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NULL CONTROLLABILITY OF A VOLUME-SURFACE REACTION-DIFFUSION EQUATION WITH DYNAMIC BOUNDARY CONDITIONS

FOUAD ET-TAHRI, SALAH-EDDINE CHORFI, LAHCEN MANIAR, AND IDRISS BOUTAAYAMOU

Abstract. This article focuses on investigating the null controllability of a volume-surface reaction-diffusion equation with dynamic boundary conditions. Notably, the reaction and diffusion coefficients depend on the state variable in the bulk and on its surface. Our main result establishes the local null controllability of the quasi-linear equation under certain conditions on the regularity of diffusion and reaction coefficients as well as initial data. To this end, we approach the problem by first addressing the question of null controllability in the framework of a inhomogeneous linearized equation. Next, we derive new estimates of both control and state, allowing us to apply a local inversion theorem to obtain the local null controllability of the quasi-linear equation.

1. Introduction and main results

The prototype model of reaction-diffusion equations with distributed control can be written as follows:

$$\begin{cases} \psi_t - \nabla \cdot (D\nabla\psi) + a(\psi) = \mathbb{1}_\omega v & \text{in } \Omega_T, \\ \psi(\cdot,0) = \psi_0 & \text{in } \Omega, \end{cases} \tag{1.1}$$

with boundary conditions. Here $\Omega \subset \mathbb{R}^d$ ($d \in \mathbb{N}$) is a bounded domain with smooth boundary $\Gamma := \partial\Omega$, $\Omega_T := \Omega \times (0,T)$, $\omega \Subset \Omega$ (the control region) a nonempty open set strictly contained in Ω, $T > 0$ the control time, v the control function, ψ_0 is the initial datum, the function a represents the reaction term, and the diffusion coefficient D can be expressed in a general form as $D := D(x,t,\psi,\nabla\psi)$. In the linear case, D varies as a function of x and t and a is linear; for semilinear equations, D depends on x and t; otherwise, the equation is quasi-linear. Reaction-diffusion equations suppose that the behavior of different populations is mainly influenced by two processes: local reactions, where populations interact with each other, and diffusion, where populations spread through a physical space. The term "population" is widely used within this framework and can incorporate examples from various fields such as biology, ecology, geology, and combustion theory. Examples of such populations may include particles, free waves on the surface of water, flames, cells, and bacteria as seen in references [27,33,36] and the cited bibliography.

Our research in this paper focuses on the null controllability of reaction-diffusion equations with dynamic boundary conditions of the surface diffusion type (generalized Wentzell type). The equation we are dealing with reads as follows:

$$\begin{cases} \psi_t - \nabla \cdot (\sigma(\psi)\nabla\psi) + a(\psi) = f + \mathbb{1}_\omega v & \text{in } \Omega_T, \\ \psi_{\Gamma,t} - \nabla_\Gamma \cdot (\delta(\psi_\Gamma)\nabla_\Gamma\psi_\Gamma) + \sigma(\psi_\Gamma)\partial_\nu\psi + b(\psi_\Gamma) = f_\Gamma & \text{on } \Gamma_T, \\ \psi_\Gamma = \psi_{|\Gamma} & \text{on } \Gamma_T, \\ (\psi(\cdot,0), \psi_\Gamma(\cdot,0)) = (\psi_0, \psi_{0,\Gamma}) & \text{in } \Omega \times \Gamma, \end{cases} \tag{1.2}$$

where $\Gamma_T := \Gamma \times (0,T)$, $\omega_T := \omega \times (0,T)$, $v \in L^2(\omega_T)$, $(\psi_0, \psi_{0,\Gamma}) \in L^2(\Omega) \times L^2(\Gamma)$ and the source terms (f, f_Γ) belong to an appropriate weight space; ∇, ∇_Γ and ∂_ν denote the gradient, tangential gradient and normal derivative associated to the outward normal ν of Ω, respectively; $\nabla\cdot$ and $\nabla_\Gamma\cdot$ denote divergence and tangential divergence respectively, $\psi_{\Gamma,t}$ is the derivative of ψ_Γ with respect to t, $\psi_{|\Gamma}$ is the trace of ψ on Γ and $\sigma, \delta, a, b : \mathbb{R} \longrightarrow \mathbb{R}$ are given such that

- The diffusion coefficients satisfy

$$\sigma, \delta \in C^3(\mathbb{R},\mathbb{R}), \quad \sigma(r) \geq \sigma_0 > 0 \quad \text{and} \quad \delta(r) \geq \delta_0 > 0 \quad \forall r \in \mathbb{R}. \tag{1.3}$$

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- The reaction terms verify

$$a, b \in C^2(\mathbb{R}, \mathbb{R}) \quad \text{and} \quad a(0) = b(0) = 0. \tag{1.4}$$

We emphasize that we do not require growth conditions on the reaction terms nor upper bounds on the diffusion coefficients. In general, the term $\psi_{\Gamma,t} - \nabla_\Gamma \cdot (\delta(\psi_\Gamma)\nabla_\Gamma \psi_\Gamma)$ models the tangential diffusive flux on the boundary which is coupled to the diffusion equation in the bulk by the co-normal derivative $\sigma(\psi_\Gamma)\partial_\nu \psi$. We treat this problem as a coupled system of dynamic equations for ψ and ψ_Γ, with side condition $\psi_{|\Gamma} = \psi_\Gamma$. For the mathematical theory of surface diffusion boundary conditions, we refer to [11, 23, 25, 38, 44, 46] and [28, 43] for the physical interpretation and derivations of the dynamic boundary condition.

