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Degeneration of families of projective hypersurfaces and Hodge conjecture

Johann Bouali

March 19, 2024

Abstract

We prove by induction on dimension the Hodge conjecture for smooth complex projective varieties. Let X be a smooth complex projective variety. Then X is birational to a possibly singular projective hypersurface, hence to a smooth projective variety E_0 which is a component of a normal crossing divisor $E = \cup_{i=0}^r E_i \subset Y$ which is the singular fiber of a pencil $f : Y \rightarrow \mathbb{A}^1$ of smooth projective hypersurfaces. Using the smooth hypersurface case ([5] theorem 1), the nearby cycle functor on mixed Hodge module with rational de Rham factor and the induction hypothesis, we prove that a Hodge class of E_0 is absolute Hodge, more precisely the locus of Hodge classes inside the algebraic vector bundle given the De Rham cohomology of the rational deformation of E_0 is defined over \mathbb{Q} . By [4] theorem 4, we get the Hodge conjecture for E_0 . By the induction hypothesis we also have the Hodge conjecture for X .

Contents

1	Introduction	1
2	Preliminaries and notations	3
2.1	Notations	3
2.2	Birational projective varieties and Hodge conjecture	5
2.3	A family of smooth projective hypersurfaces associated to a smooth projective variety	6
2.4	The nearby cycle functor and the specialization map	6
3	Hodge conjecture for smooth projective varieties	8

1 Introduction

The main result of this article is an inductive proof of the Hodge conjecture using the results of [4] and [5]. More precisely, assuming the Hodge conjecture for smooth projective varieties of dimension less or equal to $d - 1$, the property of satisfying the Hodge conjecture for connected smooth projective varieties of dimension d is a birational invariant (proposition 1). On the other hand, if X is a connected smooth projective variety of dimension d , then X is birational to a possibly singular hypersurface $Y_0 \subset \mathbb{P}_{\mathbb{C}}^{d+1}$, hence there exists (see proposition 2) a family of smooth projective hypersurfaces over $\mathbb{A}_{\mathbb{C}}^1 \setminus \{0\}$ which degenerate at 0 into a normal crossing divisor with one component birational to X , that is we have a morphism of complex algebraic varieties $f_X : Y \rightarrow \mathbb{A}_{\mathbb{C}}^1$ with

- Y a smooth complex algebraic variety, f_X flat projective,
- $Y_s := f_X^{-1}(s) \subset \mathbb{P}_{\mathbb{C}}^{d+1}$ for $s \in \mathbb{A}_{\mathbb{C}}^1 \setminus \{0\}$ are smooth projective hypersurfaces,

- $f_X = f'_X \circ \epsilon$, $f'_X : Y' \rightarrow \mathbb{A}_{\mathbb{C}}^1$ is a generic pencil of hypersurface passing through $Y_0 := f_X'^{-1}(0) \subset \mathbb{P}_{\mathbb{C}}^{d+1}$, $\epsilon : (Y, E) \rightarrow (Y', Y_0)$ is a desingularization of the pair (Y', Y_0) with $E := f_X'^{-1}(0) = \epsilon^{-1}(Y_0) = \cup_{i=0}^r E_i \subset Y$ is a normal crossing divisor, and E_0 is birational to X .

Assuming the Hodge conjecture for smooth projective varieties of dimension less or equal to $d - 1$, we then prove the Hodge conjecture for the E_i , more specifically for E_0 as follows : We consider its canonical rational deformation $\tilde{f}_X : \mathcal{Y} \rightarrow \mathbb{A}_{\mathbb{Q}}^1 \times S$ with

- \mathcal{Y} a smooth algebraic variety over \mathbb{Q} , $S \subset \mathbb{A}_{\mathbb{Q}}^s$ an open subset, \tilde{f}_X flat projective, $Y = \tilde{f}_X^{-1}(\mathbb{A}_{\mathbb{Q}}^1 \times \{t\})$, $f_X = \tilde{f}_X \times_S \{t\}$, with $t \in S(\mathbb{C})$
- $\tilde{f}_X = \tilde{f}'_X \circ \epsilon$, $\tilde{f}'_X : \mathcal{Y}' \rightarrow \mathbb{A}_{\mathbb{Q}}^1 \times S$ is the corresponding family of hypersurfaces, $\epsilon : (\mathcal{Y}, \mathcal{E}) \rightarrow (\mathcal{Y}', \mathcal{Y}_0)$ is the corresponding desingularization over \mathbb{Q} ,
- $\mathcal{E} = \cup_{i=0}^r \mathcal{E}_i := \tilde{f}_X^{-1}(\{0\} \times S) \subset \mathcal{Y}$ is a normal crossing divisor, $\tilde{f}_{X,0} : \mathcal{E}_I := \cap_{i \in I} \mathcal{E}_i \rightarrow S$ smooth projective for each $I \subset [0, \dots, r]$, $\mathcal{E}_{\mathbb{C}, t} = E$,
- $\tilde{f}_X \times_{\mathbb{A}^1} (\mathbb{A}^1 \setminus \{0\}) : \mathcal{Y}^\circ \rightarrow \mathbb{A}_{\mathbb{Q}}^1 \setminus \{0\}$ has fibers which are smooth projective hypersurfaces.

Let $p \in \mathbb{Z}$. By the hard Lefschetz theorem on $H^*(E_0)$, we may assume that $2p \leq d$. We then have then a morphism of mixed Hodge module over S (in fact of geometric variation of mixed Hodge structures over S) given by the specialization map

$$Sp_{\mathcal{E}/\tilde{f}_X} : E_{Hdg}^{2p}(\mathcal{E}/S) \rightarrow H^{2p}\psi_0 E_{Hdg}(\mathcal{Y}^\circ / ((\mathbb{A}_{\mathbb{Q}}^1 \setminus \{0\}) \times S)),$$

whose de Rham part are morphism of filtered vector bundle over S with Gauss-Manin connexion defined over \mathbb{Q} , and whose kernel of the part supported on \mathcal{E}_0 is supported a closed subset $\mathcal{F} \subset \mathcal{E}_0$ with $\text{codim}(\mathcal{F}_{t'}, \mathcal{E}_{0,t'}) = 1$ for all $t' \in S(\mathbb{C})$. The fact that the kernel is supported on a closed subset of codimension one on fibers follows from the fact that $H^{2p-1}(Y_s, \mathbb{Q}) = 0 = H_{Y_0}^{2p-1}(Y)$, by the weak Lefschetz theorem since Y_s and Y_0 are projective hypersurfaces ($2p \leq d$). Consider the exact sequence of variation of mixed Hodge structure over S

$$0 \rightarrow W_{2p-2} E_{Hdg}^{2p}(\mathcal{E}/S) \rightarrow E_{Hdg}^{2p}(\mathcal{E}/S) \xrightarrow{(i_i^*)_{0 \leq i \leq r} := q} \oplus_{i=0}^r E_{Hdg}^{2p}(\mathcal{E}_i/S) \rightarrow 0.$$

It induces the exact sequence of presheaves on $S_{\mathbb{C}}^{an}$,

$$J(W_{2p-2} E_{Hdg}^{2p}(\mathcal{E}_{\mathbb{C}}/S_{\mathbb{C}})) := \text{Ext}^1(\mathbb{Q}_S^{Hdg}, W_{2p-2} E_{Hdg}^{2p}(\mathcal{E}_{\mathbb{C}}/S_{\mathbb{C}})) \xrightarrow{\iota} \text{Ext}^1(\mathbb{Q}_S^{Hdg}, E_{Hdg}^{2p}(\mathcal{E}_{\mathbb{C}}/S_{\mathbb{C}}))$$

where $HL^{2p,p}(\mathcal{E}_{\mathbb{C}}/S_{\mathbb{C}}) \subset E_{DR}^{2p}(\mathcal{E}_{\mathbb{C}}/S_{\mathbb{C}})$, $HL^{2p,p}(\mathcal{E}_{i,\mathbb{C}}/S_{\mathbb{C}}) \subset E_{DR}^{2p}(\mathcal{E}_{i,\mathbb{C}}/S_{\mathbb{C}})$ are the locus of Hodge classes. Let $\lambda \in F^p H^{2p}(E_0^{an}, \mathbb{Q})$, where we recall $E_0 = \mathcal{E}_{0,\mathbb{C},t}$. Consider then

- $\tilde{\lambda} \in H^{2p}(E^{an}, \mathbb{Q})$, such that $q(\tilde{\lambda}) = \lambda$, and $\tilde{\lambda} := Sp_{E/f_X}(\tilde{\lambda}) \in Sp_{E/f_X}(H^{2p}(E^{an}, \mathbb{Q}))$,
- $\tilde{\lambda}_F \in F^p H^{2p}(E^{an}, \mathbb{C})$, such that $q(\tilde{\lambda}_F) = \lambda$,
- $e_\lambda := \tilde{\lambda} - \tilde{\lambda}_F \in W_{2p-2} H^{2p}(E^{an}, \mathbb{C})$, $e_\lambda \in \Gamma(\tilde{S}_{\mathbb{C}}^{an}, \pi^{*mod} E_{DR}^{2p,an}(\mathcal{E}_{\mathbb{C}}/S_{\mathbb{C}}))^\nabla$ the corresponding section where $\pi_S : \tilde{S}_{\mathbb{C}}^{an} \rightarrow S_{\mathbb{C}}^{an}$ is the universal covering, and

$$\bar{e}_\lambda := [e_\lambda] \in \Gamma(\tilde{S}_{\mathbb{C}}^{an}, \pi^{*mod} E_{DR}^{2p,an}(\mathcal{E}_{\mathbb{C}}/S_{\mathbb{C}})/F^p \oplus \mathbb{Q})^\nabla.$$

