# Degeneration of families of projective hypersurfaces and Hodge conjecture 

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# Degeneration of families of projective hypersurfaces and Hodge conjecture 

Johann Bouali

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#### Abstract

We prove by induction on dimension the Hodge conjecture for smooth complex projective varieties. Let $X$ be a smooth complex projective variety. Then $X$ is birational to a possibly singular projective hypersurface, hence to a smooth projective variety $E_{0}$ which is a component of a normal crossing divisor $E=\cup_{i=0}^{r} E_{i} \subset Y$ which is the singular fiber of a pencil $f: Y \rightarrow \mathbb{A}^{1}$ of smooth projective hypersurfaces. Using the smooth hypersurface case ([5] theorem 1), the nearby cycle functor on mixed Hodge module with rational de Rham factor and the induction hypothesis, we prove that a Hodge class of $E_{0}$ is absolute Hodge, more precisely the locus of Hodge classes inside the algebraic vector bundle given the De Rham cohomology of the rational deformation of $E_{0}$ is defined over $\mathbb{Q}$. By [4] theorem 4, we get the Hodge conjecture for $E_{0}$. By the induction hypothesis we also have the Hodge conjecture for $X$.


## Contents

## 1 Introduction 1

2 Preliminaries and notations 3
2.1 Notations . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 3
2.2 Birational projective varieties and Hodge conjecture . . . . . . . . . . . . . . . . . . . . . 5
2.3 A family of smooth projective hypersurfaces associated to a smooth projective variety . . 5
2.4 The nearby cycle functor and the specialization map . . . . . . . . . . . . . . . . . . . . . 6

3 Hodge conjecture for smooth projective varieties 8

## 1 Introduction

The main result of this article is an inductive proof of the Hodge conjecture using the results of [4] and [5]. More precisely, assuming the Hodge conjecture for smooth projective varieties of dimension less or equal to $d-1$, the property of satisfying the Hodge conjecture for connected smooth projective varieties of dimension $d$ is a birational invariant (proposition 1). On the other hand, if $X$ is a connected smooth projective variety of dimension $d$, there exists (see proposition 2) a family of smooth projective hypersurfaces $f_{X}: Y \rightarrow \mathbb{A}_{\mathbb{C}}^{1}$ with

- $Y$ a smooth complex algebraic variety, $f_{X}$ flat projective,
- $Y_{s}:=f_{X}^{-1}(s) \subset \mathbb{P}_{\mathbb{C}}^{d+1}$ for $s \in \mathbb{A}_{\mathbb{C}}^{1} \backslash\{0\}$ are smooth projective hypersurfaces,
- $E:=f_{X}^{-1}(0)=\cup_{i=0}^{r} E_{i}=\epsilon^{-1}\left(Y_{0}\right) \subset Y$ is a normal crossing divisor, $Y_{0} \subset \mathbb{P}_{\mathbb{C}}^{d+1}$ is a possibly singular hypersurface birational to $X, \epsilon:(Y, E) \rightarrow\left(Y^{\prime}, Y_{0}\right)$ is a desingularization and $E_{0}$ is birational to $X$.

Assuming the Hodge conjecture for smooth projective varieties of dimension less or equal to $d-1$, we then prove the Hodge conjecture for the $E_{i}$, more specifically for $E_{0}$ as follows: We consider its canonical rational deformation $\tilde{f}_{X}: \mathcal{Y} \rightarrow \mathbb{A}_{\mathbb{Q}}^{1} \times S$ with

- $\mathcal{Y}$ a smooth algebraic variety over $\mathbb{Q}, S \subset \mathbb{A}_{\mathbb{Q}}^{s}$ an open subset, $\tilde{f}_{X}$ flat projective
- $Y=\tilde{f}_{X}^{-1}\left(\mathbb{A}_{\mathbb{Q}}^{1} \times\{t\}\right), f_{X}=\tilde{f}_{X} \times_{S}\{t\}$, with $t \in S(\mathbb{C})$
- $\mathcal{E}=\cup_{i=0}^{r} \mathcal{E}_{i}:=\tilde{f}_{X}^{-1}(\{0\} \times S) \subset \mathcal{Y}$ is a normal crossing divisor, $\tilde{f}_{X, 0}: \mathcal{E}_{I}:=\cap_{i \in I} \mathcal{E}_{i} \rightarrow S$ smooth projective for each $I \subset[0, \ldots, r], \mathcal{E}_{\mathbb{C}, t}=E$,
- $\tilde{f}_{X} \times_{\mathbb{A}^{1}}\left(\mathbb{A}^{1} \backslash 0\right): \mathcal{Y}^{o} \rightarrow \mathbb{A}_{\mathbb{Q}}^{1} \backslash\{0\}$ has fibers which are smooth projective hypersurfaces.

Let $p \in \mathbb{Z}$. By the hard Lefchetz theorem on $H^{*}\left(E_{0}\right)$, we may assume that $2 p \leq d$. We then have then a morphism of mixed Hodge module over $S$ (in fact of geometric variation of mixed Hodge structures over $S$ ) given by the specialization map

$$
S p_{\mathcal{E} / \tilde{f}_{X}}: E_{H d g}^{2 p}(\mathcal{E} / S) \rightarrow H^{2 p} \Phi_{0} E_{H d g}\left(\mathcal{Y}^{o} /\left(\left(\mathbb{A}_{\mathbb{Q}}^{1} \backslash\{0\}\right) \times S\right)\right)
$$

whose de Rham part are morphism of filtered vector bundle over $S$ with Gauss-Manin connexion defined over $\mathbb{Q}$, and whose kernel of the part supported on $\mathcal{E}_{0}$ is supported a closed subset $\mathcal{F} \subset \mathcal{E}_{0}$ with $\operatorname{codim}\left(\mathcal{F}_{t^{\prime}}, \mathcal{E}_{0, t^{\prime}}\right)=1$ for all $t^{\prime} \in S(\mathbb{C})$. The fact that the kernel is supported on a closed subset of codimension one on fibers follows from the fact that $H^{2 p-1}\left(Y_{s}, \mathbb{Q}\right)=0=H_{Y_{0}}^{2 p-1}(Y)$, by the weak Lefchetz theorem since $Y_{s}$ and $Y_{0}$ are projective hypersurfaces $(2 p \leq d)$. Consider the exact sequence of variation of mixed Hodge structure over $S$

$$
0 \rightarrow W_{2 p-2} E_{H d g}^{2 p}(\mathcal{E} / S) \rightarrow E_{H d g}^{2 p}(\mathcal{E} / S) \xrightarrow{\left(i_{i}^{*}\right)_{0 \leq i \leq r:=q}} \oplus_{i=0}^{r} E_{H d g}^{2 p}\left(\mathcal{E}_{i} / S\right) \rightarrow 0
$$

It induces the exact sequence of presheaves on $S_{\mathbb{C}}^{a n}$,

$$
\begin{array}{r}
H L^{2 p, p}\left(\mathcal{E}_{\mathbb{C}} / S_{\mathbb{C}}\right) \xrightarrow{q} \oplus_{i=0}^{r} H L^{2 p, p}\left(\mathcal{E}_{i, \mathbb{C}} / S_{\mathbb{C}}\right) \xrightarrow{e} \\
J\left(W_{2 p-2} E_{H d g}^{2 p}\left(\mathcal{E}_{\mathbb{C}} / S_{\mathbb{C}}\right)\right):=\operatorname{Ext}^{1}\left(\mathbb{Q}_{S}^{H d g}, W_{2 p-2} E_{H d g}^{2 p}\left(\mathcal{E}_{\mathbb{C}} / S_{\mathbb{C}}\right)\right) \xrightarrow{\iota} \operatorname{Ext}^{1}\left(\mathbb{Q}_{S}^{H d g}, E_{H d g}^{2 p}\left(\mathcal{E}_{\mathbb{C}} / S_{\mathbb{C}}\right)\right)
\end{array}
$$

where $H L^{2 p, p}\left(\mathcal{E}_{\mathbb{C}} / S_{\mathbb{C}}\right) \subset E_{D R}^{2 p}\left(\mathcal{E}_{\mathbb{C}} / S_{\mathbb{C}}\right), H L^{2 p, p}\left(\mathcal{E}_{i, \mathbb{C}} / S_{\mathbb{C}}\right) \subset E_{D R}^{2 p}\left(\mathcal{E}_{i, \mathbb{C}} / S_{\mathbb{C}}\right)$ are the locus of Hodge classes. Let $\lambda \in F^{p} H^{2 p}\left(E_{0}^{a n}, \mathbb{Q}\right)$, where we recall $E_{0}=\mathcal{E}_{0, \mathbb{C}, t}$. Consider then

- $\tilde{\lambda} \in H^{2 p}\left(E^{a n}, \mathbb{Q}\right)$, such that $q(\tilde{\lambda})=\lambda$, and $\tilde{\lambda}:=S p_{E / f_{X}}(\tilde{\lambda}) \in S p_{E / f_{X}}\left(H^{2 p}\left(E^{a n}, \mathbb{Q}\right)\right)$,
- $\tilde{\lambda}_{F} \in F^{p} H^{2 p}\left(E^{a n}, \mathbb{C}\right)$, such that $q\left(\tilde{\lambda}_{F}\right)=\lambda$, and since $S$ is affine $\mu \in \Gamma\left(S_{\mathbb{C}}, F^{p} E_{D R}^{2 p}\left(\mathcal{E}_{\mathbb{C}} / S_{\mathbb{C}}\right)\right)$ such that $\mu(t)=\tilde{\lambda}_{F}$,
- $\tilde{e}(\lambda):=[\tilde{\lambda}-\mu] \in \Gamma\left(\tilde{S}_{\mathbb{C}}^{a n}, W_{2 p-2} E_{D R, a n}^{2 p}\left(\mathcal{E}_{\mathbb{C}} / S_{\mathbb{C}}\right) / F^{p} \oplus \mathbb{V}_{\mathbb{Q}}\right)$.

