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Degeneration of families of projective hypersurfaces and Hodge conjecture

Johann Bouali

January 7, 2024

Abstract

We prove by induction on dimension the Hodge conjecture for smooth complex projective varieties. Let X be a smooth complex projective variety. Then X is birational to a possibly singular projective hypersurface, hence to a smooth projective variety E_0 which is a component of a normal crossing divisor $E = \cup_{i=0}^r E_i \subset Y$ which is the singular fiber of a pencil $f : Y \rightarrow \mathbb{A}^1$ of smooth projective hypersurfaces. Using the smooth hypersurface case ([5] theorem 1), the nearby cycle functor on mixed Hodge module with rational de Rham factor, and the induction hypothesis, we prove that an Hodge class of E_0 is absolute Hodge, more precisely the locus of Hodge classes inside the algebraic vector bundle given the De Rham cohomology the rational deformation of E_0 is defined over \mathbb{Q} . By [4] theorem 4, we get the Hodge conjecture for E_0 . By the induction hypothesis we also have the Hodge conjecture for X .

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1 Introduction

The main result of this article is an inductive proof of the Hodge conjecture using the results of [4] and [5]. More precisely, assuming the Hodge conjecture for smooth projective varieties of dimension less or equal to $d - 1$, the property of satisfying the Hodge conjecture for connected smooth projective varieties of dimension d is a birational invariant (proposition 1). On the other hand, if X is a connected smooth projective variety of dimension d , there exists (see proposition 2) a family of smooth projective hypersurfaces $f_X : Y \rightarrow \mathbb{A}_{\mathbb{C}}^1$ with

- Y a smooth complex algebraic variety, f_X flat projective,
- $Y_s := f_X^{-1}(s) \subset \mathbb{P}_{\mathbb{C}}^{d+1}$ for $s \in \mathbb{A}_{\mathbb{C}}^1 \setminus 0$ are smooth projective hypersurfaces,
- $E := f_X^{-1}(0) = \cup_{i=0}^r E_i \subset Y$ is a normal crossing divisor and E_0 is birational to X .

Assuming the Hodge conjecture for smooth projective varieties of dimension less or equal to $d - 1$, we then prove the Hodge conjecture for the E_i , more specifically for E_0 as follows : We consider its canonical rational deformation $\tilde{f}_X : \mathcal{Y} \rightarrow \mathbb{A}_{\mathbb{Q}}^1 \times S$ with

- \mathcal{Y} a smooth algebraic variety over \mathbb{Q} , $S \subset \mathbb{A}_{\mathbb{Q}}^s$ an open subset, \tilde{f}_X flat projective
- $Y = \tilde{f}_X^{-1}(\mathbb{A}_{\mathbb{Q}}^1 \times \{t\})$, $f_X = \tilde{f}_X \times_S \{t\}$, with $t \in S(\mathbb{C})$
- $\mathcal{E} = \cup_{i=0}^r \mathcal{E}_i := \tilde{f}_X^{-1}(\{0\} \times S) \subset \mathcal{Y}$ is a normal crossing divisor, $\tilde{f}_{X,0} : \mathcal{E}_I := \cap_{i \in I} \mathcal{E}_i \rightarrow S$ smooth projective for each $I \subset [0, \dots, r]$, $\mathcal{E}_{\mathbb{C},t} = E$,
- $\tilde{f}_X \times_{\mathbb{A}^1} (\mathbb{A}^1 \setminus 0) \mathcal{Y}^o \rightarrow \mathbb{A}_{\mathbb{Q}}^1 \setminus 0$ has fibers which are smooth projective hypersurfaces.

We then have the exact sequence of geometric mixed Hodge module over S (in fact of geometric variation of mixed Hodge structures over S)

$$\oplus_{(i,j) \subset [1, \dots, r]}^r E_{Hdg}^{2p-2}(\mathcal{E}_{ij}/S) \xrightarrow{(i,j)!} \oplus_{i=0}^r E_{Hdg}^{2p}(\mathcal{E}_i/S) \xrightarrow{(Sp_{\mathcal{E}_i/\tilde{f}_X})_{0 \leq i \leq r}} H^{2p} \Phi_0 E_{Hdg}(\mathcal{Y}^o / ((\mathbb{A}_{\mathbb{Q}}^1 \setminus 0) \times S)).$$

whose de Rham part are morphism of filtered vector bundle over S with Gauss-Manin connexion defined over \mathbb{Q} . Since for $0 \leq i \leq r$, $E_{Hdg}^{2p}(\mathcal{E}_i/S)$ is a geometric variation of pure Hodge structures, this exact sequence split, in particular

$$E_{Hdg}^{2p}(\mathcal{E}_{0,\mathbb{C}}/S_{\mathbb{C}}) = l_{0j}!(\oplus_{j=0}^r E_{Hdg}^{2p-2}(\mathcal{E}_{0j,\mathbb{C}}/S_{\mathbb{C}})) \oplus Sp_{\mathcal{E}_0/\tilde{f}_X}(E_{Hdg}^{2p}(\mathcal{E}_{0,\mathbb{C}}/S_{\mathbb{C}})).$$

Let $\lambda \in F^p H^{2p}(E_0^{an}, \mathbb{Q})$, where we recall $E_0 = \mathcal{E}_{0,\mathbb{C},t}$. Then,

$$\lambda = \lambda^K + \lambda^G \in F^p H^{2p}(E_0^{an}, \mathbb{Q}), \lambda^K \in \oplus_{J \in R} i_{0j}! F^{p-1} H^{2p-2}(E_{0j,\mathbb{C}}^{an}, \mathbb{Q}), \lambda^G \in Sp_{E_0/f_X}(F^p H^{2p}(E_0^{an}, \mathbb{Q})).$$

