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Degeneration of families of projective hypersurfaces and Hodge conjecture

Johann Bouali

September 24, 2024

Abstract

We prove by induction on dimension the Hodge conjecture for smooth complex projective varieties. Let X be a smooth complex projective variety. Then X is birational to a possibly singular projective hypersurface, hence to a smooth projective variety E_0 which is a component of a normal crossing divisor $E = \cup_{i=0}^r E_i \subset Y$ which is the singular fiber of a pencil $f : Y \rightarrow \mathbb{A}^1$ of smooth projective hypersurfaces. Using the smooth hypersurface case ([5] theorem 1), the nearby cycle functor on mixed Hodge module with rational de Rham factor and the induction hypothesis, we prove that a Hodge class of E_0 is absolute Hodge, more precisely the locus of Hodge classes inside the algebraic vector bundle given the De Rham cohomology of the rational deformation of E_0 is defined over \mathbb{Q} . By [4] theorem 4, we get the Hodge conjecture for E_0 . By the induction hypothesis we also have the Hodge conjecture for X .

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1 Introduction

The main result of this article is an inductive proof of the Hodge conjecture using the results of [4] and [5]. More precisely, assuming the Hodge conjecture for smooth complex projective varieties of dimension less or equal to $d-1$, the property of satisfying the Hodge conjecture for connected smooth complex projective varieties of dimension d is a birational invariant (proposition 1). Let X be a connected smooth complex projective variety of dimension d . then X is birational to a possibly singular hypersurface $Y_{0_X} \subset \mathbb{P}_{\mathbb{C}}^{d+1}$ of degree $l_X := \deg(\mathbb{C}(X)/\mathbb{C}(x_1, \dots, x_d))$. Consider

$$f : \mathcal{Y}'_{\mathbb{C}} = V(\tilde{f}) \subset \mathbb{P}_{\mathbb{C}}^{d+1} \times \mathbb{P}(l_X) \rightarrow \mathbb{P}(l_X)$$

the universal family of projective hypersurfaces of degree l_X , and $0_X \in \mathbb{P}(l_X)$ the point corresponding to Y_{0_X} . Denote $\Delta \subset \mathbb{P}(l_X)$ the discriminant locus parametrizing the hypersurfaces which are singular. Let $\bar{S} := 0_X^{\mathbb{Q}} \subset \mathbb{P}(l_X)$ the \mathbb{Q} -Zariski closure of 0_X inside $\mathbb{P}(l_X)$. If Y_{0_X} is smooth, then X satisfy the hodge conjecture since Y_{0_X} satisfy the Hodge conjecture by [5] and X is birational to Y_{0_X} , using the induction

hypothesis. Hence, we may assume that Y_{0_X} is singular, that is $0_X \in \Delta$. Then, $\bar{S} \subset \Delta$ since $\Delta \subset \mathbb{P}(l_X)$ is defined over \mathbb{Q} . Let $\bar{T} \subset \mathbb{P}(l_X)$ be an irreducible closed subvariety of dimension $\dim(\bar{S}) + 1$ defined over $\bar{\mathbb{Q}}$ such that

$$\bar{S} \subset \bar{T} \text{ and } T := \bar{T} \cap (\mathbb{P}(l_X) \setminus \Delta) \neq \emptyset.$$

We take for $\bar{T} = V(f_1, \dots, f_s) \subset \mathbb{P}(l_X)$ an irreducible complete intersection defined over $\bar{\mathbb{Q}}$ which intersect Δ properly and such that

- $V(f_1) \supset \text{sing}(\Delta) \cup \bar{S}$
- for each $r \in [1, \dots, s]$, $V(f_1, \dots, f_r) \supset \text{sing}(V(f_1, \dots, f_{r-1}) \cap \Delta) \cup \bar{S}$ contain inductively the singular locus of the intersection with Δ and \bar{S} .

Then by the weak Lefschetz hyperplane theorem for homotopy groups, $T \subset \mathbb{P}(l_X) \setminus \Delta$ is not contained in a weakly special subvariety since

$$i_{T*} : \pi_1(T_{\mathbb{C}}) \rightarrow \pi_1(\mathbb{P}(l_X) \setminus \Delta), \quad (\Delta = \Delta_{\mathbb{C}})$$

is an isomorphism if $\dim \bar{T} \geq 2$ (i.e. $\dim \bar{S} \geq 1$) and surjective if $\dim \bar{T} = 1$ (i.e. $\dim \bar{S} = 0$). We have then $\bar{T} \cap \Delta = \bar{S} \cup \bar{S}'$, where $\bar{S}' \subset \bar{T}$ is a divisor. We then consider the family of algebraic varieties defined over $\bar{\mathbb{Q}}$

$$\tilde{f}_X : \mathcal{Y}_{\bar{T}^o} \xrightarrow{\epsilon} \mathcal{Y}'_{\bar{T}^o} \xrightarrow{f \times_{\mathbb{P}(l_X)} \bar{T}^o} \bar{T}^o$$

where

- $\tilde{f}_X : \mathcal{Y} \xrightarrow{\epsilon} \mathcal{Y}'_{\bar{T}} \xrightarrow{f \times_{\mathbb{P}(l_X)} \bar{T}} \bar{T}$ with $\epsilon : (\mathcal{Y}, \mathcal{E}') \rightarrow (\mathcal{Y}'_{\bar{T}}, \mathcal{Y}'_{\bar{S}} \cup \mathcal{Y}'_{\bar{S}'})$ is a desingularization over \mathbb{Q} , in particular $\mathcal{E}' = \mathcal{E} \cup \mathcal{E}''$ with $\mathcal{E} = \cup_{i=0}^r \mathcal{E}_i := \tilde{f}_X^{-1}(\bar{S}) \subset \mathcal{Y}$ which is a normal crossing divisor, in particular $\mathcal{Y}_T = \mathcal{Y}'_T$,
- $S \subset \bar{S}$ is the open subset such that S is smooth and $\tilde{f}_X \times_{\bar{T}} S : \mathcal{E}_I := \cap_{i \in I} \mathcal{E}_i \rightarrow S$ is smooth projective for each $I \subset [0, \dots, r]$, $\bar{T}^o \subset \bar{T}$ is an open subset such that $\bar{T}^o \cap \Delta = S$ and $T^o := \bar{T}^o \cap T$ is smooth, for simplicity we denote again \mathcal{E} for the open subset $\mathcal{E}_S := \tilde{f}_X^{-1}(S) \subset \mathcal{E}$,
- $t \in S(\mathbb{C})$ is the point such that $\mathcal{Y}'_{\mathbb{C}, t} = Y_{0_X}$ and $\epsilon_t : E_0 := \mathcal{E}_{0, \mathbb{C}, t} \rightarrow Y_{0_X}$ is surjective (there is at least one component of E dominant over Y_{0_X} since E is dominant over Y_{0_X} as \mathcal{E} is dominant over $\mathcal{Y}'_{\bar{S}}$), in particular E_0 is birational to Y_{0_X} , hence E_0 is birational to X . We denote $E_i := \mathcal{E}_{i, \mathbb{C}, t}$ for each $0 \leq i \leq r$ and $E := \mathcal{E}_{\mathbb{C}, t} = \cup_{i=0}^r E_i$.

Using [4] and [5], assuming the Hodge conjecture for smooth projective varieties of dimension less or equal to $d - 1$, we then prove the Hodge conjecture for the E_i , more specifically for E_0 as follows : Let $p \in \mathbb{Z}$. By the hard Lefschetz theorem on $H^*(E_0)$, we may assume that $2p \leq d$. We then have then a morphism of mixed Hodge module over S (in fact of geometric variation of mixed Hodge structures over S) given by the specialization map

$$Sp_{\mathcal{E}/\tilde{f}_X} : E_{Hdg}^{2p}(\mathcal{E}/S) \rightarrow H^{2p} \psi_S E_{Hdg}(\mathcal{Y}_{T^o}/T^o) = \psi_S E_{Hdg}^{2p}(\mathcal{Y}_{T^o}/T^o),$$

whose de Rham part are morphism of filtered vector bundle over S with Gauss-Manin connexion defined over $\bar{\mathbb{Q}}$. Moreover, we have, see lemma 1, an isomorphism of variation of mixed Hodge structures over S

$$\begin{aligned} c_{\mathcal{E}} &:= H^{2p-1}(\tilde{f}_X \circ i_{\mathcal{E}})_* c(\mathbb{Q}_{\mathcal{E}}^{Hdg}) \circ c_V : E_{Hdg, \mathcal{E}}^{2p-1}(\mathcal{Y}_{\bar{T}^o}/\bar{T}^o) \xrightarrow{\sim} \\ \phi_S E_{Hdg, \mathcal{E}}^{2p-1}(\mathcal{Y}_{T^o}/T^o) &\xrightarrow{\sim} \ker(Sp_{\mathcal{E}/\tilde{f}_X} : E_{Hdg}^{2p}(\mathcal{E}/S) \rightarrow \psi_S E_{Hdg}^{2p}(\mathcal{Y}_{T^o}/T^o)). \end{aligned}$$

Consider the exact sequence of variation of mixed Hodge structure over S

$$0 \rightarrow W_{2p-1} E_{Hdg}^{2p}(\mathcal{E}/S) \rightarrow E_{Hdg}^{2p}(\mathcal{E}/S) \xrightarrow{(i_i^*)_{0 \leq i \leq r} := q} \oplus_{i=0}^r E_{Hdg}^{2p}(\mathcal{E}_i/S) \rightarrow 0.$$

It induces the exact sequence of presheaves on $S_{\mathbb{C}}^{an}$,

$$J(W_{2p-1}E_{Hdg}^{2p}(\mathcal{E}_{\mathbb{C}}/S_{\mathbb{C}})) := \text{Ext}^1(\mathbb{Q}_S^{Hdg}, W_{2p-1}E_{Hdg}^{2p}(\mathcal{E}_{\mathbb{C}}/S_{\mathbb{C}})) \xrightarrow{\iota} \text{Ext}^1(\mathbb{Q}_S^{Hdg}, E_{Hdg}^{2p}(\mathcal{E}_{\mathbb{C}}/S_{\mathbb{C}}))$$

where $HL^{2p,p}(\mathcal{E}_{\mathbb{C}}/S_{\mathbb{C}}) \subset E_{DR}^{2p}(\mathcal{E}_{\mathbb{C}}/S_{\mathbb{C}})$, $HL^{2p,p}(\mathcal{E}_{i,\mathbb{C}}/S_{\mathbb{C}}) \subset E_{DR}^{2p}(\mathcal{E}_{i,\mathbb{C}}/S_{\mathbb{C}})$ are the locus of Hodge classes. Let $\lambda \in F^p H^{2p}(E_0^{an}, \mathbb{Q})$, where we recall $E_0 = \mathcal{E}_{0,\mathbb{C},t}$ and $E = \mathcal{E}_{\mathbb{C},t} = \cup_{i=0}^r E_i$. Consider then $\tilde{\lambda} \in H^{2p}(E^{an}, \mathbb{Q})$, such that $q(\tilde{\lambda}) = \lambda$, and

$$Sp_{\mathcal{E}/\tilde{f}_X}(\tilde{\lambda}) \in Sp_{E/f_X}(H^{2p}(E^{an}, \mathbb{Q})) \subset i_t^* \psi_S R^{2p} \tilde{f}_{X*} \mathbb{Q}_{\mathcal{Y}_{T^o, \mathbb{C}}^{an}}.$$

