



# Degeneration of families of projective hypersurfaces and Hodge conjecture

Johann Bouali

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# Degeneration of families of projective hypersurfaces and Hodge conjecture

Johann Bouali

March 19, 2024

## Abstract

We prove by induction on dimension the Hodge conjecture for smooth complex projective varieties. Let  $X$  be a smooth complex projective variety. Then  $X$  is birational to a possibly singular projective hypersurface, hence to a smooth projective variety  $E_0$  which is a component of a normal crossing divisor  $E = \cup_{i=0}^r E_i \subset Y$  which is the singular fiber of a pencil  $f : Y \rightarrow \mathbb{A}^1$  of smooth projective hypersurfaces. Using the smooth hypersurface case ([5] theorem 1), the nearby cycle functor on mixed Hodge module with rational de Rham factor and the induction hypothesis, we prove that a Hodge class of  $E_0$  is absolute Hodge, more precisely the locus of Hodge classes inside the algebraic vector bundle given the De Rham cohomology of the rational deformation of  $E_0$  is defined over  $\mathbb{Q}$ . By [4] theorem 4, we get the Hodge conjecture for  $E_0$ . By the induction hypothesis we also have the Hodge conjecture for  $X$ .

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## 1 Introduction

The main result of this article is an inductive proof of the Hodge conjecture using the results of [4] and [5]. More precisely, assuming the Hodge conjecture for smooth projective varieties of dimension less or equal to  $d - 1$ , the property of satisfying the Hodge conjecture for connected smooth projective varieties of dimension  $d$  is a birational invariant (proposition 1). On the other hand, if  $X$  is a connected smooth projective variety of dimension  $d$ , then  $X$  is birational to a possibly singular hypersurface  $Y_0 \subset \mathbb{P}_{\mathbb{C}}^{d+1}$ , hence there exists (see proposition 2) a family of smooth projective hypersurfaces over  $\mathbb{A}_{\mathbb{C}}^1 \setminus \{0\}$  which degenerate at 0 into a normal crossing divisor with one component birational to  $X$ , that is we have a morphism of complex algebraic varieties  $f_X : Y \rightarrow \mathbb{A}_{\mathbb{C}}^1$  with

- $Y$  a smooth complex algebraic variety,  $f_X$  flat projective,
- $Y_s := f_X^{-1}(s) \subset \mathbb{P}_{\mathbb{C}}^{d+1}$  for  $s \in \mathbb{A}_{\mathbb{C}}^1 \setminus \{0\}$  are smooth projective hypersurfaces,

- $f_X = f'_X \circ \epsilon$ ,  $f'_X : Y' \rightarrow \mathbb{A}_{\mathbb{C}}^1$  is a generic pencil of hypersurface passing through  $Y_0 := f_X'^{-1}(0) \subset \mathbb{P}_{\mathbb{C}}^{d+1}$ ,  $\epsilon : (Y, E) \rightarrow (Y', Y_0)$  is a desingularization of the pair  $(Y', Y_0)$  with  $E := f_X'^{-1}(0) = \epsilon^{-1}(Y_0) = \cup_{i=0}^r E_i \subset Y$  is a normal crossing divisor, and  $E_0$  is birational to  $X$ .

Assuming the Hodge conjecture for smooth projective varieties of dimension less or equal to  $d-1$ , we then prove the Hodge conjecture for the  $E_i$ , more specifically for  $E_0$  as follows : We consider its canonical rational deformation  $\tilde{f}_X : \mathcal{Y} \rightarrow \mathbb{A}_{\mathbb{Q}}^1 \times S$  with

- $\mathcal{Y}$  a smooth algebraic variety over  $\mathbb{Q}$ ,  $S \subset \mathbb{A}_{\mathbb{Q}}^s$  an open subset,  $\tilde{f}_X$  flat projective,  $Y = \tilde{f}_X^{-1}(\mathbb{A}_{\mathbb{Q}}^1 \times \{t\})$ ,  $f_X = \tilde{f}_X \times_S \{t\}$ , with  $t \in S(\mathbb{C})$
- $\tilde{f}_X = \tilde{f}'_X \circ \epsilon$ ,  $\tilde{f}'_X : \mathcal{Y}' \rightarrow \mathbb{A}_{\mathbb{Q}}^1 \times S$  is the corresponding family of hypersurfaces,  $\epsilon : (\mathcal{Y}, \mathcal{E}) \rightarrow (\mathcal{Y}', \mathcal{Y}_0)$  is the corresponding desingularization over  $\mathbb{Q}$ ,
- $\mathcal{E} = \cup_{i=0}^r \mathcal{E}_i := \tilde{f}_X^{-1}(\{0\} \times S) \subset \mathcal{Y}$  is a normal crossing divisor,  $\tilde{f}_{X,0} : \mathcal{E}_I := \cap_{i \in I} \mathcal{E}_i \rightarrow S$  smooth projective for each  $I \subset [0, \dots, r]$ ,  $\mathcal{E}_{\mathbb{C},t} = E$ ,
- $\tilde{f}_X \times_{\mathbb{A}^1} (\mathbb{A}^1 \setminus \{0\}) : \mathcal{Y}^\circ \rightarrow \mathbb{A}_{\mathbb{Q}}^1 \setminus \{0\}$  has fibers which are smooth projective hypersurfaces.

Let  $p \in \mathbb{Z}$ . By the hard Lefschetz theorem on  $H^*(E_0)$ , we may assume that  $2p \leq d$ . We then have then a morphism of mixed Hodge module over  $S$  (in fact of geometric variation of mixed Hodge structures over  $S$ ) given by the specialization map

$$Sp_{\mathcal{E}/\tilde{f}_X} : E_{Hdg}^{2p}(\mathcal{E}/S) \rightarrow H^{2p}\psi_0 E_{Hdg}(\mathcal{Y}^\circ/((\mathbb{A}_{\mathbb{Q}}^1 \setminus \{0\}) \times S)),$$

whose de Rham part are morphism of filtered vector bundle over  $S$  with Gauss-Manin connexion defined over  $\mathbb{Q}$ , and whose kernel of the part supported on  $\mathcal{E}_0$  is supported a closed subset  $\mathcal{F} \subset \mathcal{E}_0$  with  $\text{codim}(\mathcal{F}_{t'}, \mathcal{E}_{0,t'}) = 1$  for all  $t' \in S(\mathbb{C})$ . The fact that the kernel is supported on a closed subset of codimension one on fibers follows from the fact that  $H^{2p-1}(Y_s, \mathbb{Q}) = 0 = H_{Y_0}^{2p-1}(Y)$ , by the weak Lefschetz theorem since  $Y_s$  and  $Y_0$  are projective hypersurfaces ( $2p \leq d$ ). Consider the exact sequence of variation of mixed Hodge structure over  $S$

$$0 \rightarrow W_{2p-2} E_{Hdg}^{2p}(\mathcal{E}/S) \rightarrow E_{Hdg}^{2p}(\mathcal{E}/S) \xrightarrow{(i_i^*)_{0 \leq i \leq r} := q} \oplus_{i=0}^r E_{Hdg}^{2p}(\mathcal{E}_i/S) \rightarrow 0.$$

It induces the exact sequence of presheaves on  $S_{\mathbb{C}}^{an}$ ,

$$J(W_{2p-2} E_{Hdg}^{2p}(\mathcal{E}_{\mathbb{C}}/S_{\mathbb{C}})) := \text{Ext}^1(\mathbb{Q}_S^{Hdg}, W_{2p-2} E_{Hdg}^{2p}(\mathcal{E}_{\mathbb{C}}/S_{\mathbb{C}})) \xrightarrow{\iota} \text{Ext}^1(\mathbb{Q}_S^{Hdg}, E_{Hdg}^{2p}(\mathcal{E}_{\mathbb{C}}/S_{\mathbb{C}}))$$

where  $HL^{2p,p}(\mathcal{E}_{\mathbb{C}}/S_{\mathbb{C}}) \subset E_{DR}^{2p}(\mathcal{E}_{\mathbb{C}}/S_{\mathbb{C}})$ ,  $HL^{2p,p}(\mathcal{E}_{i,\mathbb{C}}/S_{\mathbb{C}}) \subset E_{DR}^{2p}(\mathcal{E}_{i,\mathbb{C}}/S_{\mathbb{C}})$  are the locus of Hodge classes. Let  $\lambda \in F^p H^{2p}(E_0^{an}, \mathbb{Q})$ , where we recall  $E_0 = \mathcal{E}_{0,\mathbb{C},t}$ . Consider then

- $\tilde{\lambda} \in H^{2p}(E^{an}, \mathbb{Q})$ , such that  $q(\tilde{\lambda}) = \lambda$ , and  $\tilde{\lambda} := Sp_{E/f_X}(\tilde{\lambda}) \in Sp_{E/f_X}(H^{2p}(E^{an}, \mathbb{Q}))$ ,
- $\tilde{\lambda}_F \in F^p H^{2p}(E^{an}, \mathbb{C})$ , such that  $q(\tilde{\lambda}_F) = \lambda$ ,
- $e_\lambda := \tilde{\lambda} - \tilde{\lambda}_F \in W_{2p-2} H^{2p}(E^{an}, \mathbb{C})$ ,  $e_\lambda \in \Gamma(\tilde{S}_{\mathbb{C}}^{an}, \pi^{*mod} E_{DR}^{2p,an}(\mathcal{E}_{\mathbb{C}}/S_{\mathbb{C}}))^{\nabla}$  the corresponding section where  $\pi_S : \tilde{S}_{\mathbb{C}}^{an} \rightarrow S_{\mathbb{C}}^{an}$  is the universal covering, and

$$\bar{e}_\lambda := [e_\lambda] \in \Gamma(\tilde{S}_{\mathbb{C}}^{an}, \pi^{*mod} E_{DR}^{2p,an}(\mathcal{E}_{\mathbb{C}}/S_{\mathbb{C}})/F^p \oplus \mathbb{Q})^{\nabla}.$$

