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# Degeneration of families of projective hypersurfaces and Hodge conjecture

Johann Bouali

March 27, 2024

## Abstract

We prove by induction on dimension the Hodge conjecture for smooth complex projective varieties. Let  $X$  be a smooth complex projective variety. Then  $X$  is birational to a possibly singular projective hypersurface, hence to a smooth projective variety  $E_0$  which is a component of a normal crossing divisor  $E = \cup_{i=0}^r E_i \subset Y$  which is the singular fiber of a pencil  $f : Y \rightarrow \mathbb{A}^1$  of smooth projective hypersurfaces. Using the smooth hypersurface case ([5] theorem 1), the nearby cycle functor on mixed Hodge module with rational de Rham factor and the induction hypothesis, we prove that a Hodge class of  $E_0$  is absolute Hodge, more precisely the locus of Hodge classes inside the algebraic vector bundle given the De Rham cohomology of the rational deformation of  $E_0$  is defined over  $\mathbb{Q}$ . By [4] theorem 4, we get the Hodge conjecture for  $E_0$ . By the induction hypothesis we also have the Hodge conjecture for  $X$ .

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## 1 Introduction

The main result of this article is an inductive proof of the Hodge conjecture using the results of [4] and [5]. More precisely, assuming the Hodge conjecture for smooth projective varieties of dimension less or equal to  $d - 1$ , the property of satisfying the Hodge conjecture for connected smooth projective varieties of dimension  $d$  is a birational invariant (proposition 1). On the other hand, if  $X$  is a connected smooth projective variety of dimension  $d$ , then  $X$  is birational to a possibly singular hypersurface  $Y_{0_X} \subset \mathbb{P}_{\mathbb{C}}^{d+1}$  of degree  $l_X := \deg(\mathbb{C}(X)/\mathbb{C}(x_1, \dots, x_d))$ . Consider

$$f : \mathcal{Y}' = V(f) \subset \mathbb{P}_{\mathbb{C}}^{d-1} \times \mathbb{P}(l_X) \rightarrow \mathbb{P}(l_X)$$

the universal family of projective hypersurfaces of degree  $l_X$ , and  $0_X \in \mathbb{P}(l_X)$  the point corresponding to  $Y_{0_X}$ . Denote  $\Delta \subset \mathbb{P}(l_X)$  the discriminant locus parametrizing the hypersurfaces which are singular. Let  $\bar{S} := 0_X^{\mathbb{Q}} \subset \mathbb{P}(l_X)$  the  $\mathbb{Q}$ -Zariski closure of  $0_X$  inside  $\mathbb{P}(l_X)$ . If  $Y_{0_X}$  is smooth, then  $X$  satisfy the hodge conjecture since  $Y_{0_X}$  satisfy the Hodge conjecture by [5] and by the induction hypothesis since  $X$

is birational to  $Y_{0_X}$ . Hence, we may assume that  $Y_{0_X}$  is singular, that is  $0_X \in \Delta$ . Then,  $\bar{S} \subset \Delta$  since  $\Delta \subset \mathbb{P}(l_X)$  is defined over  $\mathbb{Q}$ . Let  $\bar{T} \subset \mathbb{P}(l_X)$  an irreducible closed subvariety of dimension  $\dim(\bar{S}) + 1$  defined over  $\mathbb{Q}$  such that  $\bar{S} \subset \bar{T}$  and  $T := \bar{T} \cap (\mathbb{P}(l_X) \setminus \Delta) \neq \emptyset$ . Since there always exists subvarieties of  $\mathbb{P}(l_X) \setminus \Delta$  of arbitrary dimension defined over  $\mathbb{Q}$  which are not weakly special (take complete intersections which intersect  $\Delta$  transversally), we may assume that  $T \subset \mathbb{P}(l_X) \setminus \Delta$  is not contained in a weakly special subvariety. We have then  $\bar{T} \cap \Delta = \bar{S} \cup \bar{S}'$  where  $\bar{S}' \subset \Delta$  is a divisor. We then consider the family of algebraic varieties defined over  $\mathbb{Q}$

$$\tilde{f}_X : \mathcal{Y}_{\bar{T}^o} \xrightarrow{\epsilon} \mathcal{Y}'_{T^o} \xrightarrow{f \times_{\mathbb{P}(l_X)} \bar{T}^o} \bar{T}^o$$

where

- $\tilde{f}_X : \mathcal{Y} \xrightarrow{\epsilon} \mathcal{Y}'_{\bar{T}} \xrightarrow{f \times_{\mathbb{P}(l_X)} \bar{T}} \bar{T}$  with  $\epsilon : (\mathcal{Y}, \mathcal{E}') \rightarrow (\mathcal{Y}'_{\bar{T}}, \mathcal{Y}_{\bar{S}} \cup \mathcal{Y}_{\bar{S}'})$  is a desingularization over  $\mathbb{Q}$ , in particular  $\mathcal{E}' = \mathcal{E} \cup \mathcal{E}''$  with  $\mathcal{E} = \cup_{i=0}^r \mathcal{E}_i := \tilde{f}_X^{-1}(\bar{S}) \subset \mathcal{Y}$  which is a normal crossing divisor, in particular  $\mathcal{Y}_T = \mathcal{Y}'_T$ ,
- $S \subset \bar{S}$  is the open subset such that  $S$  is smooth and  $\tilde{f}_X \times_{\bar{T}} S : \mathcal{E}_I := \cap_{i \in I} \mathcal{E}_i \rightarrow S$  is smooth projective for each  $I \subset [0, \dots, r]$ ,  $\bar{T}^o \subset \bar{T}$  is an open subset such that  $\bar{T}^o \cap \Delta = S$  and  $T^o := \bar{T}^o \cap T$  is smooth,
- $t \in \mathbb{S}(\mathbb{C})$  is the point such that  $\mathcal{Y}'_{\mathbb{C}, t} = Y_{0_X}$  and  $\epsilon_t : E_0 := \mathcal{E}_{0, \mathbb{C}, t} \rightarrow Y_{0_X}$  is surjective, in particular  $E_0$  is birational to  $Y_{0_X}$ , hence  $E_0$  is birational to  $X$ .

Assuming the Hodge conjecture for smooth projective varieties of dimension less or equal to  $d - 1$ , we then prove the Hodge conjecture for the  $E_i$ , more specifically for  $E_0$  as follows : Let  $p \in \mathbb{Z}$ . By the hard Lefschetz theorem on  $H^*(E_0)$ , we may assume that  $2p \leq d$ . We then have then a morphism of mixed Hodge module over  $S$  (in fact of geometric variation of mixed Hodge structures over  $S$ ) given by the specialization map

$$Sp_{\mathcal{E}/\tilde{f}_X} : E_{Hdg}^{2p}(\mathcal{E}/S) \rightarrow H^{2p} \psi_S E_{Hdg}(\mathcal{Y}_{T^o}/T^o),$$

whose de Rham part are morphism of filtered vector bundle over  $S$  with Gauss-Manin connexion defined over  $\mathbb{Q}$ , and whose kernel of the part supported on  $\mathcal{E}_0$  is supported a closed subset  $\mathcal{F} \subset \mathcal{E}_0$  with  $\text{codim}(\mathcal{F}_{t'}, \mathcal{E}_{0, t'}) = 1$  for all  $t' \in S(\mathbb{C})$ . The fact that the kernel is supported on a closed subset of codimension one on fibers follows from the fact that  $H^{2p-1}(Y_s, \mathbb{Q}) = 0 = H_{Y_0}^{2p-1}(Y)$ , by the weak Lefschetz theorem since  $Y_s$  and  $Y_0$  are projective hypersurfaces ( $2p \leq d$ ). Consider the exact sequence of variation of mixed Hodge structure over  $S$

$$0 \rightarrow W_{2p-2} E_{Hdg}^{2p}(\mathcal{E}/S) \rightarrow E_{Hdg}^{2p}(\mathcal{E}/S) \xrightarrow{(i^*)_{0 \leq i \leq r} := q} \oplus_{i=0}^r E_{Hdg}^{2p}(\mathcal{E}_i/S) \rightarrow 0.$$