Since the monodromy of $R^{2p} \tilde{f}_{X*} \mathbb{Q}_{\mathcal{Y}_{T^o, \mathbb{C}}^{an}}$ is irreducible as $i_{T^o*} : \pi_1(T_{\mathbb{C}}^o) \rightarrow \pi_1(T_{\mathbb{C}}) \rightarrow \pi_1(\mathbb{P}(l_X) \setminus \Delta)$ is surjective, we have

$$Sp_{\mathcal{E}/\tilde{f}_X}(\ker q) := Sp_{\mathcal{E}/\tilde{f}_X}(W_{2p-1}E_{DR}^{2p}(\mathcal{E}/S)) = 0,$$

using the fact that there exists a neighborhood $V_{\bar{S} \cup \bar{S}'} \subset \bar{T}_{\mathbb{C}}$ of $\bar{S}_{\mathbb{C}} \cup \bar{S}'_{\mathbb{C}}$ in $\bar{T}_{\mathbb{C}}$ for the usual complex topology such that the inclusion $i_{\bar{S}_{\mathbb{C}} \cup \bar{S}'_{\mathbb{C}}} : \bar{S}_{\mathbb{C}} \cup \bar{S}'_{\mathbb{C}} \hookrightarrow V_{\bar{S} \cup \bar{S}'}$ admits a retraction $r : V_{\bar{S} \cup \bar{S}'} \rightarrow \bar{S}_{\mathbb{C}} \cup \bar{S}'_{\mathbb{C}}$ which is an homotopy equivalence. The locus of Hodge classes passing through λ

$$\mathbb{V}_S^p(\lambda) := \mathbb{V}_S(\lambda) \cap F^p E_{DR}^{2p}(\mathcal{E}_{0,\mathbb{C}}/S_{\mathbb{C}}) \subset E_{DR}^{2p}(\mathcal{E}_{0,\mathbb{C}}/S_{\mathbb{C}}),$$

inside the De Rham vector bundle of $\tilde{f}_X \circ i_{E_0} : \mathcal{E}_0 \rightarrow S$ thus satisfy

$$\mathbb{V}_S^p(\lambda) = q(Sp_{\mathcal{E}/\tilde{f}_X}^{-1}(\mathbb{V}_S^p(Sp_{\mathcal{E}/\tilde{f}_X}(\tilde{\lambda}))), \quad (1)$$

where

- $\mathbb{V}_S(\lambda) \subset E_{DR}^{2p}(\mathcal{E}_{0,\mathbb{C}}/S_{\mathbb{C}})$, $\mathbb{V}_S(Sp_{\mathcal{E}/\tilde{f}_X}(\tilde{\lambda})) \subset E_{DR, V_S}^{2p}(\mathcal{Y}_{T^o, \mathbb{C}}/T_{\mathbb{C}}^o)$, are the flat leaves, e.g. $\mathbb{V}_S(\lambda) := \pi_S(\lambda \times \tilde{S}_{\mathbb{C}}^{an})$ where $\pi_S : H^{2p}(E^{an}, \mathbb{C}) \times \tilde{S}_{\mathbb{C}}^{an} \rightarrow E_{DR}^{2p, an}(\mathcal{E}_{\mathbb{C}}/S_{\mathbb{C}})$ is the morphism induced by the universal covering $\pi_S : \tilde{S}_{\mathbb{C}}^{an} \rightarrow S_{\mathbb{C}}^{an}$,
- $q := q \otimes \mathbb{C} : E_{DR}^{2p}(\mathcal{E}_{\mathbb{C}}/S_{\mathbb{C}}) \rightarrow E_{DR}^{2p}(\mathcal{E}_{0,\mathbb{C}}/S_{\mathbb{C}})$ is the quotient map.
- $Sp_{\mathcal{E}/\tilde{f}_X} := Sp_{\mathcal{E}/\tilde{f}_X} \otimes \mathbb{C} : E_{DR}^{2p}(\mathcal{E}_{\mathbb{C}}/S_{\mathbb{C}}) \rightarrow i_S^{*mod} E_{DR, V_S}^{2p}(\mathcal{Y}_{T^o, \mathbb{C}}/T_{\mathbb{C}}^o)$.

Now,

- If $Sp_{\mathcal{E}/\tilde{f}_X}(\tilde{\lambda}) = 0$, we have by lemma 1 applied to $\tilde{f}_X : \mathcal{Y}_{T^o} \rightarrow \bar{T}^o$, $\tilde{\lambda} = c_{\mathcal{E}}(\tilde{\lambda})$ with $\tilde{\lambda} \in F^p H_E^{2p-1}(\mathcal{Y}_{\mathbb{C}, C}^{an}, \mathbb{Q})$, where $C \subset \bar{T}_{\mathbb{C}}^o$ is a smooth transversal slice of $S_{\mathbb{C}}$ at t in particular $\dim(C) = 1$, $C \cap S_{\mathbb{C}} = \{t\}$ and \mathcal{Y}_C is smooth, and

$$\mathbb{V}_S^p(\tilde{\lambda}) = c_{\mathcal{E}}(\mathbb{V}_S^p(\tilde{\lambda})), \quad \mathbb{V}_S^p(\tilde{\lambda}) \subset E_{DR, \mathcal{E}_{\mathbb{C}}}^{2p-1}(\mathcal{Y}_{\bar{T}^o, \mathbb{C}}/\bar{T}_{\mathbb{C}}^o),$$

with

$$c_{\mathcal{E}} : (E_{DR, \mathcal{E}_{\mathbb{C}}}^{2p-1}(\mathcal{Y}_{\bar{T}^o, \mathbb{C}}/\bar{T}_{\mathbb{C}}^o), F, W) \rightarrow (E_{DR}^{2p}(\mathcal{E}_{\mathbb{C}}/S_{\mathbb{C}}), F, W).$$

We have for each $t' \in S(\mathbb{C})$, the isomorphism

$$\begin{aligned} \oplus_{card I=2} F^p H^{2p}(\mathcal{E}_{IC, t'}^{an}, \mathbb{Q}) &= F^p \text{Gr}_{2p}^W H^{2p-2}((\mathcal{Y}_{\mathbb{C}, C'}^- \setminus \mathcal{E}_{t'})^{an}, \mathbb{Q}) \\ &\xrightarrow{c(\mathcal{Y}_{\mathbb{C}, C'} \setminus \mathcal{E}_{t'})} F^p \text{Gr}_{2p}^W H_E^{2p-1}(\mathcal{Y}_{\mathbb{C}, C'}^{an}, \mathbb{Q}), \end{aligned}$$

where $C' \subset \bar{T}_{\mathbb{C}}^o$ is a smooth transversal slice of S at t' (hence $\mathcal{Y}_{C'}$ is smooth) and $\mathcal{Y}_{C'}^-$ is a smooth compactification of $\mathcal{Y}_{C'}$. Hence, assuming the Hodge conjecture for smooth projective varieties of dimension less or equal to $d-1$,

$$\mathbb{V}_S^p(\lambda) \subset \text{Gr}_{2p}^W E_{DR, \mathcal{E}_{\mathbb{C}}}^{2p-1}(\mathcal{Y}_{\bar{T}^o, \mathbb{C}}/\bar{T}_{\mathbb{C}}^o), \quad \mathbb{V}_S^p(\lambda) = c_{\mathcal{E}}(\mathbb{V}_S^p(\lambda)) \subset E_{DR}^{2p}(\mathcal{E}_{0,\mathbb{C}}/S_{\mathbb{C}}) \subset \text{Gr}_{2p}^W E_{DR}^{2p}(\mathcal{E}_{\mathbb{C}}/S_{\mathbb{C}})$$

are defined over $\bar{\mathbb{Q}}$ and its Galois conjugates are also components of the locus of Hodge classes, since $\dim(\mathcal{E}_{IC, t'}) = d-1$ for $I \subset [0, \dots, r]$ such that $card I = 2$ and $t' \in S(\mathbb{C})$.

- Consider now the case where $Sp_{\mathcal{E}/\tilde{f}_X}(\tilde{\lambda}) \neq 0$. By [5] theorem 1, the Hodge conjecture holds for projective hypersurfaces, hence

$$\mathbb{V}_S^p(Sp_{\mathcal{E}/\tilde{f}_X}(\tilde{\lambda})) = \overline{\mathbb{V}_{T^o}^p(Sp_{\mathcal{E}/\tilde{f}_X}(\tilde{\lambda}))} \cap p^{-1}(S_{\mathbb{C}}) \subset E_{DR, V_S}^{2p}(\mathcal{Y}_{T^o, \mathbb{C}}/T_{\mathbb{C}}^o),$$

where $p := p \otimes \mathbb{C} : E_{DR, V_0}^{2p}(\mathcal{Y}_{T^o, \mathbb{C}}/T_{\mathbb{C}}^o) \rightarrow \bar{T}_{\mathbb{C}}^o$ is the projection, is an algebraic subvariety defined over $\bar{\mathbb{Q}}$ and its Galois conjugates are also components of the locus of Hodge classes, where equality follows from [6] lemma 2.11. Hence,

$$\mathbb{V}_S^p(\lambda) = q(Sp_{\mathcal{E}/\tilde{f}_X}^{-1}(\mathbb{V}_S^p(Sp_{\mathcal{E}/\tilde{f}_X}(\tilde{\lambda})))) \subset E_{DR}^{2p}(\mathcal{E}_{0\mathbb{C}}/S_{\mathbb{C}})$$

is an algebraic subvariety defined over $\bar{\mathbb{Q}}$ and its Galois conjugates are also components of the locus of Hodge classes, where the second equality follows from the fact that $Sp_{\mathcal{E}/\tilde{f}_X}(\tilde{\lambda}) \neq 0$ and (1).

The fact that $\mathbb{V}_S^p(\lambda) \subset E_{DR}^{2p}(\mathcal{E}_{0\mathbb{C}}/S_{\mathbb{C}})$ is an algebraic subvariety defined over $\bar{\mathbb{Q}}$ and its Galois conjugates are also components of the locus of Hodge classes gives, by [4] theorem 4,

$$\lambda = [Z] \in H^{2p}(E_0^{an}, \mathbb{Q}), \quad Z \in \mathcal{Z}^p(E_0).$$

We recall the key point is that since $\mathbb{V}_S^p(\lambda) \subset E_{DR}^{2p}(\mathcal{E}_{0\mathbb{C}}/S_{\mathbb{C}})$ is an algebraic subvariety defined over $\bar{\mathbb{Q}}$ and its Galois conjugate are also components of the locus of Hodge classes, we have for each $\theta \in \text{Aut}(\mathbb{C}/\bar{\mathbb{Q}})$,

$$\theta(\lambda) \in \text{Gal}(\bar{\mathbb{Q}}/\bar{\mathbb{Q}})(\mathbb{V}_S^p(\lambda)_{\theta}) \subset E_{DR}^{2p}(\mathcal{E}_{0\mathbb{C}, \theta}/S_{\mathbb{C}, \theta}),$$

which implies that $\theta(\lambda) \in H^{2p}(E_{0, \theta}^{an}, \mathbb{Q}) \subset H^{2p}(E_{0, \theta}^{an}, \mathbb{C})$, that is $\lambda \in H^{2p}(E_0^{an}, \mathbb{Q})$ is absolute Hodge and we apply [4] corollary 1. Since $p \in \mathbb{Z}$ and $\lambda \in F^p H^{2p}(E_0^{an}, \mathbb{Q})$ are arbitrary, this proves the Hodge conjecture for E_0 . Since X is birational to E_0 , the Hodge conjecture for E_0 and the induction hypothesis implies the Hodge conjecture for X .