The locus of Hodge classes passing through λ (see [6])

$$\mathbb{V}_S^p(\lambda) := \pi_S(\lambda \times \tilde{S}_{\mathbb{C}}) \cap F^p E_{DR}^{2p}(\mathcal{E}_{0\mathbb{C}}/S_{\mathbb{C}}) = \mathbb{V}_S^p(\lambda^K) \cap \mathbb{V}_S^p(\lambda^G) \subset E_{DR}^{2p}(\mathcal{E}_{0\mathbb{C}}/S_{\mathbb{C}})$$

is the intersection of two complex algebraic subvariety of the De Rham vector bundle, where $\pi_S : H^{2p}(E_0^{an}, \mathbb{C}) \times \tilde{S}_{\mathbb{C}}^{an} \rightarrow E_{DR}^{2p,an}(\mathcal{E}_{0\mathbb{C}}/S_{\mathbb{C}})$ is the morphism given by the universal covering $\pi_S : \tilde{S}_{\mathbb{C}}^{an} \rightarrow S_{\mathbb{C}}^{an}$. Now,

- by [5] theorem 1, the Hodge conjecture holds for projective hypersurfaces, hence

$$Sp_{\mathcal{E}_0/\tilde{f}_X}(\mathbb{V}_S^p(\lambda^G)) = \overline{\mathbb{V}_{(\mathbb{A}^1 \setminus 0) \times S}^p(Sp_{E_0/f_X}(\lambda^G))} \cap p^{-1}(0 \times S_{\mathbb{C}}) \subset E_{DR, V_0}^{2p}(\mathcal{Y}_{\mathbb{C}}/(\mathbb{A}_{\mathbb{C}}^1 \times S_{\mathbb{C}}))$$

is defined over \mathbb{Q} ,

- assuming the Hodge conjecture for smooth projective varieties of dimension less or equal to $d - 1$

$$\mathbb{V}_S^p(\lambda^K) = i_{0j}!(\mathbb{V}_S^p(\lambda^K)) \subset E_{DR}^{2p}(\mathcal{E}_{0\mathbb{C}}/S_{\mathbb{C}})$$

is defined over \mathbb{Q} .

Hence, $\mathbb{V}_S^p(\lambda) \subset E_{DR}^{2p}(\mathcal{E}_{0\mathbb{C}}/S_{\mathbb{C}})$ is defined over \mathbb{Q} . By [4] theorem 4, we get

$$\lambda = [Z] \in H^{2p}(E_0^{an}, \mathbb{Q}), Z \in \mathcal{Z}^p(E_0).$$

Since $p \in \mathbb{Z}$ and $\lambda \in F^p H^{2p}(E_0^{an}, \mathbb{Q})$ are arbitrary, this proves the Hodge conjecture for E_0 . Since X is birational to E_0 , the Hodge conjecture for E_0 and the induction hypothesis implies the Hodge conjecture for X .

I am grateful to professor F.Mokrane for help and support during this work.

2 Preliminaries and notations

2.1 Notations

- Denote by Top the category of topological spaces and RTop the category of ringed spaces.
- Denote by Cat the category of small categories and RCat the category of ringed topoi.
- For $\mathcal{S} \in \text{Cat}$ and $X \in \mathcal{S}$, we denote $\mathcal{S}/X \in \text{Cat}$ the category whose objects are $Y/X := (Y, f)$ with $Y \in \mathcal{S}$ and $f : Y \rightarrow X$ is a morphism in \mathcal{S} , and whose morphisms $\text{Hom}((Y', f'), (Y, f))$ consists of $g : Y' \rightarrow Y$ in \mathcal{S} such that $f \circ g = f'$.
- Let $(\mathcal{S}, O_{\mathcal{S}}) \in \text{RCat}$ a ringed topos with topology τ . For $F \in C_{O_{\mathcal{S}}}(\mathcal{S})$, we denote by $k : F \rightarrow E_{\tau}(F)$ the canonical flasque resolution in $C_{O_{\mathcal{S}}}(\mathcal{S})$ (see [3]). In particular for $X \in \mathcal{S}$, $H^*(X, E_{\tau}(F)) \xrightarrow{\sim} \mathbb{H}_{\tau}^*(X, F)$.
- For $f : \mathcal{S}' \rightarrow \mathcal{S}$ a morphism with $\mathcal{S}, \mathcal{S}' \in \text{RCat}$, endowed with topology τ and τ' respectively, we denote for $F \in C_{O_{\mathcal{S}}}(\mathcal{S})$ and each $j \in \mathbb{Z}$,

$$\begin{aligned} - f^* &:= H^j \Gamma(\mathcal{S}, k \circ \text{ad}(f^*, f_*)(F)) : \mathbb{H}^j(\mathcal{S}, F) \rightarrow \mathbb{H}^j(\mathcal{S}', f^*F), \\ - f^{*mod} &:= H^j \Gamma(\mathcal{S}, k \circ \text{ad}(f^{*mod}, f_*)(F)) : \mathbb{H}^j(\mathcal{S}, F) \rightarrow \mathbb{H}^j(\mathcal{S}', f^{*mod}F), \end{aligned}$$

the canonical maps.

