^{-1}(S) = \cup_{i=0}^r E_i \subset X$ is a normal crossing divisor and that for $I \subset [0, \dots, r]$, $f_{0,I} := f|_{E_I} : E_I := \cap_{i \in I} E_i \rightarrow S$ are smooth. We have by definition $X^\circ = X \setminus E$.

Denote for $I \subset J \subset [0, \dots, r]$, $i_{JI} : E_J \hookrightarrow E_I$ and $i_I : E_I \hookrightarrow E$, $i_E : E \hookrightarrow X$ the closed embeddings. In this particular case, c.f. [8], the distinguished triangle in $D(\text{MHM}_{k, \text{gm}}(E))$ given in (i) become

$$\begin{aligned} \mathbb{Z}_E^{\text{Hdg}} &:= i_E^* \mathbb{Z}_X^{\text{Hdg}} = ((i_{\bullet*} \Omega_{E^\bullet}, F, W), (\mathbb{Z}_{E^\bullet}, W), \alpha(E_\bullet)) \xrightarrow{sp_{E/X}} \\ \psi_E \mathbb{Z}_{X^\circ}^{\text{Hdg}} &= ((i_E^{*\text{mod}} \Omega_{X/\mathbb{A}^1 \times S}(\log E), F, W), \psi_E \mathbb{Z}_{X^\circ}, \psi_E \alpha(X)) \xrightarrow{c(\psi_E \mathbb{Z}_{X^\circ}^{\text{Hdg}})} \\ \phi_E \mathbb{Z}_{X^\circ}^{\text{Hdg}} &:= \text{Cone}(sp_{E/X}) \xrightarrow{c(\mathbb{Z}_E^{\text{Hdg}})} \mathbb{Z}_E^{\text{Hdg}}[1]. \end{aligned}$$

Applying the functor $(f \circ i_E)_*$, we obtain the generalized distinguished triangle in $D(\text{MHM}_{k, \text{gm}}(S))$

$$\begin{aligned} E_{\text{Hdg}}(E/S) &\xrightarrow{sp_{E/f} := (f \circ i_E)_* sp_{E/X}} (f \circ i_E)_*(\psi_E \mathbb{Z}_{X^\circ}^{\text{Hdg}}) = \psi_S E_{\text{Hdg}}(X^\circ / ((\mathbb{A}^1 \setminus 0) \times S)) \\ \xrightarrow{(f \circ i_E)_* c(\psi_E \mathbb{Z}_{X^\circ}^{\text{Hdg}})} &(f \circ i_E)_*(\phi_E \mathbb{Z}_{X^\circ}^{\text{Hdg}}) = \phi_S E_{\text{Hdg}}(X^\circ / ((\mathbb{A}^1 \setminus 0) \times S)) \xrightarrow{(f \circ i_E)_* c(\mathbb{Z}_X^{\text{Hdg}})} E_{\text{Hdg}}(E/S)[1]. \end{aligned}$$

If $f = f' \circ \epsilon$ where $\epsilon : (X, E) \rightarrow (X, X_S)$ is a desingularization, we have $sp_{X_S/X'} = sp_{E/X} \circ \epsilon^*$.

Proof. (i)': Follows from the exact sequence

$$\dots \rightarrow H_{X_{t'}}^l(X'_C, \mathbb{Q}) \rightarrow \mathbb{H}^l(X_{t'}, \phi_S \mathbb{Z}_{X_C^\circ}) \xrightarrow{V} \mathbb{H}^l(X_{t'}, \psi_S \mathbb{Z}_{X_C^\circ}) \rightarrow H_{X_{t'}}^{l+1}(X'_C, \mathbb{Q}) \rightarrow \dots$$

given in [8], where $C \subset T$ is a transversal slice to S passing through t' . \square

3 Hodge conjecture for smooth projective varieties

Theorem 1. *Let $X \in \text{PSmVar}(\mathbb{C})$. Then Hodge conjecture hold for X . That is, if $p \in \mathbb{Z}$ and $\lambda \in F^p H^{2p}(X^{\text{an}}, \mathbb{Q})$, $\lambda = [Z]$ with $Z \in \mathcal{Z}^p(X)$.*