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2 Preliminaries and notations

2.1 Notations

- Denote by Top the category of topological spaces and RTop the category of ringed spaces.
- Denote by Cat the category of small categories and RCat the category of ringed topos.
- For $\mathcal{S} \in \text{Cat}$ and $X \in \mathcal{S}$, we denote $\mathcal{S}/X \in \text{Cat}$ the category whose objects are $Y/X := (Y, f)$ with $Y \in \mathcal{S}$ and $f : Y \rightarrow X$ is a morphism in \mathcal{S} , and whose morphisms $\text{Hom}((Y', f'), (Y, f))$ consists of $g : Y' \rightarrow Y$ in \mathcal{S} such that $f \circ g = f'$.
- Let $(\mathcal{S}, O_{\mathcal{S}}) \in \text{RCat}$ a ringed topos with topology τ . For $F \in C_{O_{\mathcal{S}}}(\mathcal{S})$, we denote by $k : F \rightarrow E_{\tau}(F)$ the canonical flasque resolution in $C_{O_{\mathcal{S}}}(\mathcal{S})$ (see [3]). In particular for $X \in \mathcal{S}$, $H^*(X, E_{\tau}(F)) \xrightarrow{\sim} \mathbb{H}_{\tau}^*(X, F)$.
- For $f : \mathcal{S}' \rightarrow \mathcal{S}$ a morphism with $\mathcal{S}, \mathcal{S}' \in \text{RCat}$, endowed with topology τ and τ' respectively, we denote for $F \in C_{O_{\mathcal{S}}}(\mathcal{S})$ and each $j \in \mathbb{Z}$,
 - $f^* := H^j \Gamma(\mathcal{S}, k \circ \text{ad}(f^*, f_*)(F)) : \mathbb{H}^j(\mathcal{S}, F) \rightarrow \mathbb{H}^j(\mathcal{S}', f^*F)$,
 - $f^* := H^j \Gamma(\mathcal{S}, k \circ \text{ad}(f^{*mod}, f_*)(F)) : \mathbb{H}^j(\mathcal{S}, F) \rightarrow \mathbb{H}^j(\mathcal{S}', f^{*mod}F)$,

the canonical maps.

- For $m : A \rightarrow B$, $A, B \in C(\mathcal{A})$, \mathcal{A} an additive category, we denote $c(A) : \text{Cone}(m : A \rightarrow B) \rightarrow A[1]$ and $c(B) : B \rightarrow \text{Cone}(m : A \rightarrow B)$ the canonical maps.
- Denote by $\text{Sch} \subset \text{RTop}$ the subcategory of schemes (the morphisms are the morphisms of locally ringed spaces). We denote by $\text{PSch} \subset \text{Sch}$ the full subcategory of proper schemes. For a field k , we consider $\text{Sch}/k := \text{Sch}/\text{Spec } k$ the category of schemes over $\text{Spec } k$. The objects are $X := (X, a_X)$ with $X \in \text{Sch}$ and $a_X : X \rightarrow \text{Spec } k$ a morphism and the morphisms are the morphisms of schemes $f : X' \rightarrow X$ such that $f \circ a_{X'} = a_X$. We then denote by
 - $\text{Var}(k) = \text{Sch}^{ft}/k \subset \text{Sch}/k$ the full subcategory consisting of algebraic varieties over k , i.e. schemes of finite type over k ,
 - $\text{PVar}(k) \subset \text{QPVar}(k) \subset \text{Var}(k)$ the full subcategories consisting of quasi-projective varieties and projective varieties respectively,
 - $\text{PSmVar}(k) \subset \text{SmVar}(k) \subset \text{Var}(k)$, $\text{PSmVar}(k) := \text{PVar}(k) \cap \text{SmVar}(k)$, the full subcategories consisting of smooth varieties and smooth projective varieties respectively.
- Denote by $\text{AnSp}(\mathbb{C}) \subset \text{RTop}$ the subcategory of analytic spaces over \mathbb{C} , and by $\text{AnSm}(\mathbb{C}) \subset \text{AnSp}(\mathbb{C})$ the full subcategory of smooth analytic spaces (i.e. complex analytic manifold).
- For $X \in \text{Var}(k)$ and $X = \cup_{i \in I} X_i$ with $i_i : X_i \hookrightarrow X$ closed embeddings, we denote $X_\bullet \in \text{Fun}(\Delta, \text{Var}(k))$ the associated simplicial space, with for $J \subset I$, $i_{IJ} : X_I := \cap_{i \in I} X_i \hookrightarrow X_J := \cap_{i \in J} X_i$ the closed embedding.
- Let $(X, O_X) \in \text{RTop}$. We consider its De Rham complex $\Omega_X^\bullet := DR(X)(O_X)$.
 - Let $X \in \text{Sch}$. Considering its De Rham complex $\Omega_X^\bullet := DR(X)(O_X)$, we have for $j \in \mathbb{Z}$ its De Rham cohomology $H_{DR}^j(X) := \mathbb{H}^j(X, \Omega_X^\bullet)$.
 - Let $X \in \text{Var}(k)$. Considering its De Rham complex $\Omega_X^\bullet := \Omega_{X/k}^\bullet := DR(X/k)(O_X)$, we have for $j \in \mathbb{Z}$ its De Rham cohomology $H_{DR}^j(X) := \mathbb{H}^j(X, \Omega_X^\bullet)$. The differentials of $\Omega_X^\bullet := \Omega_{X/k}^\bullet$ are by definition k -linear, thus $H_{DR}^j(X) := \mathbb{H}^j(X, \Omega_X^\bullet)$ has a structure of a k vector space.
 - Let $X \in \text{AnSp}(\mathbb{C})$. Considering its De Rham complex $\Omega_X^\bullet := DR(X)(O_X)$, we have for $j \in \mathbb{Z}$ its De Rham cohomology $H_{DR}^j(X) := \mathbb{H}^j(X, \Omega_X^\bullet)$.
- For $Y \in \text{Var}(\mathbb{Q})$ and $X = V(I) = V(f_1, \dots, f_r) \subset Y_{\mathbb{C}}$, we consider $\mathcal{X} = V(\tilde{I}) = V(\tilde{f}_1, \dots, \tilde{f}_r) \subset Y \times \mathbb{A}_{\mathbb{Q}}^s$ the canonical rational deformation, where

$$\begin{aligned} \text{if } f &= \sum_{I \subset [1, \dots, d]^n} a_I x_1^{n_1} \cdots x_n^{n_n} \in \mathbb{C}[x_1, \dots, x_n], \\ \tilde{f} &= \sum_{I \subset [1, \dots, d]^n} a_I x_1^{n_1} \cdots x_n^{n_n} \in \mathbb{Q}[(a_I)_{I \subset [1, \dots, d]^n}, x_1, \dots, x_n]. \end{aligned}$$

- For $X \in \text{AnSp}(\mathbb{C})$, we denote $\alpha(X) : \mathbb{C}_X \hookrightarrow \Omega_X^\bullet$ the embedding in $C(X)$. For $X \in \text{AnSm}(\mathbb{C})$, $\alpha(X) : \mathbb{C}_X \hookrightarrow \Omega_X^\bullet$ is an equivalence usu local by Poincare lemma.
- We denote $\mathbb{I}^n := [0, 1]^n \in \text{Diff}(\mathbb{R})$ (with boundary). For $X \in \text{Top}$ and R a ring, we consider its singular cochain complex

$$C_{\text{sing}}^*(X, R) := (\mathbb{Z} \text{Hom}_{\text{Top}}(\mathbb{I}^*, X)^\vee) \otimes R$$

and for $l \in \mathbb{Z}$ its singular cohomology $H_{\text{sing}}^l(X, R) := H^n C_{\text{sing}}^*(X, R)$. For $f : X' \rightarrow X$ a continuous map with $X, X' \in \text{Top}$, we have the canonical map of complexes

$$f^* : C_{\text{sing}}^*(X, R) \rightarrow C_{\text{sing}}^*(X', R), \sigma \mapsto f^* \sigma := (\gamma \mapsto \sigma(f \circ \gamma)).$$

In particular, we get by functoriality the complex

$$C_{X,R\text{sing}}^* \in C_R(X), (U \subset X) \mapsto C_{\text{sing}}^*(U, R)$$

We recall that

- For $X \in \text{Top}$ locally contractible, e.g. $X \in \text{CW}$, and R a ring, the inclusion in $C_R(X)$ $c_X : R_X \rightarrow C_{X,R\text{sing}}^*$ is by definition an equivalence top local and that we get by the small chain theorem, for all $l \in \mathbb{Z}$, an isomorphism $H^l c_X : H^l(X, R_X) \xrightarrow{\sim} H_{\text{sing}}^l(X, R)$.
- For $X \in \text{Diff}(\mathbb{R})$, the restriction map

$$r_X : \mathbb{Z} \text{Hom}_{\text{Diff}(\mathbb{R})}(\mathbb{I}^*, X)^\vee \rightarrow C_{\text{sing}}^*(X, R), w \mapsto w : (\phi \mapsto w(\phi))$$

is a quasi-isomorphism by Whitney approximation theorem.

- Let $S \in \text{AnSm}(\mathbb{C})$ and $L \in \text{Shv}(S)$ a local system. Consider $E := L \otimes O_S \in \text{Vect}_{\mathcal{D}}(S)$ the corresponding holomorphic vector bundle with integrable connection ∇ . Let $\pi_S : \tilde{S} \rightarrow S$ be the universal covering. Consider the canonical fiber

$$L_0 := \Gamma(\tilde{S}, \pi_S^* L) = \Gamma(\tilde{S}, \pi_S^{*\text{mod}} E)^\nabla.$$

For $\lambda \in L_0$, we will consider the flat leaf

$$\mathbb{V}_S(\lambda) := \pi_S(\lambda \times \tilde{S}) \subset E, \pi_S : L_0 \times \tilde{S} \rightarrow E, \pi_S(\nu, z) := (\nu(z), \pi_S(z)).$$

Note that for $t \in S$, $\mathbb{V}_S(\lambda) \cap p_S^{-1}(t) = \pi_1(S, t)(\lambda)$ is the orbit of λ under the monodromy action, where $p_S : E \rightarrow S$ is the projection.

