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# Degeneration of families of projective hypersurfaces and Hodge conjecture

Johann Bouali

January 11, 2024

## Abstract

We prove by induction on dimension the Hodge conjecture for smooth complex projective varieties. Let  $X$  be a smooth complex projective variety. Then  $X$  is birational to a possibly singular projective hypersurface, hence to a smooth projective variety  $E_0$  which is a component of a normal crossing divisor  $E = \cup_{i=0}^r E_i \subset Y$  which is the singular fiber of a pencil  $f : Y \rightarrow \mathbb{A}^1$  of smooth projective hypersurfaces. Using the smooth hypersurface case ([5] theorem 1), the nearby cycle functor on mixed Hodge module with rational de Rham factor and the induction hypothesis, we prove that a Hodge class of  $E_0$  is absolute Hodge, more precisely the locus of Hodge classes inside the algebraic vector bundle given the De Rham cohomology of the rational deformation of  $E_0$  is defined over  $\mathbb{Q}$ . By [4] theorem 4, we get the Hodge conjecture for  $E_0$ . By the induction hypothesis we also have the Hodge conjecture for  $X$ .

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## 1 Introduction

The main result of this article is an inductive proof of the Hodge conjecture using the results of [4] and [5]. More precisely, assuming the Hodge conjecture for smooth projective varieties of dimension less or equal to  $d - 1$ , the property of satisfying the Hodge conjecture for connected smooth projective varieties of dimension  $d$  is a birational invariant (proposition 1). On the other hand, if  $X$  is a connected smooth projective variety of dimension  $d$ , there exists (see proposition 2) a family of smooth projective hypersurfaces  $f_X : Y \rightarrow \mathbb{A}_{\mathbb{C}}^1$  with

- $Y$  a smooth complex algebraic variety,  $f_X$  flat projective,
- $Y_s := f_X^{-1}(s) \subset \mathbb{P}_{\mathbb{C}}^{d+1}$  for  $s \in \mathbb{A}_{\mathbb{C}}^1 \setminus \{0\}$  are smooth projective hypersurfaces,
- $E := f_X^{-1}(0) = \cup_{i=0}^r E_i \subset Y$  is a normal crossing divisor and  $E_0$  is birational to  $X$ .

Assuming the Hodge conjecture for smooth projective varieties of dimension less or equal to  $d - 1$ , we then prove the Hodge conjecture for the  $E_i$ , more specifically for  $E_0$  as follows : We consider its canonical rational deformation  $\tilde{f}_X : \mathcal{Y} \rightarrow \mathbb{A}_{\mathbb{Q}}^1 \times S$  with

- $\mathcal{Y}$  a smooth algebraic variety over  $\mathbb{Q}$ ,  $S \subset \mathbb{A}_{\mathbb{Q}}^s$  an open subset,  $\tilde{f}_X$  flat projective
- $Y = \tilde{f}_X^{-1}(\mathbb{A}_{\mathbb{Q}}^1 \times \{t\})$ ,  $f_X = \tilde{f}_X \times_S \{t\}$ , with  $t \in S(\mathbb{C})$
- $\mathcal{E} = \cup_{i=0}^r \mathcal{E}_i := \tilde{f}_X^{-1}(\{0\} \times S) \subset \mathcal{Y}$  is a normal crossing divisor,  $\tilde{f}_{X,0} : \mathcal{E}_I := \cap_{i \in I} \mathcal{E}_i \rightarrow S$  smooth projective for each  $I \subset [0, \dots, r]$ ,  $\mathcal{E}_{\mathbb{C},t} = E$ ,
- $\tilde{f}_X \times_{\mathbb{A}^1} (\mathbb{A}^1 \setminus \{0\}) : \mathcal{Y}^\circ \rightarrow \mathbb{A}_{\mathbb{Q}}^1 \setminus \{0\}$  has fibers which are smooth projective hypersurfaces.

We then have then an injection of mixed Hodge module over  $S$  (in fact of geometric variation of mixed Hodge structures over  $S$ ) given by the specialization map

$$Sp_{\mathcal{E}/\tilde{f}_X} : E_{Hdg}^{2p}(\mathcal{E}/S) \hookrightarrow H^{2p} \Phi_0 E_{Hdg}(\mathcal{Y}^\circ / ((\mathbb{A}_{\mathbb{Q}}^1 \setminus \{0\}) \times S)),$$

whose de Rham part are morphism of filtered vector bundle over  $S$  with Gauss-Manin connexion defined over  $\mathbb{Q}$ . The injectivity of  $Sp_{\mathcal{E}/\tilde{f}_X}$  follows from the fact that  $H^{2p-1}(Y_s, \mathbb{Q}) = 0 = H_E^{2p-1}(Y)$ , by the weak Lefschetz theorem since  $Y_s$  and  $E$  are projective hypersurfaces. Consider the exact sequence of variation of mixed Hodge structure over  $S$

$$0 \rightarrow W_{2p-2} E_{Hdg}^{2p}(\mathcal{E}/S) \rightarrow E_{Hdg}^{2p}(\mathcal{E}/S) \xrightarrow{q} \oplus_{i=0}^r E_{Hdg}^{2p}(\mathcal{E}_i/S) \rightarrow 0.$$

It induces the exact sequence of presheaves on  $S_{\mathbb{C}}^{an}$ ,

$$HL^{2p,p}(\mathcal{E}_{\mathbb{C}}/S_{\mathbb{C}}) \xrightarrow{q} \oplus_{i=0}^r HL^{2p,p}(\mathcal{E}_{i,\mathbb{C}}/S_{\mathbb{C}}) \xrightarrow{e} J(W_{2p-2} E_{Hdg}^{2p}(\mathcal{E}_{\mathbb{C}}/S_{\mathbb{C}})) := \text{Ext}^1(\mathbb{Q}_S^{Hdg}, W_{2p-2} E_{Hdg}^{2p}(\mathcal{E}_{\mathbb{C}}/S_{\mathbb{C}})),$$

where  $HL^{2p,p}(\mathcal{E}_{\mathbb{C}}/S_{\mathbb{C}}) \subset E_{DR}^{2p}(\mathcal{E}_{\mathbb{C}}/S_{\mathbb{C}})$ ,  $HL^{2p,p}(\mathcal{E}_{i,\mathbb{C}}/S_{\mathbb{C}}) \subset E_{DR}^{2p}(\mathcal{E}_{i,\mathbb{C}}/S_{\mathbb{C}})$  are the locus of Hodge classes. Let  $\lambda \in F^p H^{2p}(E_0^{an}, \mathbb{Q})$ , where we recall  $E_0 = \mathcal{E}_{0,\mathbb{C},t}$ . Consider

$$\tilde{\lambda} \in H^{2p}(E^{an}, \mathbb{Q}), \text{ s.t. } q(\tilde{\lambda}) = \lambda, \text{ and } \tilde{\lambda} := Sp_{E/f_X}(\tilde{\lambda}) \in Sp_{E/f_X}(H^{2p}(E^{an}, \mathbb{Q})).$$