- For $m : A \rightarrow B$, $A, B \in C(\mathcal{A})$, \mathcal{A} an additive category, we denote $c(A) : \text{Cone}(m : A \rightarrow B) \rightarrow A[1]$ and $c(B) : B \rightarrow \text{Cone}(m : A \rightarrow B)$ the canonical maps.
- Denote by $\text{Sch} \subset \text{RTop}$ the subcategory of schemes (the morphisms are the morphisms of locally ringed spaces). We denote by $\text{PSch} \subset \text{Sch}$ the full subcategory of proper schemes. For a field k , we consider $\text{Sch}/k := \text{Sch}/\text{Spec } k$ the category of schemes over $\text{Spec } k$. The objects are $X := (X, a_X)$ with $X \in \text{Sch}$ and $a_X : X \rightarrow \text{Spec } k$ a morphism and the morphisms are the morphisms of schemes $f : X' \rightarrow X$ such that $f \circ a_{X'} = a_X$. We then denote by
 - $\text{Var}(k) = \text{Sch}^{ft}/k \subset \text{Sch}/k$ the full subcategory consisting of algebraic varieties over k , i.e. schemes of finite type over k ,
 - $\text{PVar}(k) \subset \text{QPVar}(k) \subset \text{Var}(k)$ the full subcategories consisting of quasi-projective varieties and projective varieties respectively,
 - $\text{PSmVar}(k) \subset \text{SmVar}(k) \subset \text{Var}(k)$, $\text{PSmVar}(k) := \text{PVar}(k) \cap \text{SmVar}(k)$, the full subcategories consisting of smooth varieties and smooth projective varieties respectively.
- Denote by $\text{AnSp}(\mathbb{C}) \subset \text{RTop}$ the subcategory of analytic spaces over \mathbb{C} , and by $\text{AnSm}(\mathbb{C}) \subset \text{AnSp}(\mathbb{C})$ the full subcategory of smooth analytic spaces (i.e. complex analytic manifold).
- For $X \in \text{Var}(k)$ and $X = \cup_{i \in I} X_i$ with $i_i : X_i \hookrightarrow X$ closed embeddings, we denote $X_{\bullet} \in \text{Fun}(\Delta, \text{Var}(k))$ the associated simplicial space, with for $J \subset I$, $i_{IJ} : X_I := \cap_{i \in I} X_i \hookrightarrow X_J := \cap_{i \in J} X_i$ the closed embedding.
- Let $(X, O_X) \in \text{RTop}$. We consider its De Rham complex $\Omega_X^{\bullet} := DR(X)(O_X)$.
 - Let $X \in \text{Sch}$. Considering its De Rham complex $\Omega_X^{\bullet} := DR(X)(O_X)$, we have for $j \in \mathbb{Z}$ its De Rham cohomology $H_{DR}^j(X) := \mathbb{H}^j(X, \Omega_X^{\bullet})$.
 - Let $X \in \text{Var}(k)$. Considering its De Rham complex $\Omega_X^{\bullet} := \Omega_{X/k}^{\bullet} := DR(X/k)(O_X)$, we have for $j \in \mathbb{Z}$ its De Rham cohomology $H_{DR}^j(X) := \mathbb{H}^j(X, \Omega_X^{\bullet})$. The differentials of $\Omega_X^{\bullet} := \Omega_{X/k}^{\bullet}$ are by definition k -linear, thus $H_{DR}^j(X) := \mathbb{H}^j(X, \Omega_X^{\bullet})$ has a structure of a k vector space.

- Let $X \in \text{AnSp}(\mathbb{C})$. Considering its De Rham complex $\Omega_X^\bullet := DR(X)(O_X)$, we have for $j \in \mathbb{Z}$ its De Rham cohomology $H_{DR}^j(X) := \mathbb{H}^j(X, \Omega_X^\bullet)$.
- For $X \in \text{AnSp}(\mathbb{C})$, we denote $\alpha(X) : \mathbb{C}_X \hookrightarrow \Omega_X^\bullet$ the embedding in $C(X)$. For $X \in \text{AnSm}(\mathbb{C})$, $\alpha(X) : \mathbb{C}_X \hookrightarrow \Omega_X^\bullet$ is an equivalence usu local by Poincare lemma.
- We denote $\mathbb{I}^n := [0, 1]^n \in \text{Diff}(\mathbb{R})$ (with boundary). For $X \in \text{Top}$ and R a ring, we consider its singular cochain complex

$$C_{\text{sing}}^*(X, R) := (\mathbb{Z} \text{Hom}_{\text{Top}}(\mathbb{I}^*, X)^\vee) \otimes R$$

and for $l \in \mathbb{Z}$ its singular cohomology $H_{\text{sing}}^l(X, R) := H^n C_{\text{sing}}^*(X, R)$. For $f : X' \rightarrow X$ a continuous map with $X, X' \in \text{Top}$, we have the canonical map of complexes

$$f^* : C_{\text{sing}}^*(X, R) \rightarrow C_{\text{sing}}^*(X', R), \sigma \mapsto f^* \sigma := (\gamma \mapsto \sigma(f \circ \gamma)).$$

In particular, we get by functoriality the complex

$$C_{X, R, \text{sing}}^* \in C_R(X), (U \subset X) \mapsto C_{\text{sing}}^*(U, R)$$

We recall that

- For $X \in \text{Top}$ locally contractible, e.g. $X \in \text{CW}$, and R a ring, the inclusion in $C_R(X)$ $c_X : R_X \rightarrow C_{X, R, \text{sing}}^*$ is by definition an equivalence top local and that we get by the small chain theorem, for all $l \in \mathbb{Z}$, an isomorphism $H^l c_X : H^l(X, R_X) \xrightarrow{\sim} H_{\text{sing}}^l(X, R)$.
- For $X \in \text{Diff}(\mathbb{R})$, the restriction map

$$r_X : \mathbb{Z} \text{Hom}_{\text{Diff}(\mathbb{R})}(\mathbb{I}^*, X)^\vee \rightarrow C_{\text{sing}}^*(X, R), w \mapsto w : (\phi \mapsto w(\phi))$$

is a quasi-isomorphism by Whitney approximation theorem.

2.2 Birational projective varieties and Hodge conjecture

In this subsection, we recall that assuming the Hodge conjecture for smooth projective varieties of dimension less or equal to $d - 1$, the property of satisfying the Hodge conjecture for connected smooth projective varieties of dimension d is a birational invariant :

Proposition 1. (i) *Let $X \in \text{PSmVar}(\mathbb{C})$ connected of dimension d . Let $\epsilon : \tilde{X}_Z \rightarrow X$ be the blow up of X along a smooth closed subvariety $Z \subset X$. Assume the Hodge conjecture hold for smooth complex projective varieties of dimension less or equal to $d - 1$. Then the Hodge conjecture hold for X if and only if it hold for \tilde{X}_Z .*

(ii) *Let $X, X' \in \text{PSmVar}(\mathbb{C})$ connected of dimension d . Assume the Hodge conjecture hold for smooth complex projective varieties of dimension less or equal to $d - 1$. If X is birational to X' , then the Hodge conjecture hold for X if and only if it hold for X'*

Proof. (i): Follows from the fact that

$$(\epsilon^*, \oplus_{l=1}^c i_{E^*}((-).h^{l-1})) : H^k(X, \mathbb{Q}) \oplus \oplus_{l=1}^c H^{k-2l}(E, \mathbb{Q}) \rightarrow H^k(\tilde{X}_Z, \mathbb{Q})$$

is an isomorphism of Hodge structures for each $k \in \mathbb{Z}$, where $c = \text{codim}(Z, X)$, $i_E : E \hookrightarrow \tilde{X}_Z$ is the closed embedding, $\epsilon|_E : E \rightarrow Z$ being a projective vector bundle.