Proof. The Hodge conjecture is true for curves and surfaces. Assume that the Hodge conjecture is true for smooth projective varieties of dimension less or equal to $d-1$. Let $X \in \text{PSmVar}(\mathbb{C})$ of dimension d . Up to split X into its connected components, we may assume that X is connected of dimension d . Then X is birational to a possibly singular hypersurface $Y_{0_X} \subset \mathbb{P}_{\mathbb{C}}^{d+1}$ of degree $l_X := \deg(\mathbb{C}(X)/\mathbb{C}(x_1, \dots, x_d))$. Consider

$$f : \mathcal{Y}' = V(f) \subset \mathbb{P}_{\mathbb{C}}^{d-1} \times \mathbb{P}(l_X) \rightarrow \mathbb{P}(l_X)$$

the universal family of projective hypersurfaces of degree l_X , and $0_X \in \mathbb{P}(l_X)$ the point corresponding to Y_{0_X} . Denote $\Delta \subset \mathbb{P}(l_X)$ the discriminant locus parametrizing the hypersurfaces which are singular. Let $\bar{S} := \bar{0}_X \subset \mathbb{P}(l_X)$ be the \mathbb{Q} -Zariski closure of 0_X inside $\mathbb{P}(l_X)$. If Y_{0_X} is smooth, then X satisfy the hodge conjecture since Y_{0_X} satisfy the Hodge conjecture by [5] and by the induction hypothesis since X is birational to Y_{0_X} by proposition 1(ii). Hence, we may assume that Y_{0_X} is singular, that is $0_X \in \Delta$. Then, $\bar{S} \subset \Delta$ since $\Delta \subset \mathbb{P}(l_X)$ is defined over \mathbb{Q} . Let $\bar{T} \subset \mathbb{P}(l_X)$ be an irreducible closed subvariety of dimension $\dim(\bar{S}) + 1$ defined over \mathbb{Q} such that $\bar{S} \subset \bar{T}$ and $T := \bar{T} \cap (\mathbb{P}(l_X) \setminus \Delta) \neq \emptyset$. Since there always exists subvarieties of $\mathbb{P}(l_X) \setminus \Delta$ of arbitrary dimension defined over \mathbb{Q} which are not weakly special (take complete intersections which intersect Δ transversaly), we may assume that $T \subset \mathbb{P}(l_X) \setminus \Delta$ is not contained in a weakly special subvariety. We have then $\bar{T} \cap \Delta = \bar{S} \cup \bar{S}'$ where $\bar{S}' \subset \Delta$ is a divisor. We then consider the family of algebraic varieties defined over \mathbb{Q}

$$\tilde{f}_X : \mathcal{Y}_{\bar{T}^\circ} \xrightarrow{\epsilon} \mathcal{Y}'_{\bar{T}^\circ} \xrightarrow{f \times_{\mathbb{P}(l_X)} \bar{T}^\circ} \bar{T}^\circ$$

where

- $\tilde{f}_X : \mathcal{Y} \xrightarrow{\epsilon} \mathcal{Y}'_{\bar{T}} \xrightarrow{f \times_{\mathbb{P}(l_X)} \bar{T}} \bar{T}$ with $\epsilon : (\mathcal{Y}, \mathcal{E}') \rightarrow (\mathcal{Y}'_{\bar{T}}, \mathcal{Y}_{\bar{S}} \cup \mathcal{Y}_{\bar{S}'})$ is a desingularization over \mathbb{Q} , in particular $\mathcal{E}' = \mathcal{E} \cup \mathcal{E}''$ with $\mathcal{E} = \cup_{i=0}^r \mathcal{E}_i := \tilde{f}_X^{-1}(\bar{S}) \subset \mathcal{Y}$ which is a normal crossing divisor, in particular $\mathcal{Y}_T = \mathcal{Y}'_T$,