The locus of Hodge classes passing through $\lambda$ inside the De Rham vector bundle of $\tilde{f}_{X} \circ i_{E_{0}}: \mathcal{E}_{0} \rightarrow S$ is thus by the above exact sequence an union of two components

$$
\mathbb{V}_{S}^{p}(\lambda):=\pi_{S}\left(\lambda \times \tilde{S}_{\mathbb{C}}^{a n}\right) \cap F^{p} E_{D R}^{2 p}\left(\mathcal{E}_{0, \mathbb{C}} / S_{\mathbb{C}}\right)=q\left(\mathbb{V}_{S}^{p}(\tilde{\lambda})\right) \cup p_{S, E_{D R}, \mathbb{V}}^{-1}(V(\tilde{e}(\lambda))) \subset E_{D R}^{2 p}\left(\mathcal{E}_{0, \mathbb{C}} / S_{\mathbb{C}}\right)
$$

where

- $\pi_{S}: H^{2 p}\left(E^{a n}, \mathbb{C}\right) \times \tilde{S}_{\mathbb{C}}^{a n} \rightarrow E_{D R}^{2 p, a n}\left(\mathcal{E}_{\mathbb{C}} / S_{\mathbb{C}}\right)$ is the morphism induced by the universal covering $\pi_{S}: \tilde{S}_{\mathbb{C}}^{a n} \rightarrow S_{\mathbb{C}}^{a n}$,
- $p_{S, E_{D R}, \mathbb{V}}: \mathbb{V}_{S}(\lambda) \subset E_{D R}^{2 p, a n}\left(\mathcal{E}_{0, \mathbb{C}} / S_{\mathbb{C}}\right) \rightarrow S_{\mathbb{C}}$, where $\mathbb{V}_{S}(\lambda):=\pi_{S}\left(\lambda \times \tilde{S}_{\mathbb{C}}^{a n}\right) \subset E_{D R}^{2 p, a n}\left(\mathcal{E}_{0, \mathbb{C}} / S_{\mathbb{C}}\right)$ is the flat leaf,
- $q:=q \otimes \mathbb{C}: E_{D R}^{2 p}\left(\mathcal{E}_{\mathbb{C}} / S_{\mathbb{C}}\right) \rightarrow E_{D R}^{2 p}\left(\mathcal{E}_{0, \mathbb{C}} / S_{\mathbb{C}}\right)$ is the quotient map.

Now,

- if $S p_{E / f_{X}}(\tilde{\lambda})=0$, assuming the Hodge conjecture for smooth projective varieties of dimension less or equal to $d-1, \mathbb{V}_{S}^{p}(\tilde{\lambda}) \subset E_{D R}^{2 p}\left(\mathcal{E}_{\mathbb{C}} / S_{\mathbb{C}}\right)$ is an algebraic subvariety defined over $\mathbb{Q}$, if $S p_{E / f_{X}}(\tilde{\lambda}) \neq 0$, by [5] theorem 1, the Hodge conjecture holds for projective hypersurfaces, hence

$$
S p_{\mathcal{E} / f_{X}}\left(\mathbb{V}_{S}^{p}(\tilde{\lambda})\right)=\overline{\mathbb{V}_{\left(\mathbb{A}^{1} \backslash 0\right) \times S}^{p}\left(S p_{E / f_{X}}(\tilde{\lambda})\right)} \cap p^{-1}\left(0 \times S_{\mathbb{C}}\right) \subset E_{D R, V_{0}}^{2 p}\left(\mathcal{Y}_{\mathbb{C}} /\left(\mathbb{A}_{\mathbb{C}}^{1} \times S_{\mathbb{C}}\right)\right)
$$

is an algebraic subvariety defined over $\mathbb{Q}$, where the equality follows from [6] lemma 2.11,

- assuming the Hodge conjecture for smooth projective varieties of dimension less or equal to $d-1$,

$$
V(\tilde{e}(\lambda)) \subset S_{\mathbb{C}}
$$

is an algebraic variety defined over $\mathbb{Q}$, since $\tilde{e}(\lambda)\left(t^{\prime}\right)=0$ if and only if $(\tilde{\lambda})^{\prime}-\mu\left(t^{\prime}\right) \in W_{2 p-2} H^{2 p}\left(\mathcal{E}_{\mathbb{C}, t^{\prime}}, \mathbb{C}\right)$ is an Hodge class where $(\tilde{\lambda})^{\prime} \in \pi_{S}\left(\left\{t^{\prime}\right\} \times \tilde{\lambda}\right)$.
Hence, $\mathbb{V}_{S}^{p}(\lambda) \subset E_{D R}^{2 p}\left(\mathcal{E}_{0 \mathbb{C}} / S_{\mathbb{C}}\right)$ is defined over $\mathbb{Q}$. By [4] theorem 4, we get

$$
\lambda=[Z] \in H^{2 p}\left(E_{0}^{a n}, \mathbb{Q}\right), Z \in \mathcal{Z}^{p}\left(E_{0}\right) .
$$

Since $p \in \mathbb{Z}$ and $\lambda \in F^{p} H^{2 p}\left(E_{0}^{a n}, \mathbb{Q}\right)$ are arbitrary, this proves the Hodge conjecture for $E_{0}$. Since $X$ is birational to $E_{0}$, the Hodge conjecture for $E_{0}$ and the induction hypothesis implies the Hodge conjecture for $X$.

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## 2 Preliminaries and notations

### 2.1 Notations

- Denote by Top the category of topological spaces and RTop the category of ringed spaces.
- Denote by Cat the category of small categories and RCat the category of ringed topos.
- For $\mathcal{S} \in$ Cat and $X \in \mathcal{S}$, we denote $\mathcal{S} / X \in$ Cat the category whose objects are $Y / X:=(Y, f)$ with $Y \in \mathcal{S}$ and $f: Y \rightarrow X$ is a morphism in $\mathcal{S}$, and whose morphisms $\operatorname{Hom}\left(\left(Y^{\prime}, f^{\prime}\right),(Y, f)\right)$ consists of $g: Y^{\prime} \rightarrow Y$ in $\mathcal{S}$ such that $f \circ g=f^{\prime}$.
- Let $\left(\mathcal{S}, O_{S}\right) \in$ RCat a ringed topos with topology $\tau$. For $F \in C_{O_{S}}(\mathcal{S})$, we denote by $k: F \rightarrow E_{\tau}(F)$ the canonical flasque resolution in $C_{O_{S}}(\mathcal{S})$ (see [3]). In particular for $X \in \mathcal{S}, H^{*}\left(X, E_{\tau}(F)\right) \xrightarrow{\sim}$ $\mathbb{H}_{\tau}^{*}(X, F)$.
- For $f: \mathcal{S}^{\prime} \rightarrow \mathcal{S}$ a morphism with $\mathcal{S}, \mathcal{S}^{\prime} \in$ RCat, endowed with topology $\tau$ and $\tau^{\prime}$ respectively, we denote for $F \in C_{O_{S}}(\mathcal{S})$ and each $j \in \mathbb{Z}$,

$$
\begin{aligned}
& -f^{*}:=H^{j} \Gamma\left(\mathcal{S}, k \circ \operatorname{ad}\left(f^{*}, f_{*}\right)(F)\right): \mathbb{H}^{j}(\mathcal{S}, F) \rightarrow \mathbb{H}^{j}\left(\mathcal{S}^{\prime}, f^{*} F\right), \\
& -f^{*}:=H^{j} \Gamma\left(\mathcal{S}, k \circ \operatorname{ad}\left(f^{* \bmod }, f_{*}\right)(F)\right): \mathbb{H}^{j}(\mathcal{S}, F) \rightarrow \mathbb{H}^{j}\left(\mathcal{S}^{\prime}, f^{* \bmod } F\right),
\end{aligned}
$$

the canonical maps.

- For $m: A \rightarrow B, A, B \in C(\mathcal{A}), \mathcal{A}$ an additive category, we denote $c(A): \operatorname{Cone}(m: A \rightarrow B) \rightarrow A[1]$ and $c(B): B \rightarrow \operatorname{Cone}(m: A \rightarrow B)$ the canonical maps.
- Denote by Sch $\subset$ RTop the subcategory of schemes (the morphisms are the morphisms of locally ringed spaces). We denote by PSch $\subset$ Sch the full subcategory of proper schemes. For a field $k$, we consider Sch $/ k:=\operatorname{Sch} / \operatorname{Spec} k$ the category of schemes over Spec $k$. The objects are $X:=\left(X, a_{X}\right)$ with $X \in \operatorname{Sch}$ and $a_{X}: X \rightarrow$ Spec $k$ a morphism and the morphisms are the morphisms of schemes $f: X^{\prime} \rightarrow X$ such that $f \circ a_{X^{\prime}}=a_{X}$. We then denote by
$-\operatorname{Var}(k)=\operatorname{Sch}^{f t} / k \subset \operatorname{Sch} / k$ the full subcategory consisting of algebraic varieties over $k$, i.e. schemes of finite type over $k$,
$-\mathrm{P} \operatorname{Var}(k) \subset \mathrm{QPVar}(k) \subset \operatorname{Var}(k)$ the full subcategories consisting of quasi-projective varieties and projective varieties respectively,
$-\mathrm{PSm} \operatorname{Var}(k) \subset \operatorname{Sm} \operatorname{Var}(k) \subset \operatorname{Var}(k), \operatorname{PSm} \operatorname{Var}(k):=\mathrm{P} \operatorname{Var}(k) \cap \operatorname{Sm} \operatorname{Var}(k)$, the full subcategories consisting of smooth varieties and smooth projective varieties respectively.
- Denote by $\operatorname{AnSp}(\mathbb{C}) \subset$ RTop the subcategory of analytic spaces over $\mathbb{C}$, and by $\operatorname{AnSm}(\mathbb{C}) \subset \operatorname{AnSp}(\mathbb{C})$ the full subcategory of smooth analytic spaces (i.e. complex analytic manifold).
- For $X \in \operatorname{Var}(k)$ and $X=\cup_{i \in I} X_{i}$ with $i_{i}: X_{i} \hookrightarrow X$ closed embeddings, we denote $X_{\bullet} \in$ $\operatorname{Fun}(\Delta, \operatorname{Var}(k))$ the associated simplicial space, with for $J \subset I, i_{I J}: X_{I}:=\cap_{i \in I} X_{i} \hookrightarrow X_{J}:=\cap_{i \in J} X_{i}$ the closed embedding.
- Let $\left(X, O_{X}\right) \in$ RTop. We consider its De Rham complex $\Omega_{X}^{\bullet}:=D R(X)\left(O_{X}\right)$.
- Let $X \in$ Sch. Considering its De Rham complex $\Omega_{X}^{\bullet}:=D R(X)\left(O_{X}\right)$, we have for $j \in \mathbb{Z}$ its De Rham cohomology $H_{D R}^{j}(X):=\mathbb{H}^{j}\left(X, \Omega_{X}^{\bullet}\right)$.
- Let $X \in \operatorname{Var}(k)$. Considering its De Rham complex $\Omega_{X}^{\bullet}:=\Omega_{X / k}^{\bullet}:=D R(X / k)\left(O_{X}\right)$, we have for $j \in \mathbb{Z}$ its De Rham cohomology $H_{D R}^{j}(X):=\mathbb{H}^{j}\left(X, \Omega_{X}^{\bullet}\right)$. The differentials of $\Omega_{X}^{\bullet}:=\Omega_{X / k}^{\bullet}$ are by definition $k$-linear, thus $H_{D R}^{j}(X):=\mathbb{H}^{j}\left(X, \Omega_{X}^{\bullet}\right)$ has a structure of a $k$ vector space.
- Let $X \in \operatorname{AnSp}(\mathbb{C})$. Considering its De Rham complex $\Omega_{X}^{\bullet}:=D R(X)\left(O_{X}\right)$, we have for $j \in \mathbb{Z}$ its De Rham cohomology $H_{D R}^{j}(X):=\mathbb{H}^{j}\left(X, \Omega_{X}^{\bullet}\right)$.
- For $Y \in \operatorname{Var}(\mathbb{Q})$ and $X=V(I)=V\left(f_{1}, \ldots, f_{r}\right) \subset Y_{\mathbb{C}}$, we consider $\mathcal{X}=V(\tilde{I})=V\left(\tilde{f}_{1}, \ldots, \tilde{f}_{r}\right) \subset$ $Y \times \mathbb{A}_{\mathbb{Q}}^{s}$ the canonical rational deformation, where