It induces the exact sequence of presheaves on  $S_{\mathbb{C}}^n$ ,

$$J(W_{2p-2} E_{Hdg}^{2p}(\mathcal{E}_{\mathbb{C}}/S_{\mathbb{C}})) := \text{Ext}^1(\mathbb{Q}_S^{Hdg}, W_{2p-2} E_{Hdg}^{2p}(\mathcal{E}_{\mathbb{C}}/S_{\mathbb{C}})) \xrightarrow{\iota} \text{Ext}^1(\mathbb{Q}_S^{Hdg}, E_{Hdg}^{2p}(\mathcal{E}_{\mathbb{C}}/S_{\mathbb{C}}))$$

$$HL^{2p,p}(\mathcal{E}_{\mathbb{C}}/S_{\mathbb{C}}) \xrightarrow{q} \oplus_{i=0}^r HL^{2p,p}(\mathcal{E}_{i, \mathbb{C}}/S_{\mathbb{C}}) \xrightarrow{\epsilon}$$

where  $HL^{2p,p}(\mathcal{E}_{\mathbb{C}}/S_{\mathbb{C}}) \subset E_{DR}^{2p}(\mathcal{E}_{\mathbb{C}}/S_{\mathbb{C}})$ ,  $HL^{2p,p}(\mathcal{E}_{i, \mathbb{C}}/S_{\mathbb{C}}) \subset E_{DR}^{2p}(\mathcal{E}_{i, \mathbb{C}}/S_{\mathbb{C}})$  are the locus of Hodge classes. Let  $\lambda \in F^p H^{2p}(E_0^{an}, \mathbb{Q})$ , where we recall  $E_0 = \mathcal{E}_{0, \mathbb{C}, t}$ . Consider then  $\tilde{\lambda} \in H^{2p}(E^{an}, \mathbb{Q})$ , such that  $q(\tilde{\lambda}) = \lambda$ , and  $\tilde{\lambda} := Sp_{\mathcal{E}/\tilde{f}_X}(\tilde{\lambda}) \in Sp_{E/\tilde{f}_X}(H^{2p}(E^{an}, \mathbb{Q}))$ , Since the monodromy of  $R^{2p} \tilde{f}_{X*} \mathbb{Q}_{\mathcal{Y}_{T^o, \mathbb{C}}}^{an}$  is irreducible, we have

$$Sp_{\mathcal{E}/\tilde{f}_X}(\ker q) := Sp_{\mathcal{E}/\tilde{f}_X}(W_{2p-2} E_{DR}^{2p}(\mathcal{E}/S)) = 0$$

The locus of Hodge classes passing through  $\lambda$

$$\mathbb{V}_S^p(\lambda) := \mathbb{V}_S(\lambda) \cap F^p E_{DR}^{2p}(\mathcal{E}_{0, \mathbb{C}}/S_{\mathbb{C}}) \subset E_{DR}^{2p}(\mathcal{E}_{0, \mathbb{C}}/S_{\mathbb{C}}),$$

inside the De Rham vector bundle of  $\tilde{f}_X \circ i_{E_0} : \mathcal{E}_0 \rightarrow S$  thus satisfy

$$Sp_{\mathcal{E}/\tilde{f}_X}(q^{-1}(\mathbb{V}_S^p(\lambda))) = \mathbb{V}_S^p(Sp_{\mathcal{E}/\tilde{f}_X}(\tilde{\lambda})) \subset E_{DR, V_S}^{2p}(\mathcal{Y}_{T^o, \mathbb{C}}/T_{\mathbb{C}}^o),$$

where

- $\mathbb{V}_S(\lambda) \subset E_{DR}^{2p}(\mathcal{E}_{0, \mathbb{C}}/S_{\mathbb{C}})$ ,  $\mathbb{V}_S(\tilde{\lambda}) \subset E_{DR}^{2p}(\mathcal{E}_{\mathbb{C}}/S_{\mathbb{C}})$ , are the flat leaves, e.g.  $\mathbb{V}_S(\lambda) := \pi_S(\lambda \times \tilde{S}_{\mathbb{C}}^{an})$  where  $\pi_S : H^{2p}(E^{an}, \mathbb{C}) \times \tilde{S}_{\mathbb{C}}^{an} \rightarrow E_{DR}^{2p, an}(\mathcal{E}_{\mathbb{C}}/S_{\mathbb{C}})$  is the morphism induced by the universal covering  $\pi_S : \tilde{S}_{\mathbb{C}}^{an} \rightarrow S_{\mathbb{C}}^{an}$ ,
- $q := q \otimes \mathbb{C} : E_{DR}^{2p}(\mathcal{E}_{\mathbb{C}}/S_{\mathbb{C}}) \rightarrow E_{DR}^{2p}(\mathcal{E}_{0, \mathbb{C}}/S_{\mathbb{C}})$  is the quotient map.

Now,

- if  $Sp_{E/f_X}(\tilde{\lambda}) = 0$ , assuming the Hodge conjecture for smooth projective varieties of dimension less or equal to  $d - 1$ ,  $\mathbb{V}_S^p(\tilde{\lambda}) \subset E_{DR}^{2p}(\mathcal{E}_{\mathbb{C}}/S_{\mathbb{C}})$  is an algebraic subvariety defined over  $\bar{\mathbb{Q}}$  and its Galois conjugate are also components of the locus of Hodge classes,
- if  $Sp_{E/f_X}(\tilde{\lambda}) \neq 0$ , by [5] theorem 1, the Hodge conjecture holds for projective hypersurfaces, hence

$$Sp_{\mathcal{E}/\tilde{f}_X}(\mathbb{V}_S^p(\tilde{\lambda})) = \mathbb{V}_S^p(Sp_{\mathcal{E}/\tilde{f}_X}(\tilde{\lambda})) = \overline{\mathbb{V}_{T^o}^p(Sp_{E/f_X}(\tilde{\lambda}))} \cap p^{-1}(S) \subset E_{DR, V_S}^{2p}(\mathcal{Y}_{T^o, \mathbb{C}}/T_{\mathbb{C}}^o)$$

is an algebraic subvariety defined over  $\bar{\mathbb{Q}}$  and its Galois conjugate are also components of the locus of Hodge classes, where the second equality follows from [6] lemma 2.11.

Hence,  $\mathbb{V}_S^p(\lambda) \subset E_{DR}^{2p}(\mathcal{E}_{0\mathbb{C}}/S_{\mathbb{C}})$  is an algebraic subvariety defined over  $\bar{\mathbb{Q}}$  and its Galois conjugate are also components of the locus of Hodge classes. By [4] theorem 4, we get

$$\lambda = [Z] \in H^{2p}(E_0^{an}, \mathbb{Q}), \quad Z \in \mathcal{Z}^p(E_0).$$

We recall the key point is that since  $\mathbb{V}_S^p(\lambda) \subset E_{DR}^{2p}(\mathcal{E}_{0\mathbb{C}}/S_{\mathbb{C}})$  is an algebraic subvariety defined over  $\bar{\mathbb{Q}}$  and its Galois conjugate are also components of the locus of Hodge classes, we have for each  $\theta \in \text{Aut}(\mathbb{C}, \mathbb{Q})$ ,

$$\theta(\lambda) \in \text{Gal}(\bar{\mathbb{Q}}, \mathbb{Q})(\mathbb{V}_S^p(\lambda)_{\theta}) \subset E_{DR}^{2p}(\mathcal{E}_{0\mathbb{C}, \theta}/S_{\mathbb{C}, \theta}),$$

which implies that  $\theta(\lambda) \in H^{2p}(E_{0, \theta}^{an}, \mathbb{Q}) \subset H^{2p}(E_{0, \theta}^{an}, \mathbb{C})$ , that is  $\lambda \in H^{2p}(E_0^{an}, \mathbb{Q})$  is absolute Hodge and we apply [4] corollary 1. Since  $p \in \mathbb{Z}$  and  $\lambda \in F^p H^{2p}(E_0^{an}, \mathbb{Q})$  are arbitrary, this proves the Hodge conjecture for  $E_0$ . Since  $X$  is birational to  $E_0$ , the Hodge conjecture for  $E_0$  and the induction hypothesis implies the Hodge conjecture for  $X$ .

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## 2 Preliminaries and notations

### 2.1 Notations

- Denote by Top the category of topological spaces and RTop the category of ringed spaces.
- Denote by Cat the category of small categories and RCat the category of ringed topoi.
- For  $\mathcal{S} \in \text{Cat}$  and  $X \in \mathcal{S}$ , we denote  $\mathcal{S}/X \in \text{Cat}$  the category whose objects are  $Y/X := (Y, f)$  with  $Y \in \mathcal{S}$  and  $f : Y \rightarrow X$  is a morphism in  $\mathcal{S}$ , and whose morphisms  $\text{Hom}((Y', f'), (Y, f))$  consists of  $g : Y' \rightarrow Y$  in  $\mathcal{S}$  such that  $f \circ g = f'$ .
- Let  $(\mathcal{S}, O_{\mathcal{S}}) \in \text{RCat}$  a ringed topoi with topology  $\tau$ . For  $F \in C_{O_{\mathcal{S}}}(\mathcal{S})$ , we denote by  $k : F \rightarrow E_{\tau}(F)$  the canonical flasque resolution in  $C_{O_{\mathcal{S}}}(\mathcal{S})$  (see [3]). In particular for  $X \in \mathcal{S}$ ,  $H^*(X, E_{\tau}(F)) \xrightarrow{\sim} \mathbb{H}_{\tau}^*(X, F)$ .