2.2 Birational projective varieties and Hodge conjecture

In this subsection, we recall that assuming the Hodge conjecture for smooth projective varieties of dimension less or equal to $d - 1$, the property of satisfying the Hodge conjecture for connected smooth projective varieties of dimension d is a birational invariant :

Proposition 1. (i) *Let $X \in \text{PSmVar}(\mathbb{C})$ connected of dimension d . Let $\epsilon : \tilde{X}_Z \rightarrow X$ be the blow up of X along a smooth closed subvariety $Z \subset X$. Assume the Hodge conjecture hold for smooth complex projective varieties of dimension less or equal to $d - 1$. Then the Hodge conjecture hold for X if and only if it hold for \tilde{X}_Z .*

(ii) *Let $X, X' \in \text{PSmVar}(\mathbb{C})$ connected of dimension d . Assume the Hodge conjecture hold for smooth complex projective varieties of dimension less or equal to $d - 1$. If X is birational to X' , then the Hodge conjecture hold for X if and only if it hold for X'*

Proof. (i): Follows from the fact that

$$(\epsilon^*, \oplus_{l=1}^c i_{E*}((-).h^{l-1})) : H^k(X, \mathbb{Q}) \oplus \oplus_{l=1}^c H^{k-2l}(E, \mathbb{Q}) \rightarrow H^k(\tilde{X}_Z, \mathbb{Q})$$

is an isomorphism of Hodge structures for each $k \in \mathbb{Z}$, where $c = \text{codim}(Z, X)$, $i_E : E \hookrightarrow \tilde{X}_Z$ is the closed embedding, $\epsilon|_E : E \rightarrow Z$ being a projective vector bundle.

(ii): Follows from (i) since if $\pi : X^\circ \rightarrow X'$ is a birational map, $X^\circ \subset X$ being an open subset, then X is connected to X' by a sequence of blow up of smooth projective varieties with smooth center by [1]. \square

2.3 A family of smooth projective hypersurfaces associated to a smooth projective variety

In this subsection, we recall that given a smooth complex projective variety X , there exists a family of smooth complex projective hypersurfaces which degenerates into a normal crossing divisor with one irreducible component birational to X .

Proposition 2. *Let $X \in \text{PSmVar}(\mathbb{C})$ connected of dimension d . Then there exists a family of smooth projective hypersurfaces $f_X : Y \rightarrow \mathbb{A}_{\mathbb{C}}^1$ with*

- $Y \in \text{SmVar}(\mathbb{C})$, f_X flat projective,
- $Y_s := f_X^{-1}(s) \subset \mathbb{P}_{\mathbb{C}}^{d+1}$ for $s \in \mathbb{A}_{\mathbb{C}}^1 \setminus 0$ are smooth projective hypersurfaces,
- $E := f_X^{-1}(0) = \cup_{i=0}^r E_i = \epsilon^{-1}(Y_0) \subset Y$ is a normal crossing divisor, where $Y_0 \subset \mathbb{P}_{\mathbb{C}}^{d+1}$ is a possibly singular hypersurface birational to X , $\epsilon : (Y, E) \rightarrow (Y', Y_0)$ a desingularization, and E_0 is birational to X ,

Proof. See [2] lemma 1.5.8 . It use the fact that X is birational to a possibly singular projective hypersurface Y_0 and consider the desingularization of a generic pencil $f'_X : Y' = V(f) \subset \mathbb{P}_{\mathbb{C}}^{d+1} \times \mathbb{A}_{\mathbb{C}}^1 \rightarrow \mathbb{A}_{\mathbb{C}}^1$ passing through Y_0 and $f_X = f'_X \circ \epsilon$. \square

2.4 The nearby cycle functor and the specialization map

Recall from [3] that for $k \subset \mathbb{C}$ a subfield and $S \in \text{Var}(k)$ quasi-projective, and $l : S \hookrightarrow \tilde{S}$ a closed embedding with $\tilde{S} \in \text{SmVar}(k)$, we have the full subcategory

$$\iota : MHM_{k, gm}(S) \hookrightarrow \text{PSh}_{\mathcal{D}fil, S}(\tilde{S}) \times_I P(S_{\mathbb{C}}^{an})$$

consisting of geometric mixed Hodge module whose De Rham part is defined over k , where, using the fact that the V -filtration is defined over k , $\text{PSh}_{\mathcal{D}(1,0)fil, S}(\tilde{S}) \times_I P_{fil}(S_{\mathbb{C}}^{an})$ is the category

- whose objects are $((M, F, W), (K, W), \alpha)$ where (M, F, W) is a filtered $D_{\tilde{S}}$ module supported on S , where $l : S \hookrightarrow \tilde{S}$ is a closed embedding and $\tilde{S} \in \text{SmVar}(k)$, (K, W) is a filtered Perverse sheaf on $S_{\mathbb{C}}^{an}$, and $\alpha : l_*(K, W) \otimes \mathbb{C} \rightarrow DR(\tilde{S})((M, W)^{an})$ is an isomorphism in $D_{fil}(\tilde{S}_{\mathbb{C}}^{an})$.
- whose morphism between $((M, F, W), (K, W), \alpha)$ and $((M', F, W), (K', W), \alpha')$, are couples of morphisms $(\phi : (M, F, W) \rightarrow (M', F, W), \psi : (K, W) \rightarrow (K', W))$ such that $\phi \circ \alpha = \alpha' \circ \psi$.

Then, by [3], we get for $S \in \text{Var}(k)$ quasi-projective and $l : S \hookrightarrow \tilde{S}$ is a closed embedding with $\tilde{S} \in \text{SmVar}(k)$ an embedding

$$\iota : D(MHM_{k, gm}(S)) \hookrightarrow D_{\mathcal{D}fil, S}(\tilde{S}) \times_I D(S_{\mathbb{C}}^{an})$$

consisting of mixed Hodge module whose De Rham part is defined over k , where $D_{\mathcal{D}(1,0)fil, S}(\tilde{S}) \times_I D_{fil, c}(S_{\mathbb{C}}^{an})$ is the category

- whose objects are $((M, F, W), (K, W), \alpha)$ where (M, F, W) is a complex of filtered $D_{\tilde{S}}$ module supported on S , (K, W) is a filtered complex of presheaves on $S_{\mathbb{C}}^{an}$ whose cohomology are constructible sheaves, and $\alpha : l_*(K, W) \otimes \mathbb{C} \rightarrow DR(\tilde{S})((M, W)^{an})$ is an isomorphism in $D_{fil}(\tilde{S}_{\mathbb{C}}^{an})$.
- whose morphism are given in [3],

We have then the six functors formalism :

$$\begin{aligned} D(MHM_{k, gm}(-)) : \text{Var}(k) &\mapsto \text{TriCat}, \quad S \mapsto D(MHM_{k, gm}(S)), \quad (f : T \rightarrow S) \mapsto \\ (f^*, f_*) : D(MHM_{k, gm}(S)) &\rightarrow D(MHM_{k, gm}(T)), \quad (f!, f^!) : D(MHM_{k, gm}(T)) \rightarrow D(MHM_{k, gm}(S)) \end{aligned}$$

and for $D \subset S$ a (Cartier) divisor the nearby cycle functor

$$\begin{aligned} \psi_D &: D(MHM_{k,gm}(S)) \rightarrow D(MHM_{k,gm}(D)), \\ \psi_D((M, F, W), (K, W), \alpha) &:= (\psi_D(M, F, W), \psi_D(K, W), \psi_D(\alpha)). \end{aligned}$$

For $T \in \text{Var}(k)$ and $\epsilon_\bullet : T_\bullet \rightarrow T$ a desingularization with $T_\bullet \in \text{Fun}(\Delta, \text{SmVar}(k))$, we have

$$\mathbb{Z}_T^{Hdg} := a_T^* \mathbb{Z}^{Hdg} = (\epsilon_{\bullet*} E_{zar}(O_{T_\bullet}, F, W), \epsilon_{\bullet*} E_{usu}(\mathbb{Z}_{T_\bullet^{an}}, W), \alpha(T_\bullet))$$

For $f : X \rightarrow S$ a projective morphism with $X, S \in \text{SmVar}(k)$, we denote

$$E_{Hdg}(X/S) := f_* \mathbb{Z}_X^{Hdg} := \left(\int_f (O_X, F_b), Rf_* \mathbb{Z}_{X_\mathbb{C}^{an}}, f_* \alpha(X) \right) \in D(MHM_{k,gm}(S))$$

and $E_{Hdg}^j(X/S) := H^j E_{Hdg}(X/S)$ for $j \in \mathbb{Z}$. If moreover $Z \subset X$ is a closed subset, we denote

$$E_{Hdg,Z}(X/S) := f_* \Gamma_Z^{Hdg} \mathbb{Z}_X^{Hdg} := \left(\int_f \Gamma_Z^{Hdg}(O_X, F_b), Rf_* R\Gamma_Z \mathbb{Z}_{X_\mathbb{C}^{an}}, f_* \Gamma_Z \alpha(X) \right) \in D(MHM_{k,gm}(S))$$

and $E_{Hdg,Z}^j(X/S) := H^j E_{Hdg,Z}(X/S)$ for $j \in \mathbb{Z}$. For $f : X \hookrightarrow \mathbb{P}^N \times S \xrightarrow{p} S$ a projective morphism with $X, S \in \text{Var}(k)$, S smooth, we denote

$$E_{Hdg}(X/S) := f_* \mathbb{Z}_X^{Hdg} := \left(\int_p \Gamma_X^{\vee, Hdg}(O_{\mathbb{P}^N \times S}, F), Rf_* \mathbb{Z}_{X_\mathbb{C}^{an}}, f_* \alpha(X) \right) \in D(MHM_{k,gm}(S))$$

and $E_{Hdg}^j(X/S) := H^j E_{Hdg}(X/S)$ for $j \in \mathbb{Z}$. In the particular case where $E_{Hdg}^j(X/S)$ is a variation of mixed Hodge structure, the first factor of

$$E_{Hdg}^j(X/S) := (E_{DR}^j(X/S), R^j f_* \mathbb{Z}_{X_\mathbb{C}^{an}}, H^j f_* \alpha(X)) \in MHM_{k,gm}(S)$$

is a vector bundle.

We then have the specialization map :

Definition-Proposition 1. *Let $k \subset \mathbb{C}$ be a subfield.*

(i) *Let $f' : X' \rightarrow T$ be a projective morphism with $X', T \in \text{SmVar}(k)$. Assume that*

- $f^\circ := f \otimes_T T : X^\circ \rightarrow T^\circ$ is smooth, $T^\circ \subset T$ is an open subset,
- $X'_S := f'^{-1}(S) \subset X'$, $S := T \setminus T^\circ$. We have by definition $X^\circ = X' \setminus X'_S$.