The locus of Hodge classes passing through λ inside the De Rham vector bundle of $\tilde{f}_X \circ i_{E_0} : \mathcal{E}_0 \rightarrow S$ is thus by the above exact sequence an union of two components

$$\mathbb{V}_S^p(\lambda) := \mathbb{V}_S(\lambda) \cap F^p E_{DR}^{2p}(\mathcal{E}_{0,\mathbb{C}}/S_{\mathbb{C}}) = q(\mathbb{V}_S^p(\tilde{\lambda})) \cup \mathbb{V}_V(\bar{e}_\lambda)(\lambda) \subset E_{DR}^{2p}(\mathcal{E}_{0,\mathbb{C}}/S_{\mathbb{C}}),$$

where

- $\mathbb{V}_S(\lambda) \subset E_{DR}^{2p}(\mathcal{E}_{0,\mathbb{C}}/S_{\mathbb{C}})$, $\mathbb{V}_S(\tilde{\lambda}) \subset E_{DR}^{2p}(\mathcal{E}_{\mathbb{C}}/S_{\mathbb{C}})$, $\mathbb{V}_{V(\tilde{e}(\lambda))}(\lambda) \subset E_{DR}^{2p}(\mathcal{E}_{0,\mathbb{C}}/S_{\mathbb{C}})$ are the flat leaves, e.g. $\mathbb{V}_S(\lambda) := \pi_S(\lambda \times \tilde{S}_{\mathbb{C}}^{an})$ where $\pi_S : H^{2p}(E^{an}, \mathbb{C}) \times \tilde{S}_{\mathbb{C}}^{an} \rightarrow E_{DR}^{2p,an}(\mathcal{E}_{\mathbb{C}}/S_{\mathbb{C}})$ is the morphism induced by the universal covering $\pi_S : \tilde{S}_{\mathbb{C}}^{an} \rightarrow S_{\mathbb{C}}^{an}$,
- $V(\tilde{e}\lambda) := \pi_S(V(\tilde{e}(\lambda))) \subset S_{\mathbb{C}}$, where $V(\tilde{e}\lambda) \subset \tilde{S}_{\mathbb{C}}^{an}$ is the zero locus of $\tilde{e}\lambda$,
- $q := q \otimes \mathbb{C} : E_{DR}^{2p}(\mathcal{E}_{\mathbb{C}}/S_{\mathbb{C}}) \rightarrow E_{DR}^{2p}(\mathcal{E}_{0,\mathbb{C}}/S_{\mathbb{C}})$ is the quotient map.

Now,

- if $Sp_{E/f_X}(\tilde{\lambda}) = 0$, assuming the Hodge conjecture for smooth projective varieties of dimension less or equal to $d - 1$, $\mathbb{V}_S^p(\tilde{\lambda}) \subset E_{DR}^{2p}(\mathcal{E}_{\mathbb{C}}/S_{\mathbb{C}})$ is an algebraic subvariety defined over $\bar{\mathbb{Q}}$ and its Galois conjugate are also components of the locus of Hodge classes, if $Sp_{E/f_X}(\tilde{\lambda}) \neq 0$, by [5] theorem 1, the Hodge conjecture holds for projective hypersurfaces, hence

$$Sp_{\mathcal{E}/\tilde{f}_X}(\mathbb{V}_S^p(\tilde{\lambda})) = \mathbb{V}_S^p(Sp_{E/f_X}(\tilde{\lambda})) = \overline{\mathbb{V}_{(\mathbb{A}^1 \setminus 0) \times S}^p(Sp_{E/f_X}(\tilde{\lambda}))} \cap p^{-1}(0 \times S_{\mathbb{C}}) \subset E_{DR,V_0}^{2p}(\mathcal{Y}_{\mathbb{C}}/(\mathbb{A}_{\mathbb{C}}^1 \times S_{\mathbb{C}}))$$

is an algebraic subvariety defined over $\bar{\mathbb{Q}}$ and its Galois conjugate are also components of the locus of Hodge classes, where the equality follows from [6] lemma 2.11,

- if $Sp_{E/f_X}(e\lambda) = 0$, $Sp_{E/f_X}(\tilde{\lambda}) = Sp_{E/f_X}(\tilde{\lambda}_F)$ and for $t' \in \tilde{S}_{\mathbb{C}}^{an}$, $\tilde{e}\lambda(t') = 0$ if and only if $e(\tilde{\lambda}_{t'}) = 0$ if and only if $Sp_{\mathcal{E}_{t'}/(\tilde{f}_X)_{t'}}(\tilde{\lambda}_{t'})$ is a Hodge class, if $Sp_{E/f_X}(e\lambda) \neq 0$, then considering the restrictions of

$$Sp_{E/f_X}(e\lambda) \in \Gamma(\tilde{G}_m \times \tilde{S}_{\mathbb{C}}^{an}, \pi_{S,G_m}^{*mod} E_{DR}(\mathcal{Y}_{\mathbb{C}}^o / ((\mathbb{A}_{\mathbb{C}}^1 \setminus 0) \times S_{\mathbb{C}}))^{an})^{\nabla}$$

to Lefchetz pencils $\pi_{S,G_m}^{-1}(L) \subset \tilde{G}_m \times \tilde{S}_{\mathbb{C}}^{an}$, $L \subset \overline{((\mathbb{A}^1 \setminus 0) \times S)_{\mathbb{C}}} = \mathbb{P}(l_X)$, where $\mathbb{P}(l_X)$ is the space of hypersurface over degree $l_X = \deg(\mathbb{Q}(X)/\mathbb{Q}(x_1, \dots, x_d))$ (for the equality we may assume that X is not defined over \mathbb{Q} and that $l_X > 1$), we see that $V(\tilde{e}\lambda) = \emptyset$.

Hence, $\mathbb{V}_S^p(\lambda) \subset E_{DR}^{2p}(\mathcal{E}_{0\mathbb{C}}/S_{\mathbb{C}})$ is an algebraic subvariety defined over $\bar{\mathbb{Q}}$ and its Galois conjugate are also components of the locus of Hodge classes. By [4] theorem 4, we get

$$\lambda = [Z] \in H^{2p}(E_0^{an}, \mathbb{Q}), \quad Z \in \mathcal{Z}^p(E_0).$$

We recall the key point is that since $\mathbb{V}_S^p(\lambda) \subset E_{DR}^{2p}(\mathcal{E}_{0\mathbb{C}}/S_{\mathbb{C}})$ is an algebraic subvariety defined over $\bar{\mathbb{Q}}$ and its Galois conjugate are also components of the locus of Hodge classes, we have for each $\theta \in \text{Aut}(\mathbb{C}, \mathbb{Q})$,

$$\theta(\lambda) \in \text{Gal}(\bar{\mathbb{Q}}, \mathbb{Q})(\mathbb{V}_S^p(\lambda)\theta) \subset E_{DR}^{2p}(\mathcal{E}_{0\mathbb{C},\theta}/S_{\mathbb{C},\theta}),$$

which implies that $\theta(\lambda) \in H^{2p}(E_{0,\theta}^{an}, \mathbb{Q}) \subset H^{2p}(E_{0,\theta}^{an}, \mathbb{C})$, that is $\lambda \in H^{2p}(E_0^{an}, \mathbb{Q})$ is absolute Hodge and we apply [4] corollary 1. Since $p \in \mathbb{Z}$ and $\lambda \in F^p H^{2p}(E_0^{an}, \mathbb{Q})$ are arbitrary, this proves the Hodge conjecture for E_0 . Since X is birational to E_0 , the Hodge conjecture for E_0 and the induction hypothesis implies the Hodge conjecture for X .

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2 Preliminaries and notations

2.1 Notations

- Denote by Top the category of topological spaces and RTop the category of ringed spaces.
- Denote by Cat the category of small categories and RCat the category of ringed topos.
- For $\mathcal{S} \in \text{Cat}$ and $X \in \mathcal{S}$, we denote $\mathcal{S}/X \in \text{Cat}$ the category whose objects are $Y/X := (Y, f)$ with $Y \in \mathcal{S}$ and $f : Y \rightarrow X$ is a morphism in \mathcal{S} , and whose morphisms $\text{Hom}((Y', f'), (Y, f))$ consists of $g : Y' \rightarrow Y$ in \mathcal{S} such that $f \circ g = f'$.

- Let $(\mathcal{S}, O_{\mathcal{S}}) \in \text{RCat}$ a ringed topos with topology τ . For $F \in C_{O_{\mathcal{S}}}(\mathcal{S})$, we denote by $k : F \rightarrow E_{\tau}(F)$ the canonical flasque resolution in $C_{O_{\mathcal{S}}}(\mathcal{S})$ (see [3]). In particular for $X \in \mathcal{S}$, $H^*(X, E_{\tau}(F)) \xrightarrow{\sim} \mathbb{H}_{\tau}^*(X, F)$.
- For $f : \mathcal{S}' \rightarrow \mathcal{S}$ a morphism with $\mathcal{S}, \mathcal{S}' \in \text{RCat}$, endowed with topology τ and τ' respectively, we denote for $F \in C_{O_{\mathcal{S}}}(\mathcal{S})$ and each $j \in \mathbb{Z}$,
 - $f^* := H^j \Gamma(\mathcal{S}, k \circ \text{ad}(f^*, f_*)(F)) : \mathbb{H}^j(\mathcal{S}, F) \rightarrow \mathbb{H}^j(\mathcal{S}', f^*F)$,
 - $f^* := H^j \Gamma(\mathcal{S}, k \circ \text{ad}(f^{*mod}, f_*)(F)) : \mathbb{H}^j(\mathcal{S}, F) \rightarrow \mathbb{H}^j(\mathcal{S}', f^{*mod}F)$,

the canonical maps.