- $S \subset \bar{S}$ is the open subset such that S is smooth and $\tilde{f}_X \times_{\bar{T}} S : \mathcal{E}_I := \cap_{i \in I} \mathcal{E}_i \rightarrow S$ is smooth projective for each $I \subset [0, \dots, r]$, $\bar{T}^\circ \subset \bar{T}$ is an open subset such that $\bar{T}^\circ \cap \Delta = S$ and $T^\circ := \bar{T}^\circ \cap T$ is smooth,
- $t \in \mathbb{S}(\mathbb{C})$ is the point such that $\mathcal{Y}'_{\mathbb{C}, t} = Y_{0_X}$ and $\epsilon_t : E_0 := \mathcal{E}_{0, \mathbb{C}, t} \rightarrow Y_{0_X}$ is surjective, in particular E_0 is birational to Y_{0_X} , hence E_0 is birational to X .

Denote for $I \subset [0, \dots, s]$, $i_I : \mathcal{E}_I \hookrightarrow \mathcal{E}$ and for $I \subset J \subset [0, \dots, s]$, $i_{JI} : \mathcal{E}_J \rightarrow \mathcal{E}_I$ the closed embeddings. Then, $i_\bullet : \mathcal{E}_\bullet \rightarrow \mathcal{E}$ in $\text{Fun}(\Delta, \text{Var}(\mathbb{Q}))$ is a simplicial resolution with $\mathcal{E}_\bullet \in \text{Fun}(\Delta, \text{SmVar}(\mathbb{Q}))$. We have then

$$E_{Hdg}(\mathcal{E}/S) := (\tilde{f}_X \circ i_{\bullet*} E_{zar}(\Omega_{\mathcal{E}_\bullet/S}, F_b, W), (\tilde{f}_X \circ i_{\bullet*} E_{usu}(\mathbb{Q}_{\mathcal{E}_\bullet/\mathbb{C}}^{an}, W)), \tilde{f} \circ i_{\bullet*} \alpha(\mathcal{E}_\bullet)) \in D(MHM_{\mathbb{Q}, gm}(S))$$

Then, definition-proposition 1 applied to $f_T := f \times_{\mathbb{P}(t_X)} \bar{T}^\circ : \mathcal{Y}'_{T^\circ} \rightarrow \bar{T}^\circ$ gives the distinguish in $D(MHM_{\mathbb{Q}, gm}(S))$,

$$\begin{aligned} E_{Hdg}(\mathcal{Y}'_S/S) &\xrightarrow{Sp_{\mathcal{Y}'_S/f_T} := (\tilde{f}'_X \circ i_{\mathcal{Y}'_S})_* sp_{\mathcal{Y}_S/\mathcal{Y}'_{T^\circ}}} \psi_S E_{Hdg}(\mathcal{Y}'_{T^\circ}/T^\circ) \\ &\xrightarrow{(f_T \circ i_{\mathcal{Y}'_S})_* c(\psi_{\mathcal{Y}'_S} \mathbb{Z}_{\mathcal{Y}'_{T^\circ}}^{Hdg})} \phi_S E_{Hdg}(\mathcal{Y}'_{T^\circ}/T^\circ) \xrightarrow{(f_T \circ i_{\mathcal{Y}'_S})_* c(\mathbb{Z}_{\mathcal{Y}'_S}^{Hdg})} E_{Hdg}(\mathcal{Y}'_S/S)[1] \end{aligned}$$

and definition-proposition 1 applied to $\tilde{f}_X : \mathcal{Y}_{\bar{T}^\circ} \rightarrow \bar{T}^\circ$ gives the distinguish in $D(MHM_{\mathbb{Q}, gm}(S))$,

$$\begin{aligned} E_{Hdg}(\mathcal{E}/S) &\xrightarrow{Sp_{\mathcal{E}/\tilde{f}_X} := (\tilde{f}_X \circ i_{\mathcal{E}})_* sp_{\mathcal{E}/\mathcal{Y}}} \psi_S E_{Hdg}(\mathcal{Y}_{T^\circ}/T^\circ) \\ &\xrightarrow{(\tilde{f}_X \circ i_{\mathcal{E}})_* c(\psi_{\mathcal{E}} \mathbb{Z}_{\mathcal{Y}_{T^\circ}}^{Hdg})} \phi_S E_{Hdg}(\mathcal{Y}_{T^\circ}/T^\circ) \xrightarrow{(\tilde{f}_X \circ i_{\mathcal{E}})_* c(\mathbb{Z}_{\mathcal{E}}^{Hdg})} E_{Hdg}(\mathcal{E}/S)[1] \end{aligned}$$

