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Degeneration of families of projective hypersurfaces and Hodge conjecture

Johann Bouali

March 27, 2024

Abstract

We prove by induction on dimension the Hodge conjecture for smooth complex projective varieties. Let X be a smooth complex projective variety. Then X is birational to a possibly singular projective hypersurface, hence to a smooth projective variety E_0 which is a component of a normal crossing divisor $E = \cup_{i=0}^r E_i \subset Y$ which is the singular fiber of a pencil $f : Y \rightarrow \mathbb{A}^1$ of smooth projective hypersurfaces. Using the smooth hypersurface case ([5] theorem 1), the nearby cycle functor on mixed Hodge module with rational de Rham factor and the induction hypothesis, we prove that a Hodge class of E_0 is absolute Hodge, more precisely the locus of Hodge classes inside the algebraic vector bundle given the De Rham cohomology of the rational deformation of E_0 is defined over \mathbb{Q} . By [4] theorem 4, we get the Hodge conjecture for E_0 . By the induction hypothesis we also have the Hodge conjecture for X .

Contents

1	Introduction	1
2	Preliminaries and notations	3
2.1	Notations	3
2.2	Birational projective varieties and Hodge conjecture	5
2.3	A family of smooth projective hypersurfaces associated to a smooth projective variety	6
2.4	The nearby cycle functor and the specialization map	6
3	Hodge conjecture for smooth projective varieties	8

1 Introduction

The main result of this article is an inductive proof of the Hodge conjecture using the results of [4] and [5]. More precisely, assuming the Hodge conjecture for smooth projective varieties of dimension less or equal to $d - 1$, the property of satisfying the Hodge conjecture for connected smooth projective varieties of dimension d is a birational invariant (proposition 1). On the other hand, if X is a connected smooth projective variety of dimension d , then X is birational to a possibly singular hypersurface $Y_{0_X} \subset \mathbb{P}_{\mathbb{C}}^{d+1}$ of degree $l_X := \text{deg}(\mathbb{C}(X)/\mathbb{C}(x_1, \dots, x_d))$. Consider

$$f : \mathcal{Y}' = V(f) \subset \mathbb{P}_{\mathbb{C}}^{d-1} \times \mathbb{P}(l_X) \rightarrow \mathbb{P}(l_X)$$

the universal family of projective hypersurfaces of degree l_X , and $0_X \in \mathbb{P}(l_X)$ the point corresponding to Y_{0_X} . Denote $\Delta \subset \mathbb{P}(l_X)$ the discriminant locus parametrizing the hypersurfaces which are singular. Let $\bar{S} := 0_X^{\mathbb{Q}} \subset \mathbb{P}(l_X)$ the \mathbb{Q} -Zariski closure of 0_X inside $\mathbb{P}(l_X)$. If Y_{0_X} is smooth, then X satisfy the hodge conjecture since Y_{0_X} satisfy the Hodge conjecture by [5] and by the induction hypothesis since X

is birational to Y_{0_X} . Hence, we may assume that Y_{0_X} is singular, that is $0_X \in \Delta$. Then, $\bar{S} \subset \Delta$ since $\Delta \subset \mathbb{P}(l_X)$ is defined over \mathbb{Q} . Let $\bar{T} \subset \mathbb{P}(l_X)$ an irreducible closed subvariety of dimension $\dim(\bar{S}) + 1$ defined over \mathbb{Q} such that $\bar{S} \subset \bar{T}$ and $T := \bar{T} \cap (\mathbb{P}(l_X) \setminus \Delta) \neq \emptyset$. Since there always exists subvarieties of $\mathbb{P}(l_X) \setminus \Delta$ of arbitrary dimension defined over \mathbb{Q} which are not weakly special (take complete intersections which intersect Δ transversally), we may assume that $T \subset \mathbb{P}(l_X) \setminus \Delta$ is not contained in a weakly special subvariety. We have then $\bar{T} \cap \Delta = \bar{S} \cup \bar{S}'$ where $\bar{S}' \subset \Delta$ is a divisor. We then consider the family of algebraic varieties defined over \mathbb{Q}

$$\tilde{f}_X : \mathcal{Y}_{\bar{T}^o} \xrightarrow{\epsilon} \mathcal{Y}'_{T^o} \xrightarrow{f \times_{\mathbb{P}(l_X)} \bar{T}^o} \bar{T}^o$$

where

- $\tilde{f}_X : \mathcal{Y} \xrightarrow{\epsilon} \mathcal{Y}'_{\bar{T}} \xrightarrow{f \times_{\mathbb{P}(l_X)} \bar{T}} \bar{T}$ with $\epsilon : (\mathcal{Y}, \mathcal{E}') \rightarrow (\mathcal{Y}'_{\bar{T}}, \mathcal{Y}_{\bar{S}} \cup \mathcal{Y}_{\bar{S}'})$ is a desingularization over \mathbb{Q} , in particular $\mathcal{E}' = \mathcal{E} \cup \mathcal{E}''$ with $\mathcal{E} = \cup_{i=0}^r \mathcal{E}_i := \tilde{f}_X^{-1}(\bar{S}) \subset \mathcal{Y}$ which is a normal crossing divisor, in particular $\mathcal{Y}_T = \mathcal{Y}'_T$,
- $S \subset \bar{S}$ is the open subset such that S is smooth and $\tilde{f}_X \times_{\bar{T}} S : \mathcal{E}_I := \cap_{i \in I} \mathcal{E}_i \rightarrow S$ is smooth projective for each $I \subset [0, \dots, r]$, $\bar{T}^o \subset \bar{T}$ is an open subset such that $\bar{T}^o \cap \Delta = S$ and $T^o := \bar{T}^o \cap T$ is smooth,
- $t \in \mathbb{S}(\mathbb{C})$ is the point such that $\mathcal{Y}'_{\mathbb{C}, t} = Y_{0_X}$ and $\epsilon_t : E_0 := \mathcal{E}_{0, \mathbb{C}, t} \rightarrow Y_{0_X}$ is surjective, in particular E_0 is birational to Y_{0_X} , hence E_0 is birational to X .

Assuming the Hodge conjecture for smooth projective varieties of dimension less or equal to $d - 1$, we then prove the Hodge conjecture for the E_i , more specifically for E_0 as follows : Let $p \in \mathbb{Z}$. By the hard Lefschetz theorem on $H^*(E_0)$, we may assume that $2p \leq d$. We then have then a morphism of mixed Hodge module over S (in fact of geometric variation of mixed Hodge structures over S) given by the specialization map

$$Sp_{\mathcal{E}/\tilde{f}_X} : E_{Hdg}^{2p}(\mathcal{E}/S) \rightarrow H^{2p} \psi_S E_{Hdg}(\mathcal{Y}_{T^o}/T^o),$$

whose de Rham part are morphism of filtered vector bundle over S with Gauss-Manin connexion defined over \mathbb{Q} , and whose kernel of the part supported on \mathcal{E}_0 is supported a closed subset $\mathcal{F} \subset \mathcal{E}_0$ with $\text{codim}(\mathcal{F}_{t'}, \mathcal{E}_{0, t'}) = 1$ for all $t' \in S(\mathbb{C})$. The fact that the kernel is supported on a closed subset of codimension one on fibers follows from the fact that $H^{2p-1}(Y_s, \mathbb{Q}) = 0 = H_{Y_0}^{2p-1}(Y)$, by the weak Lefschetz theorem since Y_s and Y_0 are projective hypersurfaces ($2p \leq d$). Consider the exact sequence of variation of mixed Hodge structure over S

$$0 \rightarrow W_{2p-2} E_{Hdg}^{2p}(\mathcal{E}/S) \rightarrow E_{Hdg}^{2p}(\mathcal{E}/S) \xrightarrow{(i^*)_{0 \leq i \leq r} := q} \oplus_{i=0}^r E_{Hdg}^{2p}(\mathcal{E}_i/S) \rightarrow 0.$$