- For $m : A \rightarrow B$, $A, B \in C(\mathcal{A})$, \mathcal{A} an additive category, we denote $c(A) : \text{Cone}(m : A \rightarrow B) \rightarrow A[1]$ and $c(B) : B \rightarrow \text{Cone}(m : A \rightarrow B)$ the canonical maps.
- Denote by $\text{Sch} \subset \text{RTop}$ the subcategory of schemes (the morphisms are the morphisms of locally ringed spaces). We denote by $\text{PSch} \subset \text{Sch}$ the full subcategory of proper schemes. For a field k , we consider $\text{Sch}/k := \text{Sch}/\text{Spec } k$ the category of schemes over $\text{Spec } k$. The objects are $X := (X, a_X)$ with $X \in \text{Sch}$ and $a_X : X \rightarrow \text{Spec } k$ a morphism and the morphisms are the morphisms of schemes $f : X' \rightarrow X$ such that $f \circ a_{X'} = a_X$. We then denote by
 - $\text{Var}(k) = \text{Sch}^{ft}/k \subset \text{Sch}/k$ the full subcategory consisting of algebraic varieties over k , i.e. schemes of finite type over k ,
 - $\text{PVar}(k) \subset \text{QPVar}(k) \subset \text{Var}(k)$ the full subcategories consisting of quasi-projective varieties and projective varieties respectively,
 - $\text{PSmVar}(k) \subset \text{SmVar}(k) \subset \text{Var}(k)$, $\text{PSmVar}(k) := \text{PVar}(k) \cap \text{SmVar}(k)$, the full subcategories consisting of smooth varieties and smooth projective varieties respectively.
- Denote by $\text{AnSp}(\mathbb{C}) \subset \text{RTop}$ the subcategory of analytic spaces over \mathbb{C} , and by $\text{AnSm}(\mathbb{C}) \subset \text{AnSp}(\mathbb{C})$ the full subcategory of smooth analytic spaces (i.e. complex analytic manifold).
- For $X \in \text{Var}(k)$ and $X = \cup_{i \in I} X_i$ with $i_i : X_i \hookrightarrow X$ closed embeddings, we denote $X_{\bullet} \in \text{Fun}(\Delta, \text{Var}(k))$ the associated simplicial space, with for $J \subset I$, $i_{IJ} : X_I := \cap_{i \in I} X_i \hookrightarrow X_J := \cap_{i \in J} X_i$ the closed embedding.
- Let $(X, O_X) \in \text{RTop}$. We consider its De Rham complex $\Omega_X^{\bullet} := DR(X)(O_X)$.
 - Let $X \in \text{Sch}$. Considering its De Rham complex $\Omega_X^{\bullet} := DR(X)(O_X)$, we have for $j \in \mathbb{Z}$ its De Rham cohomology $H_{DR}^j(X) := \mathbb{H}^j(X, \Omega_X^{\bullet})$.
 - Let $X \in \text{Var}(k)$. Considering its De Rham complex $\Omega_X^{\bullet} := \Omega_{X/k}^{\bullet} := DR(X/k)(O_X)$, we have for $j \in \mathbb{Z}$ its De Rham cohomology $H_{DR}^j(X) := \mathbb{H}^j(X, \Omega_X^{\bullet})$. The differentials of $\Omega_X^{\bullet} := \Omega_{X/k}^{\bullet}$ are by definition k -linear, thus $H_{DR}^j(X) := \mathbb{H}^j(X, \Omega_X^{\bullet})$ has a structure of a k vector space.
 - Let $X \in \text{AnSp}(\mathbb{C})$. Considering its De Rham complex $\Omega_X^{\bullet} := DR(X)(O_X)$, we have for $j \in \mathbb{Z}$ its De Rham cohomology $H_{DR}^j(X) := \mathbb{H}^j(X, \Omega_X^{\bullet})$.
- For $Y \in \text{Var}(\mathbb{Q})$ and $X = V(I) = V(f_1, \dots, f_r) \subset Y_{\mathbb{C}}$, we consider $\mathcal{X} = V(\tilde{I}) = V(\tilde{f}_1, \dots, \tilde{f}_r) \subset Y \times \mathbb{A}_{\mathbb{Q}}^s$ the canonical rational deformation, where

$$\begin{aligned}
 \text{if } f &= \sum_{I \subset [1, \dots, d]^n} a_I x_1^{n_1} \cdots x_n^{n_n} \in \mathbb{C}[x_1, \dots, x_n], \\
 \tilde{f} &= \sum_{I \subset [1, \dots, d]^n} a_I x_1^{n_1} \cdots x_n^{n_n} \in \mathbb{Q}[(a_I)_{I \subset [1, \dots, d]^n}, x_1, \dots, x_n].
 \end{aligned}$$

- For $X \in \text{AnSp}(\mathbb{C})$, we denote $\alpha(X) : \mathbb{C}_X \hookrightarrow \Omega_X^\bullet$ the embedding in $C(X)$. For $X \in \text{AnSm}(\mathbb{C})$, $\alpha(X) : \mathbb{C}_X \hookrightarrow \Omega_X^\bullet$ is an equivalence usu local by Poincare lemma.
- We denote $\mathbb{I}^n := [0, 1]^n \in \text{Diff}(\mathbb{R})$ (with boundary). For $X \in \text{Top}$ and R a ring, we consider its singular cochain complex

$$C_{\text{sing}}^*(X, R) := (\mathbb{Z} \text{Hom}_{\text{Top}}(\mathbb{I}^*, X)^\vee) \otimes R$$

and for $l \in \mathbb{Z}$ its singular cohomology $H_{\text{sing}}^l(X, R) := H^n C_{\text{sing}}^*(X, R)$. For $f : X' \rightarrow X$ a continuous map with $X, X' \in \text{Top}$, we have the canonical map of complexes

$$f^* : C_{\text{sing}}^*(X, R) \rightarrow C_{\text{sing}}^*(X', R), \sigma \mapsto f^* \sigma := (\gamma \mapsto \sigma(f \circ \gamma)).$$

In particular, we get by functoriality the complex

$$C_{X, R \text{ sing}}^* \in C_R(X), (U \subset X) \mapsto C_{\text{sing}}^*(U, R)$$

We recall that

- For $X \in \text{Top}$ locally contractible, e.g. $X \in \text{CW}$, and R a ring, the inclusion in $C_R(X)$ $c_X : R_X \rightarrow C_{X, R \text{ sing}}^*$ is by definition an equivalence top local and that we get by the small chain theorem, for all $l \in \mathbb{Z}$, an isomorphism $H^l c_X : H^l(X, R_X) \xrightarrow{\sim} H_{\text{sing}}^l(X, R)$.
- For $X \in \text{Diff}(\mathbb{R})$, the restriction map

$$r_X : \mathbb{Z} \text{Hom}_{\text{Diff}(\mathbb{R})}(\mathbb{I}^*, X)^\vee \rightarrow C_{\text{sing}}^*(X, R), w \mapsto w : (\phi \mapsto w(\phi))$$

is a quasi-isomorphism by Whitney approximation theorem.

- Let $S \in \text{AnSm}(\mathbb{C})$ and $L \in \text{Shv}(S)$ a local system. Consider $E := L \otimes O_S \in \text{Vect}_{\mathcal{D}}(S)$ the corresponding holomorphic vector bundle with integrable connection ∇ . Let $\pi_S : \tilde{S} \rightarrow S$ be the universal covering. Consider the canonical fiber

$$L_0 := \Gamma(\tilde{S}, \pi_S^* L) = \Gamma(\tilde{S}, \pi_S^{* \text{mod}} E)^\nabla.$$

For $\lambda \in L_0$, we will consider the flat leaf

$$\mathbb{V}_S(\lambda) := \pi_S(\lambda \times \tilde{S}) \subset E, \pi_S : L_0 \times \tilde{S} \rightarrow E, \pi_S(\nu, z) := (\nu(z), \pi_S(z)).$$

Note that for $t \in S$, $\mathbb{V}_S(\lambda) \cap p_S^{-1}(t) = \pi_1(S, t)(\lambda)$ is the orbit of λ under the monodromy action, where $p_S : E \rightarrow S$ is the projection.

2.2 Birational projective varieties and Hodge conjecture

In this subsection, we recall that assuming the Hodge conjecture for smooth projective varieties of dimension less or equal to $d - 1$, the property of satisfying the Hodge conjecture for connected smooth projective varieties of dimension d is a birational invariant :

Proposition 1. (i) *Let $X \in \text{PSmVar}(\mathbb{C})$ connected of dimension d . Let $\epsilon : \tilde{X}_Z \rightarrow X$ be the blow up of X along a smooth closed subvariety $Z \subset X$. Assume the Hodge conjecture hold for smooth complex projective varieties of dimension less or equal to $d - 1$. Then the Hodge conjecture hold for X if and only if it hold for \tilde{X}_Z .*

(ii) *Let $X, X' \in \text{PSmVar}(\mathbb{C})$ connected of dimension d . Assume the Hodge conjecture hold for smooth complex projective varieties of dimension less or equal to $d - 1$. If X is birational to X' , then the Hodge conjecture hold for X if and only if it hold for X'*

Proof. (i): Follows from the fact that

$$(\epsilon^*, \oplus_{l=1}^c i_{E*}((-).h^{l-1})) : H^k(X, \mathbb{Q}) \oplus \oplus_{l=1}^c H^{k-2l}(E, \mathbb{Q}) \rightarrow H^k(\tilde{X}_Z, \mathbb{Q})$$

is an isomorphism of Hodge structures for each $k \in \mathbb{Z}$, where $c = \text{codim}(Z, X)$, $i_E : E \hookrightarrow \tilde{X}_Z$ is the closed embedding, $\epsilon|_E : E \rightarrow Z$ being a projective vector bundle.