The locus of Hodge classes passing through  $\lambda$  inside the De Rham vector bundle of  $\tilde{f}_X \circ i_{E_0} : \mathcal{E}_0 \rightarrow S$  is thus by the above exact sequence an union of two components

$$\mathbb{V}_S^p(\lambda) := \mathbb{V}_S(\lambda) \cap F^p E_{DR}^{2p}(\mathcal{E}_{0,\mathbb{C}}/S_{\mathbb{C}}) = q(\mathbb{V}_S^p(\tilde{\lambda})) \cup \mathbb{V}_{V(\bar{e}_\lambda)}(\lambda) \subset E_{DR}^{2p}(\mathcal{E}_{0,\mathbb{C}}/S_{\mathbb{C}}),$$

where

- $\mathbb{V}_S(\lambda) \subset E_{DR}^{2p}(\mathcal{E}_{0,\mathbb{C}}/S_{\mathbb{C}})$ ,  $\mathbb{V}_S(\tilde{\lambda}) \subset E_{DR}^{2p}(\mathcal{E}_{\mathbb{C}}/S_{\mathbb{C}})$ ,  $\mathbb{V}_{V(\tilde{e}(\lambda))}(\lambda) \subset E_{DR}^{2p}(\mathcal{E}_{0,\mathbb{C}}/S_{\mathbb{C}})$  are the flat leaves, e.g.  $\mathbb{V}_S(\lambda) := \pi_S(\lambda \times \tilde{S}_{\mathbb{C}}^{an})$  where  $\pi_S : H^{2p}(E^{an}, \mathbb{C}) \times \tilde{S}_{\mathbb{C}}^{an} \rightarrow E_{DR}^{2p,an}(\mathcal{E}_{\mathbb{C}}/S_{\mathbb{C}})$  is the morphism induced by the universal covering  $\pi_S : \tilde{S}_{\mathbb{C}}^{an} \rightarrow S_{\mathbb{C}}^{an}$ ,
- $V(\bar{e}_{\lambda}) := \pi_S(V(\tilde{e}(\lambda))) \subset S_{\mathbb{C}}$ , where  $V(\bar{e}_{\lambda}) \subset \tilde{S}_{\mathbb{C}}^{an}$  is the zero locus of  $\bar{e}_{\lambda}$ ,
- $q := q \otimes \mathbb{C} : E_{DR}^{2p}(\mathcal{E}_{\mathbb{C}}/S_{\mathbb{C}}) \rightarrow E_{DR}^{2p}(\mathcal{E}_{0,\mathbb{C}}/S_{\mathbb{C}})$  is the quotient map.

Now,

- if  $Sp_{E/f_X}(\tilde{\lambda}) = 0$ , assuming the Hodge conjecture for smooth projective varieties of dimension less or equal to  $d - 1$ ,  $\mathbb{V}_S^p(\tilde{\lambda}) \subset E_{DR}^{2p}(\mathcal{E}_{\mathbb{C}}/S_{\mathbb{C}})$  is an algebraic subvariety defined over  $\bar{\mathbb{Q}}$  and its Galois conjugate are also components of the locus of Hodge classes, if  $Sp_{E/f_X}(\tilde{\lambda}) \neq 0$ , by [5] theorem 1, the Hodge conjecture holds for projective hypersurfaces, hence

$$Sp_{\mathcal{E}/\tilde{f}_X}(\mathbb{V}_S^p(\tilde{\lambda})) = \mathbb{V}_S^p(Sp_{E/f_X}(\tilde{\lambda})) = \overline{\mathbb{V}_{(\mathbb{A}^1 \setminus 0) \times S}^p(Sp_{E/f_X}(\tilde{\lambda}))} \cap p^{-1}(0 \times S_{\mathbb{C}}) \subset E_{DR,V_0}^{2p}(\mathcal{Y}_{\mathbb{C}}/(\mathbb{A}_{\mathbb{C}}^1 \times S_{\mathbb{C}}))$$

is an algebraic subvariety defined over  $\bar{\mathbb{Q}}$  and its Galois conjugate are also components of the locus of Hodge classes, where the equality follows from [6] lemma 2.11,

- if  $Sp_{E/f_X}(e_{\lambda}) = 0$ ,  $Sp_{E/f_X}(\tilde{\lambda}) = Sp_{E/f_X}(\tilde{\lambda}_F)$  and for  $t' \in \tilde{S}_{\mathbb{C}}^{an}$ ,  $\bar{e}_{\lambda}(t') = 0$  if and only if  $e(\tilde{\lambda}_{t'}) = 0$  if and only if  $Sp_{\mathcal{E}_{t'}/(\tilde{f}_X)_{t'}}(\tilde{\lambda}_{t'})$  is a Hodge class, if  $Sp_{E/f_X}(e_{\lambda}) \neq 0$ , then considering the restrictions of

$$Sp_{E/f_X}(e_{\lambda}) \in \Gamma(\tilde{G}_m \times \tilde{S}_{\mathbb{C}}^{an}, \pi_{S,G_m}^{*mod} E_{DR}(\mathcal{Y}_{\mathbb{C}}^o/((\mathbb{A}_{\mathbb{C}}^1 \setminus 0) \times S_{\mathbb{C}}))^{an})^{\nabla}$$

to Lefschetz pencils  $\pi_{S,G_m}^{-1}(L) \subset \tilde{G}_m \times \tilde{S}_{\mathbb{C}}^{an}$ ,  $L \subset \overline{((\mathbb{A}^1 \setminus 0) \times S)_{\mathbb{C}}} = \mathbb{P}(l_X)$ , where  $\mathbb{P}(l_X)$  is the space of hypersurface over degree  $l_X = \deg(\mathbb{Q}(X)/\mathbb{Q}(x_1, \dots, x_d))$  (for the equality we may assume that  $X$  is not defined over  $\mathbb{Q}$  and that  $l_X > 1$ ), we see that  $V(\bar{e}_{\lambda}) = \emptyset$ .

Hence,  $\mathbb{V}_S^p(\lambda) \subset E_{DR}^{2p}(\mathcal{E}_{0,\mathbb{C}}/S_{\mathbb{C}})$  is an algebraic subvariety defined over  $\bar{\mathbb{Q}}$  and its Galois conjugate are also components of the locus of Hodge classes. By [4] theorem 4, we get

$$\lambda = [Z] \in H^{2p}(E_0^{an}, \mathbb{Q}), \quad Z \in \mathcal{Z}^p(E_0).$$

We recall the key point is that since  $\mathbb{V}_S^p(\lambda) \subset E_{DR}^{2p}(\mathcal{E}_{0,\mathbb{C}}/S_{\mathbb{C}})$  is an algebraic subvariety defined over  $\bar{\mathbb{Q}}$  and its Galois conjugate are also components of the locus of Hodge classes, we have for each  $\theta \in \text{Aut}(\mathbb{C}, \mathbb{Q})$ ,

$$\theta(\lambda) \in \text{Gal}(\bar{\mathbb{Q}}, \mathbb{Q})(\mathbb{V}_S^p(\lambda)_{\theta}) \subset E_{DR}^{2p}(\mathcal{E}_{0,\mathbb{C},\theta}/S_{\mathbb{C},\theta}),$$

which implies that  $\theta(\lambda) \in H^{2p}(E_{0,\theta}^{an}, \mathbb{Q}) \subset H^{2p}(E_{0,\theta}^{an}, \mathbb{C})$ , that is  $\lambda \in H^{2p}(E_0^{an}, \mathbb{Q})$  is absolute Hodge and we apply [4] corollary 1. Since  $p \in \mathbb{Z}$  and  $\lambda \in F^p H^{2p}(E_0^{an}, \mathbb{Q})$  are arbitrary, this proves the Hodge conjecture for  $E_0$ . Since  $X$  is birational to  $E_0$ , the Hodge conjecture for  $E_0$  and the induction hypothesis implies the Hodge conjecture for  $X$ .

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## 2 Preliminaries and notations

### 2.1 Notations

- Denote by Top the category of topological spaces and RTop the category of ringed spaces.
- Denote by Cat the category of small categories and RCat the category of ringed topos.
- For  $\mathcal{S} \in \text{Cat}$  and  $X \in \mathcal{S}$ , we denote  $\mathcal{S}/X \in \text{Cat}$  the category whose objects are  $Y/X := (Y, f)$  with  $Y \in \mathcal{S}$  and  $f : Y \rightarrow X$  is a morphism in  $\mathcal{S}$ , and whose morphisms  $\text{Hom}((Y', f'), (Y, f))$  consists of  $g : Y' \rightarrow Y$  in  $\mathcal{S}$  such that  $f \circ g = f'$ .

- Let  $(\mathcal{S}, O_{\mathcal{S}}) \in \mathbf{RCat}$  a ringed topos with topology  $\tau$ . For  $F \in C_{O_{\mathcal{S}}}(\mathcal{S})$ , we denote by  $k : F \rightarrow E_{\tau}(F)$  the canonical flasque resolution in  $C_{O_{\mathcal{S}}}(\mathcal{S})$  (see [3]). In particular for  $X \in \mathcal{S}$ ,  $H^*(X, E_{\tau}(F)) \xrightarrow{\sim} \mathbb{H}_{\tau}^*(X, F)$ .

- For  $f : \mathcal{S}' \rightarrow \mathcal{S}$  a morphism with  $\mathcal{S}, \mathcal{S}' \in \mathbf{RCat}$ , endowed with topology  $\tau$  and  $\tau'$  respectively, we denote for  $F \in C_{O_{\mathcal{S}}}(\mathcal{S})$  and each  $j \in \mathbb{Z}$ ,

$$\begin{aligned} - f^* &:= H^j \Gamma(\mathcal{S}, k \circ \mathrm{ad}(f^*, f_*)(F)) : \mathbb{H}^j(\mathcal{S}, F) \rightarrow \mathbb{H}^j(\mathcal{S}', f^* F), \\ - f^* &:= H^j \Gamma(\mathcal{S}, k \circ \mathrm{ad}(f^{*mod}, f_*)(F)) : \mathbb{H}^j(\mathcal{S}, F) \rightarrow \mathbb{H}^j(\mathcal{S}', f^{*mod} F), \end{aligned}$$

the canonical maps.

- For  $m : A \rightarrow B$ ,  $A, B \in C(\mathcal{A})$ ,  $\mathcal{A}$  an additive category, we denote  $c(A) : \mathrm{Cone}(m : A \rightarrow B) \rightarrow A[1]$  and  $c(B) : B \rightarrow \mathrm{Cone}(m : A \rightarrow B)$  the canonical maps.