Volume-surface reaction-diffusion equations such as (1.2) can be found in the biology of cells and asymmetric cell division [13]; in the chemistry of cells to model signaling molecules [42]. The well-posedness and stability along with numerical results on the discretization are presented in the aforementioned papers. As for applications in thermomechanics, we refer to [39] and the referenced literature.

Local null controllability means that there exists a data space such that the null controllability is fulfilled for any initial data small enough in this space. We refer to [12] and the references therein for some results on the semilinear case. Recently, there has been notable progress in investigating the local null controllability of (1.1) under static boundary conditions, specifically Dirichlet conditions. In [16, 35], the authors showed the local null controllability of (1.1) when $(D, a) = (\sigma(\psi), 0)$. For the case of $(D, a) = (\sigma(\nabla\psi), 0)$, the local controllability is discussed in [18]. When considering $(D, a) = (\sigma(x, t), 0)$ or $(\sigma(|\nabla\psi|), 0)$, relevant insights are provided in [9]. Additional intriguing models, such as the thermistor problem and a Ladyzhenskaya-Smagorinsky model of turbulence, are addressed in [17, 29]. In the one-dimensional case, [6, 19, 21] analyzed an equation of similar form. For the controllability of parabolic and parabolic-elliptic equations with local and non-local nonlinearities, comprehensive discussions are available in [10, 41]. Further insights into numerical aspects can be found in [19, 20], along with the cited references.

Therefore, it is natural to extend the known results to equations with dynamic boundary conditions where the boundary diffusion also depends on the state. Concerning the null controllability and inverse problems of linear and semilinear equations with such boundary conditions, we refer to [2,3,7,31,32]. In particular, [37] dealt with the null controllability of (1.2) in the case of constant diffusion coefficients. Note that incorporating nonlinearities in both the bulk equation and boundary equation makes the null controllability of the equation rather challenging compared to the Dirichlet boundary condition. Therefore, we need to develop suitable estimates that are well-adapted to our case, since several new boundary terms need to be estimated.

Before we present the main result of this paper, we recall the definition of local null controllability associated with (1.2). We start by introducing Sobolev-type spaces needed in the sequel:

$$\mathbb{H}^k := \{(y, y_\Gamma) \in H^k(\Omega) \times H^k(\Gamma) \ : \ y_{|\Gamma} = y_\Gamma\}, \quad k = 1, 2, 3,$$

where $H^k(\Omega)$ and $H^k(\Gamma)$ are respectively the usual L^2-based Sobolev spaces over Ω and Γ.

Definition 1.1. We say that (1.2) is locally null controllable at time T if there exists $\varepsilon > 0$ such that for any $\Psi_0 := (\psi_0, \psi_{0,\Gamma}) \in \mathbb{H}^3$ and $F := (f, f_\Gamma) \in \mathbb{G}$ ($\mathbb{G}$ is introduced in Section 3) with $\|\Psi_0\|_{\mathbb{H}^3} + \|F\|_{\mathbb{G}} < \varepsilon$, there exists a control $v \in L^2(\omega_T)$ such that the solution of (1.2) satisfies $(\psi(\cdot, T), \psi_\Gamma(\cdot, T)) = (0, 0)$ in $\Omega \times \Gamma$.

The main result of this paper reads as follows:

Theorem 1.2. *Assume that $d \leq 3$ and assumptions (1.3) and (1.4) hold true, then the quasi-linear equation (1.2) is locally null controllable at any time T.*

The local null controllability of (1.2) is reformulated as a surjectivity of the mapping $\Lambda : \mathbb{X} \longrightarrow \mathbb{Y}$:

$$\Lambda(\Psi, v) := (\Lambda_1(\Psi) - \mathbb{1}_\omega v, \Lambda_2(\Psi), \Psi(\cdot, 0)),$$

where

$$\begin{aligned}
&\Psi := (\psi, \psi_\Gamma),\\
&\Lambda_1(\Psi) := \psi_t - \nabla \cdot (\sigma(\psi)\nabla\psi) + a(\psi),\\
&\Lambda_2(\Psi) := \psi_{\Gamma,t} - \nabla_\Gamma \cdot (\delta(\psi_\Gamma)\nabla_\Gamma\psi_\Gamma) + \sigma(\psi_\Gamma)\partial_\nu\psi + b(\psi_\Gamma)
\end{aligned}$$

and $\mathbb{X}, \mathbb{Y}$ are appropriate spaces of the state-control (see Section 3). More precisely, Theorem 1.2 is equivalent to

$$\exists \varepsilon > 0,\ \forall (F, \Psi_0) \in B_{\mathbb{Y}}((0,0), \varepsilon),\ \exists (\Psi, v) \in \mathbb{X},\ \text{such that } \Lambda(\Psi, v) = (F, \Psi_0).$$

To achieve this, we apply the Lyusternik-Graves Inverse Mapping Theorem in infinite dimensional spaces, whose proof can be referenced in [5].