(ii): Follows from (i) since if $\pi : X^\circ \rightarrow X'$ is a birational map, $X^\circ \subset X$ being an open subset, then X is connected to X' by a sequence of blow up of smooth projective varieties with smooth center by [1]. \square

2.3 A family of smooth projective hypersurfaces associated to a smooth projective variety

In this subsection, we recall that given a smooth complex projective variety X , there exists a family of smooth complex projective hypersurfaces which degenerates into a normal crossing divisor with one irreducible component birational to X .

Proposition 2. *Let $X \in \text{PSmVar}(\mathbb{C})$ connected of dimension d . Then there exists a family of smooth projective hypersurfaces $f_X : Y \rightarrow \mathbb{A}_{\mathbb{C}}^1$ with*

- $Y \in \text{SmVar}(\mathbb{C})$, f_X flat projective,
- $Y_s := f_X^{-1}(s) \subset \mathbb{P}_{\mathbb{C}}^{d+1}$ for $s \in \mathbb{A}_{\mathbb{C}}^1 \setminus 0$ are smooth projective hypersurfaces,
- $E := f_X^{-1}(0) = \cup_{i=0}^r E_i = \epsilon^{-1}(Y_0) \subset Y$ is a normal crossing divisor, where $Y_0 \subset \mathbb{P}_{\mathbb{C}}^{d+1}$ is a possibly singular hypersurface birational to X , $\epsilon : (Y, E) \rightarrow (Y', Y_0)$ a desingularization, and E_0 is birational to X ,

Proof. See [2] lemma 1.5.8 . It use the fact that X is birational to a possibly singular projective hypersurface Y_0 and consider the desingularization of a generic pencil $f'_X : Y' = V(f) \subset \mathbb{P}_{\mathbb{C}}^{d+1} \times \mathbb{A}_{\mathbb{C}}^1 \rightarrow \mathbb{A}_{\mathbb{C}}^1$ passing through Y_0 and $f_X = f'_X \circ \epsilon$. \square

2.4 The nearby cycle functor and the specialization map

Recall from [3] that for $k \subset \mathbb{C}$ a subfield and $S \in \text{Var}(k)$ quasi-projective, and $l : S \hookrightarrow \tilde{S}$ a closed embedding with $\tilde{S} \in \text{SmVar}(k)$, we have the full subcategory

$$\iota : \text{MHM}_{k, gm}(S) \hookrightarrow \text{PSh}_{\mathcal{D}fil, S}(\tilde{S}) \times_I P(S_{\mathbb{C}}^{an})$$

consisting of geometric mixed Hodge module whose De Rham part is defined over k , where, using the fact that the V -filtration is defined over k , $\text{PSh}_{\mathcal{D}(1,0)fil, S}(\tilde{S}) \times_I P_{fil}(S_{\mathbb{C}}^{an})$ is the category

- whose objects are $((M, F, W), (K, W), \alpha)$ where (M, F, W) is a filtered $D_{\tilde{S}}$ module supported on S , where $l : S \hookrightarrow \tilde{S}$ is a closed embedding and $\tilde{S} \in \text{SmVar}(k)$, (K, W) is a filtered Perverse sheaf on $S_{\mathbb{C}}^{an}$, and $\alpha : l_*(K, W) \otimes \mathbb{C} \rightarrow DR(\tilde{S})((M, W)^{an})$ is an isomorphism in $D_{fil}(\tilde{S}_{\mathbb{C}}^{an})$.
- whose morphism between $((M, F, W), (K, W), \alpha)$ and $((M', F, W), (K', W), \alpha')$, are couples of morphisms $(\phi : (M, F, W) \rightarrow (M', F, W), \psi : (K, W) \rightarrow (K', W))$ such that $\phi \circ \alpha = \alpha' \circ \psi$.

Then, by [3], we get for $S \in \text{Var}(k)$ quasi-projective and $l : S \hookrightarrow \tilde{S}$ is a closed embedding with $\tilde{S} \in \text{SmVar}(k)$ an embedding

$$\iota : D(\text{MHM}_{k, gm}(S)) \hookrightarrow D_{\mathcal{D}fil, S}(\tilde{S}) \times_I D(S_{\mathbb{C}}^{an})$$

consisting of mixed Hodge module whose De Rham part is defined over k , where $D_{\mathcal{D}(1,0)fil, S}(\tilde{S}) \times_I D_{fil, c}(S_{\mathbb{C}}^{an})$ is the category

- whose objects are $((M, F, W), (K, W), \alpha)$ where (M, F, W) is a complex of filtered $D_{\bar{S}}$ module supported on S , (K, W) is a filtered complex of presheaves on $S_{\mathbb{C}}^{an}$ whose cohomology are constructible sheaves, and $\alpha : l_*(K, W) \otimes \mathbb{C} \rightarrow DR(\tilde{S})((M, W)^{an})$ is an isomorphism in $D_{fil}(\tilde{S}_{\mathbb{C}}^{an})$.
- whose morphism are given in [3],

We have then the six functors formalism :

$$D(MHM_{k, gm}(-)) : \text{Var}(k) \mapsto \text{TriCat}, \quad S \mapsto D(MHM_{k, gm}(S)), \quad (f : T \rightarrow S) \mapsto (f^*, f_*) : D(MHM_{k, gm}(S)) \rightarrow D(MHM_{k, gm}(T)), \quad (f!, f^!) : D(MHM_{k, gm}(T)) \rightarrow D(MHM_{k, gm}(S))$$

and for $D \subset S$ a (Cartier) divisor the nearby cycle functor

$$\psi_D : D(MHM_{k, gm}(S)) \rightarrow D(MHM_{k, gm}(D)), \\ \psi_D((M, F, W), (K, W), \alpha) := (\psi_D(M, F, W), \psi_D(K, W), \psi_D(\alpha)).$$

For $T \in \text{Var}(k)$ and $\epsilon_{\bullet} : T_{\bullet} \rightarrow T$ a desingularization with $T_{\bullet} \in \text{Fun}(\Delta, \text{SmVar}(k))$, we have

$$\mathbb{Z}_T^{Hdg} := a_T^{*Hdg} \mathbb{Z}^{Hdg} = (\epsilon_{\bullet*} E_{zar}(O_{T_{\bullet}}, F, W), \epsilon_{\bullet*} E_{usu}(\mathbb{Z}_{T_{\bullet}^{an}}, W), \alpha(T_{\bullet}))$$

For $f : X \rightarrow S$ a projective morphism with $X, S \in \text{SmVar}(k)$, we denote

$$E_{Hdg}(X/S) := f_* \mathbb{Z}_X^{Hdg} := \left(\int f(O_X, F), Rf_* \mathbb{Z}_{X_{\mathbb{C}}^{an}}, f_* \alpha(X) \right) \in D(MHM_{k, gm}(S)).$$

For $f : X \hookrightarrow \mathbb{P}^N \times S \xrightarrow{p} S$ a projective morphism with $X, S \in \text{Var}(k)$, S smooth, we denote

$$E_{Hdg}(X/S) := f_* \mathbb{Z}_X^{Hdg} := \left(\int_p \Gamma_X^{Hdg}(O_{\mathbb{P}^N \times S}, F), Rf_* \mathbb{Z}_{X_{\mathbb{C}}^{an}}, f_* \alpha(X) \right) \in D(MHM_{k, gm}(S)).$$

We then have the specialization map :

Definition-Proposition 1. *Let k be field of characteristic zero.*

(i) *Let $f' : X' \rightarrow \mathbb{A}_k^1 \times S$ be a projective morphism with $X, S \in \text{SmVar}(k)$. Assume that*

- $f^o := f \otimes_{\mathbb{A}^1} (\mathbb{A}^1 \setminus 0) : X^o \rightarrow (\mathbb{A}^1 \setminus 0) \times S$ is smooth
- $X_0 := f^{-1}(0 \times S) \subset X'$. We have by definition $X^o = X' \setminus X_0$.

We have then, see [3], the distinguished triangles in $D(MHM_{k, gm}(X_0))$,

$$\mathbb{Z}_{X_0}^{Hdg} := i_{X_0}^* \mathbb{Z}_{X'}^{Hdg} \xrightarrow{sp_{X_0/X}} \psi_{X_0} \mathbb{Z}_{X^o}^{Hdg} := (\psi_{X_0}(O_{X'}, F), \psi_{X_0} \mathbb{Z}_{X^o}, \Phi_{X_0} \alpha(X')) \\ \xrightarrow{c(\psi_{X_0} \mathbb{Z}_{X^o}^{Hdg})} \phi_E \mathbb{Z}_{X^o}^{Hdg} := \text{Cone}(sp_{E/X}) \xrightarrow{c(\mathbb{Z}_{X_0}^{Hdg})} \mathbb{Z}_{X_0}^{Hdg}[1].$$

Applying the functor $(f' \circ i_{X_0})_$, we obtain the generalized distinguished triangle in $D(MHM_{k, gm}(S))$*

$$E_{Hdg}(X_0/S) \xrightarrow{sp_{X_0/f'} := (f' \circ i_{X_0})_* sp_{X_0/X}} \\ (f' \circ i_{X_0})_*(\psi_{X_0} \mathbb{Z}_{X^o}^{Hdg}) = \psi_0 E_{Hdg}(X^o / ((\mathbb{A}^1 \setminus 0) \times S)) \xrightarrow{(f' \circ i_{X_0})_* c(\Phi_{X_0} \mathbb{Z}_{X^o}^{Hdg})} \\ (f' \circ i_{X_0})_*(\phi_{X_0} \mathbb{Z}_{X^o}^{Hdg}) = \phi_0 E_{Hdg}(X^o / ((\mathbb{A}^1 \setminus 0) \times S)) \xrightarrow{(f' \circ i_{X_0})_* c(\mathbb{Z}_X^{Hdg})} E_{Hdg}(X_0/S)[1]$$

(i)' *If for each $t \in S(\mathbb{C})$, $H^{k-1}(X'_{(s,t)}, \mathbb{Q}) = 0$ for $s \in \mathbb{A}_{\mathbb{C}}^1 \setminus 0$ and $H_{X_0}^{k-1}(X'_{\mathbb{A}^1 \times t}, \mathbb{Q}) = 0$, then the induced map on cohomology*

$$sp_{X_0/f'} : E_{Hdg}^k(X_0/S) \rightarrow H^k \Phi_0 E_{Hdg}(X^o / ((\mathbb{A}^1 \setminus 0) \times S))$$

is injective.