- Denote by  $\mathrm{Sch} \subset \mathbf{RTop}$  the subcategory of schemes (the morphisms are the morphisms of locally ringed spaces). We denote by  $\mathrm{PSch} \subset \mathrm{Sch}$  the full subcategory of proper schemes. For a field  $k$ , we consider  $\mathrm{Sch}/k := \mathrm{Sch}/\mathrm{Spec} k$  the category of schemes over  $\mathrm{Spec} k$ . The objects are  $X := (X, a_X)$  with  $X \in \mathrm{Sch}$  and  $a_X : X \rightarrow \mathrm{Spec} k$  a morphism and the morphisms are the morphisms of schemes  $f : X' \rightarrow X$  such that  $f \circ a_{X'} = a_X$ . We then denote by

- $\mathrm{Var}(k) = \mathrm{Sch}^{ft}/k \subset \mathrm{Sch}/k$  the full subcategory consisting of algebraic varieties over  $k$ , i.e. schemes of finite type over  $k$ ,
- $\mathrm{PVar}(k) \subset \mathrm{QPVar}(k) \subset \mathrm{Var}(k)$  the full subcategories consisting of quasi-projective varieties and projective varieties respectively,
- $\mathrm{PSmVar}(k) \subset \mathrm{SmVar}(k) \subset \mathrm{Var}(k)$ ,  $\mathrm{PSmVar}(k) := \mathrm{PVar}(k) \cap \mathrm{SmVar}(k)$ , the full subcategories consisting of smooth varieties and smooth projective varieties respectively.

- Denote by  $\mathrm{AnSp}(\mathbb{C}) \subset \mathbf{RTop}$  the subcategory of analytic spaces over  $\mathbb{C}$ , and by  $\mathrm{AnSm}(\mathbb{C}) \subset \mathrm{AnSp}(\mathbb{C})$  the full subcategory of smooth analytic spaces (i.e. complex analytic manifold).

- For  $X \in \mathrm{Var}(k)$  and  $X = \cup_{i \in I} X_i$  with  $i_i : X_i \hookrightarrow X$  closed embeddings, we denote  $X_{\bullet} \in \mathrm{Fun}(\Delta, \mathrm{Var}(k))$  the associated simplicial space, with for  $J \subset I$ ,  $i_{IJ} : X_I := \cap_{i \in I} X_i \hookrightarrow X_J := \cap_{i \in J} X_i$  the closed embedding.

- Let  $(X, O_X) \in \mathbf{RTop}$ . We consider its De Rham complex  $\Omega_X^{\bullet} := DR(X)(O_X)$ .

- Let  $X \in \mathrm{Sch}$ . Considering its De Rham complex  $\Omega_X^{\bullet} := DR(X)(O_X)$ , we have for  $j \in \mathbb{Z}$  its De Rham cohomology  $H_{DR}^j(X) := \mathbb{H}^j(X, \Omega_X^{\bullet})$ .
- Let  $X \in \mathrm{Var}(k)$ . Considering its De Rham complex  $\Omega_X^{\bullet} := \Omega_{X/k}^{\bullet} := DR(X/k)(O_X)$ , we have for  $j \in \mathbb{Z}$  its De Rham cohomology  $H_{DR}^j(X) := \mathbb{H}^j(X, \Omega_X^{\bullet})$ . The differentials of  $\Omega_X^{\bullet} := \Omega_{X/k}^{\bullet}$  are by definition  $k$ -linear, thus  $H_{DR}^j(X) := \mathbb{H}^j(X, \Omega_X^{\bullet})$  has a structure of a  $k$  vector space.
- Let  $X \in \mathrm{AnSp}(\mathbb{C})$ . Considering its De Rham complex  $\Omega_X^{\bullet} := DR(X)(O_X)$ , we have for  $j \in \mathbb{Z}$  its De Rham cohomology  $H_{DR}^j(X) := \mathbb{H}^j(X, \Omega_X^{\bullet})$ .

- For  $Y \in \mathrm{Var}(\mathbb{Q})$  and  $X = V(I) = V(f_1, \dots, f_r) \subset Y_{\mathbb{C}}$ , we consider  $\mathcal{X} = V(\tilde{I}) = V(\tilde{f}_1, \dots, \tilde{f}_r) \subset Y \times \mathbb{A}_{\mathbb{Q}}^s$  the canonical rational deformation, where

$$\begin{aligned} \text{if } f &= \sum_{I \subset [1, \dots, d]^n} a_I x_1^{n_1} \cdots x_n^{n_n} \in \mathbb{C}[x_1, \dots, x_n], \\ \tilde{f} &= \sum_{I \subset [1, \dots, d]^n} a_I x_1^{n_1} \cdots x_n^{n_n} \in \mathbb{Q}[(a_I)_{I \subset [1, \dots, d]^n}, x_1, \dots, x_n]. \end{aligned}$$

- For  $X \in \text{AnSp}(\mathbb{C})$ , we denote  $\alpha(X) : \mathbb{C}_X \hookrightarrow \Omega_X^\bullet$  the embedding in  $C(X)$ . For  $X \in \text{AnSm}(\mathbb{C})$ ,  $\alpha(X) : \mathbb{C}_X \hookrightarrow \Omega_X^\bullet$  is an equivalence *usu local* by Poincare lemma.
- We denote  $\mathbb{I}^n := [0, 1]^n \in \text{Diff}(\mathbb{R})$  (with boundary). For  $X \in \text{Top}$  and  $R$  a ring, we consider its singular cochain complex

$$C_{\text{sing}}^*(X, R) := (\mathbb{Z} \text{Hom}_{\text{Top}}(\mathbb{I}^*, X)^\vee) \otimes R$$

and for  $l \in \mathbb{Z}$  its singular cohomology  $H_{\text{sing}}^l(X, R) := H^n C_{\text{sing}}^*(X, R)$ . For  $f : X' \rightarrow X$  a continuous map with  $X, X' \in \text{Top}$ , we have the canonical map of complexes

$$f^* : C_{\text{sing}}^*(X, R) \rightarrow C_{\text{sing}}^*(X', R), \sigma \mapsto f^* \sigma := (\gamma \mapsto \sigma(f \circ \gamma)).$$

In particular, we get by functoriality the complex

$$C_{X, R \text{ sing}}^* \in C_R(X), (U \subset X) \mapsto C_{\text{sing}}^*(U, R)$$

We recall that

- For  $X \in \text{Top}$  locally contractible, e.g.  $X \in \text{CW}$ , and  $R$  a ring, the inclusion in  $C_R(X)$   $c_X : R_X \rightarrow C_{X, R \text{ sing}}^*$  is by definition an equivalence *top local* and that we get by the small chain theorem, for all  $l \in \mathbb{Z}$ , an isomorphism  $H^l c_X : H^l(X, R_X) \xrightarrow{\sim} H_{\text{sing}}^l(X, R)$ .
- For  $X \in \text{Diff}(\mathbb{R})$ , the restriction map

$$r_X : \mathbb{Z} \text{Hom}_{\text{Diff}(\mathbb{R})}(\mathbb{I}^*, X)^\vee \rightarrow C_{\text{sing}}^*(X, R), w \mapsto w : (\phi \mapsto w(\phi))$$

is a quasi-isomorphism by Whitney approximation theorem.

- Let  $S \in \text{AnSm}(\mathbb{C})$  and  $L \in \text{Shv}(S)$  a local system. Consider  $E := L \otimes O_S \in \text{Vect}_{\mathcal{D}}(S)$  the corresponding holomorphic vector bundle with integrable connection  $\nabla$ . Let  $\pi_S : \tilde{S} \rightarrow S$  be the universal covering. Consider the canonical fiber

$$L_0 := \Gamma(\tilde{S}, \pi_S^* L) = \Gamma(\tilde{S}, \pi_S^{*mod} E)^\nabla.$$

For  $\lambda \in L_0$ , we will consider the flat leaf

$$\mathbb{V}_S(\lambda) := \pi_S(\lambda \times \tilde{S}) \subset E, \pi_S : L_0 \times \tilde{S} \rightarrow E, \pi_S(\nu, z) := (\nu(z), \pi_S(z)).$$

Note that for  $t \in S$ ,  $\mathbb{V}_S(\lambda) \cap p_S^{-1}(t) = \pi_1(S, t)(\lambda)$  is the orbit of  $\lambda$  under the monodromy action, where  $p_S : E \rightarrow S$  is the projection.

## 2.2 Birational projective varieties and Hodge conjecture

In this subsection, we recall that assuming the Hodge conjecture for smooth projective varieties of dimension less or equal to  $d - 1$ , the property of satisfying the Hodge conjecture for connected smooth projective varieties of dimension  $d$  is a birational invariant :

**Proposition 1.** (i) Let  $X \in \text{PSmVar}(\mathbb{C})$  connected of dimension  $d$ . Let  $\epsilon : \tilde{X}_Z \rightarrow X$  be the blow up of  $X$  along a smooth closed subvariety  $Z \subset X$ . Assume the Hodge conjecture hold for smooth complex projective varieties of dimension less or equal to  $d - 1$ . Then the Hodge conjecture hold for  $X$  if and only if it hold for  $\tilde{X}_Z$ .

(ii) Let  $X, X' \in \text{PSmVar}(\mathbb{C})$  connected of dimension  $d$ . Assume the Hodge conjecture hold for smooth complex projective varieties of dimension less or equal to  $d - 1$ . If  $X$  is birational to  $X'$ , then the Hodge conjecture hold for  $X$  if and only if it hold for  $X'$

*Proof.* (i): Follows from the fact that

$$(\epsilon^*, \oplus_{l=1}^c i_{E*}((-).h^{l-1})) : H^k(X, \mathbb{Q}) \oplus \oplus_{l=1}^c H^{k-2l}(E, \mathbb{Q}) \rightarrow H^k(\tilde{X}_Z, \mathbb{Q})$$

is an isomorphism of Hodge structures for each  $k \in \mathbb{Z}$ , where  $c = \text{codim}(Z, X)$ ,  $i_E : E \hookrightarrow \tilde{X}_Z$  is the closed embedding,  $\epsilon|_E : E \rightarrow Z$  being a projective vector bundle.