It induces the exact sequence of presheaves on $S_{\mathbb{C}}^n$,

$$J(W_{2p-2} E_{Hdg}^{2p}(\mathcal{E}_{\mathbb{C}}/S_{\mathbb{C}})) := \text{Ext}^1(\mathbb{Q}_S^{Hdg}, W_{2p-2} E_{Hdg}^{2p}(\mathcal{E}_{\mathbb{C}}/S_{\mathbb{C}})) \xrightarrow{\iota} \text{Ext}^1(\mathbb{Q}_S^{Hdg}, E_{Hdg}^{2p}(\mathcal{E}_{\mathbb{C}}/S_{\mathbb{C}}))$$

$$HL^{2p,p}(\mathcal{E}_{\mathbb{C}}/S_{\mathbb{C}}) \xrightarrow{q} \oplus_{i=0}^r HL^{2p,p}(\mathcal{E}_{i, \mathbb{C}}/S_{\mathbb{C}}) \xrightarrow{\epsilon}$$

where $HL^{2p,p}(\mathcal{E}_{\mathbb{C}}/S_{\mathbb{C}}) \subset E_{DR}^{2p}(\mathcal{E}_{\mathbb{C}}/S_{\mathbb{C}})$, $HL^{2p,p}(\mathcal{E}_{i, \mathbb{C}}/S_{\mathbb{C}}) \subset E_{DR}^{2p}(\mathcal{E}_{i, \mathbb{C}}/S_{\mathbb{C}})$ are the locus of Hodge classes. Let $\lambda \in F^p H^{2p}(E_0^{an}, \mathbb{Q})$, where we recall $E_0 = \mathcal{E}_{0, \mathbb{C}, t}$. Consider then $\tilde{\lambda} \in H^{2p}(E^{an}, \mathbb{Q})$, such that $q(\tilde{\lambda}) = \lambda$, and $\tilde{\lambda} := Sp_{\mathcal{E}/\tilde{f}_X}(\tilde{\lambda}) \in Sp_{E/\tilde{f}_X}(H^{2p}(E^{an}, \mathbb{Q}))$, Since the monodromy of $R^{2p} \tilde{f}_{X*} \mathbb{Q}_{\mathcal{Y}_{T^o, \mathbb{C}}}^{an}$ is irreducible, we have

$$Sp_{\mathcal{E}/\tilde{f}_X}(\ker q) := Sp_{\mathcal{E}/\tilde{f}_X}(W_{2p-2} E_{DR}^{2p}(\mathcal{E}/S)) = 0$$

The locus of Hodge classes passing through λ

$$\mathbb{V}_S^p(\lambda) := \mathbb{V}_S(\lambda) \cap F^p E_{DR}^{2p}(\mathcal{E}_{0, \mathbb{C}}/S_{\mathbb{C}}) \subset E_{DR}^{2p}(\mathcal{E}_{0, \mathbb{C}}/S_{\mathbb{C}}),$$

inside the De Rham vector bundle of $\tilde{f}_X \circ i_{E_0} : \mathcal{E}_0 \rightarrow S$ thus satisfy

$$Sp_{\mathcal{E}/\tilde{f}_X}(q^{-1}(\mathbb{V}_S^p(\lambda))) = \mathbb{V}_S^p(Sp_{\mathcal{E}/\tilde{f}_X}(\tilde{\lambda})) \subset E_{DR, V_S}^{2p}(\mathcal{Y}_{T^o, \mathbb{C}}/T_{\mathbb{C}}^o),$$

where

- $\mathbb{V}_S(\lambda) \subset E_{DR}^{2p}(\mathcal{E}_{0, \mathbb{C}}/S_{\mathbb{C}})$, $\mathbb{V}_S(\tilde{\lambda}) \subset E_{DR}^{2p}(\mathcal{E}_{\mathbb{C}}/S_{\mathbb{C}})$, are the flat leaves, e.g. $\mathbb{V}_S(\lambda) := \pi_S(\lambda \times \tilde{S}_{\mathbb{C}}^{an})$ where $\pi_S : H^{2p}(E^{an}, \mathbb{C}) \times \tilde{S}_{\mathbb{C}}^{an} \rightarrow E_{DR}^{2p, an}(\mathcal{E}_{\mathbb{C}}/S_{\mathbb{C}})$ is the morphism induced by the universal covering $\pi_S : \tilde{S}_{\mathbb{C}}^{an} \rightarrow S_{\mathbb{C}}^{an}$,
- $q := q \otimes \mathbb{C} : E_{DR}^{2p}(\mathcal{E}_{\mathbb{C}}/S_{\mathbb{C}}) \rightarrow E_{DR}^{2p}(\mathcal{E}_{0, \mathbb{C}}/S_{\mathbb{C}})$ is the quotient map.

Now,

- if $Sp_{E/f_X}(\tilde{\lambda}) = 0$, assuming the Hodge conjecture for smooth projective varieties of dimension less or equal to $d - 1$, $\mathbb{V}_S^p(\tilde{\lambda}) \subset E_{DR}^{2p}(\mathcal{E}_{\mathbb{C}}/S_{\mathbb{C}})$ is an algebraic subvariety defined over $\bar{\mathbb{Q}}$ and its Galois conjugate are also components of the locus of Hodge classes,
- if $Sp_{E/f_X}(\tilde{\lambda}) \neq 0$, by [5] theorem 1, the Hodge conjecture holds for projective hypersurfaces, hence

$$Sp_{\mathcal{E}/\tilde{f}_X}(\mathbb{V}_S^p(\tilde{\lambda})) = \mathbb{V}_S^p(Sp_{\mathcal{E}/\tilde{f}_X}(\tilde{\lambda})) = \overline{\mathbb{V}_{T^o}^p(Sp_{E/f_X}(\tilde{\lambda}))} \cap p^{-1}(S) \subset E_{DR, V_S}^{2p}(\mathcal{Y}_{T^o, \mathbb{C}}/T_{\mathbb{C}}^o)$$

is an algebraic subvariety defined over $\bar{\mathbb{Q}}$ and its Galois conjugate are also components of the locus of Hodge classes, where the second equality follows from [6] lemma 2.11.

Hence, $\mathbb{V}_S^p(\lambda) \subset E_{DR}^{2p}(\mathcal{E}_{0\mathbb{C}}/S_{\mathbb{C}})$ is an algebraic subvariety defined over $\bar{\mathbb{Q}}$ and its Galois conjugate are also components of the locus of Hodge classes. By [4] theorem 4, we get

$$\lambda = [Z] \in H^{2p}(E_0^{an}, \mathbb{Q}), \quad Z \in \mathcal{Z}^p(E_0).$$

We recall the key point is that since $\mathbb{V}_S^p(\lambda) \subset E_{DR}^{2p}(\mathcal{E}_{0\mathbb{C}}/S_{\mathbb{C}})$ is an algebraic subvariety defined over $\bar{\mathbb{Q}}$ and its Galois conjugate are also components of the locus of Hodge classes, we have for each $\theta \in \text{Aut}(\mathbb{C}, \mathbb{Q})$,

$$\theta(\lambda) \in \text{Gal}(\bar{\mathbb{Q}}, \mathbb{Q})(\mathbb{V}_S^p(\lambda)_{\theta}) \subset E_{DR}^{2p}(\mathcal{E}_{0\mathbb{C}, \theta}/S_{\mathbb{C}, \theta}),$$

which implies that $\theta(\lambda) \in H^{2p}(E_{0, \theta}^{an}, \mathbb{Q}) \subset H^{2p}(E_{0, \theta}^{an}, \mathbb{C})$, that is $\lambda \in H^{2p}(E_0^{an}, \mathbb{Q})$ is absolute Hodge and we apply [4] corollary 1. Since $p \in \mathbb{Z}$ and $\lambda \in F^p H^{2p}(E_0^{an}, \mathbb{Q})$ are arbitrary, this proves the Hodge conjecture for E_0 . Since X is birational to E_0 , the Hodge conjecture for E_0 and the induction hypothesis implies the Hodge conjecture for X .

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2 Preliminaries and notations

2.1 Notations

- Denote by Top the category of topological spaces and RTop the category of ringed spaces.
- Denote by Cat the category of small categories and RCat the category of ringed topoi.
- For $\mathcal{S} \in \text{Cat}$ and $X \in \mathcal{S}$, we denote $\mathcal{S}/X \in \text{Cat}$ the category whose objects are $Y/X := (Y, f)$ with $Y \in \mathcal{S}$ and $f : Y \rightarrow X$ is a morphism in \mathcal{S} , and whose morphisms $\text{Hom}((Y', f'), (Y, f))$ consists of $g : Y' \rightarrow Y$ in \mathcal{S} such that $f \circ g = f'$.
- Let $(\mathcal{S}, O_{\mathcal{S}}) \in \text{RCat}$ a ringed topoi with topology τ . For $F \in C_{O_{\mathcal{S}}}(\mathcal{S})$, we denote by $k : F \rightarrow E_{\tau}(F)$ the canonical flasque resolution in $C_{O_{\mathcal{S}}}(\mathcal{S})$ (see [3]). In particular for $X \in \mathcal{S}$, $H^*(X, E_{\tau}(F)) \xrightarrow{\sim} \mathbb{H}_{\tau}^*(X, F)$.