(ii) Let $f : X \rightarrow \mathbb{A}_k^1 \times S$ be a projective morphism with $X, S \in \text{SmVar}(k)$. Assume that

- $f^\circ := f \otimes_{\mathbb{A}^1} (\mathbb{A}^1 \setminus 0) : X^\circ \rightarrow (\mathbb{A}^1 \setminus 0) \times S$ is smooth
- $E := f^{-1}(0 \times S) = \cup_{i=0}^r E_i \subset X$ is a normal crossing divisor and that for $I \subset [0, \dots, r]$, $f_{0,I} := f|_{E_I} : E_I := \cap_{i \in I} E_i \rightarrow S$ are smooth. We have by definition $X^\circ = X \setminus E$.

Denote for $I \subset J \subset [0, \dots, r]$, $i_{J,I} : E_J \hookrightarrow E_I$ and $i_I : E_I \hookrightarrow E$, $i_E : E \hookrightarrow X$ the closed embeddings. In this particular case, c.f. [8], the distinguished triangle in $D(\text{MHM}_{k, \text{gm}}(E))$ given in (i) become

$$\begin{aligned} \mathbb{Z}_E^{\text{Hdg}} &:= i_E^* \mathbb{Z}_X^{\text{Hdg}} = ((i_{\bullet, \bullet} \Omega_{E^\bullet}^\bullet, F, W), (\mathbb{Z}_{E^\bullet}, W), \alpha(E_\bullet)) \xrightarrow{sp_{E/X}} \\ \psi_E \mathbb{Z}_{X^\circ}^{\text{Hdg}} &= ((i_E^{*\text{mod}} \Omega_{X/\mathbb{A}^1 \times S}^\bullet(\log E), F, W), \Phi_E \mathbb{Z}_{X^\circ}, \Phi_E \alpha(X)) \xrightarrow{c(\psi_E \mathbb{Z}_{X^\circ}^{\text{Hdg}})} \\ \phi_E \mathbb{Z}_{X^\circ}^{\text{Hdg}} &:= \text{Cone}(sp_{E/X}) \xrightarrow{c(\mathbb{Z}_E^{\text{Hdg}})} \mathbb{Z}_E^{\text{Hdg}}[1]. \end{aligned}$$

Applying the functor $(f \circ i_E)_*$, we obtain the generalized distinguished triangle in $D(\text{MHM}_{k, \text{gm}}(S))$

$$\begin{aligned} E_{\text{Hdg}}(E/S) &\xrightarrow{sp_{E/f} := (f \circ i_E)_* sp_{E/X}} (f \circ i_E)_*(\psi_E \mathbb{Z}_{X^\circ}^{\text{Hdg}}) = \psi_0 E_{\text{Hdg}}(X^\circ / ((\mathbb{A}^1 \setminus 0) \times S)) \\ \xrightarrow{(f \circ i_E)_* c(\psi_E \mathbb{Z}_{X^\circ}^{\text{Hdg}})} &(f \circ i_E)_*(\phi_E \mathbb{Z}_{X^\circ}^{\text{Hdg}}) = \phi_0 E_{\text{Hdg}}(X^\circ / ((\mathbb{A}^1 \setminus 0) \times S)) \xrightarrow{(f \circ i_E)_* c(\mathbb{Z}_X^{\text{Hdg}})} E_{\text{Hdg}}(E/S)[1]. \end{aligned}$$

If $f = f' \circ \epsilon$ where $\epsilon : (X, E) \rightarrow (X', X_0)$ is a desingularization, we have $sp_{X_0/X'} = sp_{E/X} \circ \epsilon^*$.

Proof. (i): Follows from the exact sequence

$$\dots \rightarrow H_{X_0, t}^l(X'_{\mathbb{A}^1 \times t}, \mathbb{Q}) \rightarrow \mathbb{H}^l(X_0, t, \phi_0 \mathbb{Z}_{X_0^\circ}^o_{(\mathbb{A}^1 \setminus 0) \times t}) \xrightarrow{V} \mathbb{H}^l(X_0, t, \psi_0 \mathbb{Z}_{X_0^\circ}^o_{(\mathbb{A}^1 \setminus 0) \times t}) \rightarrow H_{X_0, t}^{l+1}(X'_{\mathbb{A}^1 \times t}, \mathbb{Q}) \rightarrow \dots$$

given in [8]. □

3 Hodge conjecture for smooth projective varieties

Theorem 1. *Let $X \in \text{PSmVar}(\mathbb{C})$. Then Hodge conjecture hold for X . That is, if $p \in \mathbb{Z}$ and $\lambda \in F^p H^{2p}(X^{\text{an}}, \mathbb{Q})$, $\lambda = [Z]$ with $Z \in \mathcal{Z}^p(X)$.*

Proof. The Hodge conjecture is true for curves and surfaces. Assume that the Hodge conjecture is true for smooth projective varieties of dimension less or equal to $d-1$. Let $X \in \text{PSmVar}(\mathbb{C})$ of dimension d . Up to split X into its connected components, we may assume that X is connected of dimension d . Then X is birational to a possibly singular hypersurface $Y_0 \subset \mathbb{P}_{\mathbb{C}}^{d+1}$. Consider by proposition 2 a family of smooth projective hypersurfaces over $\mathbb{A}_{\mathbb{C}}^1 \setminus 0$ which degenerate at 0 into a normal crossing divisor with one component birational to X , that is a projective morphism in $\text{Var}(\mathbb{C})$ $f_X : Y = V(I) \subset \mathbb{P}_{\mathbb{C}}^N \times \mathbb{A}_{\mathbb{C}}^1 \rightarrow \mathbb{A}_{\mathbb{C}}^1$ with

- $Y \in \text{SmVar}(\mathbb{C})$, f_X flat projective,
- $Y_s := f_X^{-1}(s) \subset \mathbb{P}_{\mathbb{C}}^{d+1}$ for $s \in \mathbb{A}_{\mathbb{C}}^1 \setminus 0$ are smooth projective hypersurfaces
- $f_X = f'_X \circ \epsilon$, $f'_X : Y' = V(f) \subset \mathbb{P}_{\mathbb{C}}^{d+1} \times \mathbb{A}_{\mathbb{C}}^1 \rightarrow \mathbb{A}_{\mathbb{C}}^1$ is a generic pencil of hypersurface passing through $Y_0 := f_X^{-1}(0) \subset \mathbb{P}_{\mathbb{C}}^{d+1}$ which is birational to X , $\epsilon : (Y, E) \rightarrow (Y', Y_0)$ is a desingularization of the pair (Y', Y_0) with $E := f_X^{-1}(0) = \epsilon^{-1}(Y_0) = \cup_{i=0}^r E_i \subset Y$ is a normal crossing divisor, and E_0 is birational to X ,

and its canonical rational deformation $\tilde{f}_X : \mathcal{Y} = V(\tilde{I}) \subset \mathbb{P}_{\mathbb{Q}}^N \times \mathbb{A}_{\mathbb{Q}}^1 \times S \rightarrow \mathbb{A}_{\mathbb{Q}}^1 \times S$ with

- $\mathcal{Y} \in \text{SmVar}(\mathbb{Q})$, $S \subset \mathbb{A}_{\mathbb{Q}}^r$ an open subset, \tilde{f}_X flat projective, $Y = \tilde{f}_X^{-1}(\mathbb{A}_{\mathbb{Q}}^1 \times \{t\})$, $f_X = \tilde{f}_X \times_S \{t\}$, with $t \in S(\mathbb{C})$,

- $\tilde{f}_X = \tilde{f}'_X \circ \epsilon$, with $\tilde{f}'_X : \mathcal{Y}' = V(\tilde{f}) \subset \mathbb{P}_{\mathbb{Q}}^{d+1} \times \mathbb{A}_{\mathbb{Q}}^1 \times S \rightarrow \mathbb{A}_{\mathbb{Q}}^1 \times S$ the corresponding family of hypersurfaces and $\epsilon : (\mathcal{Y}, \mathcal{E}) \rightarrow (\mathcal{Y}', \mathcal{Y}_0)$ the corresponding desingularization over \mathbb{Q} ,
- $\mathcal{E} = \cup_{i=0}^s \mathcal{E}_i := \tilde{f}_X^{-1}(\{0\} \times S) \subset \mathcal{Y}$ is a normal crossing divisor, $\tilde{f}_{X,0} : \mathcal{E}_I := \cap_{i \in I} \mathcal{E}_i \rightarrow S$ smooth projective for each $I \subset [0, \dots, s]$, $\mathcal{E}_{\mathbb{C},t} = E$,
- $\tilde{f}_X \times_{\mathbb{A}^1} (\mathbb{A}^1 \setminus 0) : \mathcal{Y}^o \rightarrow \mathbb{A}_{\mathbb{Q}}^1 \setminus 0$ has fibers which are smooth projective hypersurfaces.