Theorem 1.3 (Lyusternik-Graves' Theorem)**.** *Let X and Y be Banach spaces and let $\Lambda : B(0,r) \subset X \to Y$ be a C^1 mapping. Let us assume that the derivative $\Lambda'(0) : X \to Y$ is onto and let us set $\xi_0 = \Lambda(0)$. Then there exist $\varepsilon > 0$, a mapping $W : B(\xi_0, \varepsilon) \subset Y \to X$ and a constant $C > 0$ satisfying:*

- *$W(z) \in B(0,r)$ and $\Lambda \circ W(z) = z\ \forall z \in B(\xi_0, \varepsilon)$,*
- *$\|W(z)\|_X \leq C\|z - \xi_0\|_Y\ \forall z \in B(\xi_0, \varepsilon)$.*

The most difficult task is therefore the choice of spaces $\mathbb{X}$ and $\mathbb{Y}$ for which mapping Λ is well defined. The surjectivity of $\Lambda'(0,0)$ is linked to the null controllability of the inhomogeneous linearized equation (around zero) of (1.2) given by

$$\begin{cases} \psi_t - \sigma(0)\Delta\psi + a'(0)\psi = \mathbb{1}_\omega v + f & \text{in } \Omega_T, \\ \psi_{\Gamma,t} - \delta(0)\Delta_\Gamma\psi_\Gamma + \sigma(0)\partial_\nu\psi + b'(0)\psi_\Gamma = f_\Gamma & \text{on } \Gamma_T, \\ \psi_\Gamma = \psi_{|\Gamma} & \text{on } \Gamma_T, \\ (\psi(\cdot,0), \psi_\Gamma(\cdot,0)) = (\psi_0, \psi_{0,\Gamma}) & \text{in } \Omega \times \Gamma. \end{cases} \tag{1.5}$$

Given suitable assumptions on f and f_Γ, equation (1.5) is null controllable. The adjoint equation of (1.5) with source terms can be expressed as follows:

$$\begin{cases} -\phi_t - \sigma(0)\Delta\phi + a'(0)\phi = f & \text{in } \Omega_T, \\ -\phi_{\Gamma,t} - \delta(0)\Delta_\Gamma\phi_\Gamma + \sigma(0)\partial_\nu\phi + b'(0)\phi_\Gamma = f_\Gamma & \text{on } \Gamma_T, \\ \phi_\Gamma = \phi_{|\Gamma} & \text{on } \Gamma_T, \\ (\phi(\cdot,T), \phi_\Gamma(\cdot,T)) = (\phi_T, \phi_{T,\Gamma}) & \text{in } \Omega \times \Gamma. \end{cases} \tag{1.6}$$

We restrict ourselves to dimensions $d \leq 3$ so that the following Sobolev embeddings are valid:

- $W^{1,4}(M) \hookrightarrow L^\infty(M)$, with continuous embedding,
- $H^2(M) \hookrightarrow W^{1,4}(M)$, with continuous embedding,
- $H^2(M) \hookrightarrow L^\infty(M)$, with continuous embedding,

where $M = \Omega$ or Γ. These are well-known results that are deduced from the general results shown in [1, Theorem 4.12] in case $M = \Omega$ and [34, Theorem 2.2] in case $M = \Gamma$.

This paper is structured as follows: Section 2 is devoted to the study of the well-posedness and null controllability of the inhomogenuous linearized equation (1.5). In Section 3, we prove the main result on local null controllability of (1.2) (Theorem 1.2). Finally, we give some conclusions and final comments in Section 4.

Notation. Throughout this paper, we adopt the following notations:

- The symbol C will stand for a generic positive constant depending on Ω, ω, s, λ, T, σ, δ, a and b (the parameters s and λ are introduced in Subsection 2.3).
- To write estimates in abbreviated form, we denote:

  $$\nabla Y := (\nabla y, \nabla_\Gamma y_\Gamma), \quad \Delta Y := (\Delta y, \Delta_\Gamma y_\Gamma), \quad Y_t := (y_t, y_{\Gamma,t}), \quad Y_{tt} := (y_{tt}, y_{\Gamma,tt}),$$

  where $Y := (y, y_\Gamma)$, ∇_Γ is the tangential gradient, Δ_Γ is the Laplace-Beltrami, Y_t and Y_{tt} respectively denote the first and second-order partial derivative of Y with respect to t.

2. Well-posedness and null controllability of (1.5)

2.1. General setting.

The natural state space for our problems is

$$\mathbb{L}^2 := L^2\left(\overline{\Omega};\ \mathrm{d}x \otimes\ \mathrm{d}S\right),$$

where $\mathrm{d}x$ denotes the Lebesgue measure on Ω and $\mathrm{d}S$ denotes the natural surface measure on Γ. This space can be identified with

$$L^2(\Omega,\ \mathrm{d}x) \otimes L^2(\Gamma,\ \mathrm{d}S).$$

It is a Hilbert space with the inner product

$$\langle Y, Z\rangle_{\mathbb{L}^2} := \langle y, z\rangle_{L^2(\Omega)} + \langle y_\Gamma, z_\Gamma\rangle_{L^2(\Gamma)}, \quad Y = (y, y_\Gamma),\ Z = (z, z_\Gamma) \in \mathbb{L}^2.$$

We will also denote

$$\mathbb{L}^\infty := L^\infty(\Omega) \otimes L^\infty(\Gamma).$$

It is a Banach space with the norm

$$\|Y\|_{\mathbb{L}^\infty} := \max\{\|y\|_{L^\infty(\Omega)}, \|y_\Gamma\|_{L^\infty(\Gamma)}\}, \quad Y = (y, y_\Gamma) \in \mathbb{L}^\infty.$$

To describe the surface heat diffusion, we need to define certain differential operators on Γ, defined locally in terms of the standard Riemannian metric on Γ, see [30]. In this paper, we will not use the local formulas that define these operators, but rather the relevant properties such as the surface divergence theorem. These operators can be defined by extensions as follows, we refer to [8]:

- We define the tangential gradient $\nabla_\Gamma y_\Gamma$ for any smooth function y_Γ on Γ by

$$\nabla_\Gamma y_\Gamma := \nabla y - (\partial_\nu y)\nu, \tag{2.1}$$

  where y is an extension of y_Γ in a neighborhood of Γ. It can be seen as the projection of the standard Euclidean gradient ∇y onto the tangent space on Γ.
- We define the tangential divergence $\mathrm{div}_\Gamma Y_\Gamma$ for any smooth vector field Y_Γ on Γ by

$$\mathrm{div}_\Gamma Y_\Gamma := \mathrm{div} Y - Y'\nu \cdot \nu, \tag{2.2}$$

  where $Y' = (\partial_i Y_j)$ and Y is an extension of Y_Γ in a neighborhood of Γ. Note that formulas (2.1) and (2.2) do not depend on the chosen extension, and $\mathrm{div}_\Gamma(Y_\Gamma)$ can be considered as a continuous linear form on $H^1(\Gamma)$

$$\mathrm{div}_\Gamma(Y_\Gamma)\ :\ H^1(\Gamma) \longrightarrow \mathbb{R}, \quad z_\Gamma \mapsto -\int_\Gamma \langle Y_\Gamma, \nabla_\Gamma z_\Gamma\rangle_\Gamma \, \mathrm{d}S,$$

  where $\langle\ ,\ \rangle_\Gamma$ is the Riemannian inner product of tangential vectors on Γ. In the following, we will denote $\nabla_\Gamma \cdot Y_\Gamma$ instead of $\mathrm{div}_\Gamma(Y_\Gamma)$ and $\cdot$ instead of $\langle\ ,\ \rangle_\Gamma$.
- The Laplace-Beltrami $\Delta_\Gamma y_\Gamma$ is defined by

$$\Delta_\Gamma y_\Gamma = \mathrm{div}_\Gamma(\nabla_\Gamma y_\Gamma) \quad \forall y_\Gamma \in H^2(\Gamma).$$

  In particular, the Stokes divergence theorem on Γ holds, see [45],

$$\int_\Gamma \Delta_\Gamma y_\Gamma z_\Gamma \, \mathrm{d}S = -\int_\Gamma \nabla_\Gamma y_\Gamma \cdot \nabla_\Gamma z_\Gamma \, \mathrm{d}S \quad \forall y_\Gamma \in H^2(\Gamma) \quad \forall z_\Gamma \in H^1(\Gamma). \tag{2.3}$$

In the one-dimensional case, we can take a look:

Remark 1. In the one-dimensional case, for example $\Omega = (0, 1)$, then $\Gamma = \{0, 1\}$. We have

- The measure $\mathrm{d}S$ on Γ can be identified with the counting measure

$$\int_\Gamma y_\Gamma \, \mathrm{d}S = y_\Gamma(0) + y_\Gamma(1).$$

- The spaces $L^2(\Gamma)$ and $L^\infty(\Gamma)$ are identified with $\mathbb{R}^2$, so $L^2(\Gamma)$ can be equipped with the inner product

$$\langle y_\Gamma, z_\Gamma\rangle_{L^2(\Gamma)} = y_\Gamma(0)z_\Gamma(0) + y_\Gamma(1)z_\Gamma(1)$$

  and $L^\infty(\Gamma)$ equipped with the following norm

$$\|y_\Gamma\|_{L^\infty(\Gamma)} = \max(|y_\Gamma(0)|, |y_\Gamma(1)|).$$

- Since Γ is a manifold of dimension 0, therefore the two tangeant spaces at 0 and 1 are trivial. Consequently, the tangential operators are trivial: $\nabla_\Gamma = 0$ and $\Delta_\Gamma = 0$.

2.2. **Well-posedness and regularity of solutions to (1.5).** In this section, we will present the well-posedness and regularity properties of solutions to (1.5) as well as to (1.6). In the sequel, we use the following notations.

Notation. For any $Y = (y, y_\Gamma)$, we will denote

$$\begin{cases} \mathbf{L}_1 Y := y_t - \sigma(0)\Delta y + a'(0)y, \\ \mathbf{L}_2 Y := y_{\Gamma,t} - \delta(0)\Delta_\Gamma y_\Gamma + \sigma(0)\partial_\nu y + b'(0)y_\Gamma, \\ \mathbf{L}_1^\star Y := -y_t - \sigma(0)\Delta y + a'(0)y, \\ \mathbf{L}_2^\star Y := -y_{\Gamma,t} - \delta(0)\Delta_\Gamma y_\Gamma + \sigma(0)\partial_\nu y + b'(0)y_\Gamma. \end{cases} \tag{2.4}$$

We also introduce the following energy space

$$\mathbb{E} := H^1(0,T;\mathbb{L}^2) \cap L^2(0,T;\mathbb{H}^2).$$

We are focused on addressing the following categories of solutions for (1.5), see [37].

Definition 2.1. Let $F = (f, f_\Gamma) \in L^2(0,T;\mathbb{L}^2)$, $\Psi_0 = (\psi_0, \psi_{0,\Gamma}) \in \mathbb{L}^2$ and $v \in L^2(\omega_T)$.