The locus of Hodge classes passing through  $\lambda$

$$\mathbb{V}_S^p(\lambda) := \pi_S(\lambda \times \tilde{S}_{\mathbb{C}}) \cap F^p E_{DR}^{2p}(\mathcal{E}_{0,\mathbb{C}}/S_{\mathbb{C}}) = q(\mathbb{V}_S^p(\tilde{\lambda})) \cup p_{S, E_{DR}}^{-1}(V(e(\lambda))) \subset E_{DR}^{2p}(\mathcal{E}_{0,\mathbb{C}}/S_{\mathbb{C}})$$

is thus an intersection the De Rham vector bundle, where  $\pi_S : H^{2p}(E^{an}, \mathbb{C}) \times \tilde{S}_{\mathbb{C}}^{an} \rightarrow E_{DR}^{2p,an}(\mathcal{E}_{\mathbb{C}}/S_{\mathbb{C}})$  is the morphism given by the universal covering  $\pi_S : \tilde{S}_{\mathbb{C}}^{an} \rightarrow S_{\mathbb{C}}^{an}$ . Now,

- by [5] theorem 1, the Hodge conjecture holds for projective hypersurfaces, hence

$$Sp_{\mathcal{E}/\tilde{f}_X}(\mathbb{V}_S^p(\tilde{\lambda})) = \overline{\mathbb{V}_{(\mathbb{A}^1 \setminus \{0\}) \times S}^p(Sp_{E/f_X}(\tilde{\lambda}))} \cap p^{-1}(0 \times S_{\mathbb{C}}) \subset E_{DR, V_0}^{2p}(\mathcal{Y}_{\mathbb{C}}/(\mathbb{A}_{\mathbb{C}}^1 \times S_{\mathbb{C}}))$$

is an algebraic subvariety defined over  $\mathbb{Q}$ , where the equality follows from [6] lemma 2.11,

- assuming the Hodge conjecture for smooth projective varieties of dimension less or equal to  $d - 1$ ,

$$\mathbb{V}(e(\tilde{\lambda})) \subset S_{\mathbb{C}}$$

is an algebraic variety defined over  $\mathbb{Q}$ .

Hence,  $\mathbb{V}_S^p(\lambda) \subset E_{DR}^{2p}(\mathcal{E}_{0,\mathbb{C}}/S_{\mathbb{C}})$  is defined over  $\mathbb{Q}$ . By [4] theorem 4, we get

$$\lambda = [Z] \in H^{2p}(E_0^{an}, \mathbb{Q}), \quad Z \in \mathcal{Z}^p(E_0).$$

Since  $p \in \mathbb{Z}$  and  $\lambda \in F^p H^{2p}(E_0^{an}, \mathbb{Q})$  are arbitrary, this proves the Hodge conjecture for  $E_0$ . Since  $X$  is birational to  $E_0$ , the Hodge conjecture for  $E_0$  and the induction hypothesis implies the Hodge conjecture for  $X$ .

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## 2 Preliminaries and notations

### 2.1 Notations

- Denote by  $\text{Top}$  the category of topological spaces and  $\text{RTop}$  the category of ringed spaces.
- Denote by  $\text{Cat}$  the category of small categories and  $\text{RCat}$  the category of ringed topoi.
- For  $\mathcal{S} \in \text{Cat}$  and  $X \in \mathcal{S}$ , we denote  $\mathcal{S}/X \in \text{Cat}$  the category whose objects are  $Y/X := (Y, f)$  with  $Y \in \mathcal{S}$  and  $f : Y \rightarrow X$  is a morphism in  $\mathcal{S}$ , and whose morphisms  $\text{Hom}((Y', f'), (Y, f))$  consists of  $g : Y' \rightarrow Y$  in  $\mathcal{S}$  such that  $f \circ g = f'$ .
- Let  $(\mathcal{S}, O_{\mathcal{S}}) \in \text{RCat}$  a ringed topos with topology  $\tau$ . For  $F \in C_{O_{\mathcal{S}}}(\mathcal{S})$ , we denote by  $k : F \rightarrow E_{\tau}(F)$  the canonical flasque resolution in  $C_{O_{\mathcal{S}}}(\mathcal{S})$  (see [3]). In particular for  $X \in \mathcal{S}$ ,  $H^*(X, E_{\tau}(F)) \xrightarrow{\sim} \mathbb{H}_{\tau}^*(X, F)$ .
- For  $f : \mathcal{S}' \rightarrow \mathcal{S}$  a morphism with  $\mathcal{S}, \mathcal{S}' \in \text{RCat}$ , endowed with topology  $\tau$  and  $\tau'$  respectively, we denote for  $F \in C_{O_{\mathcal{S}}}(\mathcal{S})$  and each  $j \in \mathbb{Z}$ ,

$$\begin{aligned} - f^* &:= H^j \Gamma(\mathcal{S}, k \circ \text{ad}(f^*, f_*)(F)) : \mathbb{H}^j(\mathcal{S}, F) \rightarrow \mathbb{H}^j(\mathcal{S}', f^*F), \\ - f^{*mod} &:= H^j \Gamma(\mathcal{S}, k \circ \text{ad}(f^{*mod}, f_*)(F)) : \mathbb{H}^j(\mathcal{S}, F) \rightarrow \mathbb{H}^j(\mathcal{S}', f^{*mod}F), \end{aligned}$$

the canonical maps.