We have then, see [3], the distinguished triangles in $D(MHM_{k,gm}(X_S))$,

$$\begin{aligned} \mathbb{Z}_{X'_S}^{Hdg} &:= i_{X'_S}^* \mathbb{Z}_{X'}^{Hdg} \xrightarrow{Sp_{X'_S/X'}} \psi_{X'_S} \mathbb{Z}_{X^\circ}^{Hdg} := (\psi_{X'_S}(O_{X'}, F), \psi_{X'_S} \mathbb{Z}_{X^\circ}, \psi_{X'_S} \alpha(X')) \\ &\xrightarrow{c(\psi_{X'_S} \mathbb{Z}_{X^\circ}^{Hdg})} \phi_{X'_S} \mathbb{Z}_{X^\circ}^{Hdg} := \text{Cone}(Sp_{X'/X_S}) \xrightarrow{c(\mathbb{Z}_{X'_S}^{Hdg})} \mathbb{Z}_{X'_S}^{Hdg}[1]. \end{aligned}$$

Applying the functor $(f' \circ i_{X'_S})_$, we obtain the generalized distinguished triangle in $D(MHM_{k,gm}(S))$*

$$\begin{aligned} E_{Hdg}(X'_S/S) &\xrightarrow{sp_{X'_S/j} := (f' \circ i_{X'_S})_* Sp_{X'_S/X'}} \\ (f' \circ i_{X'_S})_* (\psi_{X'_S} \mathbb{Z}_{X^\circ}^{Hdg}) &= \psi_0 E_{Hdg}(X^\circ/T^\circ) \xrightarrow{(f' \circ i_{X'_S})_* c(\Phi_{X_0} \mathbb{Z}_{X^\circ}^{Hdg})} \\ (f' \circ i_{X'_S})_* (\phi_{X'_S} \mathbb{Z}_{X^\circ}^{Hdg}) &= \phi_S E_{Hdg}(X^\circ/T^\circ) \xrightarrow{(f' \circ i_{X'_S})_* c(\mathbb{Z}_{X'}^{Hdg})} E_{Hdg}(X'_S/S)[1] \end{aligned}$$

(ii) Let $f : X \rightarrow T$ be a projective morphism with $X, T \in \text{SmVar}(k)$. Assume that

- $f^\circ := f \otimes_T T^\circ : X^\circ \rightarrow T^\circ$ is smooth, $T^\circ \subset T$ an open subset
- $S := T \setminus T^\circ$, $E := f^{-1}(S) = \cup_{i=0}^r E_i \subset X$ is a normal crossing divisor and that for $I \subset [0, \dots, r]$, $f_{0,I} := f|_{E_I} : E_I := \cap_{i \in I} E_i \rightarrow S$ are smooth. We have by definition $X^\circ = X \setminus E$.

Denote for $I \subset J \subset [0, \dots, r]$, $i_{JI} : E_J \hookrightarrow E_I$ and $i_I : E_I \hookrightarrow E$, $i_E : E \hookrightarrow X$ the closed embeddings. In this particular case, c.f. [8], the distinguished triangle in $D(\text{MHM}_{k, \text{gm}}(E))$ given in (i) become

$$\begin{aligned} \mathbb{Z}_E^{\text{Hdg}} &:= i_E^* \mathbb{Z}_X^{\text{Hdg}} = ((i_{\bullet\bullet} \Omega_{E^\bullet}^\bullet, F, W), (\mathbb{Z}_{E_\bullet}, W), \alpha(E_\bullet)) \xrightarrow{Sp_{E/X}} \\ \psi_E \mathbb{Z}_{X^\circ}^{\text{Hdg}} &= ((i_E^{\text{mod}} \Omega_{X/T}^\bullet(\log E), F, W), \psi_E \mathbb{Z}_{X^\circ}, \psi_E \alpha(X)) \xrightarrow{c(\psi_E \mathbb{Z}_{X^\circ}^{\text{Hdg}})} \\ \phi_E \mathbb{Z}_{X^\circ}^{\text{Hdg}} &:= \text{Cone}(Sp_{E/X}) \xrightarrow{c(\mathbb{Z}_E^{\text{Hdg}})} \mathbb{Z}_E^{\text{Hdg}}[1]. \end{aligned}$$

Recall the map $Sp_{E/X}$ is the defined as the factorization

$$\mathbb{Z}_X^{\text{Hdg}} \xrightarrow{\text{ad}(i_E^*, i_{E^\circ})} \mathbb{Z}_E^{\text{Hdg}} \xrightarrow{Sp_{E/X}} \psi_E \mathbb{Z}_{X^\circ}^{\text{Hdg}}.$$

Applying the functor $(f \circ i_E)_*$, we obtain the generalized distinguished triangle in $D(\text{MHM}_{k, \text{gm}}(S))$

$$\begin{aligned} E_{\text{Hdg}}(E/S) &\xrightarrow{Sp_{E/f} := (f \circ i_E)_* Sp_{E/X}} (f \circ i_E)_*(\psi_E \mathbb{Z}_{X^\circ}^{\text{Hdg}}) = \psi_S E_{\text{Hdg}}(X^\circ/T^\circ) \\ \xrightarrow{(f \circ i_E)_* c(\psi_E \mathbb{Z}_{X^\circ}^{\text{Hdg}})} &(f \circ i_E)_*(\phi_E \mathbb{Z}_{X^\circ}^{\text{Hdg}}) = \phi_S E_{\text{Hdg}}(X^\circ/T^\circ) \xrightarrow{(f \circ i_E)_* c(\mathbb{Z}_E^{\text{Hdg}})} E_{\text{Hdg}}(E/S)[1]. \end{aligned}$$

If $f = f' \circ \epsilon$ where $\epsilon : (X, E) \rightarrow (X, X_S)$ is a desingularization, we have $Sp_{X_S/f'} = Sp_{E/f} \circ \epsilon^*$.

Proof. See [8]. □

Lemma 1. Let $k \subset \mathbb{C}$ be a subfield. Let $f : X \rightarrow T$ be a projective morphism with $X, T \in \text{SmVar}(k)$. Assume that

- $f^\circ := f \otimes_T T^\circ : X^\circ \rightarrow T^\circ$ is smooth of relative dimension n , $T^\circ \subset T$ an open subset
- $S := T \setminus T^\circ = V(s) \subset T$ is a divisor, $E := f^{-1}(S) = \cup_{i=0}^r E_i \subset X$ is a normal crossing divisor and that for $I \subset [0, \dots, r]$, $f_{0,I} := f|_{E_I} : E_I := \cap_{i \in I} E_i \rightarrow S$ are smooth. We have by definition $X^\circ = X \setminus E$.

Denote $j_0 : E_0^\circ := E_0 \setminus \cup_{i=1}^r E_i \xrightarrow{j'_0} E_0 \xrightarrow{i_0} E$ the embedding.

(i) We have, for each $k \in \mathbb{Z}$, a canonical map in $\text{MHM}_{k, \text{gm}}(S)$

$$c_E : E_{\text{Hdg}, E}^{k-1}(X/T) \xrightarrow{c_V} \phi_S E_{\text{Hdg}}^{k-1}(X^\circ/T^\circ) \xrightarrow{H^{k-1}(f \circ i_E)_* c(\mathbb{Q}_E^{\text{Hdg}})} E_{\text{Hdg}}^k(E/S).$$

(ii) If $H^{2p-1}(X_t, \mathbb{Q}) = H^{2p-2}(X_t, \mathbb{Q}) = 0$ for $t \in T \setminus S$, then

$$\begin{aligned} c_E : E_{\text{Hdg}, E}^{k-1}(X/T) &\xrightarrow{c_V} \phi_S E_{\text{Hdg}}^{k-1}(X^\circ/T^\circ) \\ \xrightarrow{H^{k-1}(f \circ i_E)_* c(\mathbb{Q}_E^{\text{Hdg}})} &\ker(sp_{E/f} : E_{\text{Hdg}}^k(E/S) \rightarrow \psi_S E_{\text{Hdg}}^k(X^\circ/T^\circ)). \end{aligned}$$

is an isomorphism.

Proof. (i): See [8] for the definition of the map c_V . Recall that $Var = DR(S)(s)$.
(ii): Follows from the exact sequences in $MHM_{k, gm}(S)$

$$\begin{aligned} \cdots \rightarrow \psi_S E_{Hdg}^{k-2}(X^o/T^o) \rightarrow E_{Hdg, E}^{k-1}(X/T) \xrightarrow{c_V} \phi_S E_{Hdg}^{k-1}(X^o/T^o) \xrightarrow{V} \psi_S E_{Hdg}^{k-1}(X^o/T^o) \rightarrow \cdots, \\ \cdots \rightarrow \psi_S E_{Hdg}^{k-1}(X^o/T^o) \xrightarrow{H^{k-1}(f \circ i_E)_* c(\psi_E \mathbb{Q}_{X^o}^{Hdg})} \\ \phi_S E_{Hdg}^{k-1}(X^o/T^o) \xrightarrow{H^{k-1}(f \circ i_E)_* c(\mathbb{Q}_E^{Hdg})} E_{Hdg}^k(E/S) \xrightarrow{SpE/f} \psi_S E_{Hdg}^k(X^o/T^o) \rightarrow \cdots, \end{aligned}$$

given in [8] for the first one. \square

3 Hodge conjecture for smooth projective varieties

Theorem 1. *Let $X \in \text{PSmVar}(\mathbb{C})$. Then Hodge conjecture hold for X . That is, if $p \in \mathbb{Z}$ and $\lambda \in F^p H^{2p}(X^{an}, \mathbb{Q})$, $\lambda = [Z]$ with $Z \in \mathcal{Z}^p(X)$.*

Proof. The Hodge conjecture is true for curves and surfaces. Assume that the Hodge conjecture is true for smooth projective varieties of dimension less or equal to $d-1$. Let $X \in \text{PSmVar}(\mathbb{C})$ of dimension d . Up to split X into its connected components, we may assume that X is connected of dimension d . Then X is birational to a possibly singular hypersurface $Y_{0_X} \subset \mathbb{P}_{\mathbb{C}}^{d+1}$ of degree $l_X := \deg(\mathbb{C}(X)/\mathbb{C}(x_1, \dots, x_d))$. Consider

$$f : \mathcal{Y}'_{\mathbb{C}} = V(\tilde{f}) \subset \mathbb{P}_{\mathbb{C}}^{d+1} \times \mathbb{P}(l_X) \rightarrow \mathbb{P}(l_X)$$

the universal family of projective hypersurfaces of degree l_X , and $0_X \in \mathbb{P}(l_X)$ the point corresponding to Y_{0_X} . Denote $\Delta \subset \mathbb{P}(l_X)$ the discriminant locus parametrizing the hypersurfaces which are singular. Let $\bar{S} := \bar{0}_X^{\bar{\mathbb{Q}}} \subset \mathbb{P}(l_X)$ be the $\bar{\mathbb{Q}}$ -Zariski closure of 0_X inside $\mathbb{P}(l_X)$. If Y_{0_X} is smooth, then X satisfy the hodge conjecture since Y_{0_X} satisfy the Hodge conjecture ([5]) and by proposition 1(ii) and the induction hypothesis since X is birational to Y_{0_X} . Hence, we may assume that Y_{0_X} is singular, that is $0_X \in \Delta$. Then, $\bar{S} \subset \Delta$ since $\Delta \subset \mathbb{P}(l_X)$ is defined over $\bar{\mathbb{Q}}$. Let $\bar{T} \subset \mathbb{P}(l_X)$ be an irreducible closed subvariety of dimension $\dim(\bar{S}) + 1$ defined over $\bar{\mathbb{Q}}$ such that

$$\bar{S} \subset \bar{T} \text{ and } T := \bar{T} \cap (\mathbb{P}(l_X) \setminus \Delta) \neq \emptyset.$$

We take for $\bar{T} = V(f_1, \dots, f_s) \subset \mathbb{P}(l_X)$ an irreducible complete intersection defined over $\bar{\mathbb{Q}}$ which intersect Δ properly and such that

- $V(f_1) \supset \text{sing}(\Delta) \cup \bar{S}$
- for each $r \in [1, \dots, s]$, $V(f_1, \dots, f_r) \supset \text{sing}(V(f_1, \dots, f_{r-1}) \cap \Delta) \cup \bar{S}$ contain inductively the singular locus of the intersection with Δ and \bar{S} .