We have by definition the commutative diagram in $\text{SmVar}(\mathbb{Q})$

$$\begin{array}{ccccc}
\mathcal{Y}^o & \xrightarrow{j} & \mathcal{Y} & \xleftarrow{i_{\mathcal{E}}} & \mathcal{E} \\
\tilde{f}_X^o \downarrow & & \downarrow \tilde{f}_X & & \downarrow \tilde{f}_{X,0} \\
\mathbb{A}_{\mathbb{Q}}^1 \setminus \{0\} \times S & \xrightarrow{j \times I_S} & \mathbb{A}_{\mathbb{Q}}^1 \times S & \xleftarrow{i_0 \times I_S} & \{0\} \times S
\end{array}$$

whose squares are cartesian, j the open embedding and $i_{\mathcal{E}}$ the closed embedding. Denote for $I \subset [0, \dots, s]$, $i_I : \mathcal{E}_I \hookrightarrow \mathcal{E}$ and for $I \subset J \subset [0, \dots, s]$, $i_{JI} : \mathcal{E}_J \rightarrow \mathcal{E}_I$ the closed embeddings. Then, $i_{\bullet} : \mathcal{E}_{\bullet} \rightarrow \mathcal{E}$ in $\text{Fun}(\Delta, \text{Var}(\mathbb{Q}))$ is a simplicial resolution with $\mathcal{E}_{\bullet} \in \text{Fun}(\Delta, \text{SmVar}(\mathbb{Q}))$. We have then

$$E_{Hdg}(\mathcal{E}/S) := (\tilde{f}_X \circ i_{\bullet*} E_{zar}(\Omega_{\mathcal{E}_{\bullet}/S}^{\bullet}, F_b, W), (\tilde{f}_X \circ i_{\bullet*} E_{usu}(\mathbb{Q}_{\mathcal{E}_{\bullet}}^{an}, W)), \tilde{f} \circ i_{\bullet*} \alpha(\mathcal{E}_{\bullet})) \in D(MHM_{\mathbb{Q},gm}(S))$$

Then, definition-proposition 1 applied to $\tilde{f}'_X : \mathcal{Y}' \rightarrow \mathbb{A}_{\mathbb{Q}}^1 \times S$ gives the distinguish in $D(MHM_{\mathbb{Q},gm}(S))$,

$$\begin{aligned}
& E_{Hdg}(\mathcal{Y}_0/S) \xrightarrow{Sp_{\mathcal{Y}_0/\tilde{f}'_X} := (\tilde{f}'_X \circ i_{\mathcal{Y}_0})_* Sp_{\mathcal{Y}_0/\mathcal{Y}'}} \psi_0 E_{Hdg}(\mathcal{Y}^o / ((\mathbb{A}^1 \setminus 0) \times S)) \\
& \xrightarrow{(\tilde{f}'_X \circ i_{\mathcal{Y}_0})_c (\Phi_{\mathcal{Y}_0} \mathbb{Z}_{\mathcal{Y}_0}^{Hdg})} \phi_0 E_{Hdg}(\mathcal{Y}^o / ((\mathbb{A}^1 \setminus 0) \times S)) \xrightarrow{(\tilde{f}'_X \circ i_{\mathcal{Y}_0})_* c(\mathbb{Z}_{\mathcal{Y}_0}^{Hdg})} E_{Hdg}(\mathcal{Y}_0/S)[1]
\end{aligned}$$

and definition-proposition 1 applied to $\tilde{f}_X : \mathcal{Y} \rightarrow \mathbb{A}_{\mathbb{Q}}^1 \times S$ gives the distinguish in $D(MHM_{\mathbb{Q},gm}(S))$,

$$\begin{aligned}
& E_{Hdg}(\mathcal{E}/S) \xrightarrow{Sp_{\mathcal{E}/\tilde{f}_X} := (\tilde{f}_X \circ i_{\mathcal{E}})_* Sp_{\mathcal{E}/\mathcal{Y}}} \psi_0 E_{Hdg}(\mathcal{Y}^o / ((\mathbb{A}^1 \setminus 0) \times S)) \\
& \xrightarrow{(\tilde{f}_X \circ i_{\mathcal{E}})_c (\Phi_{\mathcal{E}} \mathbb{Z}_{\mathcal{Y}_0}^{Hdg})} \phi_0 E_{Hdg}(\mathcal{Y}^o / ((\mathbb{A}^1 \setminus 0) \times S)) \xrightarrow{(\tilde{f}_X \circ i_{\mathcal{E}})_* c(\mathbb{Z}_{\mathcal{E}}^{Hdg})} E_{Hdg}(\mathcal{E}/S)[1]
\end{aligned}$$

and $Sp_{\mathcal{E}/\tilde{f}_X} \circ \epsilon^* = Sp_{\mathcal{Y}_0/\tilde{f}'_X}$. Let $p \in \mathbb{Z}$. By the hard Lefschetz theorem on $H^*(E_0)$, we may assume that $2p \leq d$. Taking the cohomology to this distinguish triangle, we get in particular, for each $p \in \mathbb{Z}$, the maps in $MHM_{\mathbb{Q},gm}(S)$

- $Sp_{\mathcal{Y}_0/\tilde{f}'_X} : E_{Hdg}^{2p}(\mathcal{Y}_0/S) \hookrightarrow H^{2p} \psi_0 E_{Hdg}(\mathcal{Y}^o / ((\mathbb{A}_{\mathbb{Q}}^1 \setminus 0) \times S))$
- $Sp_{\mathcal{E}/\tilde{f}_X} : E_{Hdg}^{2p}(\mathcal{E}/S) \rightarrow H^{2p} \psi_0 E_{Hdg}(\mathcal{Y}^o / ((\mathbb{A}_{\mathbb{Q}}^1 \setminus 0) \times S))$.

The map $Sp_{\mathcal{Y}_0/\tilde{f}'_X}$ is injective by definition-proposition 1(i)' and the weak Lefschetz theorem since for $t' \in S(\mathbb{C})$ and $s \in \mathbb{A}_{\mathbb{C}}^1 \setminus 0$, $\mathcal{Y}_{(s,t')}$ and $\mathcal{Y}_{0,t'}$ are projective hypersurfaces. For $\lambda \in H^{2p}(\mathcal{E}_{0,\mathbb{C},t'}^{an}, \mathbb{Q})$, $t' \in S(\mathbb{C})$, we consider $\pi_S : \tilde{S}_{\mathbb{C}}^{an} \rightarrow S_{\mathbb{C}}^{an}$ the universal covering in $\text{AnSm}(\mathbb{C})$ and

$$\mathbb{V}_S(\lambda) := \pi_S(\lambda \times \tilde{S}_{\mathbb{C}}^{an}) \subset E_{DR}^{2p,an}(\mathcal{E}_{0,\mathbb{C}}/S_{\mathbb{C}})$$

the flat leaf induced by the universal covering (see section 2.1) and

$$\mathbb{V}_S^p(\lambda) := \mathbb{V}_S(\lambda) \cap F^p E_{DR}^{2p}(\mathcal{E}_{0,\mathbb{C}}/S_{\mathbb{C}}) \subset E_{DR}^{2p}(\mathcal{E}_{0,\mathbb{C}}/S_{\mathbb{C}}),$$

which is an algebraic variety by [6]. Note that $\mathbb{V}_S^p(\lambda) \subset HLP^{p,2p}(\mathcal{E}_{0,\mathbb{C}}/S_{\mathbb{C}})$ is the union of the irreducible components passing through λ . Similarly, for $\eta \in H^{2p}(\mathcal{E}_{\mathbb{C},t'}^{an}, \mathbb{Q})$, $t' \in S(\mathbb{C})$, we consider $\pi_S : \tilde{S}_{\mathbb{C}}^{an} \rightarrow S_{\mathbb{C}}^{an}$ the universal covering in $\text{AnSm}(\mathbb{C})$ and

$$\mathbb{V}_S(\eta) := \pi_S(\eta \times \tilde{S}_{\mathbb{C}}^{an}) \subset E_{DR}^{2p,an}(\mathcal{E}_{\mathbb{C}}/S_{\mathbb{C}})$$

the flat leaf induced by the universal covering (see section 2.1) and

$$\mathbb{V}_S^p(\eta) := \mathbb{V}_S(\eta) \cap F^p E_{DR}^{2p}(\mathcal{E}_C/S_C) \subset E_{DR}^{2p}(\mathcal{E}_C/S_C).$$

We have

$$H^{2p} E_{Hdg}(\mathcal{Y}^o/((\mathbb{A}_\mathbb{Q}^1 \setminus 0) \times S)) = ((E_{DR}^{2p}(\mathcal{Y}^o/((\mathbb{A}_\mathbb{Q}^1 \setminus 0) \times S)), F), R^{2p} \tilde{f}_{X*} \mathbb{Z}_{\mathcal{Y}^o}, \tilde{f}_{X*} \alpha(\mathcal{Y}^o)) \in MHM_{\mathbb{Q}, gm}(S)$$

By [8], the filtered vector bundle

$$(E_{DR}^{2p}(\mathcal{Y}^o/((\mathbb{A}_\mathbb{Q}^1 \setminus 0) \times S)), F) = H^{2p} \tilde{f}_{X*} E_{zar}(\Omega_{\mathcal{Y}^o/(\mathbb{A}_\mathbb{Q}^1 \setminus 0) \times S}^\bullet) \in \text{Vect}_{\mathcal{D}fil}(S)$$

extend to a filtered vector bundle

$$(E_{DR, V_0}^{2p}(\mathcal{Y}/(\mathbb{A}_\mathbb{Q}^1 \times S)), F) = H^{2p} \tilde{f}_{X*} E_{zar}(\Omega_{\mathcal{Y}/\mathbb{A}_\mathbb{Q}^1 \times S}^\bullet(\log \mathcal{E})) \in \text{Vect}_{\mathcal{D}fil}(S)$$

such that

$$H^{2p} \psi_0 E_{Hdg}(\mathcal{Y}^o/((\mathbb{A}_\mathbb{Q}^1 \setminus 0) \times S)) = (i_0^{*mod}(E_{DR, V_0}^{2p}(\mathcal{Y}/(\mathbb{A}_\mathbb{Q}^1 \times S)), F), (\psi_0 R^{2p} \tilde{f}_{X*} \mathbb{Z}_{\mathcal{Y}^o}, W), \Phi_0 \tilde{f}_{X*} \alpha(\mathcal{Y}^o)).$$