- For  $f : \mathcal{S}' \rightarrow \mathcal{S}$  a morphism with  $\mathcal{S}, \mathcal{S}' \in \text{RCat}$ , endowed with topology  $\tau$  and  $\tau'$  respectively, we denote for  $F \in C_{O_{\mathcal{S}}}(\mathcal{S})$  and each  $j \in \mathbb{Z}$ ,

$$\begin{aligned} - f^* &:= H^j \Gamma(\mathcal{S}, k \circ \text{ad}(f^*, f_*)(F)) : \mathbb{H}^j(\mathcal{S}, F) \rightarrow \mathbb{H}^j(\mathcal{S}', f^*F), \\ - f^* &:= H^j \Gamma(\mathcal{S}, k \circ \text{ad}(f^{*mod}, f_*)(F)) : \mathbb{H}^j(\mathcal{S}, F) \rightarrow \mathbb{H}^j(\mathcal{S}', f^{*mod}F), \end{aligned}$$

the canonical maps.

- For  $m : A \rightarrow B$ ,  $A, B \in C(\mathcal{A})$ ,  $\mathcal{A}$  an additive category, we denote  $c(A) : \text{Cone}(m : A \rightarrow B) \rightarrow A[1]$  and  $c(B) : B \rightarrow \text{Cone}(m : A \rightarrow B)$  the canonical maps.
- Denote by  $\text{Sch} \subset \text{RTop}$  the subcategory of schemes (the morphisms are the morphisms of locally ringed spaces). We denote by  $\text{PSch} \subset \text{Sch}$  the full subcategory of proper schemes. For a field  $k$ , we consider  $\text{Sch}/k := \text{Sch}/\text{Spec } k$  the category of schemes over  $\text{Spec } k$ . The objects are  $X := (X, a_X)$  with  $X \in \text{Sch}$  and  $a_X : X \rightarrow \text{Spec } k$  a morphism and the morphisms are the morphisms of schemes  $f : X' \rightarrow X$  such that  $f \circ a_{X'} = a_X$ . We then denote by
  - $\text{Var}(k) = \text{Sch}^{ft}/k \subset \text{Sch}/k$  the full subcategory consisting of algebraic varieties over  $k$ , i.e. schemes of finite type over  $k$ ,
  - $\text{PVar}(k) \subset \text{QPVar}(k) \subset \text{Var}(k)$  the full subcategories consisting of quasi-projective varieties and projective varieties respectively,
  - $\text{PSmVar}(k) \subset \text{SmVar}(k) \subset \text{Var}(k)$ ,  $\text{PSmVar}(k) := \text{PVar}(k) \cap \text{SmVar}(k)$ , the full subcategories consisting of smooth varieties and smooth projective varieties respectively.
- Denote by  $\text{AnSp}(\mathbb{C}) \subset \text{RTop}$  the subcategory of analytic spaces over  $\mathbb{C}$ , and by  $\text{AnSm}(\mathbb{C}) \subset \text{AnSp}(\mathbb{C})$  the full subcategory of smooth analytic spaces (i.e. complex analytic manifold).
- For  $X \in \text{Var}(k)$  and  $X = \cup_{i \in I} X_i$  with  $i_i : X_i \hookrightarrow X$  closed embeddings, we denote  $X_{\bullet} \in \text{Fun}(\Delta, \text{Var}(k))$  the associated simplicial space, with for  $J \subset I$ ,  $i_{IJ} : X_I := \cap_{i \in I} X_i \hookrightarrow X_J := \cap_{i \in J} X_i$  the closed embedding.
- Let  $(X, O_X) \in \text{RTop}$ . We consider its De Rham complex  $\Omega_X^{\bullet} := DR(X)(O_X)$ .
  - Let  $X \in \text{Sch}$ . Considering its De Rham complex  $\Omega_X^{\bullet} := DR(X)(O_X)$ , we have for  $j \in \mathbb{Z}$  its De Rham cohomology  $H_{DR}^j(X) := \mathbb{H}^j(X, \Omega_X^{\bullet})$ .
  - Let  $X \in \text{Var}(k)$ . Considering its De Rham complex  $\Omega_X^{\bullet} := \Omega_{X/k}^{\bullet} := DR(X/k)(O_X)$ , we have for  $j \in \mathbb{Z}$  its De Rham cohomology  $H_{DR}^j(X) := \mathbb{H}^j(X, \Omega_X^{\bullet})$ . The differentials of  $\Omega_X^{\bullet} := \Omega_{X/k}^{\bullet}$  are by definition  $k$ -linear, thus  $H_{DR}^j(X) := \mathbb{H}^j(X, \Omega_X^{\bullet})$  has a structure of a  $k$  vector space.
  - Let  $X \in \text{AnSp}(\mathbb{C})$ . Considering its De Rham complex  $\Omega_X^{\bullet} := DR(X)(O_X)$ , we have for  $j \in \mathbb{Z}$  its De Rham cohomology  $H_{DR}^j(X) := \mathbb{H}^j(X, \Omega_X^{\bullet})$ .
- For  $Y \in \text{Var}(\mathbb{Q})$  and  $X = V(I) = V(f_1, \dots, f_r) \subset Y_{\mathbb{C}}$ , we consider  $\mathcal{X} = V(\tilde{I}) = V(\tilde{f}_1, \dots, \tilde{f}_r) \subset Y \times \mathbb{A}_{\mathbb{Q}}^s$  the canonical rational deformation, where

$$\begin{aligned} \text{if } f &= \sum_{I \subset [1, \dots, d]^n} a_I x_1^{n_1} \cdots x_n^{n_n} \in \mathbb{C}[x_1, \dots, x_n], \\ \tilde{f} &= \sum_{I \subset [1, \dots, d]^n} a_I x_1^{n_1} \cdots x_n^{n_n} \in \mathbb{Q}[(a_I)_{I \subset [1, \dots, d]^n}, x_1, \dots, x_n]. \end{aligned}$$

- For  $X \in \text{AnSp}(\mathbb{C})$ , we denote  $\alpha(X) : \mathbb{C}_X \hookrightarrow \Omega_X^{\bullet}$  the embedding in  $C(X)$ . For  $X \in \text{AnSm}(\mathbb{C})$ ,  $\alpha(X) : \mathbb{C}_X \hookrightarrow \Omega_X^{\bullet}$  is an equivalence usu local by Poincare lemma.

- We denote  $\mathbb{I}^n := [0, 1]^n \in \text{Diff}(\mathbb{R})$  (with boundary). For  $X \in \text{Top}$  and  $R$  a ring, we consider its singular cochain complex

$$C_{\text{sing}}^*(X, R) := (\mathbb{Z} \text{Hom}_{\text{Top}}(\mathbb{I}^*, X)^\vee) \otimes R$$

and for  $l \in \mathbb{Z}$  its singular cohomology  $H_{\text{sing}}^l(X, R) := H^n C_{\text{sing}}^*(X, R)$ . For  $f : X' \rightarrow X$  a continuous map with  $X, X' \in \text{Top}$ , we have the canonical map of complexes

$$f^* : C_{\text{sing}}^*(X, R) \rightarrow C_{\text{sing}}^*(X', R), \sigma \mapsto f^* \sigma := (\gamma \mapsto \sigma(f \circ \gamma)).$$

In particular, we get by functoriality the complex

$$C_{X, R \text{sing}}^* \in C_R(X), (U \subset X) \mapsto C_{\text{sing}}^*(U, R)$$

We recall that

- For  $X \in \text{Top}$  locally contractible, e.g.  $X \in \text{CW}$ , and  $R$  a ring, the inclusion in  $C_R(X)$   $c_X : R_X \rightarrow C_{X, R \text{sing}}^*$  is by definition an equivalence top local and that we get by the small chain theorem, for all  $l \in \mathbb{Z}$ , an isomorphism  $H^l c_X : H^l(X, R_X) \xrightarrow{\sim} H_{\text{sing}}^l(X, R)$ .
- For  $X \in \text{Diff}(\mathbb{R})$ , the restriction map

$$r_X : \mathbb{Z} \text{Hom}_{\text{Diff}(\mathbb{R})}(\mathbb{I}^*, X)^\vee \rightarrow C_{\text{sing}}^*(X, R), w \mapsto w : (\phi \mapsto w(\phi))$$

is a quasi-isomorphism by Whitney approximation theorem.

- Let  $S \in \text{AnSm}(\mathbb{C})$  and  $L \in \text{Shv}(S)$  a local system. Consider  $E := L \otimes \mathcal{O}_S \in \text{Vect}_{\mathcal{D}}(S)$  the corresponding holomorphic vector bundle with integrable connection  $\nabla$ . Let  $\pi_S : \tilde{S} \rightarrow S$  be the universal covering. Consider the canonical fiber

$$L_0 := \Gamma(\tilde{S}, \pi_S^* L) = \Gamma(\tilde{S}, \pi_S^{* \text{mod}} E)^\nabla.$$

For  $\lambda \in L_0$ , we will consider the flat leaf

$$\mathbb{V}_S(\lambda) := \pi_S(\lambda \times \tilde{S}) \subset E, \pi_S : L_0 \times \tilde{S} \rightarrow E, \pi_S(\nu, z) := (\nu(z), \pi_S(z)).$$

Note that for  $t \in S$ ,  $\mathbb{V}_S(\lambda) \cap p_S^{-1}(t) = \pi_1(S, t)(\lambda)$  is the orbit of  $\lambda$  under the monodromy action, where  $p_S : E \rightarrow S$  is the projection.