(1) A distributional solution (solution by transposition) of (1.5) is a function $\Psi = (\psi, \psi_\Gamma) \in L^2(0,T;\mathbb{L}^2)$ such that for any $Z = (z, z_\Gamma) \in \mathbb{E}$ with $Z(\cdot, T) = 0$, we have

$$\begin{aligned} &\int_{\Omega_T} \psi \mathbf{L}_1^\star Z \, \mathrm{d}x \, \mathrm{d}t + \int_{\Gamma_T} \psi_\Gamma \mathbf{L}_2^\star Z \, \mathrm{d}S \, \mathrm{d}t - \int_{\omega_T} vz \, \mathrm{d}x \, \mathrm{d}t \\ &= \int_{\Omega_T} fz \, \mathrm{d}x \, \mathrm{d}t + \int_{\Gamma_T} f_\Gamma z_\Gamma \, \mathrm{d}S \, \mathrm{d}t + \int_\Omega \psi_0 z(\cdot,0) \, \mathrm{d}x + \int_\Gamma \psi_{0,\Gamma} z_\Gamma(\cdot,0) \, \mathrm{d}S. \end{aligned} \tag{2.5}$$

(2) A strong solution of (1.5) is a function $(\psi, \psi_\Gamma) \in \mathbb{E}$ fulfilling (1.5) in $L^2(0,T;\mathbb{L}^2)$.

The well-posedness and regularity properties of the solutions to (1.5) are based on semigroup theory, as studied in detail in [37].

Proposition 1. *Let $F = (f, f_\Gamma) \in L^2(0,T;\mathbb{L}^2)$ and $v \in L^2(\omega_T)$.*

(1) If $\Psi_0 = (\psi_0, \psi_{0,\Gamma}) \in \mathbb{L}^2$. Then there exists a unique distributional solution $\Psi = (\psi, \psi_\Gamma) \in C([0,T];\mathbb{L}^2)$ of (1.5). Moreover, there is a constant $C > 0$ such that

$$\|\Psi\|_{C([0,T];\mathbb{L}^2)} \le C\left(\|\Psi_0\|_{\mathbb{L}^2} + \|F\|_{L^2(0,T;\mathbb{L}^2)} + \|v\|_{L^2(\omega_T)}\right). \tag{2.6}$$

(2) If $\Psi_0 = (\psi_0, \psi_{0,\Gamma}) \in \mathbb{H}^1$. Then there exists a unique strong solution $\Psi = (\psi, \psi_\Gamma) \in \mathbb{E}$ of (1.5). Moreover, there is a constant $C > 0$ such that

$$\|\Psi\|_{\mathbb{E}} \le C\left(\|\Psi_0\|_{\mathbb{H}^1} + \|F\|_{L^2(0,T;\mathbb{L}^2)} + \|v\|_{L^2(\omega_T)}\right). \tag{2.7}$$

Proof. The proof of existence and uniqueness as well as estimates (2.6) and (2.7) can be found in [37, Propositions 2.4 and 2.5]. □

Considering the transformation $t \mapsto T - t$, one can pass from equation (1.6) to an equation similar to (1.5) without control and vice versa. We thus obtain

Corollary 1. *Let $F = (f, f_\Gamma) \in L^2(0,T;\mathbb{L}^2)$.*

(1) If $\Phi_T = (\phi_T, \phi_{T,\Gamma}) \in \mathbb{L}^2$. Then there exists a unique distributional solution $\Phi = (\phi, \phi_\Gamma) \in C([0,T];\mathbb{L}^2)$ of (1.6). Moreover, there is a constant $C > 0$ such that

$$\|\Phi\|_{C([0,T];\mathbb{L}^2)} \le C\left(\|\Phi_T\|_{\mathbb{L}^2} + \|F\|_{L^2(0,T;\mathbb{L}^2)}\right).$$

(2) If $\Phi_T = (\phi_T, \phi_{T,\Gamma}) \in \mathbb{H}^1$. Then there exists a unique strong solution $\Phi = (\phi, \phi_\Gamma) \in \mathbb{E}$ of (1.6). Moreover, there is a constant $C > 0$ such that

$$\|\Phi\|_{\mathbb{E}} \le C\left(\|\Phi_T\|_{\mathbb{H}^1} + \|F\|_{L^2(0,T;\mathbb{L}^2)}\right). \tag{2.8}$$

Remark 2. As far as the well-posedness of the quasi-linear equation is concerned (but without boundary diffusion), we refer to [14]. The well-posedness of (1.2) can be done by the Faedo-Galerkin approximation. We refer to [24, 26, 40] for similar equations.

2.3. **Null controllability of (1.5).** The aim of this section is to prove null controllability for the equation (1.5), we also prove estimates on the state and regularity on the control which require some regularity in the initial data and source terms. We consider the following positive weight functions α and ξ which depend on Ω and ω

$$\alpha(x,t)=\frac{e^{2\lambda m}-e^{\lambda(m+\eta(x))}}{t(T-t)} \quad \text{and} \quad \xi(x,t)=\frac{e^{\lambda(m+\eta(x))}}{t(T-t)}.$$

Here, $\lambda, m>1$ and $\eta=\eta(x)$ is a function in $C^2(\overline{\Omega})$ satisfying

$$\eta>0 \text{ in } \Omega, \quad \eta=0 \text{ on } \Gamma, \quad \inf_{\Omega\setminus\omega'}|\nabla\eta(x)|>0 \quad \text{and} \quad \|\eta\|_\infty=1, \tag{2.9}$$

where $\omega'\subset\subset\omega$ is a nonempty open set. The existence of such a function η satisfying (2.9) is proved in [22] if Ω is a C^2 domain. The following lemma is a Carleman estimate for the adjoint equation (1.6). The proof can be found in [37] which deals with the null controllability of linear and semilinear parabolic equations with dynamic boundary conditions.