(ii): Follows from (i) since if $\pi : X^o \rightarrow X'$ is a birational map, $X^o \subset X$ being an open subset, then X is connected to X' by a sequence of blow up of smooth projective varieties with smooth center by [1]. \square

2.3 A family of smooth projective hypersurfaces associated to a smooth projective variety

In this subsection, we recall that given a smooth complex projective variety X , there exists a family of smooth complex projective hypersurfaces which degenerates into a normal crossing divisor with one irreducible component birational to X .

Proposition 2. *Let $X \in \text{PSmVar}(\mathbb{C})$ connected. Then there exists a family of smooth projective hypersurfaces $f_X : Y \rightarrow \mathbb{A}_{\mathbb{C}}^1$ with*

- $Y \in \text{SmVar}(\mathbb{C})$, f_X flat projective,
- $Y_s := f_X^{-1}(s) \subset \mathbb{P}_{\mathbb{C}}^{d+1}$ for $s \in \mathbb{A}_{\mathbb{C}}^1 \setminus 0$ are smooth projective hypersurfaces,
- $E := f_X^{-1}(0) = \cup_{i=0}^r E_i \subset Y$ is a normal crossing divisor and E_0 is birational to X ,

Proof. See [2] lemma . It use the fact that X is birational to a possibly singular projective hypersurface Y and consider the desingularization of a generic pencil passing through Y . \square

2.4 The nearby cycle functor and the specialization map

Recall from [3] that for $k \subset \mathbb{C}$ a subfield and $S \in \text{Var}(k)$ quasi-projective, and $l : S \hookrightarrow \tilde{S}$ a closed embedding with $\tilde{S} \in \text{SmVar}(k)$, we have the full subcategory

$$\iota : MHM_{k,gm}(S) \hookrightarrow \text{PSh}_{\mathcal{D}fil,S}(\tilde{S}) \times_I P(S_{\mathbb{C}}^{an})$$

consisting of geometric mixed Hodge module whose De Rham part is defined over k , where $\text{PSh}_{\mathcal{D}(1,0)fil,S}(\tilde{S}) \times_I P_{fil}(S_{\mathbb{C}}^{an})$ is the category

- whose objects are $((M, F, W), (K, W), \alpha)$ where (M, F, W) is a filtered $D_{\tilde{S}}$ module supported on S , where $l : S \hookrightarrow \tilde{S}$ is a closed embedding and $\tilde{S} \in \text{SmVar}(k)$, (K, W) is a filtered Perverse sheaf on $S_{\mathbb{C}}^{an}$, and $\alpha : l_*(K, W) \otimes \mathbb{C} \rightarrow DR(\tilde{S})((M, W)^{an})$ is an isomorphism in $D_{fil}(\tilde{S}_{\mathbb{C}}^{an})$.
- whose morphism are,

using the fact that the V -filtration is defined over k . Then, by [3] theorem , we get for $S \in \text{Var}(k)$ quasi-projective and $l : S \hookrightarrow \tilde{S}$ is a closed embedding with $\tilde{S} \in \text{SmVar}(k)$ an embedding

$$\iota : D(MHM_{k,gm}(S)) \hookrightarrow D_{\mathcal{D}fil,S}(\tilde{S}) \times_I D(S_{\mathbb{C}}^{an})$$

consisting of mixed Hodge module whose De Rham part is defined over k , where $D_{\mathcal{D}(1,0)fil,S}(\tilde{S}) \times_I D_{fil,c}(S_{\mathbb{C}}^{an})$ is the category

- whose objects are $((M, F, W), (K, W), \alpha)$ where (M, F, W) is a complex of filtered $D_{\tilde{S}}$ module supported on S , (K, W) is a filtered complex of presheaves on $S_{\mathbb{C}}^{an}$ whose cohomology are constructible sheaves, and $\alpha : l_*(K, W) \otimes \mathbb{C} \rightarrow DR(\tilde{S})((M, W)^{an})$ is an isomorphism in $D_{fil}(\tilde{S}_{\mathbb{C}}^{an})$.
- whose morphism are,

and the six functor formalisms :

$$\begin{aligned} D(MHM_{k,gm}(-)) : \text{Var}(k) &\mapsto \text{TriCat}, \quad S \mapsto D(MHM_{k,gm}(S)), \quad (f : T \rightarrow S) \mapsto \\ (f^*, f_*) : D(MHM_{k,gm}(S)) &\rightarrow D(MHM_{k,gm}(T)), \quad (f!, f^!) : D(MHM_{k,gm}(T)) \rightarrow D(MHM_{k,gm}(S)) \end{aligned}$$

and for $D \subset S$ a (Cartier) divisor the nearby cycle functor

$$\begin{aligned} \Phi_D : D(MHM_{k,gm}(S)) &\rightarrow D(MHM_{k,gm}(D)), \\ \Phi_D((M, F, W), (K, W), \alpha) &:= (\Phi_D(M, F, W), \Phi_D(K, W), \Phi_D(\alpha)). \end{aligned}$$

For $f : X \rightarrow S$ a projective morphism with $X, S \in \text{SmVar}(k)$, we denote

$$E_{Hdg}(X/S) := f_* \mathbb{Z}_X^{Hdg} := \left(\int f(O_X, F), Rf_* \mathbb{Z}_{X_c^{an}}, f_* \alpha(X) \right) \in D(MHM_{k, gm}(S)).$$

We then have the specialization map :

Definition-Proposition 1. *Let k be field of characteristic zero. Let $f : X \rightarrow \mathbb{A}_k^1 \times S$ be a projective morphism with $X, S \in \text{SmVar}(k)$. Assume that*

- $f^\circ := f \otimes_{\mathbb{A}^1} (\mathbb{A}^1 \setminus 0) : X^\circ \rightarrow (\mathbb{A}^1 \setminus 0) \times S$ is smooth
- $E := f^{-1}(0 \times S) = \cup_{i=0}^r E_i \subset X$ is a normal crossing divisor and that for $I \subset [0, \dots, r]$, $f_{0, I} := f|_{E_I} : E_I := \cap_{i \in I} E_i \rightarrow S$ are smooth. We have by definition $X^\circ = X \setminus E$.