## 2.2 Birational projective varieties and Hodge conjecture

In this subsection, we recall that assuming the Hodge conjecture for smooth projective varieties of dimension less or equal to  $d - 1$ , the property of satisfying the Hodge conjecture for connected smooth projective varieties of dimension  $d$  is a birational invariant :

**Proposition 1.** (i) Let  $X \in \text{PSmVar}(\mathbb{C})$  connected of dimension  $d$ . Let  $\epsilon : \tilde{X}_Z \rightarrow X$  be the blow up of  $X$  along a smooth closed subvariety  $Z \subset X$ . Assume the Hodge conjecture hold for smooth complex projective varieties of dimension less or equal to  $d - 1$ . Then the Hodge conjecture hold for  $X$  if and only if it hold for  $\tilde{X}_Z$ .

(ii) Let  $X, X' \in \text{PSmVar}(\mathbb{C})$  connected of dimension  $d$ . Assume the Hodge conjecture hold for smooth complex projective varieties of dimension less or equal to  $d - 1$ . If  $X$  is birational to  $X'$ , then the Hodge conjecture hold for  $X$  if and only if it hold for  $X'$

*Proof.* (i): Follows from the fact that

$$(\epsilon^*, \oplus_{l=1}^c i_{E^*}((-).h^{l-1})) : H^k(X, \mathbb{Q}) \oplus \oplus_{l=1}^c H^{k-2l}(E, \mathbb{Q}) \rightarrow H^k(\tilde{X}_Z, \mathbb{Q})$$

is an isomorphism of Hodge structures for each  $k \in \mathbb{Z}$ , where  $c = \text{codim}(Z, X)$ ,  $i_E : E \hookrightarrow \tilde{X}_Z$  is the closed embedding,  $\epsilon|_E : E \rightarrow Z$  being a projective vector bundle.

(ii): Follows from (i) since if  $\pi : X^o \rightarrow X'$  is a birational map,  $X^o \subset X$  being an open subset, then  $X$  is connected to  $X'$  by a sequence of blow up of smooth projective varieties with smooth center by [1].  $\square$

### 2.3 A family of smooth projective hypersurfaces associated to a smooth projective variety

In this subsection, we recall that given a smooth complex projective variety  $X$ , there exists a family of smooth complex projective hypersurfaces which degenerates into a normal crossing divisor with one irreducible component birational to  $X$ .

**Proposition 2.** *Let  $X \in \text{PSmVar}(\mathbb{C})$  connected of dimension  $d$ . Then there exists a family of smooth projective hypersurfaces  $f_X : Y \rightarrow \mathbb{A}_{\mathbb{C}}^1$  with*

- $Y \in \text{SmVar}(\mathbb{C})$ ,  $f_X$  flat projective,
- $Y_s := f_X^{-1}(s) \subset \mathbb{P}_{\mathbb{C}}^{d+1}$  for  $s \in \mathbb{A}_{\mathbb{C}}^1 \setminus 0$  are smooth projective hypersurfaces,
- $E := f_X^{-1}(0) = \cup_{i=0}^r E_i = \epsilon^{-1}(Y_0) \subset Y$  is a normal crossing divisor, where  $Y_0 \subset \mathbb{P}_{\mathbb{C}}^{d+1}$  is a possibly singular hypersurface birational to  $X$ ,  $\epsilon : (Y, E) \rightarrow (Y', Y_0)$  a desingularization, and  $E_0$  is birational to  $X$ ,

*Proof.* See [2] lemma 1.5.8 . It use the fact that  $X$  is birational to a possibly singular projective hypersurface  $Y_0$  and consider the desingularization of a generic pencil  $f'_X : Y' = V(f) \subset \mathbb{P}_{\mathbb{C}}^{d+1} \times \mathbb{A}_{\mathbb{C}}^1 \rightarrow \mathbb{A}_{\mathbb{C}}^1$  passing through  $Y_0$  and  $f_X = f'_X \circ \epsilon$ .  $\square$

### 2.4 The nearby cycle functor and the specialization map

Recall from [3] that for  $k \subset \mathbb{C}$  a subfield and  $S \in \text{Var}(k)$  quasi-projective, and  $l : S \hookrightarrow \tilde{S}$  a closed embedding with  $\tilde{S} \in \text{SmVar}(k)$ , we have the full subcategory

$$\iota : MHM_{k, gm}(S) \hookrightarrow \text{PSh}_{\mathcal{D}fil, S}(\tilde{S}) \times_I P(S_{\mathbb{C}}^{an})$$

consisting of geometric mixed Hodge module whose De Rham part is defined over  $k$ , where, using the fact that the  $V$ -filtration is defined over  $k$ ,  $\text{PSh}_{\mathcal{D}(1,0)fil, S}(\tilde{S}) \times_I P_{fil}(S_{\mathbb{C}}^{an})$  is the category

- whose objects are  $((M, F, W), (K, W), \alpha)$  where  $(M, F, W)$  is a filtered  $D_{\tilde{S}}$  module supported on  $S$ , where  $l : S \hookrightarrow \tilde{S}$  is a closed embedding and  $\tilde{S} \in \text{SmVar}(k)$ ,  $(K, W)$  is a filtered Perverse sheaf on  $S_{\mathbb{C}}^{an}$ , and  $\alpha : l_*(K, W) \otimes \mathbb{C} \rightarrow DR(\tilde{S})((M, W)^{an})$  is an isomorphism in  $D_{fil}(\tilde{S}_{\mathbb{C}}^{an})$ .
- whose morphism between  $((M, F, W), (K, W), \alpha)$  and  $((M', F, W), (K', W), \alpha')$ , are couples of morphisms  $(\phi : (M, F, W) \rightarrow (M', F, W), \psi : (K, W) \rightarrow (K', W))$  such that  $\phi \circ \alpha = \alpha' \circ \psi$ .

Then, by [3], we get for  $S \in \text{Var}(k)$  quasi-projective and  $l : S \hookrightarrow \tilde{S}$  is a closed embedding with  $\tilde{S} \in \text{SmVar}(k)$  an embedding

$$\iota : D(MHM_{k, gm}(S)) \hookrightarrow D_{\mathcal{D}fil, S}(\tilde{S}) \times_I D(S_{\mathbb{C}}^{an})$$

consisting of mixed Hodge module whose De Rham part is defined over  $k$ , where  $D_{\mathcal{D}(1,0)fil, S}(\tilde{S}) \times_I D_{fil, c}(S_{\mathbb{C}}^{an})$  is the category

- whose objects are  $((M, F, W), (K, W), \alpha)$  where  $(M, F, W)$  is a complex of filtered  $D_{\tilde{S}}$  module supported on  $S$ ,  $(K, W)$  is a filtered complex of presheaves on  $S_{\mathbb{C}}^{an}$  whose cohomology are constructible sheaves, and  $\alpha : l_*(K, W) \otimes \mathbb{C} \rightarrow DR(\tilde{S})((M, W)^{an})$  is an isomorphism in  $D_{fil}(\tilde{S}_{\mathbb{C}}^{an})$ .
- whose morphism are given in [3],

We have then the six functors formalism :

$$\begin{aligned} D(MHM_{k, gm}(-)) : \text{Var}(k) &\mapsto \text{TriCat}, \quad S \mapsto D(MHM_{k, gm}(S)), \quad (f : T \rightarrow S) \mapsto \\ (f^*, f_*) : D(MHM_{k, gm}(S)) &\rightarrow D(MHM_{k, gm}(T)), \quad (f!, f^!) : D(MHM_{k, gm}(T)) \rightarrow D(MHM_{k, gm}(S)) \end{aligned}$$

and for  $D \subset S$  a (Cartier) divisor the nearby cycle functor

$$\begin{aligned} \psi_D &: D(MHM_{k,gm}(S)) \rightarrow D(MHM_{k,gm}(D)), \\ \psi_D((M, F, W), (K, W), \alpha) &:= (\psi_D(M, F, W), \psi_D(K, W), \psi_D(\alpha)). \end{aligned}$$

For  $T \in \text{Var}(k)$  and  $\epsilon_\bullet : T_\bullet \rightarrow T$  a desingularization with  $T_\bullet \in \text{Fun}(\Delta, \text{SmVar}(k))$ , we have

$$\mathbb{Z}_T^{Hdg} := a_T^{*Hdg} \mathbb{Z}^{Hdg} = (\epsilon_{\bullet*} E_{zar}(O_{T_\bullet}, F, W), \epsilon_{\bullet*} E_{usu}(\mathbb{Z}_{T_\bullet^{an}}, W), \alpha(T_\bullet))$$

For  $f : X \rightarrow S$  a projective morphism with  $X, S \in \text{SmVar}(k)$ , we denote

$$E_{Hdg}(X/S) := f_* \mathbb{Z}_X^{Hdg} := \left( \int f(O_X, F), Rf_* \mathbb{Z}_{X_\mathbb{C}^{an}}, f_* \alpha(X) \right) \in D(MHM_{k,gm}(S)).$$

For  $f : X \hookrightarrow \mathbb{P}^N \times S \xrightarrow{p} S$  a projective morphism with  $X, S \in \text{Var}(k)$ ,  $S$  smooth, we denote

$$E_{Hdg}(X/S) := f_* \mathbb{Z}_X^{Hdg} := \left( \int_p \Gamma_X^{Hdg}(O_{\mathbb{P}^N \times S}, F), Rf_* \mathbb{Z}_{X_\mathbb{C}^{an}}, f_* \alpha(X) \right) \in D(MHM_{k,gm}(S)).$$

We then have the specialization map :

**Definition-Proposition 1.** *Let  $k$  be field of characteristic zero.*

(i) *Let  $f' : X' \rightarrow T$  be a projective morphism with  $X', T \in \text{SmVar}(k)$ . Assume that*

- $f^\circ := f \otimes_T T : X^\circ \rightarrow T^\circ$  is smooth,  $T^\circ \subset T$  is an open subset,
- $X'_S := f'^{-1}(S) \subset X'$ ,  $S := T \setminus T^\circ$ . We have by definition  $X^\circ = X' \setminus X'_S$ .