(ii): Follows from (i) since if  $\pi : X^o \rightarrow X'$  is a birational map,  $X^o \subset X$  being an open subset, then  $X$  is connected to  $X'$  by a sequence of blow up of smooth projective varieties with smooth center by [1].  $\square$

## 2.3 A family of smooth projective hypersurfaces associated to a smooth projective variety

In this subsection, we recall that given a smooth complex projective variety  $X$ , there exists a family of smooth complex projective hypersurfaces which degenerates into a normal crossing divisor with one irreducible component birational to  $X$ .

**Proposition 2.** *Let  $X \in \text{PSmVar}(\mathbb{C})$  connected of dimension  $d$ . Then there exists a family of smooth projective hypersurfaces  $f_X : Y \rightarrow \mathbb{A}_{\mathbb{C}}^1$  with*

- $Y \in \text{SmVar}(\mathbb{C})$ ,  $f_X$  flat projective,
- $Y_s := f_X^{-1}(s) \subset \mathbb{P}_{\mathbb{C}}^{d+1}$  for  $s \in \mathbb{A}_{\mathbb{C}}^1 \setminus 0$  are smooth projective hypersurfaces,
- $E := f_X^{-1}(0) = \cup_{i=0}^r E_i = \epsilon^{-1}(Y_0) \subset Y$  is a normal crossing divisor, where  $Y_0 \subset \mathbb{P}_{\mathbb{C}}^{d+1}$  is a possibly singular hypersurface birational to  $X$ ,  $\epsilon : (Y, E) \rightarrow (Y', Y_0)$  a desingularization, and  $E_0$  is birational to  $X$ ,

*Proof.* See [2] lemma 1.5.8 . It use the fact that  $X$  is birational to a possibly singular projective hypersurface  $Y_0$  and consider the desingularization of a generic pencil  $f'_X : Y' = V(f) \subset \mathbb{P}_{\mathbb{C}}^{d+1} \times \mathbb{A}_{\mathbb{C}}^1 \rightarrow \mathbb{A}_{\mathbb{C}}^1$  passing through  $Y_0$  and  $f_X = f'_X \circ \epsilon$ .  $\square$

## 2.4 The nearby cycle functor and the specialization map

Recall from [3] that for  $k \subset \mathbb{C}$  a subfield and  $S \in \text{Var}(k)$  quasi-projective, and  $l : S \hookrightarrow \tilde{S}$  a closed embedding with  $\tilde{S} \in \text{SmVar}(k)$ , we have the full subcategory

$$\iota : MHM_{k, gm}(S) \hookrightarrow \text{PSh}_{\mathcal{D}fil, S}(\tilde{S}) \times_I P(S_{\mathbb{C}}^{an})$$

consisting of geometric mixed Hodge module whose De Rham part is defined over  $k$ , where, using the fact that the  $V$ -filtration is defined over  $k$ ,  $\text{PSh}_{\mathcal{D}(1,0)fil, S}(\tilde{S}) \times_I P_{fil}(S_{\mathbb{C}}^{an})$  is the category

- whose objects are  $((M, F, W), (K, W), \alpha)$  where  $(M, F, W)$  is a filtered  $D_{\tilde{S}}$  module supported on  $S$ , where  $l : S \hookrightarrow \tilde{S}$  is a closed embedding and  $\tilde{S} \in \text{SmVar}(k)$ ,  $(K, W)$  is a filtered Perverse sheaf on  $S_{\mathbb{C}}^{an}$ , and  $\alpha : l_*(K, W) \otimes \mathbb{C} \rightarrow DR(\tilde{S})((M, W)^{an})$  is an isomorphism in  $D_{fil}(\tilde{S}_{\mathbb{C}}^{an})$ .
- whose morphism between  $((M, F, W), (K, W), \alpha)$  and  $((M', F, W), (K', W), \alpha')$ , are couples of morphisms  $(\phi : (M, F, W) \rightarrow (M', F, W), \psi : (K, W) \rightarrow (K', W))$  such that  $\phi \circ \alpha = \alpha' \circ \psi$ .

Then, by [3], we get for  $S \in \text{Var}(k)$  quasi-projective and  $l : S \hookrightarrow \tilde{S}$  is a closed embedding with  $\tilde{S} \in \text{SmVar}(k)$  an embedding

$$\iota : D(MHM_{k, gm}(S)) \hookrightarrow D_{\mathcal{D}fil, S}(\tilde{S}) \times_I D(S_{\mathbb{C}}^{an})$$

consisting of mixed Hodge module whose De Rham part is defined over  $k$ , where  $D_{\mathcal{D}(1,0)fil, S}(\tilde{S}) \times_I D_{fil, c}(S_{\mathbb{C}}^{an})$  is the category

- whose objects are  $((M, F, W), (K, W), \alpha)$  where  $(M, F, W)$  is a complex of filtered  $D_{\tilde{S}}$  module supported on  $S$ ,  $(K, W)$  is a filtered complex of presheaves on  $S_{\mathbb{C}}^{an}$  whose cohomology are constructible sheaves, and  $\alpha : l_*(K, W) \otimes \mathbb{C} \rightarrow DR(\tilde{S})((M, W)^{an})$  is an isomorphism in  $D_{fil}(\tilde{S}_{\mathbb{C}}^{an})$ .
- whose morphism are given in [3],

We have then the six functors formalism :

$$D(MHM_{k,gm}(-)) : \text{Var}(k) \mapsto \text{TriCat}, \quad S \mapsto D(MHM_{k,gm}(S)), \quad (f : T \rightarrow S) \mapsto (f^*, f_*) : D(MHM_{k,gm}(S)) \rightarrow D(MHM_{k,gm}(T)), \quad (f!, f^!) : D(MHM_{k,gm}(T)) \rightarrow D(MHM_{k,gm}(S))$$

and for  $D \subset S$  a (Cartier) divisor the nearby cycle functor

$$\psi_D : D(MHM_{k,gm}(S)) \rightarrow D(MHM_{k,gm}(D)), \\ \psi_D((M, F, W), (K, W), \alpha) := (\psi_D(M, F, W), \psi_D(K, W), \psi_D(\alpha)).$$

For  $T \in \text{Var}(k)$  and  $\epsilon_{\bullet} : T_{\bullet} \rightarrow T$  a desingularization with  $T_{\bullet} \in \text{Fun}(\Delta, \text{SmVar}(k))$ , we have

$$\mathbb{Z}_T^{Hdg} := a_T^{*Hdg} \mathbb{Z}^{Hdg} = (\epsilon_{\bullet*} E_{zar}(O_{T_{\bullet}}, F, W), \epsilon_{\bullet*} E_{usu}(\mathbb{Z}_{T_{\bullet}^{an}}, W), \alpha(T_{\bullet}))$$

For  $f : X \rightarrow S$  a projective morphism with  $X, S \in \text{SmVar}(k)$ , we denote

$$E_{Hdg}(X/S) := f_* \mathbb{Z}_X^{Hdg} := \left( \int f(O_X, F), Rf_* \mathbb{Z}_{X_{\mathbb{C}}^{an}}, f_* \alpha(X) \right) \in D(MHM_{k,gm}(S)).$$

For  $f : X \hookrightarrow \mathbb{P}^N \times S \xrightarrow{p} S$  a projective morphism with  $X, S \in \text{Var}(k)$ ,  $S$  smooth, we denote

$$E_{Hdg}(X/S) := f_* \mathbb{Z}_X^{Hdg} := \left( \int_p \Gamma_X^{Hdg}(O_{\mathbb{P}^N \times S}, F), Rf_* \mathbb{Z}_{X_{\mathbb{C}}^{an}}, f_* \alpha(X) \right) \in D(MHM_{k,gm}(S)).$$

We then have the specialization map :

**Definition-Proposition 1.** *Let  $k$  be field of characteristic zero.*

(i) *Let  $f' : X' \rightarrow \mathbb{A}_k^1 \times S$  be a projective morphism with  $X, S \in \text{SmVar}(k)$ . Assume that*

- $f^o := f \otimes_{\mathbb{A}^1} (\mathbb{A}^1 \setminus 0) : X^o \rightarrow (\mathbb{A}^1 \setminus 0) \times S$  *is smooth*
- $X_0 := f^{-1}(0 \times S) \subset X'$ . *We have by definition  $X^o = X' \setminus X_0$ .*

*We have then, see [3], the distinguished triangles in  $D(MHM_{k,gm}(X_0))$ ,*

$$\mathbb{Z}_{X_0}^{Hdg} := i_{X_0}^* \mathbb{Z}_{X'}^{Hdg} \xrightarrow{sp_{X_0/X}} \psi_{X_0} \mathbb{Z}_{X^o}^{Hdg} := (\psi_{X_0}(O_{X'}, F), \psi_{X_0} \mathbb{Z}_{X^o}, \Phi_{X_0} \alpha(X')) \\ \xrightarrow{c(\psi_{X_0} \mathbb{Z}_{X^o}^{Hdg})} \phi_E \mathbb{Z}_{X^o}^{Hdg} := \text{Cone}(sp_{E/X}) \xrightarrow{c(\mathbb{Z}_{X_0}^{Hdg})} \mathbb{Z}_{X_0}^{Hdg}[1].$$

*Applying the functor  $(f' \circ i_{X_0})_*$ , we obtain the generalized distinguished triangle in  $D(MHM_{k,gm}(S))$*

$$E_{Hdg}(X_0/S) \xrightarrow{sp_{X_0/f'} := (f' \circ i_{X_0})_* sp_{X_0/X}} (f' \circ i_{X_0})_*(\psi_{X_0} \mathbb{Z}_{X^o}^{Hdg}) = \psi_0 E_{Hdg}(X^o / ((\mathbb{A}^1 \setminus 0) \times S)) \xrightarrow{(f' \circ i_{X_0})_* c(\Phi_{X_0} \mathbb{Z}_{X^o}^{Hdg})} \\ (f' \circ i_{X_0})_*(\phi_{X_0} \mathbb{Z}_{X^o}^{Hdg}) = \phi_0 E_{Hdg}(X^o / ((\mathbb{A}^1 \setminus 0) \times S)) \xrightarrow{(f' \circ i_{X_0})_* c(\mathbb{Z}_X^{Hdg})} E_{Hdg}(X_0/S)[1]$$