$$
\begin{array}{r}
\text { if } f=\sum_{I \subset[1, \ldots, d]^{n}} a_{I} x_{1}^{n_{1}} \cdots x_{n}^{n_{n}} \in \mathbb{C}\left[x_{1}, \ldots, x_{n}\right], \\
\tilde{f}=\sum_{I \subset[1, \ldots, d]^{n}} a_{I} x_{1}^{n_{1}} \cdots x_{n}^{n_{n}} \in \mathbb{Q}\left[\left(a_{I}\right)_{I \subset[1, \ldots, d]^{n}}, x_{1}, \ldots, x_{n}\right] .
\end{array}
$$

- For $X \in \operatorname{AnSp}(\mathbb{C})$, we denote $\alpha(X): \mathbb{C}_{X} \hookrightarrow \Omega_{X}^{\bullet}$ the embedding in $C(X)$. For $X \in \operatorname{AnSm}(\mathbb{C})$, $\alpha(X): \mathbb{C}_{X} \hookrightarrow \Omega_{X}^{\bullet}$ is an equivalence usu local by Poincare lemma.
- We denote $\mathbb{I}^{n}:=[0,1]^{n} \in \operatorname{Diff}(\mathbb{R})$ (with boundary). For $X \in$ Top and $R$ a ring, we consider its singular cochain complex

$$
C_{\text {sing }}^{*}(X, R):=\left(\mathbb{Z} \operatorname{Hom}_{\mathrm{Top}}\left(\mathbb{I}^{*}, X\right)^{\vee}\right) \otimes R
$$

and for $l \in \mathbb{Z}$ its singular cohomology $H_{\text {sing }}^{l}(X, R):=H^{n} C_{\text {sing }}^{*}(X, R)$. For $f: X^{\prime} \rightarrow X$ a continuous map with $X, X^{\prime} \in$ Top, we have the canonical map of complexes

$$
f^{*}: C_{\text {sing }}^{*}(X, R) \rightarrow C_{\text {sing }}^{*}(X, R), \sigma \mapsto f^{*} \sigma:=(\gamma \mapsto \sigma(f \circ \gamma))
$$

In particular, we get by functoriality the complex

$$
C_{X, R \text { sing }}^{*} \in C_{R}(X),(U \subset X) \mapsto C_{\text {sing }}^{*}(U, R)
$$

We recall that

- For $X \in$ Top locally contractible, e.g. $X \in \mathrm{CW}$, and $R$ a ring, the inclusion in $C_{R}(X)$ $c_{X}: R_{X} \rightarrow C_{X, R \operatorname{sing}}^{*}$ is by definition an equivalence top local and that we get by the small chain theorem, for all $l \in \mathbb{Z}$, an isomorphism $H^{l} c_{X}: H^{l}\left(X, R_{X}\right) \xrightarrow{\sim} H_{\text {sing }}^{l}(X, R)$.
- For $X \in \operatorname{Diff}(\mathbb{R})$, the restriction map

$$
r_{X}: \mathbb{Z} \operatorname{Hom}_{\operatorname{Diff}(\mathbb{R})}\left(\mathbb{I}^{*}, X\right)^{\vee} \rightarrow C_{\mathrm{sing}}^{*}(X, R), w \mapsto w:(\phi \mapsto w(\phi))
$$

is a quasi-isomorphism by Whitney approximation theorem.

### 2.2 Birational projective varieties and Hodge conjecture

In this subsection, we recall that assuming the Hodge conjecture for smooth projective varieties of dimension less or equal to $d-1$, the property of satisfying the Hodge conjecture for connected smooth projective varieties of dimension $d$ is a birational invariant :

Proposition 1. (i) Let $X \in \operatorname{PSmVar}(\mathbb{C})$ connected of dimension d. Let $\epsilon: \tilde{X}_{Z} \rightarrow X$ be the blow up of $X$ along a smooth closed subvariety $Z \subset X$. Assume the Hodge conjecture hold for smooth complex projective varieties of dimension less or equal to $d-1$. Then the Hodge conjecture hold for $X$ if and only if it hold for $\tilde{X}_{Z}$.
(ii) Let $X, X^{\prime} \in \operatorname{PSmVar}(\mathbb{C})$ connected of dimension d. Assume the Hodge conjecture hold for smooth complex projective varieties of dimension less or equal to $d-1$. If $X$ is birational to $X^{\prime}$, then the Hodge conjecture hold for $X$ if and only if it hold for $X^{\prime}$

Proof. (i): Follows from the fact that

$$
\left(\epsilon^{*}, \oplus_{l=1}^{c} i_{E *}\left((-) \cdot h^{l-1}\right)\right): H^{k}(X, \mathbb{Q}) \oplus \oplus_{l=1}^{c} H^{k-2 l}(E, \mathbb{Q}) \rightarrow H^{k}\left(\tilde{X}_{Z}, \mathbb{Q}\right)
$$

is an isomorphism of Hodge structures for each $k \in \mathbb{Z}$, where $c=\operatorname{codim}(Z, X), i_{E}: E \hookrightarrow \tilde{X}_{Z}$ is the closed embedding, $\epsilon_{\mid E}: E \rightarrow Z$ being a projective vector bundle.
(ii):Follows from (i) since if $\pi: X^{o} \rightarrow X^{\prime}$ is a birational map, $X^{o} \subset X$ being an open subset, then $X$ is connected to $X^{\prime}$ by a sequence of blow up of smooth projective varieties with smooth center by [1].

### 2.3 A family of smooth projective hypersurfaces associated to a smooth projective variety

In this subsection, we recall that given a smooth complex projective variety $X$, there exists a family of smooth complex projective hypersurfaces which degenerates into a normal crossing divisor with one irreducible component birational to $X$.

Proposition 2. Let $X \in \operatorname{PSm} \operatorname{Var}(\mathbb{C})$ connected of dimension $d$. Then there exists a family of smooth projective hypersurfaces $f_{X}: Y \rightarrow \mathbb{A}_{\mathbb{C}}^{1}$ with

- $Y \in \operatorname{SmVar}(\mathbb{C}), f_{X}$ flat projective,
- $Y_{s}:=f_{X}^{-1}(s) \subset \mathbb{P}_{\mathbb{C}}^{d+1}$ for $s \in \mathbb{A}_{\mathbb{C}}^{1} \backslash 0$ are smooth projective hypersurfaces,
- $E:=f_{X}^{-1}(0)=\cup_{i=0}^{r} E_{i}=\epsilon^{-1}\left(Y_{0}\right) \subset Y$ is a normal crossing divisor, where $Y_{0} \subset \mathbb{P}_{\mathbb{C}}^{d+1}$ is a possibly singular hypersurface birationnal to $X, \epsilon:(Y, E) \rightarrow\left(Y^{\prime}, Y_{0}\right)$ a desingularization, and $E_{0}$ is birational to $X$,

Proof. See [2] lemma 1.5.8. It use the fact that $X$ is birational to a possibly singular projective hypersurface $Y_{0}$ and consider the desingularization of a generic pencil $f_{X}^{\prime}: Y^{\prime}=V(f) \subset \mathbb{P}_{\mathbb{C}}^{d+1} \times \mathbb{A}_{\mathbb{C}}^{1} \rightarrow \mathbb{A}_{\mathbb{C}}^{1}$ passing through $Y_{0}$ and $f_{X}=f_{X}^{\prime} \circ \epsilon$.