- For  $m : A \rightarrow B$ ,  $A, B \in C(\mathcal{A})$ ,  $\mathcal{A}$  an additive category, we denote  $c(A) : \text{Cone}(m : A \rightarrow B) \rightarrow A[1]$  and  $c(B) : B \rightarrow \text{Cone}(m : A \rightarrow B)$  the canonical maps.
- Denote by  $\text{Sch} \subset \text{RTop}$  the subcategory of schemes (the morphisms are the morphisms of locally ringed spaces). We denote by  $\text{PSch} \subset \text{Sch}$  the full subcategory of proper schemes. For a field  $k$ , we consider  $\text{Sch}/k := \text{Sch}/\text{Spec } k$  the category of schemes over  $\text{Spec } k$ . The objects are  $X := (X, a_X)$  with  $X \in \text{Sch}$  and  $a_X : X \rightarrow \text{Spec } k$  a morphism and the morphisms are the morphisms of schemes  $f : X' \rightarrow X$  such that  $f \circ a_{X'} = a_X$ . We then denote by
  - $\text{Var}(k) = \text{Sch}^{ft}/k \subset \text{Sch}/k$  the full subcategory consisting of algebraic varieties over  $k$ , i.e. schemes of finite type over  $k$ ,
  - $\text{PVar}(k) \subset \text{QPVar}(k) \subset \text{Var}(k)$  the full subcategories consisting of quasi-projective varieties and projective varieties respectively,
  - $\text{PSmVar}(k) \subset \text{SmVar}(k) \subset \text{Var}(k)$ ,  $\text{PSmVar}(k) := \text{PVar}(k) \cap \text{SmVar}(k)$ , the full subcategories consisting of smooth varieties and smooth projective varieties respectively.
- Denote by  $\text{AnSp}(\mathbb{C}) \subset \text{RTop}$  the subcategory of analytic spaces over  $\mathbb{C}$ , and by  $\text{AnSm}(\mathbb{C}) \subset \text{AnSp}(\mathbb{C})$  the full subcategory of smooth analytic spaces (i.e. complex analytic manifold).
- For  $X \in \text{Var}(k)$  and  $X = \cup_{i \in I} X_i$  with  $i_i : X_i \hookrightarrow X$  closed embeddings, we denote  $X_{\bullet} \in \text{Fun}(\Delta, \text{Var}(k))$  the associated simplicial space, with for  $J \subset I$ ,  $i_{IJ} : X_I := \cap_{i \in I} X_i \hookrightarrow X_J := \cap_{i \in J} X_i$  the closed embedding.
- Let  $(X, O_X) \in \text{RTop}$ . We consider its De Rham complex  $\Omega_X^{\bullet} := DR(X)(O_X)$ .
  - Let  $X \in \text{Sch}$ . Considering its De Rham complex  $\Omega_X^{\bullet} := DR(X)(O_X)$ , we have for  $j \in \mathbb{Z}$  its De Rham cohomology  $H_{DR}^j(X) := \mathbb{H}^j(X, \Omega_X^{\bullet})$ .
  - Let  $X \in \text{Var}(k)$ . Considering its De Rham complex  $\Omega_X^{\bullet} := \Omega_{X/k}^{\bullet} := DR(X/k)(O_X)$ , we have for  $j \in \mathbb{Z}$  its De Rham cohomology  $H_{DR}^j(X) := \mathbb{H}^j(X, \Omega_X^{\bullet})$ . The differentials of  $\Omega_X^{\bullet} := \Omega_{X/k}^{\bullet}$  are by definition  $k$ -linear, thus  $H_{DR}^j(X) := \mathbb{H}^j(X, \Omega_X^{\bullet})$  has a structure of a  $k$  vector space.

- Let  $X \in \text{AnSp}(\mathbb{C})$ . Considering its De Rham complex  $\Omega_X^\bullet := DR(X)(O_X)$ , we have for  $j \in \mathbb{Z}$  its De Rham cohomology  $H_{DR}^j(X) := \mathbb{H}^j(X, \Omega_X^\bullet)$ .
- For  $X \in \text{AnSp}(\mathbb{C})$ , we denote  $\alpha(X) : \mathbb{C}_X \hookrightarrow \Omega_X^\bullet$  the embedding in  $C(X)$ . For  $X \in \text{AnSm}(\mathbb{C})$ ,  $\alpha(X) : \mathbb{C}_X \hookrightarrow \Omega_X^\bullet$  is an equivalence usu local by Poincare lemma.
- We denote  $\mathbb{I}^n := [0, 1]^n \in \text{Diff}(\mathbb{R})$  (with boundary). For  $X \in \text{Top}$  and  $R$  a ring, we consider its singular cochain complex

$$C_{\text{sing}}^*(X, R) := (\mathbb{Z} \text{Hom}_{\text{Top}}(\mathbb{I}^*, X)^\vee) \otimes R$$

and for  $l \in \mathbb{Z}$  its singular cohomology  $H_{\text{sing}}^l(X, R) := H^n C_{\text{sing}}^*(X, R)$ . For  $f : X' \rightarrow X$  a continuous map with  $X, X' \in \text{Top}$ , we have the canonical map of complexes

$$f^* : C_{\text{sing}}^*(X, R) \rightarrow C_{\text{sing}}^*(X', R), \sigma \mapsto f^* \sigma := (\gamma \mapsto \sigma(f \circ \gamma)).$$

In particular, we get by functoriality the complex

$$C_{X, R, \text{sing}}^* \in C_R(X), (U \subset X) \mapsto C_{\text{sing}}^*(U, R)$$

We recall that

- For  $X \in \text{Top}$  locally contractible, e.g.  $X \in \text{CW}$ , and  $R$  a ring, the inclusion in  $C_R(X)$   $c_X : R_X \rightarrow C_{X, R, \text{sing}}^*$  is by definition an equivalence top local and that we get by the small chain theorem, for all  $l \in \mathbb{Z}$ , an isomorphism  $H^l c_X : H^l(X, R_X) \xrightarrow{\sim} H_{\text{sing}}^l(X, R)$ .
- For  $X \in \text{Diff}(\mathbb{R})$ , the restriction map

$$r_X : \mathbb{Z} \text{Hom}_{\text{Diff}(\mathbb{R})}(\mathbb{I}^*, X)^\vee \rightarrow C_{\text{sing}}^*(X, R), w \mapsto w : (\phi \mapsto w(\phi))$$

is a quasi-isomorphism by Whitney approximation theorem.