(i)' *If for each  $t \in S(\mathbb{C})$ ,  $H^{k-1}(X'_{(s,t)}, \mathbb{Q}) = 0$  for  $s \in \mathbb{A}_{\mathbb{C}}^1 \setminus 0$  and  $H_{X_0}^{k-1}(X'_{\mathbb{A}^1 \times t}, \mathbb{Q}) = 0$ , then the induced map on cohomology*

$$sp_{X_0/f'} : E_{Hdg}^k(X_0/S) \rightarrow H^k \Phi_0 E_{Hdg}(X^o / ((\mathbb{A}^1 \setminus 0) \times S))$$

*is injective.*



(ii) Let  $f : X \rightarrow \mathbb{A}_k^1 \times S$  be a projective morphism with  $X, S \in \text{SmVar}(k)$ . Assume that

- $f^\circ := f \otimes_{\mathbb{A}^1} (\mathbb{A}^1 \setminus 0) : X^\circ \rightarrow (\mathbb{A}^1 \setminus 0) \times S$  is smooth
- $E := f^{-1}(0 \times S) = \cup_{i=0}^r E_i \subset X$  is a normal crossing divisor and that for  $I \subset [0, \dots, r]$ ,  $f_{0,I} := f|_{E_I} : E_I := \cap_{i \in I} E_i \rightarrow S$  are smooth. We have by definition  $X^\circ = X \setminus E$ .

Denote for  $I \subset J \subset [0, \dots, r]$ ,  $i_{JI} : E_J \hookrightarrow E_I$  and  $i_I : E_I \hookrightarrow E$ ,  $i_E : E \hookrightarrow X$  the closed embeddings. In this particular case, c.f. [8], the distinguished triangle in  $D(MHM_{k,gm}(E))$  given in (i) become

$$\begin{aligned} \mathbb{Z}_E^{Hdg} &:= i_E^* \mathbb{Z}_X^{Hdg} = ((i_{\bullet*} \Omega_{E^\bullet}^\bullet, F, W), (\mathbb{Z}_{E^\bullet}, W), \alpha(E_\bullet)) \xrightarrow{sp_{E/X}} \\ \psi_E \mathbb{Z}_{X^\circ}^{Hdg} &= ((i_E^{*mod} \Omega_{X/\mathbb{A}^1 \times S}^\bullet(\log E), F, W), \Phi_E \mathbb{Z}_{X^\circ}, \Phi_E \alpha(X)) \xrightarrow{c(\psi_E \mathbb{Z}_{X^\circ}^{Hdg})} \\ \phi_E \mathbb{Z}_{X^\circ}^{Hdg} &:= \text{Cone}(sp_{E/X}) \xrightarrow{c(\mathbb{Z}_E^{Hdg})} \mathbb{Z}_E^{Hdg}[1]. \end{aligned}$$

Applying the functor  $(f \circ i_E)_*$ , we obtain the generalized distinguished triangle in  $D(MHM_{k,gm}(S))$

$$\begin{aligned} E_{Hdg}(E/S) &\xrightarrow{sp_{E/f} := (f \circ i_E)_* sp_{E/X}} (f \circ i_E)_*(\psi_E \mathbb{Z}_{X^\circ}^{Hdg}) = \psi_0 E_{Hdg}(X^\circ / ((\mathbb{A}^1 \setminus 0) \times S)) \\ &\xrightarrow{(f \circ i_E)_* c(\psi_E \mathbb{Z}_{X^\circ}^{Hdg})} (f \circ i_E)_*(\phi_E \mathbb{Z}_{X^\circ}^{Hdg}) = \phi_0 E_{Hdg}(X^\circ / ((\mathbb{A}^1 \setminus 0) \times S)) \xrightarrow{(f \circ i_E)_* c(\mathbb{Z}_X^{Hdg})} E_{Hdg}(E/S)[1]. \end{aligned}$$

If  $f = f' \circ \epsilon$  where  $\epsilon : (X, E) \rightarrow (X', X_0)$  is a desingularization, we have  $sp_{X_0/X'} = sp_{E/X} \circ \epsilon^*$ .

*Proof.* (i): Follows from the exact sequence

$$\cdots \rightarrow H_{X_{0,t}}^l(X'_{\mathbb{A}^1 \times t}, \mathbb{Q}) \rightarrow \mathbb{H}^l(X_{0,t}, \phi_0 \mathbb{Z}_{X^\circ_{(\mathbb{A}^1 \setminus 0) \times t}}) \xrightarrow{V} \mathbb{H}^l(X_{0,t}, \psi_0 \mathbb{Z}_{X^\circ_{(\mathbb{A}^1 \setminus 0) \times t}}) \rightarrow H_{X_{0,t}}^{l+1}(X'_{\mathbb{A}^1 \times t}, \mathbb{Q}) \rightarrow \cdots$$

given in [8]. □

### 3 Hodge conjecture for smooth projective varieties

**Theorem 1.** Let  $X \in \text{PSmVar}(\mathbb{C})$ . Then Hodge conjecture hold for  $X$ . That is, if  $p \in \mathbb{Z}$  and  $\lambda \in F^p H^{2p}(X^{an}, \mathbb{Q})$ ,  $\lambda = [Z]$  with  $Z \in \mathcal{Z}^p(X)$ .

*Proof.* The Hodge conjecture is true for curves and surfaces. Assume that the Hodge conjecture is true for smooth projective varieties of dimension less or equal to  $d-1$ . Let  $X \in \text{PSmVar}(\mathbb{C})$  of dimension  $d$ . Up to split  $X$  into its connected components, we may assume that  $X$  is connected of dimension  $d$ . Then  $X$  is birational to a possibly singular hypersurface  $Y_0 \subset \mathbb{P}_{\mathbb{C}}^{d+1}$ . Consider by proposition 2 a family of smooth projective hypersurfaces over  $\mathbb{A}_{\mathbb{C}}^1 \setminus 0$  which degenerate at 0 into a normal crossing divisor with one component birational to  $X$ , that is a projective morphism in  $\text{Var}(\mathbb{C})$   $f_X : Y = V(I) \subset \mathbb{P}_{\mathbb{C}}^N \times \mathbb{A}_{\mathbb{C}}^1 \rightarrow \mathbb{A}_{\mathbb{C}}^1$  with

- $Y \in \text{SmVar}(\mathbb{C})$ ,  $f_X$  flat projective,
- $Y_s := f_X^{-1}(s) \subset \mathbb{P}_{\mathbb{C}}^{d+1}$  for  $s \in \mathbb{A}_{\mathbb{C}}^1 \setminus 0$  are smooth projective hypersurfaces
- $f_X = f'_X \circ \epsilon$ ,  $f'_X : Y' = V(f) \subset \mathbb{P}_{\mathbb{C}}^{d+1} \times \mathbb{A}_{\mathbb{C}}^1 \rightarrow \mathbb{A}_{\mathbb{C}}^1$  is a generic pencil of hypersurface passing through  $Y_0 := f_X^{-1}(0) \subset \mathbb{P}_{\mathbb{C}}^{d+1}$  which is birational to  $X$ ,  $\epsilon : (Y, E) \rightarrow (Y', Y_0)$  is a desingularization of the pair  $(Y', Y_0)$  with  $E := f_X^{-1}(0) = \epsilon^{-1}(Y_0) = \cup_{i=0}^r E_i \subset Y$  is a normal crossing divisor, and  $E_0$  is birational to  $X$ ,

and its canonical rational deformation  $\tilde{f}_X : \mathcal{Y} = V(\tilde{I}) \subset \mathbb{P}_{\mathbb{Q}}^N \times \mathbb{A}_{\mathbb{Q}}^1 \times S \rightarrow \mathbb{A}_{\mathbb{Q}}^1 \times S$  with

- $\mathcal{Y} \in \text{SmVar}(\mathbb{Q})$ ,  $S \subset \mathbb{A}_{\mathbb{Q}}^r$  an open subset,  $\tilde{f}_X$  flat projective,  $Y = \tilde{f}_X^{-1}(\mathbb{A}_{\mathbb{Q}}^1 \times \{t\})$ ,  $f_X = \tilde{f}_X \times_S \{t\}$ , with  $t \in S(\mathbb{C})$ ,

- $\tilde{f}_X = \tilde{f}'_X \circ \epsilon$ , with  $\tilde{f}'_X : \mathcal{Y}' = V(\tilde{f}) \subset \mathbb{P}_{\mathbb{Q}}^{d+1} \times \mathbb{A}_{\mathbb{Q}}^1 \times S \rightarrow \mathbb{A}_{\mathbb{Q}}^1 \times S$  the corresponding family of hypersurfaces and  $\epsilon : (\mathcal{Y}, \mathcal{E}) \rightarrow (\mathcal{Y}', \mathcal{Y}_0)$  the corresponding desingularization over  $\mathbb{Q}$ ,
- $\mathcal{E} = \cup_{i=0}^s \mathcal{E}_i := \tilde{f}_X^{-1}(\{0\} \times S) \subset \mathcal{Y}$  is a normal crossing divisor,  $\tilde{f}_{X,0} : \mathcal{E}_I := \cap_{i \in I} \mathcal{E}_i \rightarrow S$  smooth projective for each  $I \subset [0, \dots, s]$ ,  $\mathcal{E}_{\mathbb{C},t} = E$ ,
- $\tilde{f}_X \times_{\mathbb{A}^1} (\mathbb{A}^1 \setminus 0) : \mathcal{Y}^o \rightarrow \mathbb{A}_{\mathbb{Q}}^1 \setminus 0$  has fibers which are smooth projective hypersurfaces.