### 2.4 The nearby cycle functor and the specialization map

Recall from [3] that for $k \subset \mathbb{C}$ a subfield and $S \in \operatorname{Var}(k)$ quasi-projective, and $l: S \hookrightarrow \tilde{S}$ a closed embedding with $\tilde{S} \in \operatorname{Sm} \operatorname{Var}(k)$, we have the full subcategory

$$
\iota: M H M_{k, g m}(S) \hookrightarrow \operatorname{PSh}_{\mathcal{D} f i l, S}(\tilde{S}) \times_{I} P\left(S_{\mathbb{C}}^{a n}\right)
$$

consisting of geometric mixed Hodge module whose De Rham part is defined over $k$, where, using the fact that the $V$-filtration is defined over $k, \operatorname{PSh}_{\mathcal{D}(1,0) f i l, S}(\tilde{S}) \times{ }_{I} P_{f i l}\left(S_{\mathbb{C}}^{a n}\right)$ is the category

- whose objects are $((M, F, W),(K, W), \alpha)$ where $(M, F, W)$ is a filtered $D_{\tilde{S}}$ module supported on $S$, where $l: S \hookrightarrow \tilde{S}$ is a closed embedding and $\tilde{S} \in \operatorname{Sm} \operatorname{Var}(k),(K, W)$ is a filtered Perverse sheaf on $S_{\mathbb{C}}^{a n}$, and $\alpha: l_{*}(K, W) \otimes \mathbb{C} \rightarrow D R(\tilde{S})\left((M, W)^{a n}\right)$ is an isomorphism in $D_{f i l}\left(\tilde{S}_{\mathbb{C}}^{a n}\right)$.
- whose morphism between $((M, F, W),(K, W), \alpha)$ and $\left(\left(M^{\prime}, F, W\right),\left(K^{\prime}, W\right), \alpha^{\prime}\right)$, are couples of morphisms $\left(\phi:(M, F, W) \rightarrow\left(M^{\prime}, F, W\right), \psi:(K, W) \rightarrow\left(K^{\prime}, W\right)\right)$ such that $\phi \circ \alpha=\alpha^{\prime} \circ \psi$.

Then, by [3], we get for $S \in \operatorname{Var}(k)$ quasi-projective and $l: S \hookrightarrow \tilde{S}$ is a closed embedding with $\tilde{S} \in$ $\operatorname{SmVar}(k)$ an embedding

$$
\iota: D\left(M H M_{k, g m}(S)\right) \hookrightarrow D_{\mathcal{D} f i l, S}(\tilde{S}) \times_{I} D\left(S_{\mathbb{C}}^{a n}\right)
$$

consisting of mixed Hodge module whose De Rham part is defined over $k$, where $D_{\mathcal{D}(1,0) \text { fil,S }}(\tilde{S}) \times{ }_{I}$ $D_{f i l, c}\left(S_{\mathbb{C}}^{a n}\right)$ is the category

- whose objects are $((M, F, W),(K, W), \alpha)$ where $(M, F, W)$ is a complex of filtered $D_{\tilde{S}}$ module supported on $S,(K, W)$ is a filtered complex of presheaves on $S_{\mathbb{C}}^{a n}$ whose cohomology are constructible sheaves, and $\alpha: l_{*}(K, W) \otimes \mathbb{C} \rightarrow D R(\tilde{S})\left((M, W)^{a n}\right)$ is an isomorphism in $D_{f i l}\left(\tilde{S}_{\mathbb{C}}^{a n}\right)$.
- whose morphism are given in [3],

We have then the six functors formalism :

$$
\begin{array}{r}
D\left(M H M_{k, g m}(-)\right): \operatorname{Var}(k) \mapsto \text { TriCat, } S \mapsto D\left(M H M_{k, g m}(S)\right),(f: T \rightarrow S) \mapsto \\
\left(f^{*}, f_{*}\right): D\left(M H M_{k, g m}(S)\right) \rightarrow D\left(M H M_{k, g m}(T)\right),\left(f_{!}, f^{!}\right): D\left(M H M_{k, g m}(T)\right) \rightarrow D\left(M H M_{k, g m}(S)\right)
\end{array}
$$

and for $D \subset S$ a (Cartier) divisor the nearby cycle functor

$$
\begin{array}{r}
\Phi_{D}: D\left(M H M_{k, g m}(S)\right) \rightarrow D\left(M H M_{k, g m}(D)\right) \\
\Phi_{D}((M, F, W),(K, W), \alpha):=\left(\Phi_{D}(M, F, W), \Phi_{D}(K, W), \Phi_{D}(\alpha)\right)
\end{array}
$$

For $T \in \operatorname{Var}(k)$ and $\epsilon_{\bullet}: T_{\bullet} \rightarrow T$ a desingularization with $T_{\bullet} \in \operatorname{Fun}(\Delta, \operatorname{SmVar}(k))$, we have

$$
\mathbb{Z}_{T}^{H d g}:=a_{T}^{* H d g} \mathbb{Z}^{H d g}=\left(\epsilon_{\bullet *} E_{z a r}\left(O_{T_{\bullet}}, F, W\right), \epsilon_{\bullet *} E_{u s u}\left(\mathbb{Z}_{T_{\bullet}^{a n}}, W\right), \alpha\left(T_{\bullet}\right)\right)
$$

For $f: X \rightarrow S$ a projective morphism with $X, S \in \operatorname{Sm} \operatorname{Var}(k)$, we denote

$$
E_{H d g}(X / S):=f_{*} \mathbb{Z}_{X}^{H d g}:=\left(\int f\left(O_{X}, F\right), R f_{*} \mathbb{Z}_{X_{\mathbb{C}}^{a n}}, f_{*} \alpha(X)\right) \in D\left(M H M_{k, g m}(S)\right)
$$

For $f: X \hookrightarrow \mathbb{P}^{N} \times S \xrightarrow{p} S$ a projective morphism with $X, S \in \operatorname{Var}(k), S$ smooth, we denote

$$
E_{H d g}(X / S):=f_{*} \mathbb{Z}_{X}^{H d g}:=\left(\int_{p} \Gamma_{X}^{H d g}\left(O_{\mathbb{P}^{N} \times S}, F\right), R f_{*} \mathbb{Z}_{X_{\mathbb{C}}{ }^{a n}}, f_{*} \alpha(X)\right) \in D\left(M H M_{k, g m}(S)\right)
$$

We then have the specialization map :

Definition-Proposition 1. Let $k$ be field of characteristic zero.
(i) Let $f^{\prime}: X^{\prime} \rightarrow \mathbb{A}_{k}^{1} \times S$ be a projective morphism with $X, S \in \operatorname{SmVar}(k)$. Assume that
$-f^{o}:=f \otimes_{\mathbb{A}^{1}}\left(\mathbb{A}^{1} \backslash 0\right): X^{o} \rightarrow\left(\mathbb{A}^{1} \backslash 0\right) \times S$ is smooth
$-X_{0}:=f^{-1}(0 \times S) \subset X^{\prime}$. We have by definition $X^{o}=X^{\prime} \backslash X_{0}$.
We have then, see [3], the distinguished triangles in $D\left(M H M_{k, g m}\left(X_{0}\right)\right)$,

$$
\begin{aligned}
\mathbb{Z}_{X_{0}}^{H d g}:=i_{X_{0}}^{*} \mathbb{Z}_{X^{\prime}}^{H d g} & \xrightarrow{s p_{X_{0} / X}} \Phi_{X_{0}} \mathbb{Z}_{X^{o}}^{H d g}:=\left(\Phi_{X_{0}}\left(O_{X^{\prime}}, F\right), \Phi_{X_{0}} \mathbb{Z}_{X^{o}}, \Phi_{X_{0}} \alpha\left(X^{\prime}\right)\right) \\
& \xrightarrow{c\left(\Phi_{X_{0}} \mathbb{Z}_{X o}^{H d g}\right)} \phi_{E} \mathbb{Z}_{X^{o}}^{H d g}:=\operatorname{Cone}\left(s p_{E / X}\right) \xrightarrow{c\left(\mathbb{Z}_{X_{0}}^{H d g}\right)} \mathbb{Z}_{X_{0}}^{H d g}[1] .
\end{aligned}
$$

Applying the functor $\left(f^{\prime} \circ i_{X_{0}}\right)_{*}$, we obtain the generalized distinguished triangle in $D\left(M H M_{k, g m}(S)\right)$

$$
\begin{array}{r}
E_{H d g}\left(X_{0} / S\right) \xrightarrow{s p_{X_{0} / f^{\prime}}:=\left(f^{\prime} \circ i_{X_{0}}\right)_{*} s p_{X_{0} / X}} \\
\left(f^{\prime} \circ i_{X_{0}}\right)_{*}\left(\Phi_{X_{0}} \mathbb{Z}_{X^{o}}^{H d g}\right)=\Phi_{0} E_{H d g}\left(X^{o} /\left(\left(\mathbb{A}^{1} \backslash 0\right) \times S\right)\right) \xrightarrow{\left(f^{\prime} \circ i_{X_{0}}\right)_{*} c\left(\Phi_{X_{0}} \mathbb{Z}_{X o}^{H d g}\right)} \\
\left(f^{\prime} \circ i_{X_{0}}\right)_{*}\left(\phi_{X_{0}} \mathbb{Z}_{X^{o}}^{H d g}\right)=\phi_{0} E_{H d g}\left(X^{o} /\left(\left(\mathbb{A}^{1} \backslash 0\right) \times S\right)\right) \xrightarrow{\left(f^{\prime} \circ i_{X_{0}}\right)_{*} c\left(\mathbb{Z}_{X}^{H d g}\right)} E_{H d g}\left(X_{0} / S\right)[1]
\end{array}
$$

(i)' If for each $t \in S(\mathbb{C}), H^{k-1}\left(X_{(s, t)}^{\prime}, \mathbb{Q}\right)=0$ for $s \in \mathbb{A}_{\mathbb{C}}^{1} \backslash 0$ and $H_{X_{0}}^{k-1}\left(X_{\mathbb{A}^{1} \times t}^{\prime}, \mathbb{Q}\right)=0$, then the induced map on cohomology

$$
s p_{X_{0} / f^{\prime}}: E_{H d g}^{k}\left(X_{0} / S\right) \rightarrow H^{k} \Phi_{0} E_{H d g}\left(X^{o} /\left(\left(\mathbb{A}^{1} \backslash 0\right) \times S\right)\right)
$$ is injective.