- For $f : \mathcal{S}' \rightarrow \mathcal{S}$ a morphism with $\mathcal{S}, \mathcal{S}' \in \text{RCat}$, endowed with topology τ and τ' respectively, we denote for $F \in C_{O_{\mathcal{S}}}(\mathcal{S})$ and each $j \in \mathbb{Z}$,

$$\begin{aligned} - f^* &:= H^j \Gamma(\mathcal{S}, k \circ \text{ad}(f^*, f_*)(F)) : \mathbb{H}^j(\mathcal{S}, F) \rightarrow \mathbb{H}^j(\mathcal{S}', f^*F), \\ - f^* &:= H^j \Gamma(\mathcal{S}, k \circ \text{ad}(f^{*mod}, f_*)(F)) : \mathbb{H}^j(\mathcal{S}, F) \rightarrow \mathbb{H}^j(\mathcal{S}', f^{*mod}F), \end{aligned}$$

the canonical maps.

- For $m : A \rightarrow B$, $A, B \in C(\mathcal{A})$, \mathcal{A} an additive category, we denote $c(A) : \text{Cone}(m : A \rightarrow B) \rightarrow A[1]$ and $c(B) : B \rightarrow \text{Cone}(m : A \rightarrow B)$ the canonical maps.
- Denote by $\text{Sch} \subset \text{RTop}$ the subcategory of schemes (the morphisms are the morphisms of locally ringed spaces). We denote by $\text{PSch} \subset \text{Sch}$ the full subcategory of proper schemes. For a field k , we consider $\text{Sch}/k := \text{Sch}/\text{Spec } k$ the category of schemes over $\text{Spec } k$. The objects are $X := (X, a_X)$ with $X \in \text{Sch}$ and $a_X : X \rightarrow \text{Spec } k$ a morphism and the morphisms are the morphisms of schemes $f : X' \rightarrow X$ such that $f \circ a_{X'} = a_X$. We then denote by
 - $\text{Var}(k) = \text{Sch}^{ft}/k \subset \text{Sch}/k$ the full subcategory consisting of algebraic varieties over k , i.e. schemes of finite type over k ,
 - $\text{PVar}(k) \subset \text{QPVar}(k) \subset \text{Var}(k)$ the full subcategories consisting of quasi-projective varieties and projective varieties respectively,
 - $\text{PSmVar}(k) \subset \text{SmVar}(k) \subset \text{Var}(k)$, $\text{PSmVar}(k) := \text{PVar}(k) \cap \text{SmVar}(k)$, the full subcategories consisting of smooth varieties and smooth projective varieties respectively.
- Denote by $\text{AnSp}(\mathbb{C}) \subset \text{RTop}$ the subcategory of analytic spaces over \mathbb{C} , and by $\text{AnSm}(\mathbb{C}) \subset \text{AnSp}(\mathbb{C})$ the full subcategory of smooth analytic spaces (i.e. complex analytic manifold).
- For $X \in \text{Var}(k)$ and $X = \cup_{i \in I} X_i$ with $i_i : X_i \hookrightarrow X$ closed embeddings, we denote $X_{\bullet} \in \text{Fun}(\Delta, \text{Var}(k))$ the associated simplicial space, with for $J \subset I$, $i_{IJ} : X_I := \cap_{i \in I} X_i \hookrightarrow X_J := \cap_{i \in J} X_i$ the closed embedding.
- Let $(X, O_X) \in \text{RTop}$. We consider its De Rham complex $\Omega_X^{\bullet} := DR(X)(O_X)$.
 - Let $X \in \text{Sch}$. Considering its De Rham complex $\Omega_X^{\bullet} := DR(X)(O_X)$, we have for $j \in \mathbb{Z}$ its De Rham cohomology $H_{DR}^j(X) := \mathbb{H}^j(X, \Omega_X^{\bullet})$.
 - Let $X \in \text{Var}(k)$. Considering its De Rham complex $\Omega_X^{\bullet} := \Omega_{X/k}^{\bullet} := DR(X/k)(O_X)$, we have for $j \in \mathbb{Z}$ its De Rham cohomology $H_{DR}^j(X) := \mathbb{H}^j(X, \Omega_X^{\bullet})$. The differentials of $\Omega_X^{\bullet} := \Omega_{X/k}^{\bullet}$ are by definition k -linear, thus $H_{DR}^j(X) := \mathbb{H}^j(X, \Omega_X^{\bullet})$ has a structure of a k vector space.
 - Let $X \in \text{AnSp}(\mathbb{C})$. Considering its De Rham complex $\Omega_X^{\bullet} := DR(X)(O_X)$, we have for $j \in \mathbb{Z}$ its De Rham cohomology $H_{DR}^j(X) := \mathbb{H}^j(X, \Omega_X^{\bullet})$.
- For $Y \in \text{Var}(\mathbb{Q})$ and $X = V(I) = V(f_1, \dots, f_r) \subset Y_{\mathbb{C}}$, we consider $\mathcal{X} = V(\tilde{I}) = V(\tilde{f}_1, \dots, \tilde{f}_r) \subset Y \times \mathbb{A}_{\mathbb{Q}}^s$ the canonical rational deformation, where

$$\begin{aligned} \text{if } f &= \sum_{I \subset [1, \dots, d]^n} a_I x_1^{n_1} \cdots x_n^{n_n} \in \mathbb{C}[x_1, \dots, x_n], \\ \tilde{f} &= \sum_{I \subset [1, \dots, d]^n} a_I x_1^{n_1} \cdots x_n^{n_n} \in \mathbb{Q}[(a_I)_{I \subset [1, \dots, d]^n}, x_1, \dots, x_n]. \end{aligned}$$

- For $X \in \text{AnSp}(\mathbb{C})$, we denote $\alpha(X) : \mathbb{C}_X \hookrightarrow \Omega_X^{\bullet}$ the embedding in $C(X)$. For $X \in \text{AnSm}(\mathbb{C})$, $\alpha(X) : \mathbb{C}_X \hookrightarrow \Omega_X^{\bullet}$ is an equivalence usu local by Poincare lemma.

- We denote $\mathbb{I}^n := [0, 1]^n \in \text{Diff}(\mathbb{R})$ (with boundary). For $X \in \text{Top}$ and R a ring, we consider its singular cochain complex

$$C_{\text{sing}}^*(X, R) := (\mathbb{Z} \text{Hom}_{\text{Top}}(\mathbb{I}^*, X)^\vee) \otimes R$$

and for $l \in \mathbb{Z}$ its singular cohomology $H_{\text{sing}}^l(X, R) := H^n C_{\text{sing}}^*(X, R)$. For $f : X' \rightarrow X$ a continuous map with $X, X' \in \text{Top}$, we have the canonical map of complexes

$$f^* : C_{\text{sing}}^*(X, R) \rightarrow C_{\text{sing}}^*(X', R), \sigma \mapsto f^* \sigma := (\gamma \mapsto \sigma(f \circ \gamma)).$$

In particular, we get by functoriality the complex

$$C_{X, R \text{sing}}^* \in C_R(X), (U \subset X) \mapsto C_{\text{sing}}^*(U, R)$$

We recall that

- For $X \in \text{Top}$ locally contractible, e.g. $X \in \text{CW}$, and R a ring, the inclusion in $C_R(X)$ $c_X : R_X \rightarrow C_{X, R \text{sing}}^*$ is by definition an equivalence top local and that we get by the small chain theorem, for all $l \in \mathbb{Z}$, an isomorphism $H^l c_X : H^l(X, R_X) \xrightarrow{\sim} H_{\text{sing}}^l(X, R)$.
- For $X \in \text{Diff}(\mathbb{R})$, the restriction map

$$r_X : \mathbb{Z} \text{Hom}_{\text{Diff}(\mathbb{R})}(\mathbb{I}^*, X)^\vee \rightarrow C_{\text{sing}}^*(X, R), w \mapsto w : (\phi \mapsto w(\phi))$$

is a quasi-isomorphism by Whitney approximation theorem.

- Let $S \in \text{AnSm}(\mathbb{C})$ and $L \in \text{Shv}(S)$ a local system. Consider $E := L \otimes \mathcal{O}_S \in \text{Vect}_{\mathcal{D}}(S)$ the corresponding holomorphic vector bundle with integrable connection ∇ . Let $\pi_S : \tilde{S} \rightarrow S$ be the universal covering. Consider the canonical fiber

$$L_0 := \Gamma(\tilde{S}, \pi_S^* L) = \Gamma(\tilde{S}, \pi_S^{* \text{mod}} E)^\nabla.$$

For $\lambda \in L_0$, we will consider the flat leaf

$$\mathbb{V}_S(\lambda) := \pi_S(\lambda \times \tilde{S}) \subset E, \pi_S : L_0 \times \tilde{S} \rightarrow E, \pi_S(\nu, z) := (\nu(z), \pi_S(z)).$$

Note that for $t \in S$, $\mathbb{V}_S(\lambda) \cap p_S^{-1}(t) = \pi_1(S, t)(\lambda)$ is the orbit of λ under the monodromy action, where $p_S : E \rightarrow S$ is the projection.