For

$$\alpha \in H^{2p}(\mathcal{Y}_{(s, t')}^{an}, \mathbb{Q}) = \mathbb{H}^{2p}(\mathcal{E}_{t'}^{an}, \psi_0 R \tilde{f}_{X*} \mathbb{Q}_{\mathcal{Y}^o}), (s, t') \in (\mathbb{A}_\mathbb{C}^1 \setminus 0) \times S_C$$

we consider

- $\pi_{S, G_m} = \pi_S \times \pi_{G_m} : \tilde{G}_m \times \tilde{S}_C^{an} \rightarrow (\mathbb{A}_\mathbb{C}^1 \setminus 0) \times S_C^{an}$ the universal covering in $\text{AnSm}(\mathbb{C})$ and

$$\mathbb{V}_{(\mathbb{A}^1 \setminus 0) \times S}(\alpha) := \pi_{S, G_m}(\alpha \times \tilde{G}_m \times \tilde{S}_C^{an}) \subset E_{DR}^{2p, an}(\mathcal{Y}_C^o/(\mathbb{A}_\mathbb{C}^1 \setminus 0) \times S_C)$$

the flat leaf (see section 2.1) and

$$\mathbb{V}_{(\mathbb{A}^1 \setminus 0) \times S}^p(\alpha) := \mathbb{V}_{(\mathbb{A}^1 \setminus 0) \times S}(\alpha) \cap F^p E_{DR}^{2p}(\mathcal{Y}_C^o/(\mathbb{A}_\mathbb{C}^1 \setminus 0) \times S_C) \subset E_{DR}^{2p}(\mathcal{Y}_C^o/(\mathbb{A}_\mathbb{C}^1 \setminus 0) \times S_C),$$

which is an algebraic subvariety by [6],

- $\pi_S : \tilde{S}_C^{an} \rightarrow S_C^{an}$ the universal covering in $\text{AnSm}(\mathbb{C})$ and

$$\mathbb{V}_S(\alpha) := \pi_S(\alpha \times \tilde{S}_C^{an}) \subset H^{2p} \psi_0 E_{DR}(\mathcal{Y}_C^o/((\mathbb{A}_\mathbb{C}^1 \setminus 0) \times S_C))^{an}$$

the flat leaf (see section 2.1) and

$$\mathbb{V}_S^p(\alpha) := \mathbb{V}_S(\alpha) \cap F^p H^{2p} \psi_0 E_{DR}(\mathcal{Y}_C^o/((\mathbb{A}_\mathbb{C}^1 \setminus 0) \times S_C))^{an} \subset H^{2p} \psi_0 E_{DR}(\mathcal{Y}_C^o/((\mathbb{A}_\mathbb{C}^1 \setminus 0) \times S_C))^{an}.$$

Consider the exact sequence of variation of mixed Hodge structure over S

$$0 \rightarrow W_{2p-2} E_{Hdg}^{2p}(\mathcal{E}/S) \rightarrow E_{Hdg}^{2p}(\mathcal{E}/S) \xrightarrow{(i_i^*)_{0 \leq i \leq r} := q} \oplus_{i=0}^r E_{Hdg}^{2p}(\mathcal{E}_i/S) \rightarrow 0.$$

It induces the exact sequence of vector bundles over S

$$0 \rightarrow W_{2p-2} E_{DR}^{2p}(\mathcal{E}/S) \rightarrow E_{DR}^{2p}(\mathcal{E}/S) \xrightarrow{(i_i^*)_{0 \leq i \leq r} := q} \oplus_{i=0}^r E_{DR}^{2p}(\mathcal{E}_i/S) \rightarrow 0,$$

and the exact sequence of presheaves on S_C^{an} ,

$$J(W_{2p-2} E_{Hdg}^{2p}(\mathcal{E}_C/S_C)) := \text{Ext}^1(\mathbb{Q}_S^{Hdg}, W_{2p-2} E_{Hdg}^{2p}(\mathcal{E}_C/S_C)), \xrightarrow{L} \text{Ext}^1(\mathbb{Q}_S^{Hdg}, W_{2p-2} E_{Hdg}^{2p}(\mathcal{E}_C/S_C))$$

$$HL^{2p, p}(\mathcal{E}_C/S_C) \xrightarrow{q} \oplus_{i=0}^r HL^{2p, p}(\mathcal{E}_{i, C}/S_C) \xrightarrow{e}$$

where $HL^{2p, p}(\mathcal{E}_C/S_C) \subset E_{DR}^{2p}(\mathcal{E}_C/S_C)$, $HL^{2p, p}(\mathcal{E}_{i, C}/S_C) \subset E_{DR}^{2p}(\mathcal{E}_{i, C}/S_C)$ are the locus of Hodge classes. Let $\lambda \in F^p H^{2p}(E_0^{an}, \mathbb{Q})$, where we recall $E_0 = \mathcal{E}_{0, C, t}$. Consider then

- $\tilde{\lambda} \in H^{2p}(E^{an}, \mathbb{Q})$, such that $q(\tilde{\lambda}) = \lambda$, and $\tilde{\lambda} := Sp_{E/f_X}(\tilde{\lambda}) \in Sp_{E/f_X}(H^{2p}(E^{an}, \mathbb{Q}))$,
- $\tilde{\lambda}_F \in F^p H^{2p}(E^{an}, \mathbb{C})$, such that $q(\tilde{\lambda}_F) = \lambda$,
- $e_\lambda := \tilde{\lambda} - \tilde{\lambda}_F \in W_{2p-2} H^{2p}(E^{an}, \mathbb{C})$, $e_\lambda \in \Gamma(\tilde{S}_\mathbb{C}^{an}, \pi^{*mod} E_{DR}^{2p,an}(\mathcal{E}_\mathbb{C}/S_\mathbb{C}))^\nabla$ the corresponding section where $\pi_S : \tilde{S}_\mathbb{C}^{an} \rightarrow S_\mathbb{C}^{an}$ is the universal covering, and

$$\bar{e}_\lambda := [e_\lambda] \in \Gamma(\tilde{S}_\mathbb{C}^{an}, \pi^{*mod} E_{DR}^{2p,an}(\mathcal{E}_\mathbb{C}/S_\mathbb{C})/F^p \oplus \mathbb{Q})^\nabla.$$

The locus of Hodge classes passing through λ is then an union of two components

$$\mathbb{V}_S^p(\lambda) := \mathbb{V}_S(\lambda) \cap F^p E_{DR}^{2p}(\mathcal{E}_{0,\mathbb{C}}/S_\mathbb{C}) = q(\mathbb{V}_S^p(\tilde{\lambda})) \cup \mathbb{V}_{V(\bar{e}_\lambda)}(\lambda) \subset E_{DR}^{2p}(\mathcal{E}_{0,\mathbb{C}}/S_\mathbb{C}),$$

where

- $\mathbb{V}_{V(\bar{e}_\lambda)}(\lambda) = \pi_S(\lambda \times V(\bar{e}_\lambda)) \subset E_{DR}^{2p,an}(\mathcal{E}_{0,\mathbb{C}}/S_\mathbb{C})$ is the flat leaf, where $\pi_S : H^{2p}(E^{an}, \mathbb{C}) \times \tilde{S}_\mathbb{C}^{an} \rightarrow E_{DR}^{2p,an}(\mathcal{E}_\mathbb{C}/S_\mathbb{C})$ is the morphism induced by the universal covering $\pi_S : \tilde{S}_\mathbb{C}^{an} \rightarrow S_\mathbb{C}^{an}$,
- $V(\bar{e}_\lambda) := \pi_S(V(\bar{e}_\lambda)) \subset S_\mathbb{C}$, where $V(\bar{e}_\lambda) \subset \tilde{S}_\mathbb{C}^{an}$ is the zero locus of \bar{e}_λ ,
- $q := q \otimes \mathbb{C} : E_{DR}^{2p}(\mathcal{E}_\mathbb{C}/S_\mathbb{C}) \rightarrow E_{DR}^{2p}(\mathcal{E}_{0,\mathbb{C}}/S_\mathbb{C})$ is the quotient map.