We have by definition the commutative diagram in  $\text{SmVar}(\mathbb{Q})$

$$\begin{array}{ccccc}
\mathcal{Y}^o & \xrightarrow{j} & \mathcal{Y} & \xleftarrow{i_{\mathcal{E}}} & \mathcal{E} \\
\tilde{f}_X^o \downarrow & & \downarrow \tilde{f}_X & & \downarrow \tilde{f}_{X,0} \\
\mathbb{A}_{\mathbb{Q}}^1 \setminus \{0\} \times S & \xrightarrow{j \times I_S} & \mathbb{A}_{\mathbb{Q}}^1 \times S & \xleftarrow{i_0 \times I_S} & \{0\} \times S
\end{array}$$

whose squares are cartesian,  $j$  the open embedding and  $i_{\mathcal{E}}$  the closed embedding. Denote for  $I \subset [0, \dots, s]$ ,  $i_I : \mathcal{E}_I \hookrightarrow \mathcal{E}$  and for  $I \subset J \subset [0, \dots, s]$ ,  $i_{JI} : \mathcal{E}_J \rightarrow \mathcal{E}_I$  the closed embeddings. Then,  $i_{\bullet} : \mathcal{E}_{\bullet} \rightarrow \mathcal{E}$  in  $\text{Fun}(\Delta, \text{Var}(\mathbb{Q}))$  is a simplicial resolution with  $\mathcal{E}_{\bullet} \in \text{Fun}(\Delta, \text{SmVar}(\mathbb{Q}))$ . We have then

$$E_{Hdg}(\mathcal{E}/S) := (\tilde{f}_X \circ i_{\bullet*} E_{zar}(\Omega_{\mathcal{E}_{\bullet}/S}^{\bullet}, F_b, W), (\tilde{f}_X \circ i_{\bullet*} E_{usu}(\mathbb{Q}_{\mathcal{E}_{\bullet}}^{an}, W)), \tilde{f} \circ i_{\bullet*} \alpha(\mathcal{E}_{\bullet})) \in D(MHM_{\mathbb{Q},gm}(S))$$

Then, definition-proposition 1 applied to  $\tilde{f}'_X : \mathcal{Y}' \rightarrow \mathbb{A}_{\mathbb{Q}}^1 \times S$  gives the distinguish in  $D(MHM_{\mathbb{Q},gm}(S))$ ,

$$\begin{aligned}
& E_{Hdg}(\mathcal{Y}_0/S) \xrightarrow{Sp_{\mathcal{Y}_0/\tilde{f}_X} := (\tilde{f}'_X \circ i_{\mathcal{Y}_0})_* Sp_{\mathcal{Y}_0/\mathcal{Y}'}} \psi_0 E_{Hdg}(\mathcal{Y}^o / ((\mathbb{A}^1 \setminus 0) \times S)) \\
& \xrightarrow{(\tilde{f}'_X \circ i_{\mathcal{Y}_0})^* c(\Phi_{\mathcal{Y}_0} \mathbb{Z}_{\mathcal{Y}_0}^{Hdg})} \phi_0 E_{Hdg}(\mathcal{Y}^o / ((\mathbb{A}^1 \setminus 0) \times S)) \xrightarrow{(\tilde{f}'_X \circ i_{\mathcal{Y}_0})_* c(\mathbb{Z}_{\mathcal{Y}_0}^{Hdg})} E_{Hdg}(\mathcal{Y}_0/S)[1]
\end{aligned}$$

and definition-proposition 1 applied to  $\tilde{f}_X : \mathcal{Y} \rightarrow \mathbb{A}_{\mathbb{Q}}^1 \times S$  gives the distinguish in  $D(MHM_{\mathbb{Q},gm}(S))$ ,

$$\begin{aligned}
& E_{Hdg}(\mathcal{E}/S) \xrightarrow{Sp_{\mathcal{E}/\tilde{f}_X} := (\tilde{f}_X \circ i_{\mathcal{E}})_* Sp_{\mathcal{E}/\mathcal{Y}}} \psi_0 E_{Hdg}(\mathcal{Y}^o / ((\mathbb{A}^1 \setminus 0) \times S)) \\
& \xrightarrow{(\tilde{f}_X \circ i_{\mathcal{E}})^* c(\Phi_{\mathcal{E}} \mathbb{Z}_{\mathcal{Y}^o}^{Hdg})} \phi_0 E_{Hdg}(\mathcal{Y}^o / ((\mathbb{A}^1 \setminus 0) \times S)) \xrightarrow{(\tilde{f}_X \circ i_{\mathcal{E}})_* c(\mathbb{Z}_{\mathcal{E}}^{Hdg})} E_{Hdg}(\mathcal{E}/S)[1]
\end{aligned}$$

and  $Sp_{\mathcal{E}/\tilde{f}_X} \circ \epsilon^* = Sp_{\mathcal{Y}_0/\tilde{f}'_X}$ . Let  $p \in \mathbb{Z}$ . By the hard Lefschetz theorem on  $H^*(E_0)$ , we may assume that  $2p \leq d$ . Taking the cohomology to this distinguish triangle, we get in particular, for each  $p \in \mathbb{Z}$ , the maps in  $MHM_{\mathbb{Q},gm}(S)$

- $Sp_{\mathcal{Y}_0/\tilde{f}_X} : E_{Hdg}^{2p}(\mathcal{Y}_0/S) \hookrightarrow H^{2p} \psi_0 E_{Hdg}(\mathcal{Y}^o / ((\mathbb{A}_{\mathbb{Q}}^1 \setminus 0) \times S))$
- $Sp_{\mathcal{E}/\tilde{f}_X} : E_{Hdg}^{2p}(\mathcal{E}/S) \rightarrow H^{2p} \psi_0 E_{Hdg}(\mathcal{Y}^o / ((\mathbb{A}_{\mathbb{Q}}^1 \setminus 0) \times S))$ .

The map  $Sp_{\mathcal{Y}_0/\tilde{f}_X}$  is injective by definition-proposition 1(i)' and the weak Lefschetz theorem since for  $t' \in S(\mathbb{C})$  and  $s \in \mathbb{A}_{\mathbb{C}}^1 \setminus 0$ ,  $\mathcal{Y}_{(s,t')}$  and  $\mathcal{Y}_{0,t'}$  are projective hypersurfaces. For  $\lambda \in H^{2p}(\mathcal{E}_{0,\mathbb{C},t'}^{an}, \mathbb{Q})$ ,  $t' \in S(\mathbb{C})$ , we consider  $\pi_S : \tilde{S}_{\mathbb{C}}^{an} \rightarrow S_{\mathbb{C}}^{an}$  the universal covering in  $\text{AnSm}(\mathbb{C})$  and

$$\mathbb{V}_S(\lambda) := \pi_S(\lambda \times \tilde{S}_{\mathbb{C}}^{an}) \subset E_{DR}^{2p,an}(\mathcal{E}_{0,\mathbb{C}}/S_{\mathbb{C}})$$

the flat leaf induced by the universal covering (see section 2.1) and

$$\mathbb{V}_S^p(\lambda) := \mathbb{V}_S(\lambda) \cap F^p E_{DR}^{2p}(\mathcal{E}_{0,\mathbb{C}}/S_{\mathbb{C}}) \subset E_{DR}^{2p}(\mathcal{E}_{0,\mathbb{C}}/S_{\mathbb{C}}),$$

which is an algebraic variety by [6]. Note that  $\mathbb{V}_S^p(\lambda) \subset HLP^{p,2p}(\mathcal{E}_{0,\mathbb{C}}/S_{\mathbb{C}})$  is the union of the irreducible components passing through  $\lambda$ . Similarly, for  $\eta \in H^{2p}(\mathcal{E}_{\mathbb{C},t'}^{an}, \mathbb{Q})$ ,  $t' \in S(\mathbb{C})$ , we consider  $\pi_S : \tilde{S}_{\mathbb{C}}^{an} \rightarrow S_{\mathbb{C}}^{an}$  the universal covering in  $\text{AnSm}(\mathbb{C})$  and

$$\mathbb{V}_S(\eta) := \pi_S(\eta \times \tilde{S}_{\mathbb{C}}^{an}) \subset E_{DR}^{2p,an}(\mathcal{E}_{\mathbb{C}}/S_{\mathbb{C}})$$

the flat leaf induced by the universal covering (see section 2.1) and

$$\mathbb{V}_S^p(\eta) := \mathbb{V}_S(\eta) \cap F^p E_{DR}^{2p}(\mathcal{E}_{\mathbb{C}}/S_{\mathbb{C}}) \subset E_{DR}^{2p}(\mathcal{E}_{\mathbb{C}}/S_{\mathbb{C}}).$$

We have

$$H^{2p} E_{Hdg}(\mathcal{Y}^o/((\mathbb{A}_{\mathbb{Q}}^1 \setminus 0) \times S)) = ((E_{DR}^{2p}(\mathcal{Y}^o/((\mathbb{A}_{\mathbb{Q}}^1 \setminus 0) \times S)), F), R^{2p} \tilde{f}_{X*} \mathbb{Z}_{\mathcal{Y}^o}, \tilde{f}_{X*} \alpha(\mathcal{Y}^o)) \in MHM_{\mathbb{Q}, gm}(S)$$

By [8], the filtered vector bundle

$$(E_{DR}^{2p}(\mathcal{Y}^o/((\mathbb{A}_{\mathbb{Q}}^1 \setminus 0) \times S)), F) = H^{2p} \tilde{f}_{X*} E_{zar}(\Omega_{\mathcal{Y}^o/(\mathbb{A}_{\mathbb{Q}}^1 \setminus 0) \times S}^{\bullet}) \in \text{Vect}_{\mathcal{D}fil}(S)$$

extend to a filtered vector bundle

$$(E_{DR, V_0}^{2p}(\mathcal{Y}/(\mathbb{A}_{\mathbb{Q}}^1 \times S)), F) = H^{2p} \tilde{f}_{X*} E_{zar}(\Omega_{\mathcal{Y}/\mathbb{A}_{\mathbb{Q}}^1 \times S}^{\bullet}(\log \mathcal{E})) \in \text{Vect}_{\mathcal{D}fil}(S)$$

such that

$$H^{2p} \psi_0 E_{Hdg}(\mathcal{Y}^o/((\mathbb{A}_{\mathbb{Q}}^1 \setminus 0) \times S)) = (i_0^{*mod}(E_{DR, V_0}^{2p}(\mathcal{Y}/(\mathbb{A}_{\mathbb{Q}}^1 \times S)), F), (\psi_0 R^{2p} \tilde{f}_{X*} \mathbb{Z}_{\mathcal{Y}^o}, W), \Phi_0 \tilde{f}_{X*} \alpha(\mathcal{Y}^o)).$$