(ii) Let $f: X \rightarrow \mathbb{A}_{k}^{1} \times S$ be a projective morphism with $X, S \in \operatorname{SmVar}(k)$. Assume that
$-f^{o}:=f \otimes_{\mathbb{A}^{1}}\left(\mathbb{A}^{1} \backslash 0\right): X^{o} \rightarrow\left(\mathbb{A}^{1} \backslash 0\right) \times S$ is smooth
$-E:=f^{-1}(0 \times S)=\cup_{i=0}^{r} E_{i} \subset X$ is a normal crossing divisor and that for $I \subset[0, \ldots r]$, $f_{0, I}:=f_{\mid E_{I}}: E_{I}:=\cap_{i \in I} E_{i} \rightarrow S$ are smooth. We have by definition $X^{o}=X \backslash E$.
Denote for $I \subset J \subset[0, \ldots r], i_{J I}: E_{J} \hookrightarrow E_{I}$ and $i_{I}: E_{I} \hookrightarrow E, i_{E}: E \hookrightarrow X$ the closed embeddings. In this particular case, c.f. [8], the distinguished triangle in $D\left(M H M_{k, g m}(E)\right)$ given in (i) become

$$
\begin{array}{r}
\mathbb{Z}_{E}^{H d g}:=i_{E}^{*} \mathbb{Z}_{X}^{H d g}=\left(\left(i_{\bullet} \Omega_{E}^{\bullet}, F, W\right),\left(\mathbb{Z}_{E_{\bullet}}, W\right), \alpha(E \bullet)\right) \xrightarrow{s p_{E / X}} \\
\Phi_{E} \mathbb{Z}_{X^{o}}^{H d g}=\left(\left(i_{E}^{* m o d} \Omega_{X / \mathbb{A}^{\bullet} \times S}(\log E), F, W\right), \Phi_{E} \mathbb{Z}_{X^{o}}, \Phi_{E} \alpha(X)\right) \xrightarrow{c\left(\Phi_{E} \mathbb{Z}_{X^{o}}^{H d g}\right)} \\
\phi_{E} \mathbb{Z}_{X^{o}}^{H d g}:=\operatorname{Cone}\left(s p_{E / X}\right) \xrightarrow{c\left(\mathbb{Z}_{E}^{H d g}\right)} \mathbb{Z}_{E}^{H d g}[1] .
\end{array}
$$

Applying the functor $\left(f \circ i_{E}\right)_{*}$, we obtain the generalized distinguished triangle in $D\left(M H M_{k, g m}(S)\right)$

$$
\begin{gathered}
E_{H d g}(E / S) \xrightarrow{s p_{E / f}:=\left(f \circ i_{E}\right)_{*} s p_{E / X}}\left(f \circ i_{E}\right)_{*}\left(\Phi_{E} \mathbb{Z}_{X^{o}}^{H d g}\right)=\Phi_{0} E_{H d g}\left(X^{o} /\left(\left(\mathbb{A}^{1} \backslash 0\right) \times S\right)\right) \\
\xrightarrow{\left(f \circ i_{E}\right)_{*} c\left(\Phi_{E} \mathbb{Z}_{X o}^{H d g}\right)}\left(f \circ i_{E}\right)_{*}\left(\phi_{E} \mathbb{Z}_{X^{o}}^{H d g}\right)=\phi_{0} E_{H d g}\left(X^{o} /\left(\left(\mathbb{A}^{1} \backslash 0\right) \times S\right)\right) \xrightarrow{\left(f \circ i_{E}\right)_{*} c\left(\mathbb{Z}_{X}^{H d g}\right)} E_{H d g}(E / S)[1] .
\end{gathered}
$$

If $f=f^{\prime} \circ \epsilon$ where $\epsilon:(X, E) \rightarrow\left(X^{\prime}, X_{0}\right)$ is a desingularization, we have $s p_{X_{0} / X^{\prime}}=s p_{E / X} \circ \epsilon^{*}$.
Proof. (i)':Follows from the exact sequence
$\cdots \rightarrow H_{X_{0, t}}^{l}\left(X_{\mathbb{A}^{1} \times t}^{\prime}, \mathbb{Q}\right) \rightarrow \mathbb{H}^{l}\left(X_{0, t}, \phi_{0} \mathbb{Z}_{X_{\left(\mathbb{A}^{1} \backslash 0\right) \times t}^{o}}\right) \xrightarrow{V} \mathbb{H}^{l}\left(X_{0, t}, \Phi_{0} \mathbb{Z}_{X_{\left(\mathbb{A}^{1} \backslash 0\right) \times t}^{o}}\right) \rightarrow H_{X_{0, t}}^{l+1}\left(X_{\mathbb{A}^{1} \times t}^{\prime}, \mathbb{Q}\right) \rightarrow \cdots$
given in [8].

## 3 Hodge conjecture for smooth projective varieties

Theorem 1. Let $X \in \operatorname{PSmVar}(\mathbb{C})$. Then Hodge conjecture hold for $X$. That is, if $p \in \mathbb{Z}$ and $\lambda \in$ $F^{p} H^{2 p}\left(X^{a n}, \mathbb{Q}\right), \lambda=[Z]$ with $Z \in \mathcal{Z}^{p}(X)$.
Proof. Up to split $X$ into its connected components, we may assume that $X$ is connected of dimension $d$. The Hodge conjecture is true for curves and surfaces. Assume that the Hodge conjecture is true for smooth projective varieties of dimension less or equal to $d-1$. Consider by proposition 2 a family of smooth projective hypersurfaces $f_{X}: Y=V(I) \subset \mathbb{P}_{\mathbb{C}}^{N} \times \mathbb{A}_{\mathbb{C}}^{1} \rightarrow \mathbb{A}_{\mathbb{C}}^{1}$ with

- $Y \in \operatorname{Sm} \operatorname{Var}(\mathbb{C}), f_{X}$ flat projective,
- $Y_{s}:=f_{X}^{-1}(s) \subset \mathbb{P}_{\mathbb{C}}^{d+1}$ for $s \in \mathbb{A}_{\mathbb{C}}^{1} \backslash 0$ are smooth projective hypersurfaces
- $E:=f_{X}^{-1}(0)=\cup_{i=0}^{r} E_{i}=\epsilon^{-1}\left(Y_{0}\right) \subset Y$ is a normal crossing divisor, $f_{X}^{\prime}: Y^{\prime} \subset \mathbb{P}_{\mathbb{C}}^{d+1} \times \mathbb{A}_{\mathbb{C}}^{1} \rightarrow \mathbb{A}_{\mathbb{C}}^{1}$ is a generic pencil of hypersurface passing through a possibly singular hypersurface $Y_{0} \subset \mathbb{P}_{\mathbb{C}}^{d+1}$ which is birationnal to $X, f_{X}=f_{X}^{\prime} \circ \epsilon$ where $\epsilon:(Y, E) \rightarrow\left(Y^{\prime}, Y_{0}\right)$ is a desingularization, and $E_{0}$ is birational to $X$,
and its canonical rational deformation $\tilde{f}_{X}: \mathcal{Y}=V(\tilde{I}) \subset \mathbb{P}_{\mathbb{Q}}^{N} \times \mathbb{A}_{\mathbb{Q}}^{1} \times S \rightarrow \mathbb{A}_{\mathbb{Q}}^{1} \times S$ with
- $\mathcal{Y} \in \operatorname{Sm} \operatorname{Var}(\mathbb{Q}), S \subset \mathbb{A}_{\mathbb{Q}}^{r}$ an open subset, $\tilde{f}_{X}$ flat projective
- $Y=\tilde{f}_{X}^{-1}\left(\mathbb{A}_{\mathbb{Q}}^{1} \times\{t\}\right), f_{X}=\tilde{f}_{X} \times_{S}\{t\}$, with $t \in S(\mathbb{C})$
- $\mathcal{E}=\cup_{i=0}^{s} \mathcal{E}_{i}:=\tilde{f}_{X}^{-1}(\{0\} \times S) \subset \mathcal{Y}$ is a normal crossing divisor, $\tilde{f}_{X, 0}: \mathcal{E}_{I}:=\cap_{i \in I} \mathcal{E}_{i} \rightarrow S$ smooth projective for each $I \subset[0, \ldots, s], \mathcal{E}_{\mathbb{C}, t}=E$,
- $\tilde{f}_{X} \times \times_{\mathbb{A}^{1}}\left(\mathbb{A}^{1} \backslash 0\right) \mathcal{Y}^{o} \rightarrow \mathbb{A}_{\mathbb{Q}}^{1} \backslash 0$ has fibers which are smooth projective hypersurfaces.
- $\tilde{f}_{X}=\tilde{f}_{X}^{\prime} \circ \epsilon$, with $\epsilon:(\mathcal{Y}, \mathcal{E}) \rightarrow\left(\mathcal{Y}^{\prime}, \mathcal{Y}_{0}\right)$ and $\tilde{f}_{X}^{\prime}: \mathcal{Y}^{\prime}=V(\tilde{f}) \subset \mathbb{P}_{\mathbb{Q}}^{d+1} \times S \rightarrow S$.

We have by definition the commutative diagram in $\operatorname{SmVar}(\mathbb{Q})$

whose squares are cartesian, $j$ the open embedding and $i_{\mathcal{E}}$ the closed embedding. Denote for $I \subset[0, \ldots, s]$, $i_{I}: \mathcal{E}_{I} \hookrightarrow \mathcal{E}$ and for $I \subset J \subset[0, \ldots, s], i_{J I}: \mathcal{E}_{J} \rightarrow \mathcal{E}_{I}$ the closed embeddings. Then, $i_{\bullet}: \mathcal{E}_{\bullet} \rightarrow \mathcal{E}$ in $\operatorname{Fun}(\Delta, \operatorname{Var}(\mathbb{Q}))$ is a simplicial resolution with $\mathcal{E}_{\bullet} \in \operatorname{Fun}(\Delta, \operatorname{Sm} \operatorname{Var}(\mathbb{Q}))$. We have then
$E_{H d g}(\mathcal{E} / S):=\left(\tilde{f}_{X} \circ i_{\bullet *} E_{z a r}\left(\Omega_{\mathcal{E}_{\bullet} / S}^{\bullet}, F_{b}, W\right),\left(\tilde{f}_{X} \circ i_{\bullet *} E_{u s u}\left(\mathbb{Q}_{\mathcal{E}_{\bullet, C}^{a n}}, W\right)\right), \tilde{f} \circ i_{\bullet *} \alpha\left(\mathcal{E}_{\bullet}\right)\right) \in D\left(M H M_{\mathbb{Q}, g m}(S)\right)$
Then, definition-proposition 1 applied to $\tilde{f}_{X}^{\prime}: \mathcal{Y}^{\prime} \rightarrow \mathbb{A}_{\mathbb{Q}}^{1} \times S$ gives the distinguish in $D\left(M H M_{\mathbb{Q}, g m}(S)\right)$,