We have then, see [3], the distinguished triangles in  $D(MHM_{k,gm}(X_S))$ ,

$$\begin{aligned} \mathbb{Z}_{X'_S}^{Hdg} &:= i_{X'_S}^* \mathbb{Z}_{X'}^{Hdg} \xrightarrow{sp_{X'_S/X'}} \psi_{X'_S} \mathbb{Z}_{X^\circ}^{Hdg} := (\psi_{X'_S}(O_{X'}, F), \psi_{X'_S} \mathbb{Z}_{X^\circ}, \psi_{X'_S} \alpha(X')) \\ &\xrightarrow{c(\psi_{X'_S} \mathbb{Z}_{X^\circ}^{Hdg})} \phi_{X'_S} \mathbb{Z}_{X^\circ}^{Hdg} := \text{Cone}(sp_{X'/X_S}) \xrightarrow{c(\mathbb{Z}_{X'_S}^{Hdg})} \mathbb{Z}_{X'_S}^{Hdg}[1]. \end{aligned}$$

Applying the functor  $(f' \circ i_{X'_S})_*$ , we obtain the generalized distinguished triangle in  $D(MHM_{k,gm}(S))$

$$\begin{aligned} E_{Hdg}(X'_S/S) &\xrightarrow{sp_{X'_S/f} := (f \circ i_{X'_S})_* sp_{X'_S/X'}} \\ (f' \circ i_{X'_S})_* (\psi_{X'_S} \mathbb{Z}_{X^\circ}^{Hdg}) &= \psi_0 E_{Hdg}(X^\circ/T^\circ) \xrightarrow{(f' \circ i_{X'_S})_* c(\Phi_{X_0} \mathbb{Z}_{X^\circ}^{Hdg})} \\ (f' \circ i_{X'_S})_* (\phi_{X'_S} \mathbb{Z}_{X^\circ}^{Hdg}) &= \phi_S E_{Hdg}(X^\circ/T^\circ) \xrightarrow{(f' \circ i_{X'_S})_* c(\mathbb{Z}_{X'}^{Hdg})} E_{Hdg}(X'_S/S)[1] \end{aligned}$$

(i)' *If for each  $t \in T^\circ(\mathbb{C})$  we have  $H^{k-1}(X'_t, \mathbb{Q}) = 0$  and for each  $t' \in S(\mathbb{C})$  we have  $H_{X'_t'}^{k-1}(X', \mathbb{Q}) = 0$ , then the induced map on cohomology*

$$sp_{X'_S/f} : E_{Hdg}^k(X'_S/S) \rightarrow H^k \psi_S E_{Hdg}(X^\circ/T^\circ)$$

*is injective.*

(ii) *Let  $f : X \rightarrow T$  be a projective morphism with  $X, T \in \text{SmVar}(k)$ . Assume that*

- $f^\circ := f \otimes_T T^\circ : X^\circ \rightarrow T^\circ$  is smooth,  $T^\circ \subset T$  an open subset
- $S := T \setminus T^\circ$ ,  $E := f^{-1}(S) = \cup_{i=0}^r E_i \subset X$  is a normal crossing divisor and that for  $I \subset [0, \dots, r]$ ,  $f_{0,I} := f|_{E_I} : E_I := \cap_{i \in I} E_i \rightarrow S$  are smooth. We have by definition  $X^\circ = X \setminus E$ .



Denote for  $I \subset J \subset [0, \dots, r]$ ,  $i_{JI} : E_J \hookrightarrow E_I$  and  $i_I : E_I \hookrightarrow E$ ,  $i_E : E \hookrightarrow X$  the closed embeddings. In this particular case, c.f. [8], the distinguished triangle in  $D(\text{MHM}_{k, \text{gm}}(E))$  given in (i) become

$$\begin{aligned} \mathbb{Z}_E^{\text{Hdg}} &:= i_E^* \mathbb{Z}_X^{\text{Hdg}} = ((i_{\bullet*} \Omega_{E^\bullet}, F, W), (\mathbb{Z}_{E^\bullet}, W), \alpha(E_\bullet)) \xrightarrow{sp_{E/X}} \\ \psi_E \mathbb{Z}_{X^\circ}^{\text{Hdg}} &= ((i_E^{*\text{mod}} \Omega_{X/\mathbb{A}^1 \times S}(\log E), F, W), \psi_E \mathbb{Z}_{X^\circ}, \psi_E \alpha(X)) \xrightarrow{c(\psi_E \mathbb{Z}_{X^\circ}^{\text{Hdg}})} \\ \phi_E \mathbb{Z}_{X^\circ}^{\text{Hdg}} &:= \text{Cone}(sp_{E/X}) \xrightarrow{c(\mathbb{Z}_E^{\text{Hdg}})} \mathbb{Z}_E^{\text{Hdg}}[1]. \end{aligned}$$

Applying the functor  $(f \circ i_E)_*$ , we obtain the generalized distinguished triangle in  $D(\text{MHM}_{k, \text{gm}}(S))$

$$\begin{aligned} E_{\text{Hdg}}(E/S) &\xrightarrow{sp_{E/f} := (f \circ i_E)_* sp_{E/X}} (f \circ i_E)_*(\psi_E \mathbb{Z}_{X^\circ}^{\text{Hdg}}) = \psi_S E_{\text{Hdg}}(X^\circ / ((\mathbb{A}^1 \setminus 0) \times S)) \\ \xrightarrow{(f \circ i_E)_* c(\psi_E \mathbb{Z}_{X^\circ}^{\text{Hdg}})} &(f \circ i_E)_*(\phi_E \mathbb{Z}_{X^\circ}^{\text{Hdg}}) = \phi_S E_{\text{Hdg}}(X^\circ / ((\mathbb{A}^1 \setminus 0) \times S)) \xrightarrow{(f \circ i_E)_* c(\mathbb{Z}_X^{\text{Hdg}})} E_{\text{Hdg}}(E/S)[1]. \end{aligned}$$

If  $f = f' \circ \epsilon$  where  $\epsilon : (X, E) \rightarrow (X, X_S)$  is a desingularization, we have  $sp_{X_S/X'} = sp_{E/X} \circ \epsilon^*$ .

*Proof.* (i)': Follows from the exact sequence

$$\dots \rightarrow H_{X_{t'}}^l(X'_C, \mathbb{Q}) \rightarrow \mathbb{H}^l(X_{t'}, \phi_S \mathbb{Z}_{X_C^\circ}) \xrightarrow{V} \mathbb{H}^l(X_{t'}, \psi_S \mathbb{Z}_{X_C^\circ}) \rightarrow H_{X_{t'}}^{l+1}(X'_C, \mathbb{Q}) \rightarrow \dots$$

given in [8], where  $C \subset T$  is a transversal slice to  $S$  passing through  $t'$ .  $\square$

### 3 Hodge conjecture for smooth projective varieties

**Theorem 1.** *Let  $X \in \text{PSmVar}(\mathbb{C})$ . Then Hodge conjecture hold for  $X$ . That is, if  $p \in \mathbb{Z}$  and  $\lambda \in F^p H^{2p}(X^{\text{an}}, \mathbb{Q})$ ,  $\lambda = [Z]$  with  $Z \in \mathcal{Z}^p(X)$ .*