## 2.2 Birational projective varieties and Hodge conjecture

In this subsection, we recall that assuming the Hodge conjecture for smooth projective varieties of dimension less or equal to  $d - 1$ , the property of satisfying the Hodge conjecture for connected smooth projective varieties of dimension  $d$  is a birational invariant :

**Proposition 1.** (i) *Let  $X \in \text{PSmVar}(\mathbb{C})$  connected of dimension  $d$ . Let  $\epsilon : \tilde{X}_Z \rightarrow X$  be the blow up of  $X$  along a smooth closed subvariety  $Z \subset X$ . Assume the Hodge conjecture hold for smooth complex projective varieties of dimension less or equal to  $d - 1$ . Then the Hodge conjecture hold for  $X$  if and only if it hold for  $\tilde{X}_Z$ .*

(ii) *Let  $X, X' \in \text{PSmVar}(\mathbb{C})$  connected of dimension  $d$ . Assume the Hodge conjecture hold for smooth complex projective varieties of dimension less or equal to  $d - 1$ . If  $X$  is birational to  $X'$ , then the Hodge conjecture hold for  $X$  if and only if it hold for  $X'$*

*Proof.* (i): Follows from the fact that

$$(\epsilon^*, \oplus_{l=1}^c i_{E*}((-).h^{l-1})) : H^k(X, \mathbb{Q}) \oplus \oplus_{l=1}^c H^{k-2l}(E, \mathbb{Q}) \rightarrow H^k(\tilde{X}_Z, \mathbb{Q})$$

is an isomorphism of Hodge structures for each  $k \in \mathbb{Z}$ , where  $c = \text{codim}(Z, X)$ ,  $i_E : E \hookrightarrow \tilde{X}_Z$  is the closed embedding,  $\epsilon|_E : E \rightarrow Z$  being a projective vector bundle.

(ii): Follows from (i) since if  $\pi : X^o \rightarrow X'$  is a birational map,  $X^o \subset X$  being an open subset, then  $X$  is connected to  $X'$  by a sequence of blow up of smooth projective varieties with smooth center by [1].  $\square$

## 2.3 A family of smooth projective hypersurfaces associated to a smooth projective variety

In this subsection, we recall that given a smooth complex projective variety  $X$ , there exists a family of smooth complex projective hypersurfaces which degenerates into a normal crossing divisor with one irreducible component birational to  $X$ .

**Proposition 2.** *Let  $X \in \text{PSmVar}(\mathbb{C})$  connected. Then there exists a family of smooth projective hypersurfaces  $f_X : Y \rightarrow \mathbb{A}_{\mathbb{C}}^1$  with*

- $Y \in \text{SmVar}(\mathbb{C})$ ,  $f_X$  flat projective,
- $Y_s := f_X^{-1}(s) \subset \mathbb{P}_{\mathbb{C}}^{d+1}$  for  $s \in \mathbb{A}_{\mathbb{C}}^1 \setminus \{0\}$  are smooth projective hypersurfaces,
- $E := f_X^{-1}(0) = \cup_{i=0}^r E_i \subset Y$  is a normal crossing divisor and  $E_0$  is birational to  $X$ ,

*Proof.* See [2] lemma 1.5.8 . It use the fact that  $X$  is birational to a possibly singular projective hypersurface  $Y$  and consider the desingularization of a generic pencil passing through  $Y$ .  $\square$

## 2.4 The nearby cycle functor and the specialization map

Recall from [3] that for  $k \subset \mathbb{C}$  a subfield and  $S \in \text{Var}(k)$  quasi-projective, and  $l : S \hookrightarrow \tilde{S}$  a closed embedding with  $\tilde{S} \in \text{SmVar}(k)$ , we have the full subcategory

$$\iota : \text{MHM}_{k,gm}(S) \hookrightarrow \text{PSh}_{\mathcal{D}fil,S}(\tilde{S}) \times_I P(S_{\mathbb{C}}^{an})$$

consisting of geometric mixed Hodge module whose De Rham part is defined over  $k$ , where, using the fact that the  $V$ -filtration is defined over  $k$ ,  $\text{PSh}_{\mathcal{D}(1,0)fil,S}(\tilde{S}) \times_I P_{fil}(S_{\mathbb{C}}^{an})$  is the category

- whose objects are  $((M, F, W), (K, W), \alpha)$  where  $(M, F, W)$  is a filtered  $D_{\tilde{S}}$  module supported on  $S$ , where  $l : S \hookrightarrow \tilde{S}$  is a closed embedding and  $\tilde{S} \in \text{SmVar}(k)$ ,  $(K, W)$  is a filtered Perverse sheaf on  $S_{\mathbb{C}}^{an}$ , and  $\alpha : l_*(K, W) \otimes \mathbb{C} \rightarrow DR(\tilde{S})((M, W)^{an})$  is an isomorphism in  $D_{fil}(\tilde{S}_{\mathbb{C}}^{an})$ .
- whose morphism between  $((M, F, W), (K, W), \alpha)$  and  $((M', F, W), (K', W), \alpha')$ , are couples of morphisms  $(\phi : (M, F, W) \rightarrow (M', F, W), \psi : (K, W) \rightarrow (K', W))$  such that  $\phi \circ \alpha = \alpha' \circ \psi$ .

Then, by [3], we get for  $S \in \text{Var}(k)$  quasi-projective and  $l : S \hookrightarrow \tilde{S}$  is a closed embedding with  $\tilde{S} \in \text{SmVar}(k)$  an embedding

$$\iota : D(\text{MHM}_{k,gm}(S)) \hookrightarrow D_{\mathcal{D}fil,S}(\tilde{S}) \times_I D(S_{\mathbb{C}}^{an})$$

consisting of mixed Hodge module whose De Rham part is defined over  $k$ , where  $D_{\mathcal{D}(1,0)fil,S}(\tilde{S}) \times_I D_{fil,c}(S_{\mathbb{C}}^{an})$  is the category

- whose objects are  $((M, F, W), (K, W), \alpha)$  where  $(M, F, W)$  is a complex of filtered  $D_{\tilde{S}}$  module supported on  $S$ ,  $(K, W)$  is a filtered complex of presheaves on  $S_{\mathbb{C}}^{an}$  whose cohomology are constructible sheaves, and  $\alpha : l_*(K, W) \otimes \mathbb{C} \rightarrow DR(\tilde{S})((M, W)^{an})$  is an isomorphism in  $D_{fil}(\tilde{S}_{\mathbb{C}}^{an})$ .
- whose morphism are given in [3],