and $Sp_{\mathcal{E}/\tilde{f}_X} \circ \epsilon^* = Sp_{\mathcal{Y}_{T^\circ}/f_T}$. Let $p \in \mathbb{Z}$. By the hard Lefschetz theorem on $H^*(E_0)$, we may assume that $2p \leq d$. Taking the cohomology to this distinguish triangle, we get in particular, for each $p \in \mathbb{Z}$, the maps in $MHM_{\mathbb{Q}, gm}(S)$

- $Sp_{\mathcal{Y}'_S/f_T} : E_{Hdg}^{2p}(\mathcal{Y}'_S/S) \hookrightarrow H^{2p} \psi_S E_{Hdg}(\mathcal{Y}_{T^\circ}/T^\circ)$
- $Sp_{\mathcal{E}/\tilde{f}_X} : E_{Hdg}^{2p}(\mathcal{E}/S) \rightarrow H^{2p} \psi_S E_{Hdg}(\mathcal{Y}_{T^\circ}/T^\circ)$.

The map $Sp_{\mathcal{Y}'_S/f_T}$ is injective by definition-proposition 1(i)' and the weak Lefschetz theorem since for $t' \in T^\circ(\mathbb{C})$ and $t'' \in S(\mathbb{C})$, $\mathcal{Y}'_{t'}$ and $\mathcal{Y}'_{t''}$ are projective hypersurfaces. For $\lambda \in H^{2p}(\mathcal{E}_{0, \mathbb{C}, t'}^{an}, \mathbb{Q})$, $t' \in S(\mathbb{C})$, we consider $\pi_S : \tilde{S}_{\mathbb{C}}^{an} \rightarrow S_{\mathbb{C}}^{an}$ the universal covering in $\text{AnSm}(\mathbb{C})$ and

$$\mathbb{V}_S(\lambda) := \pi_S(\lambda \times \tilde{S}_{\mathbb{C}}^{an}) \subset E_{DR}^{2p, an}(\mathcal{E}_{0, \mathbb{C}}/S_{\mathbb{C}})$$

the flat leaf induced by the universal covering (see section 2.1) and

$$\mathbb{V}_S^p(\lambda) := \mathbb{V}_S(\lambda) \cap F^p E_{DR}^{2p}(\mathcal{E}_{0, \mathbb{C}}/S_{\mathbb{C}}) \subset E_{DR}^{2p}(\mathcal{E}_{0, \mathbb{C}}/S_{\mathbb{C}}),$$

which is an algebraic variety by [6]. Note that $\mathbb{V}_S^p(\lambda) \subset H^{p, 2p}(\mathcal{E}_{0, \mathbb{C}}/S_{\mathbb{C}})$ is the union of the irreducible components passing through λ . Similarly, for $\eta \in H^{2p}(\mathcal{E}_{\mathbb{C}, t'}^{an}, \mathbb{Q})$, $t' \in S(\mathbb{C})$, we consider $\pi_S : \tilde{S}_{\mathbb{C}}^{an} \rightarrow S_{\mathbb{C}}^{an}$ the universal covering in $\text{AnSm}(\mathbb{C})$ and

$$\mathbb{V}_S(\eta) := \pi_S(\eta \times \tilde{S}_{\mathbb{C}}^{an}) \subset E_{DR}^{2p, an}(\mathcal{E}_{\mathbb{C}}/S_{\mathbb{C}})$$

the flat leaf induced by the universal covering (see section 2.1) and

$$\mathbb{V}_S^p(\eta) := \mathbb{V}_S(\eta) \cap F^p E_{DR}^{2p}(\mathcal{E}_{\mathbb{C}}/S_{\mathbb{C}}) \subset E_{DR}^{2p}(\mathcal{E}_{\mathbb{C}}/S_{\mathbb{C}}).$$