Denote for $I \subset J \subset [0, \dots, r]$, $i_{JI} : E_J \hookrightarrow E_I$ and $i_I : E_I \hookrightarrow E$ the closed embeddings. We have then the generalized distinguished triangle in $D(MHM_{k, gm}(E))$, where $R := [1, \dots, r]$,

$$\begin{aligned} i_{R*} i_R^! \Phi_E \mathbb{Z}_{X^\circ}^{Hdg} &= i_{R*} \mathbb{Z}_{E_R}^{Hdg}(-r)[-2r] \xrightarrow{(\text{ad}(i_{R,J}^*, i_{R,J})_{(i_{J,I} i_I^! \Phi_E \mathbb{Z}_{X^\circ}^{Hdg})})} \dots \\ \oplus_{(i,j) \subset R} i_{ij*} i_{ij}^! \Phi_E \mathbb{Z}_{X^\circ}^{Hdg} &= \oplus_{(i,j) \subset R} i_{ij*} \mathbb{Z}_{E_{ij}}^{Hdg}(-1)[-2] \xrightarrow{(\text{ad}(i_{J,i}, i_{J,i})_{(i_i^* i_i^! \Phi_E \mathbb{Z}_{X^\circ}^{Hdg})})} \\ \oplus_{i=0}^r i_{i*} i_i^! \Phi_E \mathbb{Z}_{X^\circ}^{Hdg} &= \oplus_{i=0}^r i_{i*} \Phi_{E_i \setminus (E_i \cap (\cup_{j \in R \setminus i} E_j))} \mathbb{Z}_{E_i}^{Hdg} = \oplus_{i=0}^r i_{i*} \mathbb{Z}_{E_i}^{Hdg} \xrightarrow{(\text{ad}(i_{i,1}, i_{i,1}^!)(\Phi_E \mathbb{Z}_{X^\circ}^{Hdg}))_{0 \leq 1 \leq r}} \Phi_E \mathbb{Z}_{X^\circ}^{Hdg} \\ &\xrightarrow{c(i_{R*} i_R^! \Phi_E \mathbb{Z}_{X^\circ}^{Hdg})} i_{R*} i_R^! \Phi_E \mathbb{Z}_{X^\circ}^{Hdg}[1] \end{aligned}$$

where the second equalities follows from the fact that E_I are smooth. Applying the functor $(f \circ i_E)_*$, we obtain the generalized distinguished triangle in $D(MHM_{k, gm}(S))$

$$\begin{aligned} E_{Hdg}(E_R/S)(-r)[-2r] &\rightarrow \dots \oplus_{(i,j) \subset R} E_{Hdg}(E_{ij}/S)(-1)[-2] \xrightarrow{((f \circ i_E)_* \text{ad}(i_{J,i}, i_{J,i})_{(i_i^* i_i^! \Phi_E \mathbb{Z}_{X^\circ}^{Hdg})})} \dots \\ \oplus_{i=0}^r E_{Hdg}(E_i/S) &\xrightarrow{(sp_{f, E_i} := (f \circ i_E)_* \text{ad}(i_{i,1}, i_{i,1}^!)(\Phi_E \mathbb{Z}_{X^\circ}^{Hdg}))_{0 \leq 1 \leq r}} (f \circ i_E)_*(\Phi_E \mathbb{Z}_{X^\circ}^{Hdg}) = \Phi_0 E_{Hdg}(X^\circ / ((\mathbb{A}^1 \setminus 0) \times S)) \\ &\xrightarrow{(f \circ i_E)_* c(i_{R*} i_R^! \Phi_E \mathbb{Z}_{X^\circ}^{Hdg})} E_{Hdg}(E_R/S)(-r)[1-2r] \end{aligned}$$

Proof. Follows from the six functors formalism, and the simplicial descent for the morphism $i_\bullet : E_\bullet \rightarrow E$ in $\text{Fun}(\Delta, \text{Var}(\mathbb{C}))$, with $E_\bullet \in \text{Fun}(\Delta, \text{SmVar}(\mathbb{C}))$ (see [8]), noting that

$$\Phi_0 E_{Hdg}(X^\circ / ((\mathbb{A}^1 \setminus 0) \times S)) = ((f \circ i_E)_* E_{zar}(i_E^{*mod} \Omega_{X/\mathbb{A}^1 \times S}^\bullet(\log E)), \Phi_0 Rf_* \mathbb{Z}_{X_c^{an}}, \Phi_0 f_* \alpha(X^\circ)).$$

□

3 Hodge conjecture for smooth projective varieties

Theorem 1. *Let $X \in \text{PSmVar}(\mathbb{C})$. Then Hodge conjecture hold for X . That is, if $p \in \mathbb{Z}$ and $\lambda \in F^p H^{2p}(X^{an}, \mathbb{Q})$, $\lambda = [Z]$ with $Z \in \mathcal{Z}^p(X)$.*

Proof. Up to split X into its connected components, we may assume that X is connected of dimension d . The Hodge conjecture is true for curves and surfaces. Assume that the Hodge conjecture is true for smooth projective varieties of dimension less or equal to $d - 1$. Consider by proposition 2 a family of smooth projective hypersurfaces $f_X : Y \rightarrow \mathbb{A}_\mathbb{C}^1$ with