We have then the six functors formalism :

$$\begin{aligned} D(\text{MHM}_{k,gm}(-)) : \text{Var}(k) &\mapsto \text{TriCat}, \quad S \mapsto D(\text{MHM}_{k,gm}(S)), \quad (f : T \rightarrow S) \mapsto \\ (f^*, f_*) : D(\text{MHM}_{k,gm}(S)) &\rightarrow D(\text{MHM}_{k,gm}(T)), \quad (f!, f^!) : D(\text{MHM}_{k,gm}(T)) \rightarrow D(\text{MHM}_{k,gm}(S)) \end{aligned}$$

and for  $D \subset S$  a (Cartier) divisor the nearby cycle functor

$$\begin{aligned} \Phi_D : D(\text{MHM}_{k,gm}(S)) &\rightarrow D(\text{MHM}_{k,gm}(D)), \\ \Phi_D((M, F, W), (K, W), \alpha) &:= (\Phi_D(M, F, W), \Phi_D(K, W), \Phi_D(\alpha)). \end{aligned}$$

For  $f : X \rightarrow S$  a projective morphism with  $X, S \in \text{SmVar}(k)$ , we denote

$$E_{Hdg}(X/S) := f_* \mathbb{Z}_X^{Hdg} := \left( \int f(O_X, F), Rf_* \mathbb{Z}_{X_{\mathbb{C}^n}}, f_* \alpha(X) \right) \in D(MHM_{k, gm}(S)).$$

We then have the specialization map :

**Definition-Proposition 1.** *Let  $k$  be field of characteristic zero. Let  $f : X \rightarrow \mathbb{A}_k^1 \times S$  be a projective morphism with  $X, S \in \text{SmVar}(k)$ . Assume that*

- $f^o := f|_{\mathbb{A}^1 \setminus 0} : X^o \rightarrow (\mathbb{A}^1 \setminus 0) \times S$  is smooth
- $E := f^{-1}(0 \times S) = \cup_{i=0}^r E_i \subset X$  is a normal crossing divisor and that for  $I \subset [0, \dots, r]$ ,  $f_{0,I} := f|_{E_I} : E_I := \cap_{i \in I} E_i \rightarrow S$  are smooth. We have by definition  $X^o = X \setminus E$ .

Denote for  $I \subset J \subset [0, \dots, r]$ ,  $i_{JI} : E_J \hookrightarrow E_I$  and  $i_I : E_I \hookrightarrow E$ ,  $i_E : E \hookrightarrow X$  the closed embeddings. We have then, c.f. [8], the distinguished triangles in  $D(MHM_{k, gm}(E))$ ,

$$\begin{aligned} \mathbb{Z}_E^{Hdg} &:= i_E^* \mathbb{Z}_X^{Hdg} = ((i_{\bullet, \bullet} \Omega_{E^\bullet}^\bullet, F, W), (\mathbb{Z}_E, W), \alpha(E)) \xrightarrow{sp_{E/X}} \\ \Phi_E \mathbb{Z}_{X^o}^{Hdg} &= ((i_E^{*mod} \Omega_{X/\mathbb{A}^1 \times S}^\bullet(\log E), F, W), \Phi_E \mathbb{Z}_{X^o}, \Phi_E \alpha(X)) \xrightarrow{c(\Phi_E \mathbb{Z}_{X^o}^{Hdg})} \\ \phi_E \mathbb{Z}_{X^o}^{Hdg} &:= \text{Cone}(sp_{E/X}) \xrightarrow{c(\mathbb{Z}_E^{Hdg})} \mathbb{Z}_E^{Hdg}[1]. \end{aligned}$$

Applying the functor  $(f \circ i_E)_*$ , we obtain the generalized distinguished triangle in  $D(MHM_{k, gm}(S))$

$$\begin{aligned} E_{Hdg}(E/S) &\xrightarrow{sp_{E/f} := (f \circ i_E)_* sp_{E/X}} (f \circ i_E)_*(\Phi_E \mathbb{Z}_{X^o}^{Hdg}) = \Phi_0 E_{Hdg}(X^o / ((\mathbb{A}^1 \setminus 0) \times S)) \\ \xrightarrow{(f \circ i_E)_* c(\Phi_E \mathbb{Z}_{X^o}^{Hdg})} &(f \circ i_E)_*(\phi_E \mathbb{Z}_{X^o}^{Hdg}) = \phi_0 E_{Hdg}(X^o / ((\mathbb{A}^1 \setminus 0) \times S)) \xrightarrow{(f \circ i_E)_* c(\mathbb{Z}_X^{Hdg})} E_{Hdg}(E/S)[1] \end{aligned}$$

If for each  $t \in S(\mathbb{C})$ ,  $H^{k-1}(X_{(s,t)}, \mathbb{Q}) = 0$  for  $s \in \mathbb{A}_{\mathbb{C}}^1 \setminus 0$  and  $H_E^{k-1}(X_{\mathbb{A}^1 \times t}, \mathbb{Q}) = 0$ , then the induced map on cohomology

$$sp_{E/f} : E_{Hdg}^k(E/S) \rightarrow H^k \Phi_0 E_{Hdg}(X^o / ((\mathbb{A}^1 \setminus 0) \times S))$$

is injective.

*Proof.* Follows from the exact sequence

$$\dots \rightarrow H_{E_t}^l(X_{\mathbb{A}^1 \times t}, \mathbb{Q}) \rightarrow \mathbb{H}^l(E_t, \phi_0 \mathbb{Z}_{X^o(\mathbb{A}^1 \setminus 0) \times t}) \xrightarrow{V} \mathbb{H}^l(E_t, \Phi_0 \mathbb{Z}_{X^o(\mathbb{A}^1 \setminus 0) \times t}) \rightarrow H_{E_t}^{l+1}(X_{\mathbb{A}^1 \times t}, \mathbb{Q}) \rightarrow \dots$$

given in [8]. □

### 3 Hodge conjecture for smooth projective varieties

**Theorem 1.** *Let  $X \in \text{PSmVar}(\mathbb{C})$ . Then Hodge conjecture hold for  $X$ . That is, if  $p \in \mathbb{Z}$  and  $\lambda \in F^p H^{2p}(X^{an}, \mathbb{Q})$ ,  $\lambda = [Z]$  with  $Z \in \mathcal{Z}^p(X)$ .*

*Proof.* Up to split  $X$  into its connected components, we may assume that  $X$  is connected of dimension  $d$ . The Hodge conjecture is true for curves and surfaces. Assume that the Hodge conjecture is true for smooth projective varieties of dimension less or equal to  $d - 1$ . Consider by proposition 2 a family of smooth projective hypersurfaces  $f_X : Y \rightarrow \mathbb{A}_{\mathbb{C}}^1$  with