2.2 Birational projective varieties and Hodge conjecture

In this subsection, we recall that assuming the Hodge conjecture for smooth projective varieties of dimension less or equal to $d - 1$, the property of satisfying the Hodge conjecture for connected smooth projective varieties of dimension d is a birational invariant :

Proposition 1. (i) *Let $X \in \text{PSmVar}(\mathbb{C})$ connected of dimension d . Let $\epsilon : \tilde{X}_Z \rightarrow X$ be the blow up of X along a smooth closed subvariety $Z \subset X$. Assume the Hodge conjecture hold for smooth complex projective varieties of dimension less or equal to $d - 1$. Then the Hodge conjecture hold for X if and only if it hold for \tilde{X}_Z .*

(ii) *Let $X, X' \in \text{PSmVar}(\mathbb{C})$ connected of dimension d . Assume the Hodge conjecture hold for smooth complex projective varieties of dimension less or equal to $d - 1$. If X is birational to X' , then the Hodge conjecture hold for X if and only if it hold for X'*

Proof. (i): Follows from the fact that

$$(\epsilon^*, \oplus_{l=1}^c i_{E^*}((-).h^{l-1})) : H^k(X, \mathbb{Q}) \oplus \oplus_{l=1}^c H^{k-2l}(E, \mathbb{Q}) \rightarrow H^k(\tilde{X}_Z, \mathbb{Q})$$

is an isomorphism of Hodge structures for each $k \in \mathbb{Z}$, where $c = \text{codim}(Z, X)$, $i_E : E \hookrightarrow \tilde{X}_Z$ is the closed embedding, $\epsilon|_E : E \rightarrow Z$ being a projective vector bundle.

(ii): Follows from (i) since if $\pi : X^o \rightarrow X'$ is a birational map, $X^o \subset X$ being an open subset, then X is connected to X' by a sequence of blow up of smooth projective varieties with smooth center by [1]. \square

2.3 A family of smooth projective hypersurfaces associated to a smooth projective variety

In this subsection, we recall that given a smooth complex projective variety X , there exists a family of smooth complex projective hypersurfaces which degenerates into a normal crossing divisor with one irreducible component birational to X .

Proposition 2. *Let $X \in \text{PSmVar}(\mathbb{C})$ connected of dimension d . Then there exists a family of smooth projective hypersurfaces $f_X : Y \rightarrow \mathbb{A}_{\mathbb{C}}^1$ with*

- $Y \in \text{SmVar}(\mathbb{C})$, f_X flat projective,
- $Y_s := f_X^{-1}(s) \subset \mathbb{P}_{\mathbb{C}}^{d+1}$ for $s \in \mathbb{A}_{\mathbb{C}}^1 \setminus 0$ are smooth projective hypersurfaces,
- $E := f_X^{-1}(0) = \cup_{i=0}^r E_i = \epsilon^{-1}(Y_0) \subset Y$ is a normal crossing divisor, where $Y_0 \subset \mathbb{P}_{\mathbb{C}}^{d+1}$ is a possibly singular hypersurface birational to X , $\epsilon : (Y, E) \rightarrow (Y', Y_0)$ a desingularization, and E_0 is birational to X ,

Proof. See [2] lemma 1.5.8 . It use the fact that X is birational to a possibly singular projective hypersurface Y_0 and consider the desingularization of a generic pencil $f'_X : Y' = V(f) \subset \mathbb{P}_{\mathbb{C}}^{d+1} \times \mathbb{A}_{\mathbb{C}}^1 \rightarrow \mathbb{A}_{\mathbb{C}}^1$ passing through Y_0 and $f_X = f'_X \circ \epsilon$. \square

2.4 The nearby cycle functor and the specialization map

Recall from [3] that for $k \subset \mathbb{C}$ a subfield and $S \in \text{Var}(k)$ quasi-projective, and $l : S \hookrightarrow \tilde{S}$ a closed embedding with $\tilde{S} \in \text{SmVar}(k)$, we have the full subcategory

$$\iota : MHM_{k, gm}(S) \hookrightarrow \text{PSh}_{\mathcal{D}fil, S}(\tilde{S}) \times_I P(S_{\mathbb{C}}^{an})$$

consisting of geometric mixed Hodge module whose De Rham part is defined over k , where, using the fact that the V -filtration is defined over k , $\text{PSh}_{\mathcal{D}(1,0)fil, S}(\tilde{S}) \times_I P_{fil}(S_{\mathbb{C}}^{an})$ is the category

- whose objects are $((M, F, W), (K, W), \alpha)$ where (M, F, W) is a filtered $D_{\tilde{S}}$ module supported on S , where $l : S \hookrightarrow \tilde{S}$ is a closed embedding and $\tilde{S} \in \text{SmVar}(k)$, (K, W) is a filtered Perverse sheaf on $S_{\mathbb{C}}^{an}$, and $\alpha : l_*(K, W) \otimes \mathbb{C} \rightarrow DR(\tilde{S})((M, W)^{an})$ is an isomorphism in $D_{fil}(\tilde{S}_{\mathbb{C}}^{an})$.
- whose morphism between $((M, F, W), (K, W), \alpha)$ and $((M', F, W), (K', W), \alpha')$, are couples of morphisms $(\phi : (M, F, W) \rightarrow (M', F, W), \psi : (K, W) \rightarrow (K', W))$ such that $\phi \circ \alpha = \alpha' \circ \psi$.

Then, by [3], we get for $S \in \text{Var}(k)$ quasi-projective and $l : S \hookrightarrow \tilde{S}$ is a closed embedding with $\tilde{S} \in \text{SmVar}(k)$ an embedding

$$\iota : D(MHM_{k, gm}(S)) \hookrightarrow D_{\mathcal{D}fil, S}(\tilde{S}) \times_I D(S_{\mathbb{C}}^{an})$$

consisting of mixed Hodge module whose De Rham part is defined over k , where $D_{\mathcal{D}(1,0)fil, S}(\tilde{S}) \times_I D_{fil, c}(S_{\mathbb{C}}^{an})$ is the category

- whose objects are $((M, F, W), (K, W), \alpha)$ where (M, F, W) is a complex of filtered $D_{\tilde{S}}$ module supported on S , (K, W) is a filtered complex of presheaves on $S_{\mathbb{C}}^{an}$ whose cohomology are constructible sheaves, and $\alpha : l_*(K, W) \otimes \mathbb{C} \rightarrow DR(\tilde{S})((M, W)^{an})$ is an isomorphism in $D_{fil}(\tilde{S}_{\mathbb{C}}^{an})$.
- whose morphism are given in [3],

We have then the six functors formalism :

$$\begin{aligned} D(MHM_{k, gm}(-)) : \text{Var}(k) &\mapsto \text{TriCat}, \quad S \mapsto D(MHM_{k, gm}(S)), \quad (f : T \rightarrow S) \mapsto \\ (f^*, f_*) : D(MHM_{k, gm}(S)) &\rightarrow D(MHM_{k, gm}(T)), \quad (f!, f^!) : D(MHM_{k, gm}(T)) \rightarrow D(MHM_{k, gm}(S)) \end{aligned}$$

and for $D \subset S$ a (Cartier) divisor the nearby cycle functor

$$\begin{aligned} \psi_D &: D(MHM_{k,gm}(S)) \rightarrow D(MHM_{k,gm}(D)), \\ \psi_D((M, F, W), (K, W), \alpha) &:= (\psi_D(M, F, W), \psi_D(K, W), \psi_D(\alpha)). \end{aligned}$$

For $T \in \text{Var}(k)$ and $\epsilon_\bullet : T_\bullet \rightarrow T$ a desingularization with $T_\bullet \in \text{Fun}(\Delta, \text{SmVar}(k))$, we have

$$\mathbb{Z}_T^{Hdg} := a_T^{*Hdg} \mathbb{Z}^{Hdg} = (\epsilon_{\bullet*} E_{zar}(O_{T_\bullet}, F, W), \epsilon_{\bullet*} E_{usu}(\mathbb{Z}_{T_\bullet^{an}}, W), \alpha(T_\bullet))$$

For $f : X \rightarrow S$ a projective morphism with $X, S \in \text{SmVar}(k)$, we denote

$$E_{Hdg}(X/S) := f_* \mathbb{Z}_X^{Hdg} := \left(\int f(O_X, F), Rf_* \mathbb{Z}_{X_\mathbb{C}^{an}}, f_* \alpha(X) \right) \in D(MHM_{k,gm}(S)).$$

For $f : X \hookrightarrow \mathbb{P}^N \times S \xrightarrow{p} S$ a projective morphism with $X, S \in \text{Var}(k)$, S smooth, we denote

$$E_{Hdg}(X/S) := f_* \mathbb{Z}_X^{Hdg} := \left(\int_p \Gamma_X^{Hdg}(O_{\mathbb{P}^N \times S}, F), Rf_* \mathbb{Z}_{X_\mathbb{C}^{an}}, f_* \alpha(X) \right) \in D(MHM_{k,gm}(S)).$$

We then have the specialization map :

Definition-Proposition 1. *Let k be field of characteristic zero.*

(i) *Let $f' : X' \rightarrow T$ be a projective morphism with $X', T \in \text{SmVar}(k)$. Assume that*

- $f^\circ := f \otimes_T T : X^\circ \rightarrow T^\circ$ is smooth, $T^\circ \subset T$ is an open subset,
- $X'_S := f'^{-1}(S) \subset X'$, $S := T \setminus T^\circ$. We have by definition $X^\circ = X' \setminus X'_S$.