- $Y \in \text{SmVar}(\mathbb{C})$, f_X flat projective,
- $Y_s := f_X^{-1}(s) \subset \mathbb{P}_\mathbb{C}^{d+1}$ for $s \in \mathbb{A}_\mathbb{C}^1 \setminus 0$ are smooth projective hypersurfaces

- $E := f_X^{-1}(0) = \cup_{i=0}^r E_i \subset Y$ is a normal crossing divisor and E_0 is birational to X ,

and its canonical rational deformation $\tilde{f}_X : \mathcal{Y} \rightarrow \mathbb{A}_{\mathbb{Q}}^1 \times S$ with

- $\mathcal{Y} \in \text{SmVar}(\mathbb{Q})$, $S \subset \mathbb{A}_{\mathbb{Q}}^s$ an open subset, \tilde{f}_X flat projective
- $Y = \tilde{f}_X^{-1}(\mathbb{A}_{\mathbb{Q}}^1 \times \{t\})$, $f_X = \tilde{f}_X \times_S \{t\}$, with $t \in S(\mathbb{C})$
- $\mathcal{E} = \cup_{i=0}^r \mathcal{E}_i := \tilde{f}_X^{-1}(\{0\} \times S) \subset \mathcal{Y}$ is a normal crossing divisor, $\tilde{f}_{X,0} : \mathcal{E}_I := \cap_{i \in I} \mathcal{E}_i \rightarrow S$ smooth projective for each $I \subset [0, \dots, r]$, $\mathcal{E}_{\mathbb{C}, t} = E$,
- $\tilde{f}_X \times_{\mathbb{A}^1} (\mathbb{A}^1 \setminus 0) \mathcal{Y}^o \rightarrow \mathbb{A}_{\mathbb{Q}}^1 \setminus 0$ has fibers which are smooth projective hypersurfaces.

We have by definition the commutative diagram in $\text{SmVar}(\mathbb{Q})$

$$\begin{array}{ccccc}
\mathcal{Y}^o & \xrightarrow{j} & \mathcal{Y} & \xleftarrow{i_{\mathcal{E}}} & \mathcal{E} \\
\tilde{f}_X^o \downarrow & & \downarrow \tilde{f}_X & & \downarrow \tilde{f}_{X,0} \\
\mathbb{A}_{\mathbb{Q}}^1 \setminus \{0\} \times S & \xrightarrow{j \times I_S} & \mathbb{A}_{\mathbb{Q}}^1 \times S & \xleftarrow{i_0 \times I_S} & \{0\} \times S
\end{array}$$

whose squares are cartesian, j the open embedding and $i_{\mathcal{E}}$ the closed embedding. Denote for $I \subset [0, \dots, r]$, $i_I : \mathcal{E}_I \hookrightarrow \mathcal{E}$ and for $I \subset J \subset [0, \dots, r]$, $i_{JI} : \mathcal{E}_J \rightarrow \mathcal{E}_I$ the closed embeddings. Then, definition-proposition 1 applied to $\tilde{f}_X : \mathcal{Y} \rightarrow \mathbb{A}_{\mathbb{Q}}^1 \times S$ gives the generalized distinguish in $D(MHM_{\mathbb{Q}, gm}(S))$, where $R := [1, \dots, r]$,

$$\begin{aligned}
E_{Hdg}(\mathcal{E}_R/S)(r)[-2r] &\rightarrow \cdots \oplus_{(i,j) \in R} E_{Hdg}(\mathcal{E}_{ij}/S)(-1)[-2] \xrightarrow{(i_{JI} := (\tilde{f}_X \circ i_{\mathcal{E}})_* (\text{ad}(i_{JI}, i_{Ji}^1)(\mathbb{Z}_{E_i})))_{J=(i,j) \subset R}} \\
&\oplus_{i=0}^r E_{Hdg}(\mathcal{E}_i/S) \xrightarrow{(Sp_{\mathcal{E}_i/\tilde{f}_X} := (\tilde{f}_X \circ i_{\mathcal{E}})_* \text{ad}(i_{i!}, i_i^1)(\Phi_{\mathcal{E}, \tilde{f}_X}))_{0 \leq i \leq r}} \Phi_0 E_{Hdg}(\mathcal{Y}^o / ((\mathbb{A}_{\mathbb{Q}}^1 \setminus 0) \times S)) \\
&\xrightarrow{(\tilde{f}_X \circ i_{\mathcal{E}})_* c(i_{R!} i_R^1 \Phi_{\mathcal{E}} \mathbb{Z}_{\mathcal{Y}^o})} E_{Hdg}(\mathcal{E}_R/S)(r)[-2r+1]
\end{aligned}$$

given in definition-proposition 1. Let $p \in \mathbb{Z}$. Taking the cohomology to this distinguish triangle, we get in particular the exact sequence in $MHM_{\mathbb{Q}, gm}(S)$

$$\oplus_{(i,j) \subset R} E_{Hdg}^{2p-2}(\mathcal{E}_{ij}/S) \xrightarrow{(i_{JI})} \oplus_{i=0}^r E_{Hdg}^{2p}(\mathcal{E}_i/S) \xrightarrow{(Sp_{\mathcal{E}_i/\tilde{f}_X})_{0 \leq i \leq r}} H^{2p} \Phi_0 E_{Hdg}(\mathcal{Y}^o / ((\mathbb{A}_{\mathbb{Q}}^1 \setminus 0) \times S)).$$

Since $H^{2p} E_{Hdg}(\mathcal{E}_0/S) \in MHM_{\mathbb{Q}, gm}(S)$ is a geometric variation of pure Hodge structure, and since the category of polarized variation of pure Hodge structure is semi-simple ([7]), this exact sequence split in $MHM(S_{\mathbb{C}})$, in particular we get for $i = 0$, the splitting in $MHM(S_{\mathbb{C}})$

$$H^{2p} E_{Hdg}(\mathcal{E}_{0,\mathbb{C}}/S_{\mathbb{C}}) = i_{J0!} (\oplus_{j \in R} H^{2p-2} E_{Hdg}(\mathcal{E}_{0j,\mathbb{C}}/S_{\mathbb{C}})) \oplus Sp_{\mathcal{E}_0/\tilde{f}_X} (H^{2p} E_{Hdg}(\mathcal{E}_{0,\mathbb{C}}/S_{\mathbb{C}})).$$