$$
\begin{array}{r}
E_{H d g}\left(\mathcal{Y}_{0} / S\right) \xrightarrow{S p_{\mathcal{Y}_{0} / \tilde{f}_{X}}:=\left(\tilde{f}_{X}^{\prime} \circ i_{\mathcal{Y}_{0}}\right)_{*} s p_{\mathcal{Y}_{0} / \mathcal{Y}}} \Phi_{0} E_{H d g}\left(\mathcal{Y}^{o} /\left(\left(\mathbb{A}^{1} \backslash 0\right) \times S\right)\right) \\
\xrightarrow{\left(\tilde{f}_{X}^{\prime} \circ i \mathcal{Y}_{0}\right) c\left(\Phi_{\mathcal{Y}_{0}} \mathbb{Z}_{\mathcal{Y}^{o}}^{H d g}\right)} \phi_{0} E_{H d g}\left(\mathcal{Y}^{o} /\left(\left(\mathbb{A}^{1} \backslash 0\right) \times S\right)\right) \xrightarrow{\left(\tilde{f}_{X}^{\prime} \circ i_{\mathcal{Y}_{0}}\right)_{*} c\left(\mathbb{Z}_{\left.\mathcal{Y}_{0}\right)}^{H d g}\right.} E_{H d g}\left(\mathcal{Y}_{0} / S\right)[1]
\end{array}
$$

and definition-proposition 1 applied to $\tilde{f}_{X}: \mathcal{Y} \rightarrow \mathbb{A}_{\mathbb{Q}}^{1} \times S$ gives the distinguish in $D\left(M H M_{\mathbb{Q}, g m}(S)\right)$,

$$
\begin{array}{r}
E_{H d g}(\mathcal{E} / S) \xrightarrow{S p_{\mathcal{E} / \tilde{f}_{X}}:=\left(\tilde{f}_{X} \circ i_{\mathcal{E}}\right)_{*} s p_{\mathcal{E} / \mathcal{Y}}} \Phi_{0} E_{H d g}\left(\mathcal{Y}^{o} /\left(\left(\mathbb{A}^{1} \backslash 0\right) \times S\right)\right) \\
\xrightarrow{\left(\tilde{f}_{X} \circ i_{\mathcal{E}}\right) c\left(\Phi_{\mathcal{E}} \mathbb{Z}_{\mathcal{O}}^{H d g}\right)} \phi_{0} E_{H d g}\left(\mathcal{Y}^{0} /\left(\left(\mathbb{A}^{1} \backslash 0\right) \times S\right)\right) \xrightarrow{\left(\tilde{f}_{X} \circ i_{\mathcal{E}}\right)_{*} c\left(\mathbb{Z}_{\mathcal{E})}^{H d g}\right.} E_{H d g}(\mathcal{E} / S)[1]
\end{array}
$$

and $S p_{\mathcal{E} / \tilde{f}_{X}} \circ \epsilon^{*}=S p_{\mathcal{Y}_{0} / \tilde{f}_{X}^{\prime}}$. Let $p \in \mathbb{Z}$. By the hard Lefchetz theorem on $H^{*}\left(E_{0}\right)$, we may assume that $2 p \leq d$. Taking the cohomology to this distinguish triangle, we get in particular, for each $p \in \mathbb{Z}$, the maps in $M H M_{\mathbb{Q}, g m}(S)$

- $S p_{\mathcal{Y}_{0} / \tilde{f}_{X}}: E_{H d g}^{2 p}\left(\mathcal{Y}_{0} / S\right) \hookrightarrow H^{2 p} \Phi_{0} E_{H d g}\left(\mathcal{Y}^{o} /\left(\left(\mathbb{A}_{\mathbb{Q}}^{1} \backslash 0\right) \times S\right)\right)$
- $S p_{\mathcal{E} / \tilde{f}_{X}}: E_{H d g}^{2 p}(\mathcal{E} / S) \hookrightarrow H^{2 p} \Phi_{0} E_{H d g}\left(\mathcal{Y}^{o} /\left(\left(\mathbb{A}_{\mathbb{Q}}^{1} \backslash 0\right) \times S\right)\right)$.

The map $S p_{\mathcal{Y}_{0} / \tilde{f}_{X}}$ is injective by definition-proposition $1(\mathrm{i})^{\prime}$ and the weak Lefchetz theorem since for $t^{\prime} \in S(\mathbb{C})$ and $s \in \mathbb{A}_{\mathbb{C}}^{1} \backslash 0, \mathcal{Y}_{\left(s, t^{\prime}\right)}$ and $\mathcal{Y}_{0, t^{\prime}}$ are projective hypersurfaces. For $\lambda \in H^{2 p}\left(\mathcal{E}_{0, \mathbb{C}, t^{\prime}}^{a n}, \mathbb{Q}\right), t^{\prime} \in S(\mathbb{C})$, we consider $\pi_{S}: \tilde{S}_{\mathbb{C}}^{a n} \rightarrow S_{\mathbb{C}}^{a n}$ the universal covering in $\operatorname{AnSm}(\mathbb{C})$ and

$$
\mathbb{V}_{S}(\lambda):=\pi_{S}\left(\lambda \times \tilde{S}_{\mathbb{C}}^{a n}\right) \subset E_{D R}^{2 p, a n}\left(\mathcal{E}_{0, \mathbb{C}} / S_{\mathbb{C}}\right)
$$

with

$$
\begin{array}{r}
\pi_{S}: \tilde{S}_{\mathbb{C}}^{a n} \times H^{2 p}\left(\mathcal{E}_{0, t^{\prime}}^{a n}, \mathbb{C}\right) \rightarrow E_{D R}^{2 p, a n}\left(\mathcal{E}_{0, \mathbb{C}} / S_{\mathbb{C}}\right) \\
\left(D, D^{*}\right) \rightarrow\left(\mathbb{A}_{\mathbb{C}}^{r}, S_{\mathbb{C}}^{a n}\right), \pi_{S}\left(z_{1}, \cdots, z_{r}, h\right):=\left(e^{2 i \pi z_{1}}, \cdots, e^{2 i \pi z_{r}}, e^{-\left(z_{1} N_{1}+\cdots+z_{r} N_{r}\right)} h\right)
\end{array}
$$

the flat leaf induced by the universal covering and

$$
\mathbb{V}_{S}^{p}(\lambda):=\mathbb{V}_{S}(\lambda) \cap F^{p} E_{D R}^{2 p}\left(\mathcal{E}_{0, \mathbb{C}} / S_{\mathbb{C}}\right) \subset E_{D R}^{2 p}\left(\mathcal{E}_{0, \mathbb{C}} / S_{\mathbb{C}}\right)
$$

which is an algebraic variety by [6]. Note that

- $\mathbb{V}_{S}(\lambda) \cap p_{S}^{-1}\left(t^{\prime}\right)=\pi_{1}\left(S_{\mathbb{C}}^{a n}, t^{\prime}\right)(\lambda)$ is the orbit of $\lambda$ under the monodromy action, where $p_{S}$ : $E_{D R}^{2 p}\left(\mathcal{E}_{0, \mathbb{C}} / S_{\mathbb{C}}\right) \rightarrow S$ is the projection,
- $\mathbb{V}_{S}^{p}(\lambda) \subset H L^{p, 2 p}\left(\mathcal{E}_{0, \mathbb{C}} / S_{\mathbb{C}}\right)$ is the union of the irreducible components passing through $\lambda$.

Similarly, for $\eta \in H^{2 p}\left(\mathcal{E}_{\mathbb{C}, t^{\prime}}^{a n}, \mathbb{Q}\right), t^{\prime} \in S(\mathbb{C})$, we consider $\pi_{S}: \tilde{S}_{\mathbb{C}}^{a n} \rightarrow S_{\mathbb{C}}^{a n}$ the universal covering in AnSm( $\left.\mathbb{C}\right)$ and

$$
\mathbb{V}_{S}(\eta):=\pi_{S}\left(\eta \times \tilde{S}_{\mathbb{C}}^{a n}\right) \subset E_{D R}^{2 p, a n}\left(\mathcal{E}_{\mathbb{C}} / S_{\mathbb{C}}\right)
$$

with

$$
\begin{array}{r}
\pi_{S}: \tilde{S}_{\mathbb{C}}^{a n} \times H^{2 p}\left(\mathcal{E}_{t^{\prime}}^{a n}, \mathbb{C}\right) \rightarrow E_{D R}^{2 p, a n}\left(\mathcal{E}_{\mathbb{C}} / S_{\mathbb{C}}\right) \\
\left(D, D^{*}\right) \rightarrow\left(\mathbb{A}_{\mathbb{C}}^{r}, S_{\mathbb{C}}^{a n}\right), \pi_{S}\left(z_{1}, \cdots, z_{r}, h\right):=\left(e^{2 i \pi z_{1}}, \cdots, e^{2 i \pi z_{r}}, e^{-\left(z_{1} N_{1}+\cdots+z_{r} N_{r}\right)} h\right)
\end{array}
$$

the flat leaf induced by the universal covering and

$$
\mathbb{V}_{S}^{p}(\eta):=\mathbb{V}_{S}(\eta) \cap F^{p} E_{D R}^{2 p}\left(\mathcal{E}_{\mathbb{C}} / S_{\mathbb{C}}\right) \subset E_{D R}^{2 p}\left(\mathcal{E}_{\mathbb{C}} / S_{\mathbb{C}}\right)
$$