- $Y \in \text{SmVar}(\mathbb{C})$ ,  $f_X$  flat projective,
- $Y_s := f_X^{-1}(s) \subset \mathbb{P}_{\mathbb{C}}^{d+1}$  for  $s \in \mathbb{A}_{\mathbb{C}}^1 \setminus 0$  are smooth projective hypersurfaces

- $E := f_X^{-1}(0) = \cup_{i=0}^r E_i \subset Y$  is a normal crossing divisor and  $E_0$  is birational to  $X$ ,

and its canonical rational deformation  $\tilde{f}_X : \mathcal{Y} \rightarrow \mathbb{A}_{\mathbb{Q}}^1 \times S$  with

- $\mathcal{Y} \in \text{SmVar}(\mathbb{Q})$ ,  $S \subset \mathbb{A}_{\mathbb{Q}}^s$  an open subset,  $\tilde{f}_X$  flat projective
- $Y = \tilde{f}_X^{-1}(\mathbb{A}_{\mathbb{Q}}^1 \times \{t\})$ ,  $f_X = \tilde{f}_X \times_S \{t\}$ , with  $t \in S(\mathbb{C})$
- $\mathcal{E} = \cup_{i=0}^r \mathcal{E}_i := \tilde{f}_X^{-1}(\{0\} \times S) \subset \mathcal{Y}$  is a normal crossing divisor,  $\tilde{f}_{X,0} : \mathcal{E}_I := \cap_{i \in I} \mathcal{E}_i \rightarrow S$  smooth projective for each  $I \subset [0, \dots, r]$ ,  $\mathcal{E}_{\mathbb{C}, t} = E$ ,
- $\tilde{f}_X \times_{\mathbb{A}^1} (\mathbb{A}^1 \setminus 0) \mathcal{Y}^o \rightarrow \mathbb{A}_{\mathbb{Q}}^1 \setminus 0$  has fibers which are smooth projective hypersurfaces.

We have by definition the commutative diagram in  $\text{SmVar}(\mathbb{Q})$

$$\begin{array}{ccccc}
\mathcal{Y}^o & \xrightarrow{j} & \mathcal{Y} & \xleftarrow{i_{\mathcal{E}}} & \mathcal{E} \\
\tilde{f}_X^o \downarrow & & \downarrow \tilde{f}_X & & \downarrow \tilde{f}_{X,0} \\
\mathbb{A}_{\mathbb{Q}}^1 \setminus \{0\} \times S & \xrightarrow{j \times I_S} & \mathbb{A}_{\mathbb{Q}}^1 \times S & \xleftarrow{i_0 \times I_S} & \{0\} \times S
\end{array}$$

whose squares are cartesian,  $j$  the open embedding and  $i_{\mathcal{E}}$  the closed embedding. Denote for  $I \subset [0, \dots, r]$ ,  $i_I : \mathcal{E}_I \hookrightarrow \mathcal{E}$  and for  $I \subset J \subset [0, \dots, r]$ ,  $i_{JI} : \mathcal{E}_J \rightarrow \mathcal{E}_I$  the closed embeddings. Then, definition-proposition 1 applied to  $\tilde{f}_X : \mathcal{Y} \rightarrow \mathbb{A}_{\mathbb{Q}}^1 \times S$  gives the distinguish in  $D(MHM_{\mathbb{Q}, gm}(S))$ ,

$$\begin{array}{c}
E_{Hdg}(\mathcal{E}/S) \xrightarrow{sp_{\mathcal{E}/\tilde{f}_X} := (\tilde{f}_X \circ i_{\mathcal{E}})_* sp_{\mathcal{E}/\mathcal{Y}}} \Phi_0 E_{Hdg}(\mathcal{Y}^o / ((\mathbb{A}^1 \setminus 0) \times S)) \\
\downarrow (\tilde{f}_X \circ i_{\mathcal{E}})_c (\Phi_{\mathcal{E}} \mathbb{Z}_{\mathcal{Y}^o}^{Hdg}) \quad \downarrow (f \circ i_E)_* c(\mathbb{Z}_{\mathcal{E}}^{Hdg}) \\
\phi_0 E_{Hdg}(\mathcal{Y}^o / ((\mathbb{A}^1 \setminus 0) \times S)) \xrightarrow{\quad} E_{Hdg}(E/S)[1]
\end{array}$$

given in definition-proposition 1. Let  $p \in \mathbb{Z}$ . Taking the cohomology to this distinguish triangle, we get in particular, for each  $p \in \mathbb{Z}$ , a map in  $MHM_{\mathbb{Q}, gm}(S)$

$$Sp_{\mathcal{E}/\tilde{f}_X} : E_{Hdg}^{2p}(\mathcal{E}/S) \hookrightarrow H^{2p} \Phi_0 E_{Hdg}(\mathcal{Y}^o / ((\mathbb{A}_{\mathbb{Q}}^1 \setminus 0) \times S)).$$

which is injective by definition-proposition 1 and the weak Lefschetz theorem since for  $t' \in S(\mathbb{C})$  and  $s \in \mathbb{A}_{\mathbb{C}}^1 \setminus 0$ ,  $\mathcal{Y}_{(s, t')}$  and  $E_{t'}$  are projective hypersurfaces. For  $\lambda \in H^{2p}(\mathcal{E}_{0, \mathbb{C}, t'}^{an}, \mathbb{Q})$ ,  $t' \in S(\mathbb{C})$ , we consider  $\pi_S : \tilde{S}_{\mathbb{C}}^{an} \rightarrow S_{\mathbb{C}}^{an}$  the universal covering in  $\text{AnSm}(\mathbb{C})$  and

$$\mathbb{V}_S(\lambda) := \pi_S(\lambda) \subset E_{DR}^{2p, an}(\mathcal{E}_{0, \mathbb{C}}/S_{\mathbb{C}})$$

with  $\pi_S : \tilde{S}_{\mathbb{C}}^{an} \times H^{2p}(\mathcal{E}_{0, t'}^{an}, \mathbb{C}) \rightarrow E_{DR}^{2p, an}(\mathcal{E}_{0, \mathbb{C}}/S_{\mathbb{C}})$  the universal covering and

$$\mathbb{V}_S^p(\lambda) := \mathbb{V}_S(\lambda) \cap F^p E_{DR}^{2p}(\mathcal{E}_{0, \mathbb{C}}/S_{\mathbb{C}}) \subset E_{DR}^{2p}(\mathcal{E}_{0, \mathbb{C}}/S_{\mathbb{C}}),$$

which is an algebraic variety by [6]. We have

$$H^{2p} E_{Hdg}(\mathcal{Y}^o / ((\mathbb{A}_{\mathbb{Q}}^1 \setminus 0) \times S)) = ((E_{DR}^{2p}(\mathcal{Y}^o / ((\mathbb{A}_{\mathbb{Q}}^1 \setminus 0) \times S)), F), R^{2p} \tilde{f}_X^* \mathbb{Z}_{\mathcal{Y}^o}, \tilde{f}_X^* \alpha(\mathcal{Y}^o)) \in MHM_{\mathbb{Q}, gm}(S)$$