Then by the weak Lefchetz hyperplane theorem for homotopy groups, $T \subset \mathbb{P}(l_X) \setminus \Delta$ is not contained in a weakly special subvariety since

$$i_{T*} : \pi_1(T_{\mathbb{C}}) \rightarrow \pi_1(\mathbb{P}(l_X) \setminus \Delta)$$

is an isomorphism if $\dim \bar{T} \geq 2$ and surjective if $\dim \bar{T} = 1$. We have then $\bar{T} \cap \Delta = \bar{S} \cup \bar{S}'$, where $\bar{S}' \subset \bar{T}$ is a divisor. We then consider the family of algebraic varieties defined over \mathbb{Q}

$$\tilde{f}_X : \mathcal{Y}_{\bar{T}^o} \xrightarrow{\epsilon} \mathcal{Y}'_{\bar{T}^o} \xrightarrow{f \times_{\mathbb{P}(l_X)} \bar{T}^o} \bar{T}^o$$

where

- $\tilde{f}_X : \mathcal{Y} \xrightarrow{\epsilon} \mathcal{Y}'_{\bar{T}} \xrightarrow{f \times_{\mathbb{P}(l_X)} \bar{T}} \bar{T}$ with $\epsilon : (\mathcal{Y}, \mathcal{E}') \rightarrow (\mathcal{Y}'_{\bar{T}}, \mathcal{Y}'_{\bar{S}} \cup \mathcal{Y}'_{\bar{S}'})$ is a desingularization over \mathbb{Q} , in particular $\mathcal{E}' = \mathcal{E} \cup \mathcal{E}''$ with $\mathcal{E} = \cup_{i=0}^r \mathcal{E}_i := \tilde{f}_X^{-1}(\bar{S}) \subset \mathcal{Y}$ which is a normal crossing divisor, in particular $\mathcal{Y}_T = \mathcal{Y}'_T$,

- $S \subset \bar{S}$ is the open subset such that S is smooth and $\tilde{f}_X \times_{\bar{T}} S : \mathcal{E}_I := \cap_{i \in I} \mathcal{E}_i \rightarrow S$ is smooth projective for each $I \subset [0, \dots, r]$, $\bar{T}^o \subset \bar{T}$ is an open subset such that $\bar{T}^o \cap \Delta = S$ and $T^o := \bar{T}^o \cap T$ is smooth, for simplicity we denote again \mathcal{E} for the open subset $\mathcal{E}_S := \tilde{f}_X^{-1}(S) \subset \mathcal{E}$,
- $t \in S(\mathbb{C})$ is the point such that $\mathcal{Y}'_{\mathbb{C},t} = Y_{0_X}$ and $\epsilon_t : E_0 := \mathcal{E}_{0,\mathbb{C},t} \rightarrow Y_{0_X}$ is surjective (there is at least one component of E dominant over Y_{0_X} since E is dominant over Y_{O_X} as \mathcal{E} is dominant over \mathcal{Y}'_S), in particular E_0 is birational to Y_{0_X} , hence E_0 is birational to X . We denote $E_i := \mathcal{E}_{i,\mathbb{C},t}$ for each $0 \leq i \leq r$ and $E := \mathcal{E}_{\mathbb{C},t} = \cup_{i=0}^r E_i$.

We then prove, assuming the Hodge conjecture for smooth projective varieties of dimension less or equal to $d-1$, the Hodge conjecture for the E_i , more specifically for E_0 . Denote for $I \subset [0, \dots, s]$, $i_I : \mathcal{E}_I \hookrightarrow \mathcal{E}$ and for $I \subset J \subset [0, \dots, s]$, $i_{JI} : \mathcal{E}_J \rightarrow \mathcal{E}_I$ the closed embeddings. Then, $i_\bullet : \mathcal{E}_\bullet \rightarrow \mathcal{E}$ in $\text{Fun}(\Delta, \text{Var}(\mathbb{Q}))$ is a simplicial resolution with $\mathcal{E}_\bullet \in \text{Fun}(\Delta, \text{SmVar}(\mathbb{Q}))$. We have then

$$E_{Hdg}(\mathcal{E}/S) := (\tilde{f}_X \circ i_{\bullet*} E_{zar}(\Omega_{\mathcal{E}_\bullet/S}^\bullet, F_b, W), (\tilde{f}_X \circ i_{\bullet*} E_{usu}(\mathbb{Q}_{\mathcal{E}_\bullet}^{an}, W)), \tilde{f} \circ i_{\bullet*} \alpha(\mathcal{E}_\bullet)) \in D(MHM_{\mathbb{Q},gm}(S))$$

Then, definition-proposition 1 applied to $f_T = f \times_{\mathbb{P}(t_X)} \bar{T}^o : \mathcal{Y}'_{\bar{T}^o} \rightarrow \bar{T}^o$ gives the distinguish in $D(MHM_{\mathbb{Q},gm}(S))$,

$$\begin{aligned} E_{Hdg}(\mathcal{Y}'_S/S) &\xrightarrow{Sp_{\mathcal{Y}'_S/f_T} := (f_T \circ i_{\mathcal{Y}'_S})_* Sp_{\mathcal{Y}_S/\mathcal{Y}'_{T^o}}} \psi_S E_{Hdg}(\mathcal{Y}'_{T^o}/T^o) \\ &\xrightarrow{(f_T \circ i_{\mathcal{Y}'_S})_* c(\psi_{\mathcal{Y}'_S} \mathbb{Z}_{\mathcal{Y}'_{T^o}}^{Hdg})} \phi_S E_{Hdg}(\mathcal{Y}'_{T^o}/T^o) \xrightarrow{(f_T \circ i_{\mathcal{Y}'_S})_* c(\mathbb{Z}_{\mathcal{Y}'_S}^{Hdg})} E_{Hdg}(\mathcal{Y}'_S/S)[1] \end{aligned}$$

and definition-proposition 1 applied to $\tilde{f}_X : \mathcal{Y}_{\bar{T}^o} \rightarrow \bar{T}^o$ gives the distinguish in $D(MHM_{\mathbb{Q},gm}(S))$,

$$\begin{aligned} E_{Hdg}(\mathcal{E}/S) &\xrightarrow{Sp_{\mathcal{E}/\tilde{f}_X} := (\tilde{f}_X \circ i_{\mathcal{E}})_* Sp_{\mathcal{E}/\mathcal{Y}}} \psi_S E_{Hdg}(\mathcal{Y}_{T^o}/T^o) \\ &\xrightarrow{(\tilde{f}_X \circ i_{\mathcal{E}})_* c(\psi_{\mathcal{E}} \mathbb{Z}_{\mathcal{Y}_{T^o}}^{Hdg})} \phi_S E_{Hdg}(\mathcal{Y}_{T^o}/T^o) \xrightarrow{(\tilde{f}_X \circ i_{\mathcal{E}})_* c(\mathbb{Z}_{\mathcal{E}}^{Hdg})} E_{Hdg}(\mathcal{E}/S)[1] \end{aligned}$$

and $Sp_{\mathcal{E}/\tilde{f}_X} \circ \epsilon^* = Sp_{\mathcal{Y}_{T^o}/f_T}$. Let $p \in \mathbb{Z}$. By the hard Lefschetz theorem on $H^*(E_0)$, we may assume that $2p \leq d$. Taking the cohomology to this distinguish triangle, we get in particular, for each $p \in \mathbb{Z}$, the maps in $MHM_{\mathbb{Q},gm}(S)$

- $Sp_{\mathcal{Y}'_S/f_T} : E_{Hdg}^{2p}(\mathcal{Y}'_S/S) \rightarrow H^{2p} \psi_S E_{Hdg}(\mathcal{Y}_{T^o}/T^o)$
- $Sp_{\mathcal{E}/\tilde{f}_X} : E_{Hdg}^{2p}(\mathcal{E}/S) \rightarrow H^{2p} \psi_S E_{Hdg}(\mathcal{Y}_{T^o}/T^o)$.

Moreover, we have, see lemma 1, an isomorphism of variation of mixed Hodge structures over S

$$\begin{aligned} c_{\mathcal{E}} := H^{2p-1}(\tilde{f}_X \circ i_{\mathcal{E}})_* c(\mathbb{Q}_{\mathcal{E}}^{Hdg}) \circ c_V : E_{Hdg,\mathcal{E}}^{2p-1}(\mathcal{Y}_{\bar{T}^o}/\bar{T}^o) &\xrightarrow{\sim} \\ \phi_S E_{Hdg,\mathcal{E}}^{2p-1}(\mathcal{Y}_{T^o}/T^o) &\xrightarrow{\sim} \ker(Sp_{\mathcal{E}/\tilde{f}_X} : E_{Hdg}^{2p}(\mathcal{E}/S) \rightarrow \psi_S E_{Hdg}^{2p}(\mathcal{Y}_{T^o}/T^o)). \end{aligned}$$

For $\lambda \in H^{2p}(\mathcal{E}_{0,\mathbb{C},t'}^{an}, \mathbb{Q})$, $t' \in S(\mathbb{C})$, we consider $\pi_S : \tilde{S}_{\mathbb{C}}^{an} \rightarrow S_{\mathbb{C}}^{an}$ the universal covering in $\text{AnSm}(\mathbb{C})$ and

$$\mathbb{V}_S(\lambda) := \pi_S(\lambda \times \tilde{S}_{\mathbb{C}}^{an}) \subset E_{DR}^{2p,an}(\mathcal{E}_{0,\mathbb{C}}/S_{\mathbb{C}})$$

the flat leaf induced by the universal covering (see section 2.1) and

$$\mathbb{V}_S^p(\lambda) := \mathbb{V}_S(\lambda) \cap F^p E_{DR}^{2p}(\mathcal{E}_{0,\mathbb{C}}/S_{\mathbb{C}}) \subset E_{DR}^{2p}(\mathcal{E}_{0,\mathbb{C}}/S_{\mathbb{C}}),$$

which is an algebraic variety by [6] (finite over S). Note that $\mathbb{V}_S^p(\lambda) \subset H L^{p,2p}(\mathcal{E}_{0,\mathbb{C}}/S_{\mathbb{C}})$ is the union of the irreducible components passing through λ . Similarly, for $\eta \in H^{2p}(\mathcal{E}_{\mathbb{C},t'}^{an}, \mathbb{Q})$, $t' \in S(\mathbb{C})$, we consider $\pi_S : \tilde{S}_{\mathbb{C}}^{an} \rightarrow S_{\mathbb{C}}^{an}$ the universal covering in $\text{AnSm}(\mathbb{C})$ and

$$\mathbb{V}_S(\eta) := \pi_S(\eta \times \tilde{S}_{\mathbb{C}}^{an}) \subset E_{DR}^{2p,an}(\mathcal{E}_{\mathbb{C}}/S_{\mathbb{C}})$$

the flat leaf induced by the universal covering (see section 2.1) and

$$\mathbb{V}_S^p(\eta) := \mathbb{V}_S(\eta) \cap F^p E_{DR}^{2p}(\mathcal{E}_{\mathbb{C}}/S_{\mathbb{C}}) \subset E_{DR}^{2p}(\mathcal{E}_{\mathbb{C}}/S_{\mathbb{C}}).$$