We have

$$H^{2p}E_{Hdg}(\mathcal{Y}_{T^o}/T^o) = ((E_{DR}^{2p}(\mathcal{Y}_{T^o}/T^o), F), R^{2p}\tilde{f}_{X*}\mathbb{Z}_{\mathcal{Y}_{T^o}}, \tilde{f}_{X*}\alpha(\mathcal{Y}_{T^o})) \in MHM_{\mathbb{Q}, gm}(T^o)$$

By [8], the filtered vector bundle

$$(E_{DR}^{2p}(\mathcal{Y}_{T^o}/T^o), F) = H^{2p}\tilde{f}_{X*}E_{zar}(\Omega_{\mathcal{Y}_{T^o}/T^o}^\bullet) \in \text{Vect}_{\mathcal{D}fil}(T^o)$$

extend to a filtered vector bundle

$$(E_{DR, V_S}^{2p}(\mathcal{Y}_{T^o}/T^o), F) := H^{2p}\tilde{f}_{X*}E_{zar}(\Omega_{\mathcal{Y}_{T^o}/T^o}^\bullet(\log \mathcal{E})) \in \text{Vect}_{\mathcal{D}fil}(\bar{T}^o)$$

such that

$$H^{2p}\psi_S E_{Hdg}(\mathcal{Y}_{T^o}/T^o) = (i_S^{*mod}(E_{DR, V_S}^{2p}(\mathcal{Y}_{T^o}/T^o), F), (\psi_S R^{2p}\tilde{f}_{X*}\mathbb{Z}_{\mathcal{Y}^o}, W), \psi_S \tilde{f}_{X*}\alpha(\mathcal{Y}_{T^o})).$$

For

$$\alpha \in H^{2p}(\mathcal{Y}_{t''}^{an}, \mathbb{Q}) = \mathbb{H}^{2p}(\mathcal{E}_{t''}^{an}, \psi_S R\tilde{f}_{X*}\mathbb{Q}_{\mathcal{Y}_{T^o}}), t' \in S_{\mathbb{C}}, t'' \in T^o(\mathbb{C}),$$

we consider

- $\pi_{T^o} : \tilde{T}^{o, an} \rightarrow T_{\mathbb{C}}^{o, an}$ the universal covering in $\text{AnSm}(\mathbb{C})$, the factorization $\pi_S : \tilde{S}_{\mathbb{C}}^{an} \rightarrow \tilde{T}^{o, an} \xrightarrow{\pi_{T^o}} T_{\mathbb{C}}^{o, an}$ and

$$\mathbb{V}_{T^o}(\alpha) := \pi_{T^o}(\alpha \times \tilde{T}^{o, an}) \subset E_{DR}^{2p, an}(\mathcal{Y}_{T^o, \mathbb{C}}/T_{\mathbb{C}}^o)$$

the flat leaf (see section 2.1) and

$$\mathbb{V}_{T^o}^p(\alpha) := \mathbb{V}_{T^o}(\alpha) \cap F^p E_{DR}^{2p}(\mathcal{Y}_{T^o, \mathbb{C}}/T_{\mathbb{C}}^o) \subset E_{DR}^{2p}(\mathcal{Y}_{T^o, \mathbb{C}}/T_{\mathbb{C}}^o),$$

which is an algebraic subvariety by [6],

- $\pi_S : \tilde{S}_{\mathbb{C}}^{an} \rightarrow S_{\mathbb{C}}^{an}$ the universal covering in $\text{AnSm}(\mathbb{C})$ and

$$\mathbb{V}_S(\alpha) := \pi_S(\alpha \times \tilde{S}_{\mathbb{C}}^{an}) \subset H^{2p}\psi_S E_{DR}(\mathcal{Y}_{T^o, \mathbb{C}}/T_{\mathbb{C}}^o)^{an}$$

the flat leaf (see section 2.1) and

$$\mathbb{V}_S^p(\alpha) := \mathbb{V}_S(\alpha) \cap F^p H^{2p}\psi_S E_{DR}(\mathcal{Y}_{T^o, \mathbb{C}}/T_{\mathbb{C}}^o)^{an} \subset H^{2p}\psi_S E_{DR}(\mathcal{Y}_{T^o, \mathbb{C}}/T_{\mathbb{C}}^o)^{an}.$$