By [8], the filtered vector bundle

$$(E_{DR}^{2p}(\mathcal{Y}^o / ((\mathbb{A}_{\mathbb{Q}}^1 \setminus 0) \times S)), F) = H^{2p} \tilde{f}_X^* E_{zar}(\Omega_{\mathcal{Y}^o / ((\mathbb{A}_{\mathbb{Q}}^1 \setminus 0) \times S)}^{\bullet}) \in \text{Vect}_{\mathcal{D}fil}(S)$$

extend to a filtered vector bundle

$$(E_{DR, V_0}^{2p}(\mathcal{Y} / (\mathbb{A}_{\mathbb{Q}}^1 \times S)), F) = H^{2p} \tilde{f}_X^* E_{zar}(\Omega_{\mathcal{Y} / \mathbb{A}_{\mathbb{Q}}^1 \times S}^{\bullet}(\log \mathcal{E})) \in \text{Vect}_{\mathcal{D}fil}(S)$$



such that

$$H^{2p}\Phi_0 E_{Hdg}(\mathcal{Y}^\circ/((\mathbb{A}_\mathbb{Q}^1 \setminus 0) \times S)) = (i_0^{*mod}(E_{DR,V_0}^{2p}(\mathcal{Y}/(\mathbb{A}_\mathbb{Q}^1 \setminus 0) \times S)), F), (\Phi_0 R^{2p} \tilde{f}_{X*} \mathbb{Z}_{\mathcal{Y}^\circ}, W), (\Phi_0 \tilde{f}_{X*} \alpha(\mathcal{Y}^\circ)).$$

For

$$\alpha \in H^{2p}(\mathcal{Y}_{(s,t')}^{an}, \mathbb{Q}) = \mathbb{H}^{2p}(\mathcal{E}_{t'}^{an}, \Phi_0 R \tilde{f}_{X*} \mathbb{Q}_{\mathcal{Y}^\circ}), (s, t') \in (\mathbb{A}_\mathbb{C}^1 \setminus 0) \times S_\mathbb{C}$$

we consider

- $\pi_{S,G_m} = \pi_S \times \pi_{G_m} : \tilde{G}_m \times \tilde{S}_\mathbb{C}^{an} \rightarrow (\mathbb{A}_\mathbb{C}^1 \setminus 0) \times S_\mathbb{C}^{an}$  the universal covering in  $\text{AnSm}(\mathbb{C})$  and

$$\mathbb{V}_{(\mathbb{A}^1 \setminus 0) \times S}(\alpha) := \pi_{S,G_m}(\alpha) \subset E_{DR}^{2p,an}(\mathcal{Y}_\mathbb{C}^\circ/(\mathbb{A}_\mathbb{C}^1 \setminus 0) \times S_\mathbb{C})$$

with  $\pi_{S,G_m} : \tilde{S}_\mathbb{C}^{an} \times \tilde{G}_m \times H^{2p}(\mathcal{Y}_{(s,t')}^{an}, \mathbb{C}) \rightarrow E_{DR}^{2p,an}(\mathcal{Y}_\mathbb{C}^\circ/S_\mathbb{C})$  and

$$\mathbb{V}_{(\mathbb{A}^1 \setminus 0) \times S}^p(\alpha) := \mathbb{V}_{(\mathbb{A}^1 \setminus 0) \times S}(\alpha) \cap F^p E_{DR}^{2p}(\mathcal{Y}_\mathbb{C}^\circ/(\mathbb{A}_\mathbb{C}^1 \setminus 0) \times S_\mathbb{C}) \subset E_{DR}^{2p}(\mathcal{Y}_\mathbb{C}^\circ/(\mathbb{A}_\mathbb{C}^1 \setminus 0) \times S_\mathbb{C}),$$

which is an algebraic subvariety by [6],

- $\pi_S : \tilde{S}_\mathbb{C}^{an} \rightarrow S_\mathbb{C}^{an}$  the universal covering in  $\text{AnSm}(\mathbb{C})$  and

$$\mathbb{V}_S(\alpha) := \pi_S(\alpha) \subset H^{2p}\Phi_0 E_{DR}(\mathcal{Y}_\mathbb{C}^\circ/((\mathbb{A}_\mathbb{C}^1 \setminus 0) \times S_\mathbb{C}))^{an}$$

with  $\pi_S : \tilde{S}_\mathbb{C}^{an} \times H^{2p}(\mathcal{Y}_{(s,t')}^{an}, \mathbb{C}) \rightarrow H^{2p}\Phi_0 E_{DR}(\mathcal{Y}_\mathbb{C}^\circ/((\mathbb{A}_\mathbb{C}^1 \setminus 0) \times S_\mathbb{C}))^{an}$  and

$$\mathbb{V}_S^p(\alpha) := \mathbb{V}_S(\alpha) \cap F^p H^{2p}\Phi_0 E_{DR}(\mathcal{Y}_\mathbb{C}^\circ/((\mathbb{A}_\mathbb{C}^1 \setminus 0) \times S_\mathbb{C}))^{an} \subset H^{2p}\Phi_0 E_{DR}(\mathcal{Y}_\mathbb{C}^\circ/((\mathbb{A}_\mathbb{C}^1 \setminus 0) \times S_\mathbb{C}))^{an}.$$

Consider the exact sequence of variation of mixed Hodge structure over  $S$

$$0 \rightarrow W_{2p-2} E_{Hdg}^{2p}(\mathcal{E}/S) \rightarrow E_{Hdg}^{2p}(\mathcal{E}/S) \xrightarrow{q} \bigoplus_{i=0}^r E_{Hdg}^{2p}(\mathcal{E}_i/S) \rightarrow 0.$$