We have

$$H^{2p} E_{Hdg}(\mathcal{Y}_{T^o}/T^o) = ((E_{DR}^{2p}(\mathcal{Y}_{T^o}/T^o), F), R^{2p} \tilde{f}_{X*} \mathbb{Q}_{\mathcal{Y}_{T^o, \mathbb{C}}^{an}}, \tilde{f}_{X*} \alpha(\mathcal{Y}_{T^o})) \in MHM_{\mathbb{Q}, gm}(T^o)$$

By [8], the filtered vector bundle

$$(E_{DR}^{2p}(\mathcal{Y}_{T^o}/T^o), F) = H^{2p} \tilde{f}_{X*} E_{zar}(\Omega_{\mathcal{Y}_{T^o}/T^o}^{\bullet}, F_b) \in \text{Vect}_{\mathcal{D}fil}(T^o)$$

extend to a filtered vector bundle

$$(E_{DR, V_S}^{2p}(\mathcal{Y}_{T^o}/T^o), F) := H^{2p} \tilde{f}_{X*} E_{zar}(\Omega_{\mathcal{Y}_{T^o}/T^o}^{\bullet}(\log \mathcal{E}), F_b) \in \text{Vect}_{\mathcal{D}fil}(\bar{T}^o)$$

such that

$$H^{2p} \psi_S E_{Hdg}(\mathcal{Y}_{T^o}/T^o) = (i_S^{*mod}(E_{DR, V_S}^{2p}(\mathcal{Y}_{T^o}/T^o), F, W), (\psi_S R^{2p} \tilde{f}_{X*} \mathbb{Q}_{\mathcal{Y}_{T^o, \mathbb{C}}^{an}}, W), \psi_S \tilde{f}_{X*} \alpha(\mathcal{Y}_{T^o}))$$

and

$$\begin{aligned} Sp_{\mathcal{E}/\tilde{f}_X} &:= (Sp_{\mathcal{E}/\tilde{f}_X}, Sp_{\mathcal{E}/\tilde{f}_X}) : ((E_{DR}^{2p}(\mathcal{E}/S), F, W), (R^{2p}(\tilde{f}_X \circ i_{\bullet})_* \mathbb{Q}_{\mathcal{E}, \mathbb{C}}^{an}, W), (\tilde{f}_X \circ i_{\bullet})_* \alpha(\mathcal{E}_{\bullet})) \\ &\rightarrow (i_S^{*mod}(E_{DR, V_S}^{2p}(\mathcal{Y}_{T^o}/T^o), F, W) = ((E_{DR, V_S}^{2p}(\mathcal{Y}_{T^o}/T^o) \cap p^{-1}(S), F), W), \\ &\quad (\psi_S R^{2p} \tilde{f}_{X*} \mathbb{Q}_{\mathcal{Y}_{T^o, \mathbb{C}}^{an}}, W), \psi_S \tilde{f}_{X*} \alpha(\mathcal{Y}_{T^o})) \end{aligned}$$

where $p : E_{DR, V_S}^{2p}(\mathcal{Y}_{T^o}/T^o) \rightarrow \bar{T}^o$ is the projection. For

$$\alpha \in H^{2p}(\mathcal{Y}_{t''}^{an}, \mathbb{Q}) = i_{t'}^* \psi_S R^{2p} \tilde{f}_{X*} \mathbb{Q}_{\mathcal{Y}_{T^o, \mathbb{C}}^{an}}, t' \in S_{\mathbb{C}}, t'' \in T^o(\mathbb{C}),$$

we consider

- $\pi_{T^o} : \tilde{T}^{o, an} \rightarrow T_{\mathbb{C}}^{o, an}$ the universal covering in $\text{AnSm}(\mathbb{C})$, the factorization

$$\pi_S : \tilde{S}_{\mathbb{C}}^{an} \rightarrow \tilde{T}_{\mathbb{C}}^{o, an} \xrightarrow{\pi_{T^o}} \bar{T}_{\mathbb{C}}^{o, an}$$

and

$$\mathbb{V}_{T^o}(\alpha) := \pi_{T^o}(\alpha \times \tilde{T}_{\mathbb{C}}^{o, an}) \subset E_{DR}^{2p, an}(\mathcal{Y}_{T^o, \mathbb{C}}/T_{\mathbb{C}}^o)$$

the flat leaf (see section 2.1) and

$$\mathbb{V}_{T^o}^p(\alpha) := \mathbb{V}_{T^o}(\alpha) \cap F^p E_{DR}^{2p}(\mathcal{Y}_{T^o, \mathbb{C}}/T_{\mathbb{C}}^o) \subset E_{DR}^{2p}(\mathcal{Y}_{T^o, \mathbb{C}}/T_{\mathbb{C}}^o),$$

which is an algebraic subvariety (finite over T^o) by [6],

- $\pi_S : \tilde{S}_{\mathbb{C}}^{an} \rightarrow S_{\mathbb{C}}^{an}$ the universal covering in $\text{AnSm}(\mathbb{C})$ and

$$\mathbb{V}_S(\alpha) := \pi_S(\alpha \times \tilde{S}_{\mathbb{C}}^{an}) \subset H^{2p} \psi_S E_{DR}(\mathcal{Y}_{T^o, \mathbb{C}}/T_{\mathbb{C}}^o)^{an} = i_S^{*mod}(E_{DR, V_S}^{2p}(\mathcal{Y}_{T^o, \mathbb{C}}/T_{\mathbb{C}}^o))^{an}$$

the flat leaf (see section 2.1) and

$$\mathbb{V}_S^p(\alpha) := \mathbb{V}_S(\alpha) \cap F^p H^{2p} \psi_S E_{DR}(\mathcal{Y}_{T^o, \mathbb{C}}/T_{\mathbb{C}}^o)^{an} \subset H^{2p} \psi_S E_{DR}(\mathcal{Y}_{T^o, \mathbb{C}}^{\prime}/T_{\mathbb{C}}^o)^{an}.$$

Consider the exact sequence of variation of mixed Hodge structure over S

$$0 \rightarrow W_{2p-1}E_{Hdg}^{2p}(\mathcal{E}/S) \rightarrow E_{Hdg}^{2p}(\mathcal{E}/S) \xrightarrow{(i_i^*)_{0 \leq i \leq r} := q} \bigoplus_{i=0}^r E_{Hdg}^{2p}(\mathcal{E}_i/S) \rightarrow 0.$$

It induces the exact sequence of vector bundles over S

$$0 \rightarrow W_{2p-1}E_{DR}^{2p}(\mathcal{E}/S) \rightarrow E_{DR}^{2p}(\mathcal{E}/S) \xrightarrow{(i_i^*)_{0 \leq i \leq r} := q} \bigoplus_{i=0}^r E_{DR}^{2p}(\mathcal{E}_i/S) \rightarrow 0,$$

and the exact sequence of presheaves on $S_{\mathbb{C}}^{an}$,

$$J(W_{2p-1}E_{Hdg}^{2p}(\mathcal{E}_{\mathbb{C}}/S_{\mathbb{C}})) := \text{Ext}^1(\mathbb{Q}_S^{Hdg}, W_{2p-1}E_{Hdg}^{2p}(\mathcal{E}_{\mathbb{C}}/S_{\mathbb{C}})) \xrightarrow{q} \text{Ext}^1(\mathbb{Q}_S^{Hdg}, W_{2p-1}E_{Hdg}^{2p}(\mathcal{E}_{\mathbb{C}}/S_{\mathbb{C}})) \xrightarrow{q} \bigoplus_{i=0}^r \text{Ext}^1(\mathbb{Q}_S^{Hdg}, W_{2p-1}E_{Hdg}^{2p}(\mathcal{E}_i/S_{\mathbb{C}})) \xrightarrow{e} \bigoplus_{i=0}^r \text{Ext}^1(\mathbb{Q}_S^{Hdg}, W_{2p-1}E_{Hdg}^{2p}(\mathcal{E}_i/S_{\mathbb{C}}))$$

where $HL^{2p,p}(\mathcal{E}_{\mathbb{C}}/S_{\mathbb{C}}) \subset E_{DR}^{2p}(\mathcal{E}_{\mathbb{C}}/S_{\mathbb{C}})$, $HL^{2p,p}(\mathcal{E}_{i,\mathbb{C}}/S_{\mathbb{C}}) \subset E_{DR}^{2p}(\mathcal{E}_{i,\mathbb{C}}/S_{\mathbb{C}})$ are the locus of Hodge classes. Let $\lambda \in F^p H^{2p}(E_0^{an}, \mathbb{Q})$, where we recall $E_0 = \mathcal{E}_{0,\mathbb{C},t}$ and $E = \mathcal{E}_{\mathbb{C},t} = \bigcup_{i=0}^r \mathcal{E}_i$. Consider then $\tilde{\lambda} \in H^{2p}(E^{an}, \mathbb{Q})$, such that $q(\tilde{\lambda}) = \lambda$, and

$$Sp_{\mathcal{E}/\tilde{f}_X}(\tilde{\lambda}) \in Sp_{E/f_X}(H^{2p}(E^{an}, \mathbb{Q})) \subset i_t^* \psi_S R^{2p} \tilde{f}_X^* \mathbb{Q}_{\mathcal{Y}_{T^o, \mathbb{C}}^{an}}.$$