We have then, see [3], the distinguished triangles in $D(MHM_{k,gm}(X_S))$,

$$\begin{aligned} \mathbb{Z}_{X'_S}^{Hdg} &:= i_{X'_S}^* \mathbb{Z}_{X'}^{Hdg} \xrightarrow{sp_{X'_S/X'}} \psi_{X'_S} \mathbb{Z}_{X^\circ}^{Hdg} := (\psi_{X'_S}(O_{X'}, F), \psi_{X'_S} \mathbb{Z}_{X^\circ}, \psi_{X'_S} \alpha(X')) \\ &\xrightarrow{c(\psi_{X'_S} \mathbb{Z}_{X^\circ}^{Hdg})} \phi_{X'_S} \mathbb{Z}_{X^\circ}^{Hdg} := \text{Cone}(sp_{X'/X_S}) \xrightarrow{c(\mathbb{Z}_{X'_S}^{Hdg})} \mathbb{Z}_{X'_S}^{Hdg}[1]. \end{aligned}$$

Applying the functor $(f' \circ i_{X'_S})_*$, we obtain the generalized distinguished triangle in $D(MHM_{k,gm}(S))$

$$\begin{aligned} E_{Hdg}(X'_S/S) &\xrightarrow{sp_{X'_S/f} := (f' \circ i_{X'_S})_* sp_{X'_S/X'}} \\ (f' \circ i_{X'_S})_* (\psi_{X'_S} \mathbb{Z}_{X^\circ}^{Hdg}) &= \psi_0 E_{Hdg}(X^\circ/T^\circ) \xrightarrow{(f' \circ i_{X'_S})_* c(\Phi_{X_0} \mathbb{Z}_{X^\circ}^{Hdg})} \\ (f' \circ i_{X'_S})_* (\phi_{X'_S} \mathbb{Z}_{X^\circ}^{Hdg}) &= \phi_S E_{Hdg}(X^\circ/T^\circ) \xrightarrow{(f' \circ i_{X'_S})_* c(\mathbb{Z}_{X'}^{Hdg})} E_{Hdg}(X'_S/S)[1] \end{aligned}$$

(i)' *If for each $t \in T^\circ(\mathbb{C})$ we have $H^{k-1}(X'_t, \mathbb{Q}) = 0$ and for each $t' \in S(\mathbb{C})$ we have $H_{X'_{t'}}^{k-1}(X', \mathbb{Q}) = 0$, then the induced map on cohomology*

$$sp_{X'_S/f} : E_{Hdg}^k(X'_S/S) \rightarrow H^k \psi_S E_{Hdg}(X^\circ/T^\circ)$$

is injective.

(ii) *Let $f : X \rightarrow T$ be a projective morphism with $X, T \in \text{SmVar}(k)$. Assume that*

- $f^\circ := f \otimes_T T^\circ : X^\circ \rightarrow T^\circ$ is smooth, $T^\circ \subset T$ an open subset
- $S := T \setminus T^\circ$, $E := f^{-1}(S) = \cup_{i=0}^r E_i \subset X$ is a normal crossing divisor and that for $I \subset [0, \dots, r]$, $f_{0,I} := f|_{E_I} : E_I := \cap_{i \in I} E_i \rightarrow S$ are smooth. We have by definition $X^\circ = X \setminus E$.

Denote for $I \subset J \subset [0, \dots, r]$, $i_{JI} : E_J \hookrightarrow E_I$ and $i_I : E_I \hookrightarrow E$, $i_E : E \hookrightarrow X$ the closed embeddings. In this particular case, c.f. [8], the distinguished triangle in $D(\text{MHM}_{k, \text{gm}}(E))$ given in (i) become

$$\begin{aligned} \mathbb{Z}_E^{\text{Hdg}} &:= i_E^* \mathbb{Z}_X^{\text{Hdg}} = ((i_{\bullet*} \Omega_{E^\bullet}, F, W), (\mathbb{Z}_{E^\bullet}, W), \alpha(E_\bullet)) \xrightarrow{sp_{E/X}} \\ \psi_E \mathbb{Z}_{X^\circ}^{\text{Hdg}} &= ((i_E^{*\text{mod}} \Omega_{X/\mathbb{A}^1 \times S}(\log E), F, W), \psi_E \mathbb{Z}_{X^\circ}, \psi_E \alpha(X)) \xrightarrow{c(\psi_E \mathbb{Z}_{X^\circ}^{\text{Hdg}})} \\ \phi_E \mathbb{Z}_{X^\circ}^{\text{Hdg}} &:= \text{Cone}(sp_{E/X}) \xrightarrow{c(\mathbb{Z}_E^{\text{Hdg}})} \mathbb{Z}_E^{\text{Hdg}}[1]. \end{aligned}$$

Applying the functor $(f \circ i_E)_*$, we obtain the generalized distinguished triangle in $D(\text{MHM}_{k, \text{gm}}(S))$

$$\begin{aligned} E_{\text{Hdg}}(E/S) &\xrightarrow{sp_{E/f} := (f \circ i_E)_* sp_{E/X}} (f \circ i_E)_*(\psi_E \mathbb{Z}_{X^\circ}^{\text{Hdg}}) = \psi_S E_{\text{Hdg}}(X^\circ / ((\mathbb{A}^1 \setminus 0) \times S)) \\ \xrightarrow{(f \circ i_E)_* c(\psi_E \mathbb{Z}_{X^\circ}^{\text{Hdg}})} &(f \circ i_E)_*(\phi_E \mathbb{Z}_{X^\circ}^{\text{Hdg}}) = \phi_S E_{\text{Hdg}}(X^\circ / ((\mathbb{A}^1 \setminus 0) \times S)) \xrightarrow{(f \circ i_E)_* c(\mathbb{Z}_X^{\text{Hdg}})} E_{\text{Hdg}}(E/S)[1]. \end{aligned}$$

If $f = f' \circ \epsilon$ where $\epsilon : (X, E) \rightarrow (X, X_S)$ is a desingularization, we have $sp_{X_S/X'} = sp_{E/X} \circ \epsilon^*$.

Proof. (i)': Follows from the exact sequence

$$\dots \rightarrow H_{X_{t'}}^l(X'_C, \mathbb{Q}) \rightarrow \mathbb{H}^l(X_{t'}, \phi_S \mathbb{Z}_{X_C^\circ}) \xrightarrow{V} \mathbb{H}^l(X_{t'}, \psi_S \mathbb{Z}_{X_C^\circ}) \rightarrow H_{X_{t'}}^{l+1}(X'_C, \mathbb{Q}) \rightarrow \dots$$

given in [8], where $C \subset T$ is a transversal slice to S passing through t' . \square

3 Hodge conjecture for smooth projective varieties

Theorem 1. *Let $X \in \text{PSmVar}(\mathbb{C})$. Then Hodge conjecture hold for X . That is, if $p \in \mathbb{Z}$ and $\lambda \in F^p H^{2p}(X^{\text{an}}, \mathbb{Q})$, $\lambda = [Z]$ with $Z \in \mathcal{Z}^p(X)$.*