It induces the exact sequence of presheaves on  $S_\mathbb{C}^{an}$ ,

$$HL^{2p,p}(\mathcal{E}_\mathbb{C}/S_\mathbb{C}) \xrightarrow{q} \bigoplus_{i=0}^r HL^{2p,p}(\mathcal{E}_{i,\mathbb{C}}/S_\mathbb{C}) \xrightarrow{e} J(W_{2p-2} E_{Hdg}^{2p}(\mathcal{E}_\mathbb{C}/S_\mathbb{C})) := \text{Ext}^1(\mathbb{Q}_S^{Hdg}, W_{2p-2} E_{Hdg}^{2p}(\mathcal{E}_\mathbb{C}/S_\mathbb{C})),$$

where  $HL^{2p,p}(\mathcal{E}_\mathbb{C}/S_\mathbb{C}) \subset E_{DR}^{2p}(\mathcal{E}_\mathbb{C}/S_\mathbb{C})$ ,  $HL^{2p,p}(\mathcal{E}_{i,\mathbb{C}}/S_\mathbb{C}) \subset E_{DR}^{2p}(\mathcal{E}_{i,\mathbb{C}}/S_\mathbb{C})$  are the locus of Hodge classes. Let  $\lambda \in F^p H^{2p}(E_0^{an}, \mathbb{Q})$ , where we recall  $E_0 = \mathcal{E}_{0,\mathbb{C},t}$ . Consider

$$\tilde{\lambda} \in H^{2p}(E^{an}, \mathbb{Q}), \text{ s.t. } q(\tilde{\lambda}) = \lambda, \text{ and } \tilde{\lambda} := Sp_{E/f_X}(\tilde{\lambda}) \in Sp_{E/f_X}(H^{2p}(E^{an}, \mathbb{Q})).$$

The locus of Hodge classes passing through  $\lambda$

$$\mathbb{V}_S^p(\lambda) := \pi_S(\lambda \times \tilde{S}_\mathbb{C}) \cap F^p E_{DR}^{2p}(\mathcal{E}_{0,\mathbb{C}}/S_\mathbb{C}) = q(\mathbb{V}_S^p(\tilde{\lambda})) \cup p_{S,E_{DR}}^{-1}(V(e(\lambda))) \subset E_{DR}^{2p}(\mathcal{E}_{0,\mathbb{C}}/S_\mathbb{C})$$

is thus an intersection the De Rham vector bundle, where  $\pi_S : H^{2p}(E^{an}, \mathbb{C}) \times \tilde{S}_\mathbb{C}^{an} \rightarrow E_{DR}^{2p,an}(\mathcal{E}_\mathbb{C}/S_\mathbb{C})$  is the morphism given by the universal covering  $\pi_S : \tilde{S}_\mathbb{C}^{an} \rightarrow S_\mathbb{C}^{an}$ . Now,

- we have, by [6],

$$Sp_{\mathcal{E}/\tilde{f}_X}(\mathbb{V}_S^p(\tilde{\lambda})) = \overline{\mathbb{V}_{(\mathbb{A}^1 \setminus 0) \times S}^p(Sp_{E/f_X}(\tilde{\lambda}))} \cap p^{-1}(0 \times S_\mathbb{C}) \subset E_{DR,V_0}^{2p}(\mathcal{Y}_\mathbb{C}/(\mathbb{A}_\mathbb{C}^1 \times S_\mathbb{C}))$$

where  $p : E_{DR,V_0}^{2p}(\mathcal{Y}_\mathbb{C}/(\mathbb{A}_\mathbb{C}^1 \times S_\mathbb{C})) \rightarrow \mathbb{A}_\mathbb{C}^1 \times S_\mathbb{C}$  is the projection (the first equality is obvious since  $sp_{\mathcal{E}/\tilde{f}_X}$  is a morphism of variation of mixed Hodge structures over  $S$ , whereas the inclusion  $\supset$  follows from [6] lemma 2.11 and the invariant cycle theorem of a geometric variation of Hodge structures by strictness of the Hodge filtration), but by [5] theorem 1, the Hodge conjecture holds for projective hypersurfaces hence

$$\mathbb{V}_{(\mathbb{A}^1 \setminus 0) \times S}^p(Sp_{E/f_X}(\tilde{\lambda})) \subset E_{DR}^{2p}(\mathcal{Y}_\mathbb{C}^\circ/((\mathbb{A}_\mathbb{C}^1 \setminus 0) \times S_\mathbb{C}))$$

is defined over  $\mathbb{Q}$ , hence  $\mathbb{V}_S^p(\tilde{\lambda}) \subset E_{DR}^{2p}(\mathcal{E}_\mathbb{C}/S_\mathbb{C})$  is defined over  $\mathbb{Q}$ ,

- assuming the Hodge conjecture for smooth projective varieties of dimension less or equal to  $d - 1$ ,

$$\mathbb{V}(e(\tilde{\lambda})) \subset S_{\mathbb{C}}$$

is an algebraic variety defined over  $\mathbb{Q}$ .

Hence  $\mathbb{V}_S^p(\lambda) \subset E_{DR}^{2p}(\mathcal{E}_{0\mathbb{C}}/S_{\mathbb{C}})$  is defined over  $\mathbb{Q}$ . Thus, by [4] theorem 4,

$$\lambda = [Z] \in H^{2p}(E_0^{an}, \mathbb{Q}), \text{ with } Z \in \mathcal{Z}^p(E_0).$$

Since  $p \in \mathbb{Z}$  and  $\lambda \in F^p H^{2p}(E_0^{an}, \mathbb{Q})$  are arbitrary, the Hodge conjecture holds for  $E_0$ . Proposition 1(ii) and the fact that the Hodge conjecture is true for smooth projective varieties of dimension less or equal to  $d - 1$  by induction hypothesis then implies that Hodge conjecture holds for  $X$ , since  $X$  is birational to  $E_0$  and  $X, E_0 \in \text{PSmVar}(\mathbb{C})$  are connected of dimension  $d$ .  $\square$

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LAGA UMR CNRS 7539  
 Université Paris 13, Sorbonne Paris Cité, 99 av Jean-Baptiste Clement,  
 93430 Villetaneuse, France,  
 bouali@math.univ-paris13.fr