For

$$\alpha \in H^{2p}(\mathcal{Y}_{(s, t')}^{an}, \mathbb{Q}) = \mathbb{H}^{2p}(\mathcal{E}_{t'}^{an}, \psi_0 R \tilde{f}_{X*} \mathbb{Q}_{\mathcal{Y}^o}), (s, t') \in (\mathbb{A}_{\mathbb{C}}^1 \setminus 0) \times S_{\mathbb{C}}$$

we consider

- $\pi_{S, G_m} = \pi_S \times \pi_{G_m} : \tilde{G}_m \times \tilde{S}_{\mathbb{C}}^{an} \rightarrow (\mathbb{A}_{\mathbb{C}}^1 \setminus 0) \times S_{\mathbb{C}}^{an}$  the universal covering in  $\text{AnSm}(\mathbb{C})$  and

$$\mathbb{V}_{(\mathbb{A}^1 \setminus 0) \times S}(\alpha) := \pi_{S, G_m}(\alpha \times \tilde{G}_m \times \tilde{S}_{\mathbb{C}}^{an}) \subset E_{DR}^{2p, an}(\mathcal{Y}_{\mathbb{C}}^o/(\mathbb{A}_{\mathbb{C}}^1 \setminus 0) \times S_{\mathbb{C}})$$

the flat leaf (see section 2.1) and

$$\mathbb{V}_{(\mathbb{A}^1 \setminus 0) \times S}^p(\alpha) := \mathbb{V}_{(\mathbb{A}^1 \setminus 0) \times S}(\alpha) \cap F^p E_{DR}^{2p}(\mathcal{Y}_{\mathbb{C}}^o/(\mathbb{A}_{\mathbb{C}}^1 \setminus 0) \times S_{\mathbb{C}}) \subset E_{DR}^{2p}(\mathcal{Y}_{\mathbb{C}}^o/(\mathbb{A}_{\mathbb{C}}^1 \setminus 0) \times S_{\mathbb{C}}),$$

which is an algebraic subvariety by [6],

- $\pi_S : \tilde{S}_{\mathbb{C}}^{an} \rightarrow S_{\mathbb{C}}^{an}$  the universal covering in  $\text{AnSm}(\mathbb{C})$  and

$$\mathbb{V}_S(\alpha) := \pi_S(\alpha \times \tilde{S}_{\mathbb{C}}^{an}) \subset H^{2p} \psi_0 E_{DR}(\mathcal{Y}_{\mathbb{C}}^o/((\mathbb{A}_{\mathbb{C}}^1 \setminus 0) \times S_{\mathbb{C}}))^{an}$$

the flat leaf (see section 2.1) and

$$\mathbb{V}_S^p(\alpha) := \mathbb{V}_S(\alpha) \cap F^p H^{2p} \psi_0 E_{DR}(\mathcal{Y}_{\mathbb{C}}^o/((\mathbb{A}_{\mathbb{C}}^1 \setminus 0) \times S_{\mathbb{C}}))^{an} \subset H^{2p} \psi_0 E_{DR}(\mathcal{Y}_{\mathbb{C}}^o/((\mathbb{A}_{\mathbb{C}}^1 \setminus 0) \times S_{\mathbb{C}}))^{an}.$$

Consider the exact sequence of variation of mixed Hodge structure over  $S$

$$0 \rightarrow W_{2p-2} E_{Hdg}^{2p}(\mathcal{E}/S) \rightarrow E_{Hdg}^{2p}(\mathcal{E}/S) \xrightarrow{(i_i^*)_{0 \leq i \leq r} := q} \oplus_{i=0}^r E_{Hdg}^{2p}(\mathcal{E}_i/S) \rightarrow 0.$$

It induces the exact sequence of vector bundles over  $S$

$$0 \rightarrow W_{2p-2} E_{DR}^{2p}(\mathcal{E}/S) \rightarrow E_{DR}^{2p}(\mathcal{E}/S) \xrightarrow{(i_i^*)_{0 \leq i \leq r} := q} \oplus_{i=0}^r E_{DR}^{2p}(\mathcal{E}_i/S) \rightarrow 0,$$

and the exact sequence of presheaves on  $S_{\mathbb{C}}^{an}$ ,

$$J(W_{2p-2} E_{Hdg}^{2p}(\mathcal{E}_{\mathbb{C}}/S_{\mathbb{C}})) := \text{Ext}^1(\mathbb{Q}_S^{Hdg}, W_{2p-2} E_{Hdg}^{2p}(\mathcal{E}_{\mathbb{C}}/S_{\mathbb{C}})), \xrightarrow{\iota} \text{Ext}^1(\mathbb{Q}_S^{Hdg}, W_{2p-2} E_{Hdg}^{2p}(\mathcal{E}_{\mathbb{C}}/S_{\mathbb{C}}))$$

where  $HL^{2p, p}(\mathcal{E}_{\mathbb{C}}/S_{\mathbb{C}}) \subset E_{DR}^{2p}(\mathcal{E}_{\mathbb{C}}/S_{\mathbb{C}})$ ,  $HL^{2p, p}(\mathcal{E}_{i, \mathbb{C}}/S_{\mathbb{C}}) \subset E_{DR}^{2p}(\mathcal{E}_{i, \mathbb{C}}/S_{\mathbb{C}})$  are the locus of Hodge classes. Let  $\lambda \in F^p H^{2p}(E_0^{an}, \mathbb{Q})$ , where we recall  $E_0 = \mathcal{E}_{0, \mathbb{C}, t}$ . Consider then

- $\tilde{\lambda} \in H^{2p}(E^{an}, \mathbb{Q})$ , such that  $q(\tilde{\lambda}) = \lambda$ , and  $\tilde{\lambda} := Sp_{E/f_X}(\tilde{\lambda}) \in Sp_{E/f_X}(H^{2p}(E^{an}, \mathbb{Q}))$ ,
- $\tilde{\lambda}_F \in F^p H^{2p}(E^{an}, \mathbb{C})$ , such that  $q(\tilde{\lambda}_F) = \lambda$ ,
- $e_\lambda := \tilde{\lambda} - \tilde{\lambda}_F \in W_{2p-2} H^{2p}(E^{an}, \mathbb{C})$ ,  $e_\lambda \in \Gamma(\tilde{S}_{\mathbb{C}}^{an}, \pi^{*mod} E_{DR}^{2p,an}(\mathcal{E}_{\mathbb{C}}/S_{\mathbb{C}}))^{\nabla}$  the corresponding section where  $\pi_S : \tilde{S}_{\mathbb{C}}^{an} \rightarrow S_{\mathbb{C}}^{an}$  is the universal covering, and

$$\bar{e}_\lambda := [e_\lambda] \in \Gamma(\tilde{S}_{\mathbb{C}}^{an}, \pi^{*mod} E_{DR}^{2p,an}(\mathcal{E}_{\mathbb{C}}/S_{\mathbb{C}})/F^p \oplus \mathbb{Q})^{\nabla}.$$

The locus of Hodge classes passing through  $\lambda$  is then an union of two components

$$\mathbb{V}_S^p(\lambda) := \mathbb{V}_S(\lambda) \cap F^p E_{DR}^{2p}(\mathcal{E}_{0,\mathbb{C}}/S_{\mathbb{C}}) = q(\mathbb{V}_S^p(\tilde{\lambda})) \cup \mathbb{V}_{V(\bar{e}_\lambda)}(\lambda) \subset E_{DR}^{2p}(\mathcal{E}_{0,\mathbb{C}}/S_{\mathbb{C}}),$$

where

- $\mathbb{V}_{V(\bar{e}_\lambda)}(\lambda) = \pi_S(\lambda \times V(\bar{e}_\lambda)) \subset E_{DR}^{2p,an}(\mathcal{E}_{0,\mathbb{C}}/S_{\mathbb{C}})$  is the flat leaf, where  $\pi_S : H^{2p}(E^{an}, \mathbb{C}) \times \tilde{S}_{\mathbb{C}}^{an} \rightarrow E_{DR}^{2p,an}(\mathcal{E}_{\mathbb{C}}/S_{\mathbb{C}})$  is the morphism induced by the universal covering  $\pi_S : \tilde{S}_{\mathbb{C}}^{an} \rightarrow S_{\mathbb{C}}^{an}$ ,
- $V(\bar{e}_\lambda) := \pi_S(V(\bar{e}_\lambda)) \subset S_{\mathbb{C}}$ , where  $V(\bar{e}_\lambda) \subset \tilde{S}_{\mathbb{C}}^{an}$  is the zero locus of  $\bar{e}_\lambda$ ,
- $q := q \otimes \mathbb{C} : E_{DR}^{2p}(\mathcal{E}_{\mathbb{C}}/S_{\mathbb{C}}) \rightarrow E_{DR}^{2p}(\mathcal{E}_{0,\mathbb{C}}/S_{\mathbb{C}})$  is the quotient map.