Proof. The Hodge conjecture is true for curves and surfaces. Assume that the Hodge conjecture is true for smooth projective varieties of dimension less or equal to $d-1$. Let $X \in \text{PSmVar}(\mathbb{C})$ of dimension d . Up to split X into its connected components, we may assume that X is connected of dimension d . Then X is birational to a possibly singular hypersurface $Y_{0_X} \subset \mathbb{P}_{\mathbb{C}}^{d+1}$ of degree $l_X := \deg(\mathbb{C}(X)/\mathbb{C}(x_1, \dots, x_d))$. Consider

$$f : \mathcal{Y}' = V(f) \subset \mathbb{P}_{\mathbb{C}}^{d-1} \times \mathbb{P}(l_X) \rightarrow \mathbb{P}(l_X)$$

the universal family of projective hypersurfaces of degree l_X , and $0_X \in \mathbb{P}(l_X)$ the point corresponding to Y_{0_X} . Denote $\Delta \subset \mathbb{P}(l_X)$ the discriminant locus parametrizing the hypersurfaces which are singular. Let $\bar{S} := 0_X^{\mathbb{Q}} \subset \mathbb{P}(l_X)$ be the \mathbb{Q} -Zariski closure of 0_X inside $\mathbb{P}(l_X)$. If Y_{0_X} is smooth, then X satisfy the hodge conjecture since Y_{0_X} satisfy the Hodge conjecture by [5] and by the induction hypothesis since X is birational to Y_{0_X} by proposition 1(ii). Hence, we may assume that Y_{0_X} is singular, that is $0_X \in \Delta$. Then, $\bar{S} \subset \Delta$ since $\Delta \subset \mathbb{P}(l_X)$ is defined over \mathbb{Q} . Let $\bar{T} \subset \mathbb{P}(l_X)$ be an irreducible closed subvariety of dimension $\dim(\bar{S}) + 1$ defined over \mathbb{Q} such that $\bar{S} \subset \bar{T}$ and $T := \bar{T} \cap (\mathbb{P}(l_X) \setminus \Delta) \neq \emptyset$. Since there always exists subvarieties of $\mathbb{P}(l_X) \setminus \Delta$ of arbitrary dimension defined over \mathbb{Q} which are not weakly special (take complete intersections which intersect Δ transversaly), we may assume that $T \subset \mathbb{P}(l_X) \setminus \Delta$ is not contained in a weakly special subvariety. We have then $\bar{T} \cap \Delta = \bar{S} \cup \bar{S}'$ where $\bar{S}' \subset \Delta$ is a divisor. We then consider the family of algebraic varieties defined over \mathbb{Q}

$$\tilde{f}_X : \mathcal{Y}_{\bar{T}^\circ} \xrightarrow{\epsilon} \mathcal{Y}'_{\bar{T}^\circ} \xrightarrow{f \times_{\mathbb{P}(l_X)} \bar{T}^\circ} \bar{T}^\circ$$

where

- $\tilde{f}_X : \mathcal{Y} \xrightarrow{\epsilon} \mathcal{Y}'_{\bar{T}} \xrightarrow{f \times_{\mathbb{P}(l_X)} \bar{T}} \bar{T}$ with $\epsilon : (\mathcal{Y}, \mathcal{E}') \rightarrow (\mathcal{Y}'_{\bar{T}}, \mathcal{Y}_{\bar{S}} \cup \mathcal{Y}_{\bar{S}'})$ is a desingularization over \mathbb{Q} , in particular $\mathcal{E}' = \mathcal{E} \cup \mathcal{E}''$ with $\mathcal{E} = \cup_{i=0}^r \mathcal{E}_i := \tilde{f}_X^{-1}(\bar{S}) \subset \mathcal{Y}$ which is a normal crossing divisor, in particular $\mathcal{Y}_T = \mathcal{Y}'_T$,

- $S \subset \bar{S}$ is the open subset such that S is smooth and $\tilde{f}_X \times_{\bar{T}} S : \mathcal{E}_I := \cap_{i \in I} \mathcal{E}_i \rightarrow S$ is smooth projective for each $I \subset [0, \dots, r]$, $\bar{T}^\circ \subset \bar{T}$ is an open subset such that $\bar{T}^\circ \cap \Delta = S$ and $T^\circ := \bar{T}^\circ \cap T$ is smooth,
- $t \in \mathbb{S}(\mathbb{C})$ is the point such that $\mathcal{Y}'_{\mathbb{C}, t} = Y_{0_X}$ and $\epsilon_t : E_0 := \mathcal{E}_{0, \mathbb{C}, t} \rightarrow Y_{0_X}$ is surjective, in particular E_0 is birational to Y_{0_X} , hence E_0 is birational to X .

Denote for $I \subset [0, \dots, s]$, $i_I : \mathcal{E}_I \hookrightarrow \mathcal{E}$ and for $I \subset J \subset [0, \dots, s]$, $i_{JI} : \mathcal{E}_J \rightarrow \mathcal{E}_I$ the closed embeddings. Then, $i_\bullet : \mathcal{E}_\bullet \rightarrow \mathcal{E}$ in $\text{Fun}(\Delta, \text{Var}(\mathbb{Q}))$ is a simplicial resolution with $\mathcal{E}_\bullet \in \text{Fun}(\Delta, \text{SmVar}(\mathbb{Q}))$. We have then

$$E_{Hdg}(\mathcal{E}/S) := (\tilde{f}_X \circ i_{\bullet*} E_{zar}(\Omega_{\mathcal{E}_\bullet/S}, F_b, W), (\tilde{f}_X \circ i_{\bullet*} E_{usu}(\mathbb{Q}_{\mathcal{E}_\bullet/\mathbb{C}}^{an}, W)), \tilde{f} \circ i_{\bullet*} \alpha(\mathcal{E}_\bullet)) \in D(MHM_{\mathbb{Q}, gm}(S))$$

Then, definition-proposition 1 applied to $f_T := f \times_{\mathbb{P}(t_X)} \bar{T}^\circ : \mathcal{Y}'_{T^\circ} \rightarrow \bar{T}^\circ$ gives the distinguish in $D(MHM_{\mathbb{Q}, gm}(S))$,

$$\begin{aligned} E_{Hdg}(\mathcal{Y}'_S/S) &\xrightarrow{Sp_{\mathcal{Y}'_S/f_T} := (\tilde{f}'_X \circ i_{\mathcal{Y}'_S})_* sp_{\mathcal{Y}_S/\mathcal{Y}'_{T^\circ}}} \psi_S E_{Hdg}(\mathcal{Y}'_{T^\circ}/T^\circ) \\ &\xrightarrow{(f_T \circ i_{\mathcal{Y}'_S})_* c(\psi_{\mathcal{Y}'_S} \mathbb{Z}_{\mathcal{Y}'_{T^\circ}}^{Hdg})} \phi_S E_{Hdg}(\mathcal{Y}'_{T^\circ}/T^\circ) \xrightarrow{(f_T \circ i_{\mathcal{Y}'_S})_* c(\mathbb{Z}_{\mathcal{Y}'_S}^{Hdg})} E_{Hdg}(\mathcal{Y}'_S/S)[1] \end{aligned}$$

and definition-proposition 1 applied to $\tilde{f}_X : \mathcal{Y}_{\bar{T}^\circ} \rightarrow \bar{T}^\circ$ gives the distinguish in $D(MHM_{\mathbb{Q}, gm}(S))$,

$$\begin{aligned} E_{Hdg}(\mathcal{E}/S) &\xrightarrow{Sp_{\mathcal{E}/\tilde{f}_X} := (\tilde{f}_X \circ i_{\mathcal{E}})_* sp_{\mathcal{E}/\mathcal{Y}}} \psi_S E_{Hdg}(\mathcal{Y}_{T^\circ}/T^\circ) \\ &\xrightarrow{(\tilde{f}_X \circ i_{\mathcal{E}})_* c(\psi_{\mathcal{E}} \mathbb{Z}_{\mathcal{Y}_{T^\circ}}^{Hdg})} \phi_S E_{Hdg}(\mathcal{Y}_{T^\circ}/T^\circ) \xrightarrow{(\tilde{f}_X \circ i_{\mathcal{E}})_* c(\mathbb{Z}_{\mathcal{E}}^{Hdg})} E_{Hdg}(\mathcal{E}/S)[1] \end{aligned}$$

and $Sp_{\mathcal{E}/\tilde{f}_X} \circ \epsilon^* = Sp_{\mathcal{Y}_{T^\circ}/f_T}$. Let $p \in \mathbb{Z}$. By the hard Lefschetz theorem on $H^*(E_0)$, we may assume that $2p \leq d$. Taking the cohomology to this distinguish triangle, we get in particular, for each $p \in \mathbb{Z}$, the maps in $MHM_{\mathbb{Q}, gm}(S)$

- $Sp_{\mathcal{Y}'_S/f_T} : E_{Hdg}^{2p}(\mathcal{Y}'_S/S) \hookrightarrow H^{2p} \psi_S E_{Hdg}(\mathcal{Y}_{T^\circ}/T^\circ)$
- $Sp_{\mathcal{E}/\tilde{f}_X} : E_{Hdg}^{2p}(\mathcal{E}/S) \rightarrow H^{2p} \psi_S E_{Hdg}(\mathcal{Y}_{T^\circ}/T^\circ)$.