Lemma 2.2. *There are constants $C_1>0$ and $\lambda_1, s_1\geq 1$ such that for any $s\geq s_1$, any $\lambda\geq\lambda_1$ and any $(\phi,\phi_\Gamma)\in\mathbb{E}$, we have the following estimate*

$$\begin{aligned}
&\int_{\Omega_T} e^{-2s\alpha}\left[s^{-1}\xi^{-1}\left(|\phi_t|^2+|\Delta\phi|^2\right)+s\lambda^2\xi|\nabla\phi|^2+s^3\lambda^4\xi^3|\phi|^2\right]\,dx\,dt\\
&+\int_{\Gamma_T} e^{-2s\alpha}\left[s^{-1}\xi^{-1}\left(|\phi_{\Gamma,t}|^2+|\Delta_\Gamma\phi_\Gamma|^2\right)+s\lambda\xi|\nabla_\Gamma\phi_\Gamma|^2+s^3\lambda^3\xi^3|\phi_\Gamma|^2\right]\,dS\,dt\\
&+s\lambda\int_{\Gamma_T} e^{-2s\alpha}\xi|\partial_\nu\phi|^2\,dS\,dt\leq C_1\left(s^3\lambda^4\int_{\omega'\times(0,T)} e^{-2s\alpha}\xi^3|\phi|^2\,dx\,dt\right.\\
&\left.+\int_{\Omega_T} e^{-2s\alpha}|\mathbf{L}_1^*\Phi|^2\,dx\,dt+\int_{\Gamma_T} e^{-2s\alpha}|\mathbf{L}_2^*\Phi|^2\,dS\,dt\right).
\end{aligned}\tag{2.10}$$

Furthermore, C_1 and λ_1 only depend on Ω and ω, and s_1 can be chosen of the form $s_1=C(T+T^2)$, where C only depends on Ω, ω, $\sigma(0)$, $\delta(0)$, $a'(0)$ and $b'(0)$.

We will deduce a Carleman estimate similar to (2.10) with functions blowing up only at $t=T$. Define the new weight functions:

$$\beta(x,t)=\frac{e^{2\lambda m}-e^{\lambda(m+\eta(x))}}{\ell(t)} \quad \text{and} \quad \zeta(x,t)=\frac{e^{\lambda(m+\eta(x))}}{\ell(t)}, \quad (x,t)\in\overline{\Omega}\times(0,T),$$

where the function ℓ is given by

$$\ell(t)=\begin{cases}\frac{T^2}{4} & \text{if } t\in[0,T/2]\\ t(T-t) & \text{if } t\in[T/2,T].\end{cases}$$

Note that $\ell\in C^1([0,T])$. An estimate with such weights is given in the following result. In the proof, we will use Lemma 2.2 and energy estimate (2.8).

Proposition 2. *There exist constants $\lambda_1, s_1\geq 1$ such that for any $s\geq s_1$, any $\lambda\geq\lambda_1$, there exists a constant $C:=C(s,\lambda,T)>0$ which satisfies the following property: for any $\Phi=(\phi,\phi_\Gamma)\in\mathbb{E}$, we have the following estimate:*

$$\begin{aligned}
&\int_{\Omega_T} e^{-2s\beta}\left[\ell(t)\left(|\phi_t|^2+|\Delta\phi|^2\right)+\ell^{-1}(t)|\nabla\phi|^2+\ell^{-3}(t)|\phi|^2\right]\,dx\,dt\\
&+\int_{\Gamma_T} e^{-2s\beta}\left[\ell(t)\left(|\phi_{\Gamma,t}|^2+|\Delta_\Gamma\phi_\Gamma|^2\right)+\ell^{-1}(t)|\nabla_\Gamma\phi_\Gamma|^2+\ell^{-3}(t)|\phi_\Gamma|^2\right]\,dS\,dt\\
&+\int_{\Gamma_T} e^{-2s\beta}\ell^{-1}(t)|\partial_\nu\phi|^2\,dS\,dt+\int_\Omega|\phi(\cdot,0)|^2\,dx+\int_\Gamma|\phi_\Gamma(\cdot,0)|^2\,dS\\
&\leq C\left(\int_{\omega'\times(0,T)} e^{-2s\beta}\ell^{-3}(t)|\phi|^2\,dx\,dt+\int_{\Omega_T} e^{-2s\beta}|\mathbf{L}_1^*\Phi|^2\,dx\,dt+\int_{\Gamma_T} e^{-2s\beta}|\mathbf{L}_2^*\Phi|^2\,dS\,dt\right),
\end{aligned}\tag{2.11}$$

with s_1 and λ_1 as in Lemma 2.2.

Proof. To do so, we split the left-hand side of (2.11) into two parts, corresponding to the restrictions of ϕ on $\overline{\Omega}\times(0,T/2)$ and $\overline{\Omega}\times(T/2,T)$. Firstly, on $\overline{\Omega}\times(T/2,T)$. Using Carleman estimate (2.10), $\alpha=\beta$ and $\xi=e^{\lambda(m+\eta(x))}\ell^{-1}$ in $\overline{\Omega}\times(T/2,T)$, we obtain