Indeed, let $\lambda_{t'} \in \mathbb{V}_S(\lambda)$, $t' \in \tilde{S}_\mathbb{C}^{an}$. If $\lambda_{t'} \in \mathbb{V}_S^p(\lambda)$, then $\iota(e(\lambda_{t'})) = 0$. If $\lambda_{t'} \in \mathbb{V}_S^p(\lambda) \setminus q(\mathbb{V}_S^p(\tilde{\lambda}))$, then

$$[e_\lambda(t')] = e(\lambda_{t'}) \neq 0 \in W_{2p-2} H^{2p}(\mathcal{E}_{\pi(t')}, \mathbb{C}) / ((W_{2p-2} \cap F^p) \oplus W_{2p-2} \mathbb{Q})$$

Hence $\lambda_{t'} \in \mathbb{V}_S^p(\lambda) \setminus q(\mathbb{V}_S^p(\tilde{\lambda}))$

$$[e_\lambda(t')] = \iota(e(\lambda_{t'})) = 0 \in H^{2p}(\mathcal{E}_{\pi(t')}, \mathbb{C}) / (F^p \oplus \mathbb{Q}).$$

Now,

- if $Sp_{E/f_X}(\tilde{\lambda}) = 0$, since $Sp_{\mathcal{E}/\tilde{f}_X} \circ \epsilon^* = Sp_{\mathcal{Y}_0/\tilde{f}'_X}$ and $Sp_{\mathcal{Y}_0/\tilde{f}'_X}$ is injective, we have $\tilde{\lambda}_{t'} = k_{t_*} \tilde{\lambda}_{t'}$ with $\tilde{\lambda}_{t'} \in F^p H_{\mathcal{F}_c, t'}^{2p}(\mathcal{E}_{\mathbb{C}, t'}^{an}, \mathbb{Q})$, where $k : \mathcal{F} \rightarrow \mathcal{E}$ is the exceptional locus of $\epsilon : \mathcal{E}_0 \rightarrow \mathcal{Y}_0$ (note that $\epsilon^* : H^{2p}(\mathcal{Y}_{0, t'}^{an}, \mathbb{Q}) \rightarrow H^{2p}(\mathcal{E}_{0, t'}^{an}, \mathbb{Q})$ is injective by Poincare Lefschetz duality), hence, assuming the Hodge conjecture for smooth projective varieties of dimension less or equal to $d-1$, $\mathbb{V}_S^p(\tilde{\lambda}) \subset E_{DR}^{2p}(\mathcal{E}_\mathbb{C}/S_\mathbb{C})$ is defined over \mathbb{Q} and its Galois conjugates are also components of the locus of Hodge classes, if $Sp_{E/f_X}(\tilde{\lambda}) \neq 0$, we have,

$$Sp_{\mathcal{E}/\tilde{f}_X}(\mathbb{V}_S^p(\tilde{\lambda})) = \mathbb{V}_S^p(Sp_{E/f_X}(\tilde{\lambda})) = \overline{\mathbb{V}_{(\mathbb{A}^1 \setminus 0) \times S}(Sp_{E/f_X}(\tilde{\lambda}))} \cap p^{-1}(0 \times S_\mathbb{C}) \subset E_{DR, V_0}^{2p}(\mathcal{Y}_\mathbb{C}/(\mathbb{A}_\mathbb{C}^1 \times S_\mathbb{C}))$$

where $p : E_{DR, V_0}^{2p}(\mathcal{Y}_\mathbb{C}/(\mathbb{A}_\mathbb{C}^1 \times S_\mathbb{C})) \rightarrow \mathbb{A}_\mathbb{C}^1 \times S_\mathbb{C}$ is the projection (the first equality is obvious since $Sp_{\mathcal{E}/\tilde{f}_X}$ is a morphism of variation of mixed Hodge structures over S , whereas for the second equality the inclusion \supset follows from [6] lemma 2.11 which state the invariance of $\tilde{\lambda}$ under the monodromy

$$\text{Im}(\pi_1(p(\mathbb{V}^p(\tilde{\lambda})) \cap ((\mathbb{A}_\mathbb{C}^1 \setminus 0) \times D^*)) \rightarrow \pi_1((\mathbb{A}_\mathbb{C}^1 \setminus 0) \times D^*)), (D, D^*) \rightarrow (\mathbb{A}_\mathbb{C}^r, S_\mathbb{C}^{an}))$$

but by [5] theorem 1, the Hodge conjecture holds for projective hypersurfaces hence

$$\mathbb{V}_{(\mathbb{A}^1 \setminus 0) \times S}^p(Sp_{E/f_X}(\tilde{\lambda})) \subset E_{DR}^{2p}(\mathcal{Y}_\mathbb{C}/((\mathbb{A}_\mathbb{C}^1 \setminus 0) \times S_\mathbb{C}))$$

is defined over \mathbb{Q} and its Galois conjugates are also components of the locus of Hodge classes, hence $\mathbb{V}_S^p(\tilde{\lambda}) \subset E_{DR}^{2p}(\mathcal{E}_\mathbb{C}/S_\mathbb{C})$ is defined over \mathbb{Q} and its Galois conjugates are also components of the locus of Hodge classes,

- if $Sp_{E/f_X}(e_\lambda) = 0$, $Sp_{E/f_X}(\tilde{\lambda}) = Sp_{E/f_X}(\tilde{\lambda}_F)$ and for $t' \in \tilde{S}_\mathbb{C}^{an}$, $e_\lambda(t') = 0$ if and only if $e(\tilde{\lambda}_{t'}) = 0$ if and only if $Sp_{\mathcal{E}_{t'}/(\tilde{f}_X)_{t'}}(\tilde{\lambda}_{t'})$ is a Hodge class, if $Sp_{E/f_X}(e_\lambda) \neq 0$, consider

$$Sp_{E/f_X}(e_\lambda) \in \Gamma(\tilde{G}_m \times \tilde{S}_\mathbb{C}^{an}, \pi_{S, G_m}^{*mod} E_{DR}(\mathcal{Y}_\mathbb{C}^o / ((\mathbb{A}_\mathbb{C}^1 \setminus 0) \times S_\mathbb{C}))^{an})^\nabla$$

and

$$[Sp_{E/f_X}(e_\lambda)] \in \Gamma(\tilde{G}_m \times \tilde{S}_\mathbb{C}^{an}, \pi_{S, G_m}^{*mod} E_{DR}(\mathcal{Y}_\mathbb{C}^o / ((\mathbb{A}_\mathbb{C}^1 \setminus 0) \times S_\mathbb{C}))^{an})^\nabla / (F^p \oplus \mathbb{Q}),$$

we may assume that X is not defined over $\bar{\mathbb{Q}}$ (otherwise Y_0 is defined over $\bar{\mathbb{Q}}$ and S is a finite set), we may also assume that X is not rational i.e. $l_X = \deg(\mathbb{Q}(X)/\mathbb{Q}(x_1, \dots, x_d)) > 1$, then $((\mathbb{A}^1 \setminus 0) \times S)_\mathbb{C} = \mathbb{P}(l_X)$ where $\mathbb{P}(l_X)$ is the space of hypersurface over degree l_X , consider then a Lefschetz pencil $L \subset ((\mathbb{A}^1 \setminus 0) \times S)_\mathbb{C} = \mathbb{P}(l_X)$ and denote $L^\circ := L \cap (\mathbb{A}^1 \setminus 0) \times S)_\mathbb{C}$, we have the factorization $\pi_L : \tilde{L}^\circ \xrightarrow{i_L} \pi_{S, G_m}^{-1}(L) \xrightarrow{\pi_{S, G_m}} L$ of the universal covering of L° and consider

$$Sp_{E/f_X}(\widehat{e_\lambda})_{|\tilde{L}^\circ} \in \Gamma(L^\circ, E_{DR}^{2p}(\tilde{Y}'_{L^\circ}/L^\circ)), u \in \tilde{L}^\circ \mapsto Sp_{E/f_X}(\widehat{e_\lambda})(u) := e^{2i\pi N u} Sp_{E/f_X}(e_\lambda)(u),$$

for $u \in \tilde{L}^\circ$, if $Sp_{E/f_X}(e_\lambda)(u) = 0 \in H^{2p}(\mathcal{Y}_u, \mathbb{C}) / (F^p \oplus \mathbb{Q})$ then $Sp_{E/f_X}(\widehat{e_\lambda})_{|\tilde{L}^\circ}(u) = 0 \in H^{2p}(\mathcal{Y}_u, \mathbb{C}) / (F^{p+1} \oplus \mathbb{Q})$, consider

$$[Sp_{E/f_X}(\widehat{e_\lambda})_{|\tilde{L}^\circ}] \in \Gamma(L^\circ, E_{DR}^{2p}(\tilde{Y}'_{L^\circ}/L^\circ) / F^{p+1} \oplus \mathbb{Q}),$$

since $H^{2p+1}(\mathcal{Y}'_L, \mathbb{Q})_{pr} = 0$ as $\mathcal{Y}'_L = \mathbb{P}^N_{B(L)}$ is the blow up of \mathbb{P}^N in the base locus of L , we get $H^1(L^\circ, R^{2p}(\tilde{f}_X)_{|L^*} \mathbb{Q}) = 0$, hence

$$[Sp_{E/f_X}(\widehat{e_\lambda})_{|\tilde{L}^\circ}] \in H^{2p}(\mathcal{Y}'_L, \mathbb{C}) / (F^{p+1} \oplus \mathbb{Q})$$

is a constant which is non zero by the irreducibility of the monodromy representation, hence $V([Sp_{E/f_X}(e_\lambda)]) \cap i_L(\tilde{L}^\circ) = \emptyset$, hence $V(\tilde{e}_\lambda) = \emptyset$ since $\tilde{G}_m \times \tilde{S}_\mathbb{C}^{an} = \cup_{L \subset \mathbb{P}(l_X)} i_L(\tilde{L}^\circ)$.

Hence $\mathbb{V}_S^p(\lambda) \subset E_{DR}^{2p}(\mathcal{E}_{0\mathbb{C}}/S_\mathbb{C})$ is defined over $\bar{\mathbb{Q}}$ and its Galois conjugates are also components of the locus of Hodge classes. Thus, by [4] theorem 4,

$$\lambda = [Z] \in H^{2p}(E_0^{an}, \mathbb{Q}), \text{ with } Z \in \mathcal{Z}^p(E_0).$$

Since $p \in \mathbb{Z}$ and $\lambda \in F^p H^{2p}(E_0^{an}, \mathbb{Q})$ are arbitrary, the Hodge conjecture holds for E_0 . Proposition 1(ii) and the fact that the Hodge conjecture is true for smooth projective varieties of dimension less or equal to $d-1$ by induction hypothesis then implies that Hodge conjecture holds for X , since X is birational to E_0 and $X, E_0 \in \text{PSmVar}(\mathbb{C})$ are connected of dimension d . \square

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LAGA UMR CNRS 7539
Université Paris 13, Sorbonne Paris Cité, 99 av Jean-Baptiste Clement,
93430 Villetaneuse, France,
bouali@math.univ-paris13.fr