*Proof.* The Hodge conjecture is true for curves and surfaces. Assume that the Hodge conjecture is true for smooth projective varieties of dimension less or equal to  $d-1$ . Let  $X \in \text{PSmVar}(\mathbb{C})$  of dimension  $d$ . Up to split  $X$  into its connected components, we may assume that  $X$  is connected of dimension  $d$ . Then  $X$  is birational to a possibly singular hypersurface  $Y_{0_X} \subset \mathbb{P}_{\mathbb{C}}^{d+1}$  of degree  $l_X := \deg(\mathbb{C}(X)/\mathbb{C}(x_1, \dots, x_d))$ . Consider

$$f : \mathcal{Y}' = V(f) \subset \mathbb{P}_{\mathbb{C}}^{d-1} \times \mathbb{P}(l_X) \rightarrow \mathbb{P}(l_X)$$

the universal family of projective hypersurfaces of degree  $l_X$ , and  $0_X \in \mathbb{P}(l_X)$  the point corresponding to  $Y_{0_X}$ . Denote  $\Delta \subset \mathbb{P}(l_X)$  the discriminant locus parametrizing the hypersurfaces which are singular. Let  $\bar{S} := \bar{0}_X \subset \mathbb{P}(l_X)$  be the  $\mathbb{Q}$ -Zariski closure of  $0_X$  inside  $\mathbb{P}(l_X)$ . If  $Y_{0_X}$  is smooth, then  $X$  satisfy the hodge conjecture since  $Y_{0_X}$  satisfy the Hodge conjecture by [5] and by the induction hypothesis since  $X$  is birational to  $Y_{0_X}$  by proposition 1(ii). Hence, we may assume that  $Y_{0_X}$  is singular, that is  $0_X \in \Delta$ . Then,  $\bar{S} \subset \Delta$  since  $\Delta \subset \mathbb{P}(l_X)$  is defined over  $\mathbb{Q}$ . Let  $\bar{T} \subset \mathbb{P}(l_X)$  be an irreducible closed subvariety of dimension  $\dim(\bar{S}) + 1$  defined over  $\mathbb{Q}$  such that  $\bar{S} \subset \bar{T}$  and  $T := \bar{T} \cap (\mathbb{P}(l_X) \setminus \Delta) \neq \emptyset$ . Since there always exists subvarieties of  $\mathbb{P}(l_X) \setminus \Delta$  of arbitrary dimension defined over  $\mathbb{Q}$  which are not weakly special (take complete intersections which intersect  $\Delta$  transversaly), we may assume that  $T \subset \mathbb{P}(l_X) \setminus \Delta$  is not contained in a weakly special subvariety. We have then  $\bar{T} \cap \Delta = \bar{S} \cup \bar{S}'$  where  $\bar{S}' \subset \Delta$  is a divisor. We then consider the family of algebraic varieties defined over  $\mathbb{Q}$

$$\tilde{f}_X : \mathcal{Y}_{\bar{T}^\circ} \xrightarrow{\epsilon} \mathcal{Y}'_{\bar{T}^\circ} \xrightarrow{f \times_{\mathbb{P}(l_X)} \bar{T}^\circ} \bar{T}^\circ$$

where

- $\tilde{f}_X : \mathcal{Y} \xrightarrow{\epsilon} \mathcal{Y}'_{\bar{T}} \xrightarrow{f \times_{\mathbb{P}(l_X)} \bar{T}} \bar{T}$  with  $\epsilon : (\mathcal{Y}, \mathcal{E}') \rightarrow (\mathcal{Y}'_{\bar{T}}, \mathcal{Y}_{\bar{S}} \cup \mathcal{Y}_{\bar{S}'})$  is a desingularization over  $\mathbb{Q}$ , in particular  $\mathcal{E}' = \mathcal{E} \cup \mathcal{E}''$  with  $\mathcal{E} = \cup_{i=0}^r \mathcal{E}_i := \tilde{f}_X^{-1}(\bar{S}) \subset \mathcal{Y}$  which is a normal crossing divisor, in particular  $\mathcal{Y}_T = \mathcal{Y}'_T$ ,

- $S \subset \bar{S}$  is the open subset such that  $S$  is smooth and  $\tilde{f}_X \times_{\bar{T}} S : \mathcal{E}_I := \cap_{i \in I} \mathcal{E}_i \rightarrow S$  is smooth projective for each  $I \subset [0, \dots, r]$ ,  $\bar{T}^\circ \subset \bar{T}$  is an open subset such that  $\bar{T}^\circ \cap \Delta = S$  and  $T^\circ := \bar{T}^\circ \cap T$  is smooth,
- $t \in \mathbb{S}(\mathbb{C})$  is the point such that  $\mathcal{Y}'_{\mathbb{C}, t} = Y_{0_X}$  and  $\epsilon_t : E_0 := \mathcal{E}_{0, \mathbb{C}, t} \rightarrow Y_{0_X}$  is surjective, in particular  $E_0$  is birational to  $Y_{0_X}$ , hence  $E_0$  is birational to  $X$ .

Denote for  $I \subset [0, \dots, s]$ ,  $i_I : \mathcal{E}_I \hookrightarrow \mathcal{E}$  and for  $I \subset J \subset [0, \dots, s]$ ,  $i_{JI} : \mathcal{E}_J \rightarrow \mathcal{E}_I$  the closed embeddings. Then,  $i_\bullet : \mathcal{E}_\bullet \rightarrow \mathcal{E}$  in  $\text{Fun}(\Delta, \text{Var}(\mathbb{Q}))$  is a simplicial resolution with  $\mathcal{E}_\bullet \in \text{Fun}(\Delta, \text{SmVar}(\mathbb{Q}))$ . We have then

$$E_{Hdg}(\mathcal{E}/S) := (\tilde{f}_X \circ i_{\bullet*} E_{zar}(\Omega_{\mathcal{E}_\bullet/S}, F_b, W), (\tilde{f}_X \circ i_{\bullet*} E_{usu}(\mathbb{Q}_{\mathcal{E}_\bullet/\mathbb{C}}^{an}, W)), \tilde{f} \circ i_{\bullet*} \alpha(\mathcal{E}_\bullet)) \in D(MHM_{\mathbb{Q}, gm}(S))$$

Then, definition-proposition 1 applied to  $f_T := f \times_{\mathbb{P}(t_X)} \bar{T}^\circ : \mathcal{Y}'_{T^\circ} \rightarrow \bar{T}^\circ$  gives the distinguish in  $D(MHM_{\mathbb{Q}, gm}(S))$ ,

$$\begin{aligned} E_{Hdg}(\mathcal{Y}'_S/S) &\xrightarrow{Sp_{\mathcal{Y}'_S/f_T} := (\tilde{f}'_X \circ i_{\mathcal{Y}'_S})_* sp_{\mathcal{Y}_S/\mathcal{Y}'_{T^\circ}}} \psi_S E_{Hdg}(\mathcal{Y}'_{T^\circ}/T^\circ) \\ &\xrightarrow{(f_T \circ i_{\mathcal{Y}'_S})_* c(\psi_{\mathcal{Y}'_S} \mathbb{Z}_{\mathcal{Y}'_{T^\circ}}^{Hdg})} \phi_S E_{Hdg}(\mathcal{Y}'_{T^\circ}/T^\circ) \xrightarrow{(f_T \circ i_{\mathcal{Y}'_S})_* c(\mathbb{Z}_{\mathcal{Y}'_S}^{Hdg})} E_{Hdg}(\mathcal{Y}'_S/S)[1] \end{aligned}$$

and definition-proposition 1 applied to  $\tilde{f}_X : \mathcal{Y}_{\bar{T}^\circ} \rightarrow \bar{T}^\circ$  gives the distinguish in  $D(MHM_{\mathbb{Q}, gm}(S))$ ,

$$\begin{aligned} E_{Hdg}(\mathcal{E}/S) &\xrightarrow{Sp_{\mathcal{E}/\tilde{f}_X} := (\tilde{f}_X \circ i_{\mathcal{E}})_* sp_{\mathcal{E}/\mathcal{Y}}} \psi_S E_{Hdg}(\mathcal{Y}_{T^\circ}/T^\circ) \\ &\xrightarrow{(\tilde{f}_X \circ i_{\mathcal{E}})_* c(\psi_{\mathcal{E}} \mathbb{Z}_{\mathcal{Y}_{T^\circ}}^{Hdg})} \phi_S E_{Hdg}(\mathcal{Y}_{T^\circ}/T^\circ) \xrightarrow{(\tilde{f}_X \circ i_{\mathcal{E}})_* c(\mathbb{Z}_{\mathcal{E}}^{Hdg})} E_{Hdg}(\mathcal{E}/S)[1] \end{aligned}$$

and  $Sp_{\mathcal{E}/\tilde{f}_X} \circ \epsilon^* = Sp_{\mathcal{Y}_{T^\circ}/f_T}$ . Let  $p \in \mathbb{Z}$ . By the hard Lefschetz theorem on  $H^*(E_0)$ , we may assume that  $2p \leq d$ . Taking the cohomology to this distinguish triangle, we get in particular, for each  $p \in \mathbb{Z}$ , the maps in  $MHM_{\mathbb{Q}, gm}(S)$

- $Sp_{\mathcal{Y}'_S/f_T} : E_{Hdg}^{2p}(\mathcal{Y}'_S/S) \hookrightarrow H^{2p} \psi_S E_{Hdg}(\mathcal{Y}_{T^\circ}/T^\circ)$
- $Sp_{\mathcal{E}/\tilde{f}_X} : E_{Hdg}^{2p}(\mathcal{E}/S) \rightarrow H^{2p} \psi_S E_{Hdg}(\mathcal{Y}_{T^\circ}/T^\circ)$ .