$$\begin{aligned}
&\int_\Omega\int_{T/2}^T e^{-2s\beta}\left[\ell(t)\left(|\phi_t|^2+|\Delta\phi|^2\right)+\ell^{-1}(t)|\nabla\phi|^2+\ell^{-3}(t)|\phi|^2\right]\,\mathrm{d}x\,\mathrm{d}t\\
&+\int_\Gamma\int_{T/2}^T e^{-2s\beta}\left[\ell(t)\left(|\phi_{\Gamma,t}|^2+|\Delta_\Gamma\phi_\Gamma|^2\right)+\ell^{-1}(t)|\nabla_\Gamma\phi_\Gamma|^2+\ell^{-3}(t)|\phi_\Gamma|^2\right]\,\mathrm{d}S\,\mathrm{d}t\\
&+\int_\Gamma\int_{T}^{T/2} e^{-2s\beta}\ell^{-1}(t)|\partial_\nu\phi|^2\,\mathrm{d}S\,\mathrm{d}t\\
&\le C\left(\int_{\omega'\times(0,T)} e^{-2s\alpha}\xi^3|\phi|^2\,\mathrm{d}x\,\mathrm{d}t+\int_{\Omega_T} e^{-2s\alpha}|\mathbf{L}_1^*\Phi|^2\,\mathrm{d}x\,\mathrm{d}t+\int_{\Gamma_T} e^{-2s\alpha}|\mathbf{L}_2^*\Phi|^2\,\mathrm{d}S\,\mathrm{d}t\right).
\end{aligned}\tag{2.12}$$

In accordance with the definition of ℓ, we have $\ell(t)\ge t(T-t)$ on $[0,T]$, then

$$e^{-2s\alpha}\le e^{-2s\beta}\quad\text{on }\ \overline{\Omega}\times(0,T).\tag{2.13}$$

On the other hand, one has

$$e^{-2s\alpha}\xi^3\le Ce^{-2s\beta}\ell^{-3}\quad\text{on }\ \overline{\Omega}\times(0,T),\tag{2.14}$$

where C only depending on s, λ and T. Indeed, on $\overline{\Omega}\times(T/2,T)$, we have $e^{-2s\alpha}\xi^3=e^{3\lambda(m+\eta(x))}e^{-2s\beta}\ell^{-3}$ and on $\overline{\Omega}\times(0,T/2)$, $e^{-2s(\alpha-\beta)}\left(\xi\ell\right)^3$ is bounded with respect to x and this bound admits 0 as a limit when $t\searrow 0$. Hence $e^{-2s(\alpha-\beta)}\left(\xi\ell\right)^3$ is bounded on $\overline{\Omega}\times(0,T/2)$. Considering (2.12)-(2.14), we obtain

$$\begin{aligned}
&\int_\Omega\int_{T/2}^T e^{-2s\beta}\left[\ell(t)\left(|\phi_t|^2+|\Delta\phi|^2\right)+\ell^{-1}(t)|\nabla\phi|^2+\ell^{-3}(t)|\phi|^2\right]\,\mathrm{d}x\,\mathrm{d}t\\
&+\int_\Gamma\int_{T/2}^T e^{-2s\beta}\left[\ell(t)\left(|\phi_{\Gamma,t}|^2+|\Delta_\Gamma\phi_\Gamma|^2\right)+\ell^{-1}(t)|\nabla_\Gamma\phi_\Gamma|^2+\ell^{-3}(t)|\phi_\Gamma|^2\right]\,\mathrm{d}S\,\mathrm{d}t\\
&\le C\left(\int_{\omega'\times(0,T)} e^{-2s\beta}\ell^{-3}(t)|\phi|^2\,\mathrm{d}x\,\mathrm{d}t+\int_{\Omega_T} e^{-2s\beta}|\mathbf{L}_1^*\Phi|^2\,\mathrm{d}x\,\mathrm{d}t+\int_{\Gamma_T} e^{-2s\beta}|\mathbf{L}_2^*\Phi|^2\,\mathrm{d}S\,\mathrm{d}t\right).
\end{aligned}\tag{2.15}$$

Secondly, on $\overline{\Omega}\times(0,T/2)$. Let us define a function $\vartheta\in C^1([0,T])$ such that

$$\vartheta(t)=1\quad\text{in}\quad[0,T/2]\quad\text{and}\quad\vartheta(t)=0\quad\text{in}\quad[3T/4,T]\,.$$

Put $Y:=(y,y_\Gamma)$ defined by $Y:=\vartheta\,\Phi$, then Y is the strong solution of the equation

$$\begin{cases}
\mathbf{L}_1^\star Y=\vartheta\mathbf{L}_1^\star\Phi-\vartheta'\phi & \text{in }\Omega_T,\\
\mathbf{L}_2^\star Y=\vartheta\mathbf{L}_2^\star\Phi-\vartheta'\phi_\Gamma & \text{on }\Gamma_T\\
y_\Gamma=y_{|\Gamma} & \text{on }\Gamma_T,\\
(y(\cdot,T),y_\Gamma(\cdot,T))=(0,0) & \text{in }\Omega\times\Gamma.
\end{cases}$$

Using the continuity of the normal derivative from $H^2(\Omega)$ to $L^2(\Gamma)$ and energy estimate (2.8), one has

$$\begin{aligned}
&\|Y(\cdot,0)\|^2_{\mathbb{L}^2}+\|Y\|^2_{L^2(0,T;\mathbb{L}^2)}+\|Y_t\|^2_{L^2(0,T;\mathbb{L}^2)}+\|\nabla Y\|^2_{L^2(0,T;\mathbb{L}^2)}\\
&+\|\Delta Y\|^2_{L^2(0,T;\mathbb{L}^2)}+\|\partial_\nu y\|^2_{L^2(0,T;L^2(\Gamma))}\le C\|Y\|^2_{\mathbb{E}}\\
&\le C\|(\vartheta\mathbf{L}_1^\star\Phi,\vartheta\mathbf{L}_2^\star\Phi)-\vartheta'\Phi\|^2_{L^2(0,T;\mathbb{L}^2)}.
\end{aligned}\tag{2.16}$$