Consider the exact sequence of variation of mixed Hodge structure over S

$$0 \rightarrow W_{2p-2}E_{Hdg}^{2p}(\mathcal{E}/S) \rightarrow E_{Hdg}^{2p}(\mathcal{E}/S) \xrightarrow{(i_i^*)_{0 \leq i \leq r} := q} \oplus_{i=0}^r E_{Hdg}^{2p}(\mathcal{E}_i/S) \rightarrow 0.$$

It induces the exact sequence of vector bundles over S

$$0 \rightarrow W_{2p-2}E_{DR}^{2p}(\mathcal{E}/S) \rightarrow E_{DR}^{2p}(\mathcal{E}/S) \xrightarrow{(i_i^*)_{0 \leq i \leq r} := q} \oplus_{i=0}^r E_{DR}^{2p}(\mathcal{E}_i/S) \rightarrow 0,$$

and the exact sequence of presheaves on $S_{\mathbb{C}}^{an}$,

$$J(W_{2p-2}E_{Hdg}^{2p}(\mathcal{E}_{\mathbb{C}}/S_{\mathbb{C}})) := \text{Ext}^1(\mathbb{Q}_S^{Hdg}, W_{2p-2}E_{Hdg}^{2p}(\mathcal{E}_{\mathbb{C}}/S_{\mathbb{C}})), \xrightarrow{L} \text{Ext}^1(\mathbb{Q}_S^{Hdg}, W_{2p-2}E_{Hdg}^{2p}(\mathcal{E}_{\mathbb{C}}/S_{\mathbb{C}}))$$

$$HL^{2p, p}(\mathcal{E}_{\mathbb{C}}/S_{\mathbb{C}}) \xrightarrow{q} \oplus_{i=0}^r HL^{2p, p}(\mathcal{E}_{i, \mathbb{C}}/S_{\mathbb{C}}) \xrightarrow{e}$$

where $HL^{2p, p}(\mathcal{E}_{\mathbb{C}}/S_{\mathbb{C}}) \subset E_{DR}^{2p}(\mathcal{E}_{\mathbb{C}}/S_{\mathbb{C}})$, $HL^{2p, p}(\mathcal{E}_{i, \mathbb{C}}/S_{\mathbb{C}}) \subset E_{DR}^{2p}(\mathcal{E}_{i, \mathbb{C}}/S_{\mathbb{C}})$ are the locus of Hodge classes. Let $\lambda \in F^p H^{2p}(E_0^{an}, \mathbb{Q})$, where we recall $E_0 = \mathcal{E}_{0, \mathbb{C}, t}$. Consider then $\tilde{\lambda} \in H^{2p}(E^{an}, \mathbb{Q})$, such that $q(\tilde{\lambda}) = \lambda$, and $\tilde{\lambda} := Sp_{\mathcal{E}/\tilde{f}_X}(\tilde{\lambda}) \in Sp_{E/\tilde{f}_X}(H^{2p}(E^{an}, \mathbb{Q}))$, Since the monodromy of $R^{2p}\tilde{f}_{X*}\mathbb{Q}_{\mathcal{Y}_{T^o, \mathbb{C}}}^{an}$ is irreducible, we have

$$Sp_{\mathcal{E}/\tilde{f}_X}(\ker q) := Sp_{\mathcal{E}/\tilde{f}_X}(W_{2p-2}E_{DR}^{2p}(\mathcal{E}/S)) = 0$$