The map $Sp_{\mathcal{Y}'_S/f_T}$ is injective by definition-proposition 1(i)' and the weak Lefschetz theorem since for $t' \in T^\circ(\mathbb{C})$ and $t'' \in S(\mathbb{C})$, $\mathcal{Y}'_{t'}$ and $\mathcal{Y}'_{t''}$ are projective hypersurfaces. For $\lambda \in H^{2p}(\mathcal{E}_{0, \mathbb{C}, t'}^{an}, \mathbb{Q})$, $t' \in S(\mathbb{C})$, we consider $\pi_S : \tilde{S}_{\mathbb{C}}^{an} \rightarrow S_{\mathbb{C}}^{an}$ the universal covering in $\text{AnSm}(\mathbb{C})$ and

$$\mathbb{V}_S(\lambda) := \pi_S(\lambda \times \tilde{S}_{\mathbb{C}}^{an}) \subset E_{DR}^{2p, an}(\mathcal{E}_{0, \mathbb{C}}/S_{\mathbb{C}})$$

the flat leaf induced by the universal covering (see section 2.1) and

$$\mathbb{V}_S^p(\lambda) := \mathbb{V}_S(\lambda) \cap F^p E_{DR}^{2p}(\mathcal{E}_{0, \mathbb{C}}/S_{\mathbb{C}}) \subset E_{DR}^{2p}(\mathcal{E}_{0, \mathbb{C}}/S_{\mathbb{C}}),$$

which is an algebraic variety by [6]. Note that $\mathbb{V}_S^p(\lambda) \subset H^{p, 2p}(\mathcal{E}_{0, \mathbb{C}}/S_{\mathbb{C}})$ is the union of the irreducible components passing through λ . Similarly, for $\eta \in H^{2p}(\mathcal{E}_{\mathbb{C}, t'}^{an}, \mathbb{Q})$, $t' \in S(\mathbb{C})$, we consider $\pi_S : \tilde{S}_{\mathbb{C}}^{an} \rightarrow S_{\mathbb{C}}^{an}$ the universal covering in $\text{AnSm}(\mathbb{C})$ and

$$\mathbb{V}_S(\eta) := \pi_S(\eta \times \tilde{S}_{\mathbb{C}}^{an}) \subset E_{DR}^{2p, an}(\mathcal{E}_{\mathbb{C}}/S_{\mathbb{C}})$$

the flat leaf induced by the universal covering (see section 2.1) and

$$\mathbb{V}_S^p(\eta) := \mathbb{V}_S(\eta) \cap F^p E_{DR}^{2p}(\mathcal{E}_{\mathbb{C}}/S_{\mathbb{C}}) \subset E_{DR}^{2p}(\mathcal{E}_{\mathbb{C}}/S_{\mathbb{C}}).$$

We have

$$H^{2p}E_{Hdg}(\mathcal{Y}_{T^o}/T^o) = ((E_{DR}^{2p}(\mathcal{Y}_{T^o}/T^o), F), R^{2p}\tilde{f}_{X*}\mathbb{Z}_{\mathcal{Y}_{T^o}}, \tilde{f}_{X*}\alpha(\mathcal{Y}_{T^o})) \in MHM_{\mathbb{Q}, gm}(T^o)$$

By [8], the filtered vector bundle

$$(E_{DR}^{2p}(\mathcal{Y}_{T^o}/T^o), F) = H^{2p}\tilde{f}_{X*}E_{zar}(\Omega_{\mathcal{Y}_{T^o}/T^o}^\bullet) \in \text{Vect}_{\mathcal{D}fil}(T^o)$$

extend to a filtered vector bundle

$$(E_{DR, V_S}^{2p}(\mathcal{Y}_{T^o}/T^o), F) := H^{2p}\tilde{f}_{X*}E_{zar}(\Omega_{\mathcal{Y}_{T^o}/T^o}^\bullet(\log \mathcal{E})) \in \text{Vect}_{\mathcal{D}fil}(\bar{T}^o)$$

such that

$$H^{2p}\psi_S E_{Hdg}(\mathcal{Y}_{T^o}/T^o) = (i_S^{*mod}(E_{DR, V_S}^{2p}(\mathcal{Y}_{T^o}/T^o), F), (\psi_S R^{2p}\tilde{f}_{X*}\mathbb{Z}_{\mathcal{Y}^o}, W), \psi_S \tilde{f}_{X*}\alpha(\mathcal{Y}_{T^o})).$$

For

$$\alpha \in H^{2p}(\mathcal{Y}_{t''}^{an}, \mathbb{Q}) = \mathbb{H}^{2p}(\mathcal{E}_{t''}^{an}, \psi_S R\tilde{f}_{X*}\mathbb{Q}_{\mathcal{Y}_{T^o}}), t' \in S_{\mathbb{C}}, t'' \in T^o(\mathbb{C}),$$

we consider

- $\pi_{T^o} : \tilde{T}^{o, an} \rightarrow T_{\mathbb{C}}^{o, an}$ the universal covering in $\text{AnSm}(\mathbb{C})$, the factorization $\pi_S : \tilde{S}_{\mathbb{C}}^{an} \rightarrow \tilde{T}^{o, an} \xrightarrow{\pi_{T^o}} T_{\mathbb{C}}^{o, an}$ and

$$\mathbb{V}_{T^o}(\alpha) := \pi_{T^o}(\alpha \times \tilde{T}^{o, an}) \subset E_{DR}^{2p, an}(\mathcal{Y}_{T^o, \mathbb{C}}/T_{\mathbb{C}}^o)$$

the flat leaf (see section 2.1) and

$$\mathbb{V}_{T^o}^p(\alpha) := \mathbb{V}_{T^o}(\alpha) \cap F^p E_{DR}^{2p}(\mathcal{Y}_{T^o, \mathbb{C}}/T_{\mathbb{C}}^o) \subset E_{DR}^{2p}(\mathcal{Y}_{T^o, \mathbb{C}}/T_{\mathbb{C}}^o),$$

which is an algebraic subvariety by [6],

- $\pi_S : \tilde{S}_{\mathbb{C}}^{an} \rightarrow S_{\mathbb{C}}^{an}$ the universal covering in $\text{AnSm}(\mathbb{C})$ and

$$\mathbb{V}_S(\alpha) := \pi_S(\alpha \times \tilde{S}_{\mathbb{C}}^{an}) \subset H^{2p}\psi_S E_{DR}(\mathcal{Y}_{T^o, \mathbb{C}}/T_{\mathbb{C}}^o)^{an}$$

the flat leaf (see section 2.1) and

$$\mathbb{V}_S^p(\alpha) := \mathbb{V}_S(\alpha) \cap F^p H^{2p}\psi_S E_{DR}(\mathcal{Y}_{T^o, \mathbb{C}}/T_{\mathbb{C}}^o)^{an} \subset H^{2p}\psi_S E_{DR}(\mathcal{Y}_{T^o, \mathbb{C}}/T_{\mathbb{C}}^o)^{an}.$$

Consider the exact sequence of variation of mixed Hodge structure over S

$$0 \rightarrow W_{2p-2}E_{Hdg}^{2p}(\mathcal{E}/S) \rightarrow E_{Hdg}^{2p}(\mathcal{E}/S) \xrightarrow{(i_i^*)_{0 \leq i \leq r} := q} \oplus_{i=0}^r E_{Hdg}^{2p}(\mathcal{E}_i/S) \rightarrow 0.$$

It induces the exact sequence of vector bundles over S

$$0 \rightarrow W_{2p-2}E_{DR}^{2p}(\mathcal{E}/S) \rightarrow E_{DR}^{2p}(\mathcal{E}/S) \xrightarrow{(i_i^*)_{0 \leq i \leq r} := q} \oplus_{i=0}^r E_{DR}^{2p}(\mathcal{E}_i/S) \rightarrow 0,$$

and the exact sequence of presheaves on $S_{\mathbb{C}}^{an}$,

$$J(W_{2p-2}E_{Hdg}^{2p}(\mathcal{E}_{\mathbb{C}}/S_{\mathbb{C}})) := \text{Ext}^1(\mathbb{Q}_S^{Hdg}, W_{2p-2}E_{Hdg}^{2p}(\mathcal{E}_{\mathbb{C}}/S_{\mathbb{C}})), \xrightarrow{L} \text{Ext}^1(\mathbb{Q}_S^{Hdg}, W_{2p-2}E_{Hdg}^{2p}(\mathcal{E}_{\mathbb{C}}/S_{\mathbb{C}}))$$

$$HL^{2p, p}(\mathcal{E}_{\mathbb{C}}/S_{\mathbb{C}}) \xrightarrow{q} \oplus_{i=0}^r HL^{2p, p}(\mathcal{E}_{i, \mathbb{C}}/S_{\mathbb{C}}) \xrightarrow{e}$$