The map  $Sp_{\mathcal{Y}'_S/f_T}$  is injective by definition-proposition 1(i)' and the weak Lefschetz theorem since for  $t' \in T^\circ(\mathbb{C})$  and  $t'' \in S(\mathbb{C})$ ,  $\mathcal{Y}'_{t'}$  and  $\mathcal{Y}'_{t''}$  are projective hypersurfaces. For  $\lambda \in H^{2p}(\mathcal{E}_{0, \mathbb{C}, t'}^{an}, \mathbb{Q})$ ,  $t' \in S(\mathbb{C})$ , we consider  $\pi_S : \tilde{S}_{\mathbb{C}}^{an} \rightarrow S_{\mathbb{C}}^{an}$  the universal covering in  $\text{AnSm}(\mathbb{C})$  and

$$\mathbb{V}_S(\lambda) := \pi_S(\lambda \times \tilde{S}_{\mathbb{C}}^{an}) \subset E_{DR}^{2p, an}(\mathcal{E}_{0, \mathbb{C}}/S_{\mathbb{C}})$$

the flat leaf induced by the universal covering (see section 2.1) and

$$\mathbb{V}_S^p(\lambda) := \mathbb{V}_S(\lambda) \cap F^p E_{DR}^{2p}(\mathcal{E}_{0, \mathbb{C}}/S_{\mathbb{C}}) \subset E_{DR}^{2p}(\mathcal{E}_{0, \mathbb{C}}/S_{\mathbb{C}}),$$

which is an algebraic variety by [6]. Note that  $\mathbb{V}_S^p(\lambda) \subset H L^{p, 2p}(\mathcal{E}_{0, \mathbb{C}}/S_{\mathbb{C}})$  is the union of the irreducible components passing through  $\lambda$ . Similarly, for  $\eta \in H^{2p}(\mathcal{E}_{\mathbb{C}, t'}^{an}, \mathbb{Q})$ ,  $t' \in S(\mathbb{C})$ , we consider  $\pi_S : \tilde{S}_{\mathbb{C}}^{an} \rightarrow S_{\mathbb{C}}^{an}$  the universal covering in  $\text{AnSm}(\mathbb{C})$  and

$$\mathbb{V}_S(\eta) := \pi_S(\eta \times \tilde{S}_{\mathbb{C}}^{an}) \subset E_{DR}^{2p, an}(\mathcal{E}_{\mathbb{C}}/S_{\mathbb{C}})$$

the flat leaf induced by the universal covering (see section 2.1) and

$$\mathbb{V}_S^p(\eta) := \mathbb{V}_S(\eta) \cap F^p E_{DR}^{2p}(\mathcal{E}_{\mathbb{C}}/S_{\mathbb{C}}) \subset E_{DR}^{2p}(\mathcal{E}_{\mathbb{C}}/S_{\mathbb{C}}).$$

We have

$$H^{2p}E_{Hdg}(\mathcal{Y}_{T^o}/T^o) = ((E_{DR}^{2p}(\mathcal{Y}_{T^o}/T^o), F), R^{2p}\tilde{f}_{X*}\mathbb{Z}_{\mathcal{Y}_{T^o}}, \tilde{f}_{X*}\alpha(\mathcal{Y}_{T^o})) \in MHM_{\mathbb{Q}, gm}(T^o)$$

By [8], the filtered vector bundle

$$(E_{DR}^{2p}(\mathcal{Y}_{T^o}/T^o), F) = H^{2p}\tilde{f}_{X*}E_{zar}(\Omega_{\mathcal{Y}_{T^o}/T^o}^\bullet) \in \text{Vect}_{\mathcal{D}fil}(T^o)$$

extend to a filtered vector bundle

$$(E_{DR, V_S}^{2p}(\mathcal{Y}_{T^o}/T^o), F) := H^{2p}\tilde{f}_{X*}E_{zar}(\Omega_{\mathcal{Y}_{T^o}/T^o}^\bullet(\log \mathcal{E})) \in \text{Vect}_{\mathcal{D}fil}(\bar{T}^o)$$

such that

$$H^{2p}\psi_S E_{Hdg}(\mathcal{Y}_{T^o}/T^o) = (i_S^{*mod}(E_{DR, V_S}^{2p}(\mathcal{Y}_{T^o}/T^o), F), (\psi_S R^{2p}\tilde{f}_{X*}\mathbb{Z}_{\mathcal{Y}^o}, W), \psi_S \tilde{f}_{X*}\alpha(\mathcal{Y}_{T^o})).$$

For

$$\alpha \in H^{2p}(\mathcal{Y}_{t''}^{an}, \mathbb{Q}) = \mathbb{H}^{2p}(\mathcal{E}_{t''}^{an}, \psi_S R\tilde{f}_{X*}\mathbb{Q}_{\mathcal{Y}_{T^o}}), t' \in S_{\mathbb{C}}, t'' \in T^o(\mathbb{C}),$$

we consider

- $\pi_{T^o} : \tilde{T}^{o, an} \rightarrow T_{\mathbb{C}}^{o, an}$  the universal covering in  $\text{AnSm}(\mathbb{C})$ , the factorization  $\pi_S : \tilde{S}_{\mathbb{C}}^{an} \rightarrow \tilde{T}^{o, an} \xrightarrow{\pi_{T^o}} T_{\mathbb{C}}^{o, an}$  and

$$\mathbb{V}_{T^o}(\alpha) := \pi_{T^o}(\alpha \times \tilde{T}^{o, an}) \subset E_{DR}^{2p, an}(\mathcal{Y}_{T^o, \mathbb{C}}/T_{\mathbb{C}}^o)$$

the flat leaf (see section 2.1) and

$$\mathbb{V}_{T^o}^p(\alpha) := \mathbb{V}_{T^o}(\alpha) \cap F^p E_{DR}^{2p}(\mathcal{Y}_{T^o, \mathbb{C}}/T_{\mathbb{C}}^o) \subset E_{DR}^{2p}(\mathcal{Y}_{T^o, \mathbb{C}}/T_{\mathbb{C}}^o),$$

which is an algebraic subvariety by [6],

- $\pi_S : \tilde{S}_{\mathbb{C}}^{an} \rightarrow S_{\mathbb{C}}^{an}$  the universal covering in  $\text{AnSm}(\mathbb{C})$  and

$$\mathbb{V}_S(\alpha) := \pi_S(\alpha \times \tilde{S}_{\mathbb{C}}^{an}) \subset H^{2p}\psi_S E_{DR}(\mathcal{Y}_{T^o, \mathbb{C}}/T_{\mathbb{C}}^o)^{an}$$

the flat leaf (see section 2.1) and

$$\mathbb{V}_S^p(\alpha) := \mathbb{V}_S(\alpha) \cap F^p H^{2p}\psi_S E_{DR}(\mathcal{Y}_{T^o, \mathbb{C}}/T_{\mathbb{C}}^o)^{an} \subset H^{2p}\psi_S E_{DR}(\mathcal{Y}_{T^o, \mathbb{C}}/T_{\mathbb{C}}^o)^{an}.$$

Consider the exact sequence of variation of mixed Hodge structure over  $S$

$$0 \rightarrow W_{2p-2}E_{Hdg}^{2p}(\mathcal{E}/S) \rightarrow E_{Hdg}^{2p}(\mathcal{E}/S) \xrightarrow{(i_i^*)_{0 \leq i \leq r} := q} \oplus_{i=0}^r E_{Hdg}^{2p}(\mathcal{E}_i/S) \rightarrow 0.$$