We have
$H^{2 p} E_{H d g}\left(\mathcal{Y}^{o} /\left(\left(\mathbb{A}_{\mathbb{Q}}^{1} \backslash 0\right) \times S\right)\right)=\left(\left(E_{D R}^{2 p}\left(\mathcal{Y}^{o} /\left(\left(\mathbb{A}_{\mathbb{Q}}^{1} \backslash 0\right) \times S\right)\right), F\right), R^{2 p} \tilde{f}_{X *} \mathbb{Z}_{\mathcal{Y}^{o}}, \tilde{f}_{X *} \alpha\left(\mathcal{Y}^{o}\right)\right) \in M H M_{\mathbb{Q}, g m}(S)$
By [8], the filtered vector bundle

$$
\left(E_{D R}^{2 p}\left(\mathcal{Y}^{o} /\left(\left(\mathbb{A}_{\mathbb{Q}}^{1} \backslash 0\right) \times S\right)\right), F\right)=H^{2 p} \tilde{f}_{X *} E_{z a r}\left(\Omega_{\mathcal{Y}^{o} /\left(\mathbb{A}_{\mathbb{Q}}^{1} \backslash 0\right) \times S}\right) \in \operatorname{Vect}_{\mathcal{D} f i l}(S)
$$

extend to a filtered vector bundle

$$
\left(E_{D R, V_{0}}^{2 p}\left(\mathcal{Y} /\left(\mathbb{A}_{\mathbb{Q}}^{1} \times S\right)\right), F\right)=H^{2 p} \tilde{f}_{X *} E_{z a r}\left(\Omega_{\mathcal{Y} / \mathbb{A}_{\mathbb{Q}}^{1} \times S}(\log \mathcal{E}) \in \operatorname{Vect}_{\mathcal{D} f i l}(S)\right.
$$

such that
$H^{2 p} \Phi_{0} E_{H d g}\left(\mathcal{Y}^{o} /\left(\left(\mathbb{A}_{\mathbb{Q}}^{1} \backslash 0\right) \times S\right)\right)=\left(i_{0}^{* \bmod }\left(E_{D R, V_{0}}^{2 p}\left(\mathcal{Y} /\left(\mathbb{A}_{\mathbb{Q}}^{1} \times S\right)\right), F\right),\left(\Phi_{0} R^{2 p} \tilde{f}_{X *} \mathbb{Z}_{\mathcal{Y}^{o}}, W\right), \Phi_{0} \tilde{f}_{X *} \alpha\left(\mathcal{Y}^{o}\right)\right)$.
For

$$
\alpha \in H^{2 p}\left(\mathcal{Y}_{\left(s, t^{\prime}\right)}^{a n}, \mathbb{Q}\right)=\mathbb{H}^{2 p}\left(\mathcal{E}_{t^{\prime}}^{a n}, \Phi_{0} R \tilde{f}_{X *} \mathbb{Q}_{\mathcal{Y}^{o}}\right),\left(s, t^{\prime}\right) \in\left(\mathbb{A}_{\mathbb{C}}^{1} \backslash 0\right) \times S_{\mathbb{C}}
$$

we consider

- $\pi_{S, G_{m}}=\pi_{S} \times \pi_{G_{m}}: \tilde{G}_{m} \times \tilde{S}_{\mathbb{C}}^{a n} \rightarrow\left(\mathbb{A}_{\mathbb{C}}^{1} \backslash 0\right) \times S_{\mathbb{C}}^{a n}$ the universal covering in $\operatorname{AnSm}(\mathbb{C})$ and

$$
\mathbb{V}_{\left(\mathbb{A}^{1} \backslash 0\right) \times S}(\alpha):=\pi_{S, G_{m}}\left(\alpha \times \tilde{G}_{m} \times \tilde{S}_{\mathbb{C}}^{a n}\right) \subset E_{D R}^{2 p, a n}\left(\mathcal{Y}_{\mathbb{C}}^{o} /\left(\mathbb{A}_{\mathbb{C}}^{1} \backslash 0\right) \times S_{\mathbb{C}}\right)
$$

with

$$
\begin{array}{r}
\pi_{S, G_{m}}: \tilde{S}_{\mathbb{C}}^{a n} \times \tilde{G}_{m} \times H^{2 p}\left(\mathcal{Y}_{\left(s, t^{\prime}\right)}^{a n}, \mathbb{C}\right) \rightarrow E_{D R}^{2 p, a n}\left(\mathcal{Y}_{\mathbb{C}}^{o} / S_{\mathbb{C}}\right),\left(D, D^{*}\right) \rightarrow\left(\mathbb{A}_{\mathbb{C}}^{r+1},\left(\mathbb{A}_{\mathbb{C}}^{1} \backslash 0\right) \times S_{\mathbb{C}}^{a n}\right), \\
\pi_{S, G_{m}}\left(z_{1}, \cdots, z_{r+1}, h\right):=\left(e^{2 i \pi z_{1}}, \cdots, e^{2 i \pi z_{r+1}}, e^{-\left(z_{1} N_{1}+\cdots+z_{r} N_{r+1}\right)} h\right)
\end{array}
$$

and

$$
\mathbb{V}_{\left(\mathbb{A}^{1} \backslash 0\right) \times S}^{p}(\alpha):=\mathbb{V}_{\left(\mathbb{A}^{1} \backslash 0\right) \times S}(\alpha) \cap F^{p} E_{D R}^{2 p}\left(\mathcal{Y}_{\mathbb{C}}^{o} /\left(\mathbb{A}_{\mathbb{C}}^{1} \backslash 0\right) \times S_{\mathbb{C}}\right) \subset E_{D R}^{2 p}\left(\mathcal{Y}_{\mathbb{C}}^{o} /\left(\mathbb{A}_{\mathbb{C}}^{1} \backslash 0\right) \times S_{\mathbb{C}}\right),
$$

which is an algebraic subvariety by [6],

- $\pi_{S}: \tilde{S}_{\mathbb{C}}^{a n} \rightarrow S_{\mathbb{C}}^{a n}$ the universal covering in $\operatorname{AnSm}(\mathbb{C})$ and

$$
\mathbb{V}_{S}(\alpha):=\pi_{S}\left(\alpha \times \tilde{S}_{\mathbb{C}}^{a n}\right) \subset H^{2 p} \Phi_{0} E_{D R}\left(\mathcal{Y}_{\mathbb{C}}^{o} /\left(\left(\mathbb{A}_{\mathbb{C}}^{1} \backslash 0\right) \times S_{\mathbb{C}}\right)\right)^{a n}
$$

with

$$
\begin{array}{r}
\pi_{S}: \tilde{S}_{\mathbb{C}}^{a n} \times H^{2 p}\left(\mathcal{Y}_{\left(s, t^{\prime}\right)}^{a n}, \mathbb{C}\right) \rightarrow H^{2 p} \Phi_{0} E_{D R}\left(\mathcal{Y}_{\mathbb{C}}^{o} /\left(\left(\mathbb{A}_{\mathbb{C}}^{1} \backslash 0\right) \times S_{\mathbb{C}}\right)\right)^{a n}, \\
\left(D, D^{*}\right) \rightarrow\left(\mathbb{A}_{\mathbb{C}}^{r}, S_{\mathbb{C}}^{a n}\right), \pi_{S}\left(z_{1}, \cdots, z_{r}, h\right):=\left(e^{2 i \pi z_{1}}, \cdots, e^{2 i \pi z_{r}}, e^{-\left(z_{1} N_{1}+\cdots+z_{r} N_{r}\right)} h\right)
\end{array}
$$

and

$$
\mathbb{V}_{S}^{p}(\alpha):=\mathbb{V}_{S}(\alpha) \cap F^{p} H^{2 p} \Phi_{0} E_{D R}\left(\mathcal{Y}_{\mathbb{C}}^{o} /\left(\left(\mathbb{A}_{\mathbb{C}}^{1} \backslash 0\right) \times S_{\mathbb{C}}\right)\right)^{a n} \subset H^{2 p} \Phi_{0} E_{D R}\left(\mathcal{Y}_{\mathbb{C}}^{o} /\left(\left(\mathbb{A}_{\mathbb{C}}^{1} \backslash 0\right) \times S_{\mathbb{C}}\right)\right)^{a n}
$$

Consider the exact sequence of variation of mixed Hodge structure over $S$

$$
0 \rightarrow W_{2 p-2} E_{H d g}^{2 p}(\mathcal{E} / S) \rightarrow E_{H d g}^{2 p}(\mathcal{E} / S) \xrightarrow{\left(i_{i}^{*}\right)_{0 \leq i \leq r}:=q} \oplus_{i=0}^{r} E_{H d g}^{2 p}\left(\mathcal{E}_{i} / S\right) \rightarrow 0
$$

It induces the exact sequence of vector bundles over $S$

$$
0 \rightarrow W_{2 p-2} E_{D R}^{2 p}(\mathcal{E} / S) \rightarrow E_{D R}^{2 p}(\mathcal{E} / S) \xrightarrow{\left(i_{i}^{*}\right)_{0 \leq i \leq r}:=q} \oplus_{i=0}^{r} E_{D R}^{2 p}\left(\mathcal{E}_{i} / S\right) \rightarrow 0,
$$

and the exact sequence of presheaves on $S_{\mathrm{C}}^{a n}$,

$$
\begin{array}{r}
H L^{2 p, p}\left(\mathcal{E}_{\mathbb{C}} / S_{\mathbb{C}}\right) \xrightarrow{q} \oplus_{i=0}^{r} H L^{2 p, p}\left(\mathcal{E}_{i, \mathbb{C}} / S_{\mathbb{C}}\right) \xrightarrow{e} \\
J\left(W_{2 p-2} E_{H d g}^{2 p}\left(\mathcal{E}_{\mathbb{C}} / S_{\mathbb{C}}\right)\right):=\operatorname{Ext}^{1}\left(\mathbb{Q}_{S}^{H d g}, W_{2 p-2} E_{H d g}^{2 p}\left(\mathcal{E}_{\mathbb{C}} / S_{\mathbb{C}}\right)\right) \stackrel{\xrightarrow{\rightarrow} \operatorname{Ext}^{1}\left(\mathbb{Q}_{S}^{H d g}, W_{2 p-2} E_{H d g}^{2 p}\left(\mathcal{E}_{\mathbb{C}} / S_{\mathbb{C}}\right)\right)}{ }
\end{array}
$$

where $H L^{2 p, p}\left(\mathcal{E}_{\mathbb{C}} / S_{\mathbb{C}}\right) \subset E_{D R}^{2 p}\left(\mathcal{E}_{\mathbb{C}} / S_{\mathbb{C}}\right), H L^{2 p, p}\left(\mathcal{E}_{i, \mathbb{C}} / S_{\mathbb{C}}\right) \subset E_{D R}^{2 p}\left(\mathcal{E}_{i, \mathrm{C}} / S_{\mathbb{C}}\right)$ are the locus of Hodge classes. Let $\lambda \in F^{p} H^{2 p}\left(E_{0}^{a n}, \mathbb{Q}\right)$, where we recall $E_{0}=\mathcal{E}_{0, \mathrm{C}, t}$. Consider then