For $\lambda \in H^{2p}(\mathcal{E}_{0,\mathbb{C},t'}^{an}, \mathbb{Q})$, $t' \in S(\mathbb{C})$, we consider $\pi_S : \tilde{S}_{\mathbb{C}}^{an} \rightarrow S_{\mathbb{C}}^{an}$ the universal covering in $\text{AnSm}(\mathbb{C})$ and

$$\mathbb{V}_S(\lambda) := \pi_S(\lambda) \subset E_{DR}^{2p, an}(\mathcal{E}_{0,\mathbb{C}}/S_{\mathbb{C}})$$

with $\pi_S : \tilde{S}_{\mathbb{C}}^{an} \times H^{2p}(\mathcal{E}_{0,t'}^{an}, \mathbb{C}) \rightarrow E_{DR}^{2p, an}(\mathcal{E}_{0,\mathbb{C}}/S_{\mathbb{C}})$ the universal covering and

$$\mathbb{V}_S^p(\lambda) := \mathbb{V}_S(\lambda) \cap F^p E_{DR}^{2p}(\mathcal{E}_{0,\mathbb{C}}/S_{\mathbb{C}}) \subset E_{DR}^{2p}(\mathcal{E}_{0,\mathbb{C}}/S_{\mathbb{C}}),$$

which is an algebraic subvariety by [6]. We have

$$H^{2p} E_{Hdg}(\mathcal{Y}^o / ((\mathbb{A}_{\mathbb{Q}}^1 \setminus 0) \times S)) = ((E_{DR}^{2p}(\mathcal{Y}^o / ((\mathbb{A}_{\mathbb{Q}}^1 \setminus 0) \times S)), F), R^{2p} \tilde{f}_X^* \mathbb{Z}_{\mathcal{Y}^o}, \tilde{f}_X^* \alpha(\mathcal{Y}^o)) \in MHM_{\mathbb{Q}, gm}(S)$$

By [8], the filtered vector bundle

$$(E_{DR}^{2p}(\mathcal{Y}^\circ/((\mathbb{A}_\mathbb{Q}^1 \setminus 0) \times S)), F) = H^{2p} \tilde{f}_{X*} E_{zar}(\Omega_{\mathcal{Y}^\circ/(\mathbb{A}_\mathbb{Q}^1 \setminus 0) \times S}^\bullet) \in \text{Vect}_{\mathcal{D}fil}(S)$$

extend to a filtered vector bundle

$$(E_{DR, V_0}^{2p}(\mathcal{Y}/(\mathbb{A}_\mathbb{Q}^1 \times S)), F) = H^{2p} \tilde{f}_{X*} E_{zar}(\Omega_{\mathcal{Y}/\mathbb{A}_\mathbb{Q}^1 \times S}^\bullet(\log \mathcal{E})) \in \text{Vect}_{\mathcal{D}fil}(S)$$

such that

$$H^{2p} \Phi_0 E_{Hdg}(\mathcal{Y}^\circ/((\mathbb{A}_\mathbb{Q}^1 \setminus 0) \times S)) = (i_0^{*mod}(E_{DR, V_0}^{2p}(\mathcal{Y}/(\mathbb{A}_\mathbb{Q}^1 \times S)), F), (\Phi_0 R^{2p} \tilde{f}_{X*} \mathbb{Z}_{\mathcal{Y}^\circ}, W), \Phi_0 \tilde{f}_{X*} \alpha(\mathcal{Y}^\circ)).$$

For

$$\alpha \in H^{2p}(\mathcal{Y}_{(s, t')}^{an}, \mathbb{Q}) = \mathbb{H}^{2p}(\mathcal{E}_{t'}^{an}, \Phi_0 R \tilde{f}_{X*} \mathbb{Q}_{\mathcal{Y}^\circ}), (s, t') \in (\mathbb{A}_\mathbb{C}^1 \setminus 0) \times S_{\mathbb{C}}$$

we consider

- $\pi_{S, G_m} = \pi_S \times \pi_{G_m} : \tilde{G}_m \times \tilde{S}_{\mathbb{C}}^{an} \rightarrow (\mathbb{A}_\mathbb{C}^1 \setminus 0) \times S_{\mathbb{C}}^{an}$ the universal covering in $\text{AnSm}(\mathbb{C})$ and

$$\mathbb{V}_{(\mathbb{A}^1 \setminus 0) \times S}(\alpha) := \pi_{S, G_m}(\alpha) \subset E_{DR}^{2p, an}(\mathcal{Y}_{\mathbb{C}}^\circ/(\mathbb{A}_\mathbb{C}^1 \setminus 0) \times S_{\mathbb{C}})$$

with $\pi_{S, G_m} : \tilde{S}_{\mathbb{C}}^{an} \times \tilde{G}_m \times H^{2p}(\mathcal{Y}_{(s, t')}^{an}, \mathbb{C}) \rightarrow E_{DR}^{2p, an}(\mathcal{Y}_{\mathbb{C}}^\circ/S_{\mathbb{C}})$ and

$$\mathbb{V}_{(\mathbb{A}^1 \setminus 0) \times S}^p(\alpha) := \mathbb{V}_{(\mathbb{A}^1 \setminus 0) \times S}(\alpha) \cap F^p E_{DR}^{2p}(\mathcal{Y}_{\mathbb{C}}^\circ/(\mathbb{A}_\mathbb{C}^1 \setminus 0) \times S_{\mathbb{C}}) \subset E_{DR}^{2p}(\mathcal{Y}_{\mathbb{C}}^\circ/(\mathbb{A}_\mathbb{C}^1 \setminus 0) \times S_{\mathbb{C}})$$