It induces the exact sequence of vector bundles over  $S$

$$0 \rightarrow W_{2p-2}E_{DR}^{2p}(\mathcal{E}/S) \rightarrow E_{DR}^{2p}(\mathcal{E}/S) \xrightarrow{(i_i^*)_{0 \leq i \leq r} := q} \oplus_{i=0}^r E_{DR}^{2p}(\mathcal{E}_i/S) \rightarrow 0,$$

and the exact sequence of presheaves on  $S_{\mathbb{C}}^{an}$ ,

$$J(W_{2p-2}E_{Hdg}^{2p}(\mathcal{E}_{\mathbb{C}}/S_{\mathbb{C}})) := \text{Ext}^1(\mathbb{Q}_S^{Hdg}, W_{2p-2}E_{Hdg}^{2p}(\mathcal{E}_{\mathbb{C}}/S_{\mathbb{C}})), \xrightarrow{L} \text{Ext}^1(\mathbb{Q}_S^{Hdg}, W_{2p-2}E_{Hdg}^{2p}(\mathcal{E}_{\mathbb{C}}/S_{\mathbb{C}}))$$

$$HL^{2p, p}(\mathcal{E}_{\mathbb{C}}/S_{\mathbb{C}}) \xrightarrow{q} \oplus_{i=0}^r HL^{2p, p}(\mathcal{E}_{i, \mathbb{C}}/S_{\mathbb{C}}) \xrightarrow{e}$$

where  $HL^{2p, p}(\mathcal{E}_{\mathbb{C}}/S_{\mathbb{C}}) \subset E_{DR}^{2p}(\mathcal{E}_{\mathbb{C}}/S_{\mathbb{C}})$ ,  $HL^{2p, p}(\mathcal{E}_{i, \mathbb{C}}/S_{\mathbb{C}}) \subset E_{DR}^{2p}(\mathcal{E}_{i, \mathbb{C}}/S_{\mathbb{C}})$  are the locus of Hodge classes. Let  $\lambda \in F^p H^{2p}(E_0^{an}, \mathbb{Q})$ , where we recall  $E_0 = \mathcal{E}_{0, \mathbb{C}, t}$ . Consider then  $\tilde{\lambda} \in H^{2p}(E^{an}, \mathbb{Q})$ , such that  $q(\tilde{\lambda}) = \lambda$ , and  $\tilde{\lambda} := Sp_{\mathcal{E}/\tilde{f}_X}(\tilde{\lambda}) \in Sp_{E/\tilde{f}_X}(H^{2p}(E^{an}, \mathbb{Q}))$ , Since the monodromy of  $R^{2p}\tilde{f}_{X*}\mathbb{Q}_{\mathcal{Y}_{T^o, \mathbb{C}}}^{an}$  is irreducible, we have

$$Sp_{\mathcal{E}/\tilde{f}_X}(\ker q) := Sp_{\mathcal{E}/\tilde{f}_X}(W_{2p-2}E_{DR}^{2p}(\mathcal{E}/S)) = 0$$

The locus of Hodge classes passing through  $\lambda$

$$\mathbb{V}_S^p(\lambda) := \mathbb{V}_S(\lambda) \cap F^p E_{DR}^{2p}(\mathcal{E}_{0,\mathbb{C}}/S_{\mathbb{C}}) \subset E_{DR}^{2p}(\mathcal{E}_{0,\mathbb{C}}/S_{\mathbb{C}}),$$

inside the De Rham vector bundle of  $\tilde{f}_X \circ i_{E_0} : \mathcal{E}_0 \rightarrow S$  thus satisfy

$$Sp_{\mathcal{E}/\tilde{f}_X}(q^{-1}(\mathbb{V}_S^p(\lambda))) = \mathbb{V}_S^p(Sp_{\mathcal{E}/\tilde{f}_X}(\tilde{\lambda})) \subset E_{DR, V_S}^{2p}(\mathcal{Y}_{T^o, \mathbb{C}}/T_{\mathbb{C}}^o),$$

where

- $\mathbb{V}_S(\lambda) \subset E_{DR}^{2p}(\mathcal{E}_{0,\mathbb{C}}/S_{\mathbb{C}})$ ,  $\mathbb{V}_S(\tilde{\lambda}) \subset E_{DR}^{2p}(\mathcal{E}_{\mathbb{C}}/S_{\mathbb{C}})$ , are the flat leaves, e.g.  $\mathbb{V}_S(\lambda) := \pi_S(\lambda \times \tilde{S}_{\mathbb{C}}^{an})$  where  $\pi_S : H^{2p}(E^{an}, \mathbb{C}) \times \tilde{S}_{\mathbb{C}}^{an} \rightarrow E_{DR}^{2p, an}(\mathcal{E}_{\mathbb{C}}/S_{\mathbb{C}})$  is the morphism induced by the universal covering  $\pi_S : \tilde{S}_{\mathbb{C}}^{an} \rightarrow S_{\mathbb{C}}^{an}$ ,
- $q := q \otimes \mathbb{C} : E_{DR}^{2p}(\mathcal{E}_{\mathbb{C}}/S_{\mathbb{C}}) \rightarrow E_{DR}^{2p}(\mathcal{E}_{0,\mathbb{C}}/S_{\mathbb{C}})$  is the quotient map.

Now,

- if  $Sp_{\mathcal{E}/\tilde{f}_X}(\tilde{\lambda}) = 0$ , since  $Sp_{\mathcal{E}/\tilde{f}_X} \circ \epsilon^* = Sp_{\mathcal{Y}_0/\tilde{f}'_X}$  and  $Sp_{\mathcal{Y}_0/\tilde{f}'_X}$  is injective, we have  $\tilde{\lambda}_{t'} = k_{t*} \tilde{\lambda}_{t'}$  with  $\tilde{\lambda}_{t'} \in F^p H_{\mathcal{F}, t'}^{2p}(\mathcal{E}_{\mathbb{C}, t'}^{an}, \mathbb{Q})$ , where  $k : \mathcal{F} \rightarrow \mathcal{E}$  is the exceptional locus of  $\epsilon : \mathcal{E}_0 \rightarrow \mathcal{Y}_0$  (note that  $\epsilon^* : H^{2p}(\mathcal{Y}_{0, t'}^{an}, \mathbb{Q}) \rightarrow H^{2p}(\mathcal{E}_{0, t'}^{an}, \mathbb{Q})$  is injective by Poincare Lefschetz duality), hence, assuming the Hodge conjecture for smooth projective varieties of dimension less or equal to  $d-1$ ,  $\mathbb{V}_S^p(\tilde{\lambda}) \subset E_{DR}^{2p}(\mathcal{E}_{\mathbb{C}}/S_{\mathbb{C}})$  is defined over  $\bar{\mathbb{Q}}$  and its Galois conjugates are also components of the locus of Hodge classes,
- if  $Sp_{\tilde{E}/\tilde{f}_X}(\tilde{\lambda}) \neq 0$ , we have,

$$Sp_{\mathcal{E}/\tilde{f}_X}(\mathbb{V}_S^p(\tilde{\lambda})) = \mathbb{V}_S^p(Sp_{\mathcal{E}/\tilde{f}_X}(\tilde{\lambda})) = \overline{\mathbb{V}_{T^o}^p(Sp_{\mathcal{E}/\tilde{f}_X}(\tilde{\lambda}))} \cap p^{-1}(S_{\mathbb{C}}) \subset E_{DR, V_S}^{2p}(\mathcal{Y}_{T^o, \mathbb{C}}/T_{\mathbb{C}}^o)$$

where  $p : E_{DR, V_0}^{2p}(\mathcal{Y}_{T^o, \mathbb{C}}/T_{\mathbb{C}}^o) \rightarrow \bar{T}_{\mathbb{C}}^o$  is the projection (the first equality is obvious since  $Sp_{\mathcal{E}/\tilde{f}_X}$  is a morphism of variation of mixed Hodge structures over  $S$ , whereas for the second equality the inclusion  $\supset$  follows from [6] lemma 2.11 which state the invariance of  $\tilde{\lambda}$  under the monodromy

$$\text{Im}(\pi_1(p(\mathbb{V}_{T^o}^p(\tilde{\lambda})) \cap D^*) \rightarrow \pi_1(D^*)), (D, D^*) \rightarrow (\bar{T}_{\mathbb{C}}^{o, an}, \bar{T}_{\mathbb{C}}^{o, an} \setminus S_{\mathbb{C}}^{an})$$

but by [5] theorem 1, the Hodge conjecture holds for projective hypersurfaces hence

$$\mathbb{V}_{T^o}^p(Sp_{\mathcal{E}/\tilde{f}_X}(\tilde{\lambda})) \subset E_{DR}^{2p}(\mathcal{Y}_{T^o, \mathbb{C}}/T_{\mathbb{C}}^o)$$

is defined over  $\bar{\mathbb{Q}}$  and its Galois conjugates are also components of the locus of Hodge classes, hence  $\mathbb{V}_S^p(\tilde{\lambda}) \subset E_{DR}^{2p}(\mathcal{E}_{\mathbb{C}}/S_{\mathbb{C}})$  is defined over  $\bar{\mathbb{Q}}$  and its Galois conjugates are also components of the locus of Hodge classes,

Hence  $\mathbb{V}_S^p(\lambda) \subset E_{DR}^{2p}(\mathcal{E}_{0,\mathbb{C}}/S_{\mathbb{C}})$  is defined over  $\bar{\mathbb{Q}}$  and its Galois conjugates are also components of the locus of Hodge classes. Thus, by [4] theorem 4,

$$\lambda = [Z] \in H^{2p}(E_0^{an}, \mathbb{Q}), \text{ with } Z \in \mathcal{Z}^p(E_0).$$

Since  $p \in \mathbb{Z}$  and  $\lambda \in F^p H^{2p}(E_0^{an}, \mathbb{Q})$  are arbitrary, the Hodge conjecture holds for  $E_0$ . Proposition 1(ii) and the fact that the Hodge conjecture is true for smooth projective varieties of dimension less or equal to  $d-1$  by induction hypothesis then implies that Hodge conjecture holds for  $X$ , since  $X$  is birational to  $E_0$  and  $X, E_0 \in \text{PSmVar}(\mathbb{C})$  are connected of dimension  $d$ .  $\square$

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