The locus of Hodge classes passing through λ

$$\mathbb{V}_S^p(\lambda) := \mathbb{V}_S(\lambda) \cap F^p E_{DR}^{2p}(\mathcal{E}_{0,\mathbb{C}}/S_{\mathbb{C}}) \subset E_{DR}^{2p}(\mathcal{E}_{0,\mathbb{C}}/S_{\mathbb{C}}),$$

inside the De Rham vector bundle of $\tilde{f}_X \circ i_{E_0} : \mathcal{E}_0 \rightarrow S$ thus satisfy

$$Sp_{\mathcal{E}/\tilde{f}_X}(q^{-1}(\mathbb{V}_S^p(\lambda))) = \mathbb{V}_S^p(Sp_{\mathcal{E}/\tilde{f}_X}(\tilde{\lambda})) \subset E_{DR, V_S}^{2p}(\mathcal{Y}_{T^o, \mathbb{C}}/T_{\mathbb{C}}^o),$$

where

- $\mathbb{V}_S(\lambda) \subset E_{DR}^{2p}(\mathcal{E}_{0,\mathbb{C}}/S_{\mathbb{C}})$, $\mathbb{V}_S(\tilde{\lambda}) \subset E_{DR}^{2p}(\mathcal{E}_{\mathbb{C}}/S_{\mathbb{C}})$, are the flat leaves, e.g. $\mathbb{V}_S(\lambda) := \pi_S(\lambda \times \tilde{S}_{\mathbb{C}}^{an})$ where $\pi_S : H^{2p}(E^{an}, \mathbb{C}) \times \tilde{S}_{\mathbb{C}}^{an} \rightarrow E_{DR}^{2p, an}(\mathcal{E}_{\mathbb{C}}/S_{\mathbb{C}})$ is the morphism induced by the universal covering $\pi_S : \tilde{S}_{\mathbb{C}}^{an} \rightarrow S_{\mathbb{C}}^{an}$,
- $q := q \otimes \mathbb{C} : E_{DR}^{2p}(\mathcal{E}_{\mathbb{C}}/S_{\mathbb{C}}) \rightarrow E_{DR}^{2p}(\mathcal{E}_{0,\mathbb{C}}/S_{\mathbb{C}})$ is the quotient map.

Now,

- if $Sp_{\mathcal{E}/\tilde{f}_X}(\tilde{\lambda}) = 0$, since $Sp_{\mathcal{E}/\tilde{f}_X} \circ \epsilon^* = Sp_{\mathcal{Y}_0/\tilde{f}'_X}$ and $Sp_{\mathcal{Y}_0/\tilde{f}'_X}$ is injective, we have $\tilde{\lambda}_{t'} = k_{t*} \tilde{\lambda}_{t'}$ with $\tilde{\lambda}_{t'} \in F^p H_{\mathcal{F}, t'}^{2p}(\mathcal{E}_{\mathbb{C}, t'}^{an}, \mathbb{Q})$, where $k : \mathcal{F} \rightarrow \mathcal{E}$ is the exceptional locus of $\epsilon : \mathcal{E}_0 \rightarrow \mathcal{Y}_0$ (note that $\epsilon^* : H^{2p}(\mathcal{Y}_{0, t'}^{an}, \mathbb{Q}) \rightarrow H^{2p}(\mathcal{E}_{0, t'}^{an}, \mathbb{Q})$ is injective by Poincare Lefschetz duality), hence, assuming the Hodge conjecture for smooth projective varieties of dimension less or equal to $d-1$, $\mathbb{V}_S^p(\tilde{\lambda}) \subset E_{DR}^{2p}(\mathcal{E}_{\mathbb{C}}/S_{\mathbb{C}})$ is defined over $\bar{\mathbb{Q}}$ and its Galois conjugates are also components of the locus of Hodge classes,
- if $Sp_{\tilde{E}/\tilde{f}_X}(\tilde{\lambda}) \neq 0$, we have,

$$Sp_{\mathcal{E}/\tilde{f}_X}(\mathbb{V}_S^p(\tilde{\lambda})) = \mathbb{V}_S^p(Sp_{\mathcal{E}/\tilde{f}_X}(\tilde{\lambda})) = \overline{\mathbb{V}_{T^o}^p(Sp_{\mathcal{E}/\tilde{f}_X}(\tilde{\lambda}))} \cap p^{-1}(S_{\mathbb{C}}) \subset E_{DR, V_S}^{2p}(\mathcal{Y}_{T^o, \mathbb{C}}/T_{\mathbb{C}}^o)$$