which is a algebraic subvariety by [6],

- $\pi_S : \tilde{S}_{\mathbb{C}}^{an} \rightarrow S_{\mathbb{C}}^{an}$ the universal covering in $\text{AnSm}(\mathbb{C})$ and

$$\mathbb{V}_S(\alpha) := \pi_S(\alpha) \subset H^{2p} \Phi_0 E_{DR}(\mathcal{Y}_{\mathbb{C}}^\circ/((\mathbb{A}_\mathbb{C}^1 \setminus 0) \times S_{\mathbb{C}}))^{an}$$

with $\pi_S : \tilde{S}_{\mathbb{C}}^{an} \times H^{2p}(\mathcal{Y}_{(s, t')}^{an}, \mathbb{C}) \rightarrow H^{2p} \Phi_0 E_{DR}(\mathcal{Y}_{\mathbb{C}}^\circ/((\mathbb{A}_\mathbb{C}^1 \setminus 0) \times S_{\mathbb{C}}))^{an}$ and

$$\mathbb{V}_S^p(\alpha) := \mathbb{V}_S(\alpha) \cap F^p H^{2p} \Phi_0 E_{DR}(\mathcal{Y}_{\mathbb{C}}^\circ/((\mathbb{A}_\mathbb{C}^1 \setminus 0) \times S_{\mathbb{C}}))^{an} \subset H^{2p} \Phi_0 E_{DR}(\mathcal{Y}_{\mathbb{C}}^\circ/((\mathbb{A}_\mathbb{C}^1 \setminus 0) \times S_{\mathbb{C}}))^{an}.$$

Let $\lambda \in F^p H^{2p}(E_0^{an}, \mathbb{Q})$, where we recall $E_0 = \mathcal{E}_{0, \mathbb{C}, t}$. Then,

$$\lambda = \lambda^K + \lambda^G \in F^p H^{2p}(E_0^{an}, \mathbb{Q}), \lambda^K \in \bigoplus_{j \in \text{Ri}0_j} (F^{p-1} H^{2p-2}(E_{0_j}^{an}, \mathbb{Q})), \lambda^G \in Sp_{E_0/f_X}(F^p H^{2p}(E_0^{an}, \mathbb{Q}))$$

and

$$\mathbb{V}_S^p(\lambda) = \mathbb{V}_S^p(\lambda^K) \cap \mathbb{V}_S^p(\lambda^G) \subset E_{DR}^{2p}(\mathcal{E}_{0\mathbb{C}}/S_{\mathbb{C}}).$$

Now,

- we have, by [6],

$$\begin{aligned} Sp_{\mathcal{E}_0/\tilde{f}_X}(\mathbb{V}_S^p(\lambda^G)) &= \mathbb{V}_S^p(Sp_{E_0/f_X}(\lambda^G)) = \overline{\mathbb{V}_{(\mathbb{A}^1 \setminus 0) \times S}^p(Sp_{E_0/f_X}(\lambda^G))} \cap p^{-1}(0 \times S_{\mathbb{C}}) \\ &\subset E_{DR, V_0}^{2p}(\mathcal{Y}_{\mathbb{C}}/(\mathbb{A}_\mathbb{C}^1 \times S_{\mathbb{C}})) \end{aligned}$$

where $p : E_{DR, V_0}^{2p}(\mathcal{Y}_{\mathbb{C}}/(\mathbb{A}_\mathbb{C}^1 \times S_{\mathbb{C}})) \rightarrow \mathbb{A}_\mathbb{C}^1 \times S_{\mathbb{C}}$ is the projection (the first equality is obvious since $sp_{\mathcal{E}_0/\tilde{f}_X}$ is a morphism of variation of mixed Hodge structures over S , for the second equality, the inclusion \subset is obvious, whereas the inclusion \supset follows from [6] lemma 2.11 and the invariant cycle theorem of a geometric variation of Hodge structures by strictness of the Hodge filtration), but by [5] theorem 1, the Hodge conjecture holds for projective hypersurfaces hence

$$\mathbb{V}_{(\mathbb{A}^1 \setminus 0) \times S}^p(Sp(\lambda^G)) \subset E_{DR}^{2p}(\mathcal{Y}_{\mathbb{C}}^\circ/((\mathbb{A}_\mathbb{C}^1 \setminus 0) \times S_{\mathbb{C}}))$$

is defined over \mathbb{Q} , hence $\mathbb{V}_S^p(\lambda^G) \subset E_{DR}^{2p}(\mathcal{E}_{0, \mathbb{C}}/S_{\mathbb{C}})$ is defined over \mathbb{Q} ,

- since the Hodge conjecture is true for smooth projective varieties of dimension less or equal to $d - 1$ by induction hypothesis

$$\mathbb{V}_S^p(\lambda^K) = i_{0,J!}(\mathbb{V}_S^p(\lambda^K)) \subset E_{DR}^{2p}(\mathcal{E}_{0\mathbb{C}}/S_{\mathbb{C}})$$

is defined over \mathbb{Q} .

Hence $\mathbb{V}_S^p(\lambda) \subset E_{DR}^{2p}(\mathcal{E}_{0\mathbb{C}}/S_{\mathbb{C}})$ is defined over \mathbb{Q} . Thus, by [4] theorem 4,

$$\lambda = [Z] \in H^{2p}(E_0^{an}, \mathbb{Q}), \text{ with } Z \in \mathcal{Z}^p(E_0).$$

Since $p \in \mathbb{Z}$ and $\lambda \in F^p H^{2p}(E_0^{an}, \mathbb{Q})$ are arbitrary, the Hodge conjecture holds for E_0 . Proposition 1(ii) and the fact that the Hodge conjecture is true for smooth projective varieties of dimension less or equal to $d - 1$ by induction hypothesis then implies that Hodge conjecture holds for X , since X is birational to E_0 and $X, E_0 \in \text{PSmVar}(\mathbb{C})$ are connected of dimension d . \square

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