where $HL^{2p, p}(\mathcal{E}_{\mathbb{C}}/S_{\mathbb{C}}) \subset E_{DR}^{2p}(\mathcal{E}_{\mathbb{C}}/S_{\mathbb{C}})$, $HL^{2p, p}(\mathcal{E}_{i, \mathbb{C}}/S_{\mathbb{C}}) \subset E_{DR}^{2p}(\mathcal{E}_{i, \mathbb{C}}/S_{\mathbb{C}})$ are the locus of Hodge classes. Let $\lambda \in F^p H^{2p}(E_0^{an}, \mathbb{Q})$, where we recall $E_0 = \mathcal{E}_{0, \mathbb{C}, t}$. Consider then $\tilde{\lambda} \in H^{2p}(E^{an}, \mathbb{Q})$, such that $q(\tilde{\lambda}) = \lambda$, and $\tilde{\lambda} := Sp_{\mathcal{E}/\tilde{f}_X}(\tilde{\lambda}) \in Sp_{E/\tilde{f}_X}(H^{2p}(E^{an}, \mathbb{Q}))$, Since the monodromy of $R^{2p}\tilde{f}_{X*}\mathbb{Q}_{\mathcal{Y}_{T^o, \mathbb{C}}}^{an}$ is irreducible, we have

$$Sp_{\mathcal{E}/\tilde{f}_X}(\ker q) := Sp_{\mathcal{E}/\tilde{f}_X}(W_{2p-2}E_{DR}^{2p}(\mathcal{E}/S)) = 0$$

The locus of Hodge classes passing through λ

$$\mathbb{V}_S^p(\lambda) := \mathbb{V}_S(\lambda) \cap F^p E_{DR}^{2p}(\mathcal{E}_{0,\mathbb{C}}/S_{\mathbb{C}}) \subset E_{DR}^{2p}(\mathcal{E}_{0,\mathbb{C}}/S_{\mathbb{C}}),$$

inside the De Rham vector bundle of $\tilde{f}_X \circ i_{E_0} : \mathcal{E}_0 \rightarrow S$ thus satisfy

$$Sp_{\mathcal{E}/\tilde{f}_X}(q^{-1}(\mathbb{V}_S^p(\lambda))) = \mathbb{V}_S^p(Sp_{\mathcal{E}/\tilde{f}_X}(\tilde{\lambda})) \subset E_{DR, V_S}^{2p}(\mathcal{Y}_{T^o, \mathbb{C}}/T_{\mathbb{C}}^o),$$

where

- $\mathbb{V}_S(\lambda) \subset E_{DR}^{2p}(\mathcal{E}_{0,\mathbb{C}}/S_{\mathbb{C}})$, $\mathbb{V}_S(\tilde{\lambda}) \subset E_{DR}^{2p}(\mathcal{E}_{\mathbb{C}}/S_{\mathbb{C}})$, are the flat leaves, e.g. $\mathbb{V}_S(\lambda) := \pi_S(\lambda \times \tilde{S}_{\mathbb{C}}^{an})$ where $\pi_S : H^{2p}(E^{an}, \mathbb{C}) \times \tilde{S}_{\mathbb{C}}^{an} \rightarrow E_{DR}^{2p, an}(\mathcal{E}_{\mathbb{C}}/S_{\mathbb{C}})$ is the morphism induced by the universal covering $\pi_S : \tilde{S}_{\mathbb{C}}^{an} \rightarrow S_{\mathbb{C}}^{an}$,
- $q := q \otimes \mathbb{C} : E_{DR}^{2p}(\mathcal{E}_{\mathbb{C}}/S_{\mathbb{C}}) \rightarrow E_{DR}^{2p}(\mathcal{E}_{0,\mathbb{C}}/S_{\mathbb{C}})$ is the quotient map.

Now,

- if $Sp_{\mathcal{E}/\tilde{f}_X}(\tilde{\lambda}) = 0$, since $Sp_{\mathcal{E}/\tilde{f}_X} \circ \epsilon^* = Sp_{\mathcal{Y}_0/\tilde{f}'_X}$ and $Sp_{\mathcal{Y}_0/\tilde{f}'_X}$ is injective, we have $\tilde{\lambda}_{t'} = k_{t*} \tilde{\lambda}_{t'}$ with $\tilde{\lambda}_{t'} \in F^p H_{\mathcal{F}, t'}^{2p}(\mathcal{E}_{\mathbb{C}, t'}^{an}, \mathbb{Q})$, where $k : \mathcal{F} \rightarrow \mathcal{E}$ is the exceptional locus of $\epsilon : \mathcal{E}_0 \rightarrow \mathcal{Y}_0$ (note that $\epsilon^* : H^{2p}(\mathcal{Y}_{0, t'}^{an}, \mathbb{Q}) \rightarrow H^{2p}(\mathcal{E}_{0, t'}^{an}, \mathbb{Q})$ is injective by Poincare Lefschetz duality), hence, assuming the Hodge conjecture for smooth projective varieties of dimension less or equal to $d-1$, $\mathbb{V}_S^p(\tilde{\lambda}) \subset E_{DR}^{2p}(\mathcal{E}_{\mathbb{C}}/S_{\mathbb{C}})$ is defined over $\bar{\mathbb{Q}}$ and its Galois conjugates are also components of the locus of Hodge classes,
- if $Sp_{\tilde{E}/\tilde{f}_X}(\tilde{\lambda}) \neq 0$, we have,

$$Sp_{\mathcal{E}/\tilde{f}_X}(\mathbb{V}_S^p(\tilde{\lambda})) = \mathbb{V}_S^p(Sp_{\mathcal{E}/\tilde{f}_X}(\tilde{\lambda})) = \overline{\mathbb{V}_{T^o}^p(Sp_{\mathcal{E}/\tilde{f}_X}(\tilde{\lambda}))} \cap p^{-1}(S_{\mathbb{C}}) \subset E_{DR, V_S}^{2p}(\mathcal{Y}_{T^o, \mathbb{C}}/T_{\mathbb{C}}^o)$$

where $p : E_{DR, V_0}^{2p}(\mathcal{Y}_{T^o, \mathbb{C}}/T_{\mathbb{C}}^o) \rightarrow \bar{T}_{\mathbb{C}}^o$ is the projection (the first equality is obvious since $Sp_{\mathcal{E}/\tilde{f}_X}$ is a morphism of variation of mixed Hodge structures over S , whereas for the second equality the inclusion \supset follows from [6] lemma 2.11 which state the invariance of $\tilde{\lambda}$ under the monodromy

$$\text{Im}(\pi_1(p(\mathbb{V}_{T^o}^p(\tilde{\lambda})) \cap D^*) \rightarrow \pi_1(D^*)), (D, D^*) \rightarrow (\bar{T}_{\mathbb{C}}^{o, an}, \bar{T}_{\mathbb{C}}^{o, an} \setminus S_{\mathbb{C}}^{an})$$

but by [5] theorem 1, the Hodge conjecture holds for projective hypersurfaces hence

$$\mathbb{V}_{T^o}^p(Sp_{\mathcal{E}/\tilde{f}_X}(\tilde{\lambda})) \subset E_{DR}^{2p}(\mathcal{Y}_{T^o, \mathbb{C}}/T_{\mathbb{C}}^o)$$

is defined over $\bar{\mathbb{Q}}$ and its Galois conjugates are also components of the locus of Hodge classes, hence $\mathbb{V}_S^p(\tilde{\lambda}) \subset E_{DR}^{2p}(\mathcal{E}_{\mathbb{C}}/S_{\mathbb{C}})$ is defined over $\bar{\mathbb{Q}}$ and its Galois conjugates are also components of the locus of Hodge classes,

Hence $\mathbb{V}_S^p(\lambda) \subset E_{DR}^{2p}(\mathcal{E}_{0,\mathbb{C}}/S_{\mathbb{C}})$ is defined over $\bar{\mathbb{Q}}$ and its Galois conjugates are also components of the locus of Hodge classes. Thus, by [4] theorem 4,

$$\lambda = [Z] \in H^{2p}(E_0^{an}, \mathbb{Q}), \text{ with } Z \in \mathcal{Z}^p(E_0).$$

Since $p \in \mathbb{Z}$ and $\lambda \in F^p H^{2p}(E_0^{an}, \mathbb{Q})$ are arbitrary, the Hodge conjecture holds for E_0 . Proposition 1(ii) and the fact that the Hodge conjecture is true for smooth projective varieties of dimension less or equal to $d-1$ by induction hypothesis then implies that Hodge conjecture holds for X , since X is birational to E_0 and $X, E_0 \in \text{PSmVar}(\mathbb{C})$ are connected of dimension d . \square

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