Since the monodromy of $R^{2p} \tilde{f}_X^* \mathbb{Q}_{\mathcal{Y}_{T^o, \mathbb{C}}^{an}}$ is irreducible by Picard Lefschetz theory as $i_{T^o*} : \pi_1(T_{\mathbb{C}}^o) \rightarrow \pi_1(T_{\mathbb{C}}) \xrightarrow{\sim} \pi_1(\mathbb{P}(l_X) \setminus \Delta)$ is surjective, we have

$$Sp_{\mathcal{E}/\tilde{f}_X}(\ker q) := Sp_{\mathcal{E}/\tilde{f}_X}(W_{2p-1}E_{DR}^{2p}(\mathcal{E}/S)) = 0,$$

using the fact that there exists a neighborhood $V_{\bar{S} \cup \bar{S}'} \subset \bar{T}_{\mathbb{C}}$ of $\bar{S}_{\mathbb{C}} \cup \bar{S}'_{\mathbb{C}}$ in $\bar{T}_{\mathbb{C}}$ for the usual complex topology such that the inclusion $i_{\bar{S}_{\mathbb{C}} \cup \bar{S}'_{\mathbb{C}}} : \bar{S}_{\mathbb{C}} \cup \bar{S}'_{\mathbb{C}} \hookrightarrow V_{\bar{S} \cup \bar{S}'}$ admits a retraction $r : V_{\bar{S} \cup \bar{S}'} \rightarrow \bar{S}_{\mathbb{C}} \cup \bar{S}'_{\mathbb{C}}$ which is an homotopy equivalence. The locus of Hodge classes passing through λ

$$\mathbb{V}_S^p(\lambda) := \mathbb{V}_S(\lambda) \cap F^p E_{DR}^{2p}(\mathcal{E}_{0,\mathbb{C}}/S_{\mathbb{C}}) \subset E_{DR}^{2p}(\mathcal{E}_{0,\mathbb{C}}/S_{\mathbb{C}}),$$

inside the De Rham vector bundle of $\tilde{f}_X \circ i_{E_0} : \mathcal{E}_0 \rightarrow S$ thus satisfy

$$\mathbb{V}_S^p(\lambda) = q(Sp_{\mathcal{E}/\tilde{f}_X}^{-1}(\mathbb{V}_S^p(Sp_{\mathcal{E}/\tilde{f}_X}(\tilde{\lambda}))) \subset E_{DR}^{2p}(\mathcal{E}_{0,\mathbb{C}}/S_{\mathbb{C}}), \quad (2)$$

where

- $\mathbb{V}_S(\lambda) \subset E_{DR}^{2p}(\mathcal{E}_{0,\mathbb{C}}/S_{\mathbb{C}})$, $\mathbb{V}_S(Sp_{\mathcal{E}/\tilde{f}_X}(\tilde{\lambda})) \subset E_{DR, V_S}^{2p}(\mathcal{Y}_{T^o, \mathbb{C}}/T_{\mathbb{C}}^o)$, are the flat leaves, e.g. $\mathbb{V}_S(\lambda) := \pi_S(\lambda \times \tilde{S}_{\mathbb{C}}^{an})$ where $\pi_S : H^{2p}(E^{an}, \mathbb{C}) \times \tilde{S}_{\mathbb{C}}^{an} \rightarrow E_{DR}^{2p, an}(\mathcal{E}_{\mathbb{C}}/S_{\mathbb{C}})$ is the morphism induced by the universal covering $\pi_S : \tilde{S}_{\mathbb{C}}^{an} \rightarrow S_{\mathbb{C}}^{an}$,
- $q := q \otimes \mathbb{C} : E_{DR}^{2p}(\mathcal{E}_{\mathbb{C}}/S_{\mathbb{C}}) \rightarrow E_{DR}^{2p}(\mathcal{E}_{0,\mathbb{C}}/S_{\mathbb{C}})$ is the quotient map.
- $Sp_{\mathcal{E}/\tilde{f}_X} := Sp_{\mathcal{E}/\tilde{f}_X} \otimes \mathbb{C} : E_{DR}^{2p}(\mathcal{E}_{\mathbb{C}}/S_{\mathbb{C}}) \rightarrow i_S^{*mod} E_{DR, V_S}^{2p}(\mathcal{Y}_{T^o, \mathbb{C}}/T_{\mathbb{C}}^o)$.

Now,

- If $Sp_{\mathcal{E}/\tilde{f}_X}(\tilde{\lambda}) = 0$, we have by lemma 1 applied to $\tilde{f}_X : \mathcal{Y}_{\bar{T}^o} \rightarrow \bar{T}^o$, $\tilde{\lambda} = c_{\mathcal{E}}(\tilde{\lambda})$ with $\tilde{\lambda} \in F^p H_E^{2p-1}(\mathcal{Y}_{\bar{T}^o, \mathbb{C}}^{an}, \mathbb{Q})$ where $C \subset \bar{T}_{\mathbb{C}}^o$ is a smooth transversal slice, hence $\dim(C) = 1$ and \mathcal{Y}_C is smooth, such that $C \cap S_{\mathbb{C}} = \{t\}$ and

$$\mathbb{V}_S^p(\tilde{\lambda}) = c_{\mathcal{E}}(\mathbb{V}_S^p(\tilde{\lambda})), \quad \mathbb{V}_S^p(\tilde{\lambda}) \subset E_{DR, \mathcal{E}_{\mathbb{C}}}^{2p-1}(\mathcal{Y}_{T^o, \mathbb{C}}/\bar{T}_{\mathbb{C}}^o)$$

with

$$c_{\mathcal{E}} = (c_{\mathcal{E}}, c_{\mathcal{E}}) : ((E_{DR, \mathcal{E}}^{2p-1}(\mathcal{Y}_{\bar{T}^o}/\bar{T}^o), F, W), (R^{2p-1} \tilde{f}_X^* \circ \Gamma_{\mathcal{E}} \mathbb{Q}_{\mathcal{Y}_{T^o, \mathbb{C}}^{an}}, W), \alpha(\mathcal{Y}_{\bar{T}^o})) \rightarrow ((E_{DR}^{2p}(\mathcal{E}/S), F, W), (R^{2p}(\tilde{f}_X \circ i_{\mathcal{E}})_* \mathbb{Q}_{\mathcal{E}_{\mathbb{C}}^{an}}, W), \alpha(\mathcal{E}_{\bullet}))$$

We have for each $t' \in S(\mathbb{C})$, the isomorphism

$$\begin{aligned} \bigoplus_{\text{card} I=2} F^p H^{2p}(E_{IC, t'}^{an}, \mathbb{Q}) &= F^p \text{Gr}_{2p}^W H^{2p-2}((\mathcal{Y}_{\mathbb{C}, C'}^- \setminus E)^{an}, \mathbb{Q}) \xrightarrow{c(\mathbb{Z}(\mathcal{Y}_C \setminus \mathcal{E}_{C, t'}))} \\ &F^p \text{Gr}_{2p}^W H_E^{2p-1}(\mathcal{Y}_{\mathbb{C}, C'}^{an}, \mathbb{Q}) = F^p \text{Gr}_{2p}^W H_E^{2p-1}(\mathcal{Y}_{\mathbb{C}, C'}^{an}, \mathbb{Q}) \end{aligned}$$

where $C \subset \bar{T}_{\mathbb{C}}^o$ is a smooth transversal slice, hence $\dim(C') = 1$ and $\mathcal{Y}_{C'}$ is smooth, such that $C' \cap S_{\mathbb{C}} = \{t'\}$ and $\bar{\mathcal{Y}}_{C'} \in \text{PSmVar}(\mathbb{C})$ is a smooth compactification of $\mathcal{Y}_{C'}$, the last equality follows from excision (note that E is proper). Hence, assuming the Hodge conjecture for smooth projective varieties of dimension less or equal to $d-1$,

$$\mathbb{V}_S^p(\lambda) \subset \text{Gr}_{2p}^W E_{DR, \mathcal{E}_{\mathbb{C}}}^{2p-1}(\mathcal{Y}_{\bar{T}^o, \mathbb{C}}/\bar{T}_{\mathbb{C}}^o), \quad \mathbb{V}_S^p(\lambda) = c_{\mathcal{E}}(\mathbb{V}_S^p(\lambda)) \subset E_{DR}^{2p}(\mathcal{E}_{0\mathbb{C}}/S_{\mathbb{C}}) \subset \text{Gr}_{2p}^W E_{DR}^{2p}(\mathcal{E}_{\mathbb{C}}/S_{\mathbb{C}})$$

are defined over $\bar{\mathbb{Q}}$ and its Galois conjugates are also components of the locus of Hodge classes, since $\dim(E_I) = d-1$ for $I \subset [0, \dots, r]$ such that $\text{card} I = 2$.

- Consider now the case where $Sp_{\mathcal{E}/\bar{f}_X}(\tilde{\lambda}) \neq 0$. We have the key equality

$$\mathbb{V}_S^p(Sp_{\mathcal{E}/\bar{f}_X}(\tilde{\lambda})) = \overline{\mathbb{V}_{T^o}^p(Sp_{\mathcal{E}/\bar{f}_X}(\tilde{\lambda}))} \cap p^{-1}(S_{\mathbb{C}}) \subset E_{DR, V_S}^{2p}(\mathcal{Y}_{T^o, \mathbb{C}}/T_{\mathbb{C}}^o)$$

where $p := p \otimes \mathbb{C} : E_{DR, V_S}^{2p}(\mathcal{Y}_{T^o, \mathbb{C}}/T_{\mathbb{C}}^o) \rightarrow \bar{T}_{\mathbb{C}}^o$ is the projection, the inclusion \subset is obvious whereas the inclusion \supset follows from [6] lemma 2.11 which state the invariance of $\tilde{\lambda}$ under the monodromy

$$\text{Im}(\pi_1(p(\mathbb{V}_{T^o}^p(\tilde{\lambda})) \cap D^*) \rightarrow \pi_1(D^*)), (D, D^*) \rightarrow (\bar{T}_{\mathbb{C}}^{o, an}, \bar{T}_{\mathbb{C}}^{o, an} \setminus S_{\mathbb{C}}^{an}).$$

But by [5] theorem 1, the Hodge conjecture holds for projective hypersurfaces hence

$$\mathbb{V}_{T^o}^p(Sp_{\mathcal{E}/\bar{f}_X}(\tilde{\lambda})) \subset E_{DR}^{2p}(\mathcal{Y}_{T^o, \mathbb{C}}/T_{\mathbb{C}}^o)$$

is an algebraic subvariety defined over $\bar{\mathbb{Q}}$ and its Galois conjugates are also components of the locus of Hodge classes. Hence,

$$\mathbb{V}_S^p(\lambda) = q(Sp_{\mathcal{E}/\bar{f}_X}^{-1}(\mathbb{V}_S^p(Sp_{\mathcal{E}/\bar{f}_X}(\tilde{\lambda})))) \subset E_{DR}^{2p}(\mathcal{E}_{0\mathbb{C}}/S_{\mathbb{C}})$$

is an algebraic subvariety defined over $\bar{\mathbb{Q}}$ and its Galois conjugates are also components of the locus of Hodge classes.

The fact that $\mathbb{V}_S^p(\lambda) \subset E_{DR}^{2p}(\mathcal{E}_{0\mathbb{C}}/S_{\mathbb{C}})$ is defined over $\bar{\mathbb{Q}}$ and its Galois conjugates are also components of the locus of Hodge classes gives, by [4] theorem 4,

$$\lambda = [Z] \in H^{2p}(E_0^{an}, \mathbb{Q}), \text{ with } Z \in \mathcal{Z}^p(E_0).$$

Since $p \in \mathbb{Z}$ and $\lambda \in F^p H^{2p}(E_0^{an}, \mathbb{Q})$ are arbitrary, the Hodge conjecture holds for E_0 . Proposition 1(ii) and the fact that the Hodge conjecture is true for smooth complex projective varieties of dimension less or equal to $d-1$ by induction hypothesis then implies that Hodge conjecture holds for X , since X is birational to E_0 and $X, E_0 \in \text{PSmVar}(\mathbb{C})$ are connected of dimension d . \square

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