where $p : E_{DR, V_0}^{2p}(\mathcal{Y}_{T^o, \mathbb{C}}/T_{\mathbb{C}}^o) \rightarrow \bar{T}_{\mathbb{C}}^o$ is the projection (the first equality is obvious since $Sp_{\mathcal{E}/\tilde{f}_X}$ is a morphism of variation of mixed Hodge structures over S , whereas for the second equality the inclusion \supset follows from [6] lemma 2.11 which state the invariance of $\tilde{\lambda}$ under the monodromy

$$\text{Im}(\pi_1(p(\mathbb{V}_{T^o}^p(\tilde{\lambda})) \cap D^*) \rightarrow \pi_1(D^*)), (D, D^*) \rightarrow (\bar{T}_{\mathbb{C}}^{o, an}, \bar{T}_{\mathbb{C}}^{o, an} \setminus S_{\mathbb{C}}^{an})$$

but by [5] theorem 1, the Hodge conjecture holds for projective hypersurfaces hence

$$\mathbb{V}_{T^o}^p(Sp_{\mathcal{E}/\tilde{f}_X}(\tilde{\lambda})) \subset E_{DR}^{2p}(\mathcal{Y}_{T^o, \mathbb{C}}/T_{\mathbb{C}}^o)$$

is defined over $\bar{\mathbb{Q}}$ and its Galois conjugates are also components of the locus of Hodge classes, hence $\mathbb{V}_S^p(\tilde{\lambda}) \subset E_{DR}^{2p}(\mathcal{E}_{\mathbb{C}}/S_{\mathbb{C}})$ is defined over $\bar{\mathbb{Q}}$ and its Galois conjugates are also components of the locus of Hodge classes,

Hence $\mathbb{V}_S^p(\lambda) \subset E_{DR}^{2p}(\mathcal{E}_{0,\mathbb{C}}/S_{\mathbb{C}})$ is defined over $\bar{\mathbb{Q}}$ and its Galois conjugates are also components of the locus of Hodge classes. Thus, by [4] theorem 4,

$$\lambda = [Z] \in H^{2p}(E_0^{an}, \mathbb{Q}), \text{ with } Z \in \mathcal{Z}^p(E_0).$$

Since $p \in \mathbb{Z}$ and $\lambda \in F^p H^{2p}(E_0^{an}, \mathbb{Q})$ are arbitrary, the Hodge conjecture holds for E_0 . Proposition 1(ii) and the fact that the Hodge conjecture is true for smooth projective varieties of dimension less or equal to $d-1$ by induction hypothesis then implies that Hodge conjecture holds for X , since X is birational to E_0 and $X, E_0 \in \text{PSmVar}(\mathbb{C})$ are connected of dimension d . \square

References

- [1] D.Abramovich, K.Karu, K.Matsuki, J.Włodarczyk, *Torification and compactification of birational maps*, AMS, 15, p531-572, 1999.
- [2] J.Ayoub, *The motivic vanishing cycles and the conservation conjecture*, Algebraic cycles and motives Vol 1, 3-54, London Math. Soc., Cambridge University Press, 2007.
- [3] J.Bouali, *The De Rham, complex and p-adic Hodge realization functor for relative motives of algebraic varieties over a field of characteristic zero*, preprint, arxiv 2203.10875, 2022.
- [4] J.Bouali, *De Rham logarithmic classes and Tate conjecture*, preprint, arxiv 2303.09932, hal 04034328, 2023.
- [5] J.Bouali, *Hodge conjecture for projective hypersurfaces*, preprint, arxiv 2312.09268, hal 04312948, 2023.
- [6] E.Catani, P.Deligne, A.Kaplan, *On the locus of Hodge classes*, AMS Volume 8, Number 2, 1995.
- [7] P.Deligne, *Theorie de Hodge II*, Pub. Math. IHES, Tome 40, p5-57, 1971.
- [8] C.Peters, J.Steenbrink, *Mixed Hodge Structures*, Springer Verlag, Volume 52, 2008.

LAGA UMR CNRS 7539
Université Paris 13, Sorbonne Paris Cité, 99 av Jean-Baptiste Clement,
93430 Villetaneuse, France,
bouali@math.univ-paris13.fr