Indeed, let  $\lambda_{t'} \in \mathbb{V}_S(\lambda)$ ,  $t' \in \tilde{S}_{\mathbb{C}}^{an}$ . If  $\lambda_{t'} \in \mathbb{V}_S^p(\lambda)$ , then  $\iota(e(\lambda_{t'})) = 0$ . If  $\lambda_{t'} \in \mathbb{V}_S^p(\lambda) \setminus q(\mathbb{V}_S^p(\tilde{\lambda}))$ , then

$$[e_\lambda(t')] = e(\lambda_{t'}) \neq 0 \in W_{2p-2} H^{2p}(\mathcal{E}_{\pi(t')}, \mathbb{C}) / ((W_{2p-2} \cap F^p) \oplus W_{2p-2} \mathbb{Q})$$

Hence  $\lambda_{t'} \in \mathbb{V}_S^p(\lambda) \setminus q(\mathbb{V}_S^p(\tilde{\lambda}))$

$$[e_\lambda(t')] = \iota(e(\lambda_{t'})) = 0 \in H^{2p}(\mathcal{E}_{\pi(t')}, \mathbb{C}) / (F^p \oplus \mathbb{Q}).$$

Now,

- if  $Sp_{E/f_X}(\tilde{\lambda}) = 0$ , since  $Sp_{\mathcal{E}/\tilde{f}_X} \circ \epsilon^* = Sp_{\mathcal{Y}_0/\tilde{f}_X}$  and  $Sp_{\mathcal{Y}_0/\tilde{f}_X}$  is injective, we have  $\tilde{\lambda}_{t'} = k_{t*} \tilde{\lambda}_{t'}$  with  $\tilde{\lambda}_{t'} \in F^p H_{\mathcal{F}_{\mathbb{C},t'}}^{2p}(\mathcal{E}_{\mathbb{C},t'}^{an}, \mathbb{Q})$ , where  $k : \mathcal{F} \rightarrow \mathcal{E}$  is the exceptional locus of  $\epsilon : \mathcal{E}_0 \rightarrow \mathcal{Y}_0$  (note that  $\epsilon^* : H^{2p}(\mathcal{Y}_{0,t'}^{an}, \mathbb{Q}) \rightarrow H^{2p}(\mathcal{E}_{0,t'}^{an}, \mathbb{Q})$  is injective by Poincare Lefschetz duality), hence, assuming the Hodge conjecture for smooth projective varieties of dimension less or equal to  $d-1$ ,  $\mathbb{V}_S^p(\tilde{\lambda}) \subset E_{DR}^{2p}(\mathcal{E}_{\mathbb{C}}/S_{\mathbb{C}})$  is defined over  $\mathbb{Q}$  and its Galois conjugates are also components of the locus of Hodge classes, if  $Sp_{E/f_X}(\tilde{\lambda}) \neq 0$ , we have,

$$Sp_{\mathcal{E}/\tilde{f}_X}(\mathbb{V}_S^p(\tilde{\lambda})) = \mathbb{V}_S^p(Sp_{E/f_X}(\tilde{\lambda})) = \overline{\mathbb{V}_{(\mathbb{A}^1 \setminus 0) \times S}^p(Sp_{E/f_X}(\tilde{\lambda}))} \cap p^{-1}(0 \times S_{\mathbb{C}}) \subset E_{DR,V_0}^{2p}(\mathcal{Y}_{\mathbb{C}}/(\mathbb{A}_{\mathbb{C}}^1 \times S_{\mathbb{C}}))$$

where  $p : E_{DR,V_0}^{2p}(\mathcal{Y}_{\mathbb{C}}/(\mathbb{A}_{\mathbb{C}}^1 \times S_{\mathbb{C}})) \rightarrow \mathbb{A}_{\mathbb{C}}^1 \times S_{\mathbb{C}}$  is the projection (the first equality is obvious since  $Sp_{\mathcal{E}/\tilde{f}_X}$  is a morphism of variation of mixed Hodge structures over  $S$ , whereas for the second equality the inclusion  $\supset$  follows from [6] lemma 2.11 which state the invariance of  $\tilde{\lambda}$  under the monodromy

$$\text{Im}(\pi_1(p(\mathbb{V}^p(\tilde{\lambda})) \cap ((\mathbb{A}_{\mathbb{C}}^1 \setminus 0) \times D^*)) \rightarrow \pi_1((\mathbb{A}_{\mathbb{C}}^1 \setminus 0) \times D^*)), (D, D^*) \rightarrow (\mathbb{A}_{\mathbb{C}}^r, S_{\mathbb{C}}^{an}))$$

but by [5] theorem 1, the Hodge conjecture holds for projective hypersurfaces hence

$$\mathbb{V}_{(\mathbb{A}^1 \setminus 0) \times S}^p(Sp_{E/f_X}(\tilde{\lambda})) \subset E_{DR}^{2p}(\mathcal{Y}_{\mathbb{C}}/((\mathbb{A}_{\mathbb{C}}^1 \setminus 0) \times S_{\mathbb{C}}))$$

is defined over  $\bar{\mathbb{Q}}$  and its Galois conjugates are also components of the locus of Hodge classes, hence  $\mathbb{V}_S^p(\tilde{\lambda}) \subset E_{DR}^{2p}(\mathcal{E}_{\mathbb{C}}/S_{\mathbb{C}})$  is defined over  $\mathbb{Q}$  and its Galois conjugates are also components of the locus of Hodge classes,

- if  $Sp_{E/f_X}(e_\lambda) = 0$ ,  $Sp_{E/f_X}(\tilde{\lambda}) = Sp_{E/f_X}(\tilde{\lambda}_F)$  and for  $t' \in \tilde{S}_\mathbb{C}^{an}$ ,  $e_\lambda(t') = 0$  if and only if  $e(\tilde{\lambda}_{t'}) = 0$  if and only if  $Sp_{\mathcal{E}_{t'}/(\tilde{f}_X)_{t'}}(\tilde{\lambda}_{t'})$  is a Hodge class, if  $Sp_{E/f_X}(e_\lambda) \neq 0$ , consider

$$Sp_{E/f_X}(e_\lambda) \in \Gamma(\tilde{G}_m \times \tilde{S}_\mathbb{C}^{an}, \pi_{S, G_m}^{*mod} E_{DR}(\mathcal{Y}_\mathbb{C}^o / ((\mathbb{A}_\mathbb{C}^1 \setminus 0) \times S_\mathbb{C}))^{an})^\nabla$$

and

$$[Sp_{E/f_X}(e_\lambda)] \in \Gamma(\tilde{G}_m \times \tilde{S}_\mathbb{C}^{an}, \pi_{S, G_m}^{*mod} E_{DR}(\mathcal{Y}_\mathbb{C}^o / ((\mathbb{A}_\mathbb{C}^1 \setminus 0) \times S_\mathbb{C}))^{an})^\nabla / (F^p \oplus \mathbb{Q}),$$

we may assume that  $X$  is not defined over  $\bar{\mathbb{Q}}$  (otherwise  $Y_0$  is defined over  $\bar{\mathbb{Q}}$  and  $S$  is a finite set), we may also assume that  $X$  is not rational i.e.  $l_X = \deg(\mathbb{Q}(X)/\mathbb{Q}(x_1, \dots, x_d)) > 1$ , then  $((\mathbb{A}^1 \setminus 0) \times S)_\mathbb{C} = \mathbb{P}(l_X)$  where  $\mathbb{P}(l_X)$  is the space of hypersurface over degree  $l_X$ , consider then a Lefschetz pencil  $L \subset ((\mathbb{A}^1 \setminus 0) \times S)_\mathbb{C} = \mathbb{P}(l_X)$  and denote  $L^\circ := L \cap (\mathbb{A}^1 \setminus 0) \times S)_\mathbb{C}$ , we have the factorization  $\pi_L : \tilde{L}^\circ \xrightarrow{i_L} \pi_{S, G_m}^{-1}(L) \xrightarrow{\pi_{S, G_m}} L$  of the universal covering of  $L^\circ$  and consider

$$Sp_{E/f_X}(\widehat{e_\lambda})|_{\tilde{L}^\circ} \in \Gamma(L^\circ, E_{DR}^{2p}(\tilde{Y}'_{L^\circ}/L^\circ)), \quad u \in \tilde{L}^\circ \mapsto Sp_{E/f_X}(\widehat{e_\lambda})(u) := e^{2i\pi N u} Sp_{E/f_X}(e_\lambda)(u),$$

for  $u \in \tilde{L}^\circ$ , if  $Sp_{E/f_X}(e_\lambda)(u) = 0 \in H^{2p}(\mathcal{Y}_u, \mathbb{C})/(F^p \oplus \mathbb{Q})$  then  $Sp_{E/f_X}(\widehat{e_\lambda})|_{\tilde{L}^\circ}(u) = 0 \in H^{2p}(\mathcal{Y}_u, \mathbb{C})/(F^{p+1} \oplus \mathbb{Q})$ , consider

$$[Sp_{E/f_X}(\widehat{e_\lambda})|_{\tilde{L}^\circ}] \in \Gamma(L^\circ, E_{DR}^{2p}(\tilde{Y}'_{L^\circ}/L^\circ)/F^{p+1} \oplus \mathbb{Q}),$$

since  $H^{2p+1}(\mathcal{Y}'_L, \mathbb{Q})_{pr} = 0$  as  $\mathcal{Y}'_L = \mathbb{P}^N_{B(L)}$  is the blow up of  $\mathbb{P}^N$  in the base locus of  $L$ , we get  $H^1(L^\circ, R^{2p}(\tilde{f}_X)|_{L^*} \mathbb{Q}) = 0$ , hence

$$[Sp_{E/f_X}(\widehat{e_\lambda})|_{\tilde{L}^\circ}] \in H^{2p}(\mathcal{Y}'_L, \mathbb{C})/(F^{p+1} \oplus \mathbb{Q})$$

is a constant which is non zero by the irreducibility of the monodromy representation, hence  $V([Sp_{E/f_X}(e_\lambda)]) \cap i_L(\tilde{L}^\circ) = \emptyset$ , hence  $V(\tilde{e}_\lambda) = \emptyset$  since  $\tilde{G}_m \times \tilde{S}_\mathbb{C}^{an} = \cup_{L \subset \mathbb{P}(l_X)} i_L(\tilde{L}^\circ)$ .

Hence  $\mathbb{V}_S^p(\lambda) \subset E_{DR}^{2p}(\mathcal{E}_{0\mathbb{C}}/S_\mathbb{C})$  is defined over  $\bar{\mathbb{Q}}$  and its Galois conjugates are also components of the locus of Hodge classes. Thus, by [4] theorem 4,

$$\lambda = [Z] \in H^{2p}(E_0^{an}, \mathbb{Q}), \text{ with } Z \in \mathcal{Z}^p(E_0).$$

Since  $p \in \mathbb{Z}$  and  $\lambda \in F^p H^{2p}(E_0^{an}, \mathbb{Q})$  are arbitrary, the Hodge conjecture holds for  $E_0$ . Proposition 1(ii) and the fact that the Hodge conjecture is true for smooth projective varieties of dimension less or equal to  $d-1$  by induction hypothesis then implies that Hodge conjecture holds for  $X$ , since  $X$  is birational to  $E_0$  and  $X, E_0 \in \text{PSmVar}(\mathbb{C})$  are connected of dimension  $d$ .  $\square$

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