- $\tilde{\lambda} \in H^{2 p}\left(E^{a n}, \mathbb{Q}\right)$, such that $q(\tilde{\lambda})=\lambda$, and $\tilde{\lambda}:=S p_{E / f_{X}}(\tilde{\lambda}) \in S p_{E / f_{X}}\left(H^{2 p}\left(E^{a n}, \mathbb{Q}\right)\right)$,
- $\tilde{\lambda}_{F} \in F^{p} H^{2 p}\left(E^{a n}, \mathbb{C}\right)$, such that $q\left(\tilde{\lambda}_{F}\right)=\lambda$, and since $S$ is affine $\mu \in \Gamma\left(S_{\mathbb{C}}, F^{p} E_{D R}^{2 p}\left(\mathcal{E}_{\mathbb{C}} / S_{\mathbb{C}}\right)\right)$ such that $\mu(t)=\tilde{\lambda}_{F}$,
- $\tilde{e}(\lambda):=[\tilde{\lambda}-\mu] \in \Gamma\left(\tilde{S}_{\mathbb{C}}^{a n}, W_{2 p-2} E_{D R, a n}^{2 p}\left(\mathcal{E}_{\mathbb{C}} / S_{\mathbb{C}}\right) / F^{p} \oplus \mathbb{V}_{\mathbb{Q}}\right)$.

The locus of Hodge classes passing through $\lambda$ is then an union of two components

$$
\mathbb{V}_{S}^{p}(\lambda):=\pi_{S}\left(\lambda \times \tilde{S}_{\mathbb{C}}\right) \cap F^{p} E_{D R}^{2 p}\left(\mathcal{E}_{0, \mathbb{C}} / S_{\mathbb{C}}\right)=q\left(\mathbb{V}_{S}^{p}(\tilde{\lambda})\right) \cup p_{S, E_{D R}, \mathbb{V}}^{-1}(V(\tilde{e}(\lambda))) \subset E_{D R}^{2 p}\left(\mathcal{E}_{0, \mathbb{C}} / S_{\mathbb{C}}\right),
$$

where

- $\pi_{S}: H^{2 p}\left(E^{a n}, \mathbb{C}\right) \times \tilde{S}_{\mathbb{C}}^{a n} \rightarrow E_{D R}^{2 p, a n}\left(\mathcal{E}_{\mathbb{C}} / S_{\mathbb{C}}\right)$ is the morphism given by the universal covering $\pi_{S}$ : $\tilde{S}_{\mathbb{C}}^{a n} \rightarrow S_{\mathbb{C}}^{a n}$
- $p_{S, E_{D R}, \mathbb{V}}: \mathbb{V}_{S}(\lambda) \subset E_{D R}^{2 p}\left(\mathcal{E}_{0, \mathbb{C}} / S_{\mathbb{C}}\right) \xrightarrow{p_{S}} S_{\mathbb{C}}$,
- $q:=q \otimes \mathbb{C}: E_{D R}^{2 p}\left(\mathcal{E}_{\mathbb{C}} / S_{\mathbb{C}}\right) \rightarrow E_{D R}^{2 p}\left(\mathcal{E}_{0, \mathbb{C}} / S_{\mathbb{C}}\right)$ is the quotient map.

Indeed, let $\lambda_{t^{\prime}}^{\prime} \in \mathbb{V}_{S}(\lambda), t^{\prime} \in S(\mathbb{C})$. If $\lambda_{t^{\prime}}^{\prime} \in \mathbb{V}_{S}^{p}(\lambda)$, then either $e\left(\lambda_{t^{\prime}}^{\prime}\right)=0$ and $\tilde{\lambda}_{t^{\prime}}^{\prime} \in \mathbb{V}_{S}(\tilde{\lambda}) \cap q^{-1}\left(\lambda_{t^{\prime}}^{\prime}\right)$ is a Hodge class by the above exact sequence, or $e\left(\lambda_{t^{\prime}}^{\prime}\right) \neq 0$ and $\iota\left(e\left(\lambda_{t^{\prime}}^{\prime}\right)\right)=0$ hence $\tilde{e}(\lambda)\left(t^{\prime}\right)=0$. On the other hand, if $\tilde{e}(\lambda)\left(t^{\prime}\right)=0, \iota\left(e\left(\lambda_{t^{\prime}}^{\prime}\right)\right)=0$ hence by the above exact sequence $\lambda_{t^{\prime}}^{\prime} \in \mathbb{V}_{S}^{p}(\lambda)$. Now,

- if $S p_{E / f_{X}}(\tilde{\lambda})=0$, since $S p_{\mathcal{E} / \tilde{f}_{X}} \circ \epsilon^{*}=S p_{\mathcal{Y}_{0} / \tilde{f}_{X}^{\prime}}$ and $S p_{\mathcal{Y}_{0} / \tilde{f}_{X}^{\prime}}$ is injective, we have $\tilde{\lambda}_{t^{\prime}}=k_{t *} \tilde{\lambda}_{t^{\prime}}$ with $\tilde{\lambda}_{t^{\prime}} \in F^{p} H_{\mathcal{F}_{\mathcal{C}, t^{\prime}}}^{2 p}\left(\mathcal{E}_{\mathbb{C}, t^{\prime}}^{a n}, \mathbb{Q}\right)$, where $k: \mathcal{F} \rightarrow \mathcal{E}$ is the exceptional locus of $\epsilon: \mathcal{E}_{0} \rightarrow \mathcal{Y}_{0}$, hence, assuming the Hodge conjecture for smooth projective varieties of dimension less or equal to $d-1$, $\mathbb{V}_{S}^{p}(\tilde{\lambda}) \subset E_{D R}^{2 p}\left(\mathcal{E}_{\mathbb{C}} / S_{\mathbb{C}}\right)$ is defined over $\mathbb{Q}$, if $S p_{E / f_{X}}(\tilde{\lambda}) \neq 0$, we have, by $[6]$,

$$
S p_{\mathcal{E} / \tilde{f}_{X}}\left(\mathbb{V}_{S}^{p}(\tilde{\lambda})\right)=\mathbb{V}_{S}^{p}\left(S p_{E / f_{X}}(\tilde{\lambda})\right)=\overline{\mathbb{V}_{\left(\mathbb{A}^{1} \backslash 0\right) \times S}^{p}\left(S p_{E / f_{X}}(\tilde{\lambda})\right)} \cap p^{-1}\left(0 \times S_{\mathbb{C}}\right) \subset E_{D R, V_{0}}^{2 p}\left(\mathcal{Y}_{\mathbb{C}} /\left(\mathbb{A}_{\mathbb{C}}^{1} \times S_{\mathbb{C}}\right)\right)
$$

where $p: E_{D R, V_{0}}^{2 p}\left(\mathcal{Y}_{\mathbb{C}} /\left(\mathbb{A}_{\mathbb{C}}^{1} \times S_{\mathbb{C}}\right)\right) \rightarrow \mathbb{A}_{\mathbb{C}}^{1} \times S_{\mathbb{C}}$ is the projection (the first equality is obvious since $S p_{\mathcal{E} / \tilde{f}_{X}}$ is a morphism of variation of mixed Hodge structures over $S$, whereas for the second equality the inclusion $\supset$ follows from [6] lemma 2.11 and the invariant cycle theorem of a geometric variation of Hodge structures by strictness of the Hodge filtration), but by [5] theorem 1, the Hodge conjecture holds for projective hypersurfaces hence

$$
\mathbb{V}_{\left(\mathbb{A}^{1} \backslash 0\right) \times S}^{p}\left(S p_{E / f_{X}}(\tilde{\lambda})\right) \subset E_{D R}^{2 p}\left(\mathcal{Y}_{\mathbb{C}}^{o} /\left(\left(\mathbb{A}_{\mathbb{C}}^{1} \backslash 0\right) \times S_{\mathbb{C}}\right)\right)
$$

is defined over $\mathbb{Q}$, hence $\mathbb{V}_{S}^{p}(\tilde{\lambda}) \subset E_{D R}^{2 p}\left(\mathcal{E}_{\mathbb{C}} / S_{\mathbb{C}}\right)$ is defined over $\mathbb{Q}$,

- assuming the Hodge conjecture for smooth projective varieties of dimension less or equal to $d-1$,

$$
V(\tilde{e}(\lambda)) \subset S_{\mathbb{C}}
$$

is an algebraic variety defined over $\mathbb{Q}$, since $\tilde{e}(\lambda)\left(t^{\prime}\right)=0$ if and only if $\left((\tilde{\lambda})^{\prime}-\mu\left(t^{\prime}\right)\right) \in W_{2 p-2} H^{2 p}\left(\mathcal{E}_{\mathbb{C}, t^{\prime}}, \mathbb{C}\right)$ is a Hodge class where $(\tilde{\lambda})^{\prime} \in \pi_{S}\left(\left\{t^{\prime}\right\} \times \tilde{\lambda}\right)$.

Hence $\mathbb{V}_{S}^{p}(\lambda) \subset E_{D R}^{2 p}\left(\mathcal{E}_{0 \mathbb{C}} / S_{\mathbb{C}}\right)$ is defined over $\mathbb{Q}$. Thus, by [4] theorem 4 ,

$$
\lambda=[Z] \in H^{2 p}\left(E_{0}^{a n}, \mathbb{Q}\right), \text { with } Z \in \mathcal{Z}^{p}\left(E_{0}\right)
$$

Since $p \in \mathbb{Z}$ and $\lambda \in F^{p} H^{2 p}\left(E_{0}^{a n}, \mathbb{Q}\right)$ are arbitrary, the Hodge conjecture holds for $E_{0}$. Proposition 1(ii) and the fact that the Hodge conjecture is true for smooth projective varieties of dimension less or equal to $d-1$ by induction hypothesis then implies that Hodge conjecture holds for $X$, since $X$ is birational to $E_{0}$ and $X, E_{0} \in \operatorname{PSmVar}(\mathbb{C})$ are connected of dimension $d$.

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