The definition of ϑ and (2.16) imply that

$$\begin{aligned}
&\|\Phi(\cdot,0)\|^2_{\mathbb{L}^2}+\|\Phi\|^2_{L^2(0,T/2;\mathbb{L}^2)}+\|\Phi_t\|^2_{L^2(0,T/2;\mathbb{L}^2)}+\|\nabla\Phi\|^2_{L^2(0,T/2;\mathbb{L}^2)}\\
&\quad+\|\Delta\Phi\|^2_{L^2(0,T/2;\mathbb{L}^2)}+\|\partial_\nu\phi\|^2_{L^2(0,T/2;L^2(\Gamma))}\\
&\le C\left(\|(\mathbf{L}_1^\star\Phi,\mathbf{L}_2^\star\Phi)\|^2_{L^2(0,3T/4;\mathbb{L}^2)}+\|\Phi\|^2_{L^2(T/2,3T/4;\mathbb{L}^2)}\right).
\end{aligned}\tag{2.17}$$

Using the boundedness from above and from below of the weight functions β and ℓ in $\overline{\Omega}\times(0,3T/4)$ and (2.17), we get

$$\begin{aligned}
&\int_\Omega\int_0^{T/2} e^{-2s\beta}\left[\ell(t)\left(|\phi_t|^2+|\Delta\phi|^2\right)+\ell^{-1}(t)|\nabla\phi|^2+\ell^{-3}(t)|\phi|^2\right]\,\mathrm{d}x\,\mathrm{d}t\\
&+\int_\Gamma\int_0^{T/2} e^{-2s\beta}\left[\ell(t)\left(|\phi_{\Gamma,t}|^2+|\Delta_\Gamma\phi_\Gamma|^2\right)+\ell^{-1}(t)|\nabla_\Gamma\phi_\Gamma|^2+\ell^{-3}(t)|\phi_\Gamma|^2\right]\,\mathrm{d}S\,\mathrm{d}t\\
&+\int_\Gamma\int_0^{T/2} e^{-2s\beta}\ell^{-1}(t)|\partial_\nu\phi|^2\,\mathrm{d}S\,\mathrm{d}t+\|\phi(\cdot,0)\|^2_{L^2(\Omega)}+\|\phi_\Gamma(\cdot,0)\|^2_{L^2(\Gamma)}\\
&\le C\left(\int_\Omega\int_0^{3T/4} e^{-2s\beta}|\mathbf{L}_1^*\Phi|^2\,\mathrm{d}x\,\mathrm{d}t+\int_\Gamma\int_0^{3T/4} e^{-2s\beta}|\mathbf{L}_2^*\Phi|^2\,\mathrm{d}S\,\mathrm{d}t\right.\\
&\left.+\int_\Omega\int_{T/2}^{3T/4} e^{-2s\beta}\ell^{-3}(t)|\phi|^2\,\mathrm{d}x\,\mathrm{d}t+\int_\Omega\int_{T/2}^{3T/4} e^{-2s\beta}\ell^{-3}(t)|\phi_\Gamma|^2\,\mathrm{d}x\,\mathrm{d}S\right),
\end{aligned}$$

which, combined with (2.15), yields (2.11). □

From now on, we fix $s=s_1$, $\lambda=\lambda_1$, and introduce the following weights, which we will need in the sequel.

Remark 3. Taking $m>1$ large enough for instance $m>\frac{\log(5e^\lambda-4)}{\lambda}$, we have that

$$\hat\beta(t)<(5/4)\check\beta(t)\quad\forall(x,t)\in\overline{\Omega}\times(0,T),$$

where

$$\hat\beta(t):=\max_{x\in\overline{\Omega}}\beta(x,t),\quad \check\beta(t):=\min_{x\in\overline{\Omega}}\beta(x,t).$$

Notation. Let us introduce the notations

$$\begin{aligned}
&\gamma(t):=\hat\beta/5,\quad \mu(t):=e^{5s\gamma(t)}\ell^{3/2},\quad \mu_0(t):=e^{4s\gamma(t)},\\
&\mu_1(t):=\mu_0(t)\ell^{3/2},\quad \mu_k(t):=e^{3s\gamma(t)}\ell^{(2k+1)/2}\quad\forall k\in\{2,\cdots,5\}.
\end{aligned}$$

Remark 4. We have the following elementary estimates

$$\begin{aligned}
&\mu_3\mu_1^{-2}=\mu^{-1}\ell^2,\quad(\mu_3\mu_1^{-2})_t\le C\mu^{-1},\quad|\mu_{3,t}|\le C\mu_1,\\
&\mu_0\le C\mu,\quad\mu\le C\mu_5^2,\quad\mu_k\le C\mu_{k-1}\quad\forall k\in\{1,\cdots,5\},\\
&|\mu_k\mu_{k,t}|\le C\mu_{k-1}^2\quad\forall k\in\{2,\cdots,5\}.
\end{aligned}$$

Proposition 3. *Let $F=(f,f_\Gamma)$, $\Psi_0=(\psi_0,\psi_{\Gamma,0})$ such that $\mu F\in L^2(0,T;\mathbb{L}^2)$ and $\Psi_0\in\mathbb{L}^2$. Then, there exists a control v such that the solution Ψ of (1.5) corresponding to v, F and Ψ_0, satisfies*

$$\|\mu_0\Psi\|^2_{L^2(0,T;\mathbb{L}^2)}+\|\mu_1 v\|^2_{L^2(\omega_T)}\le C\left(\|\mu F\|^2_{L^2(0,T;\mathbb{L}^2)}+\|\Psi_0\|^2_{\mathbb{L}^2}\right).\tag{2.18}$$

In particular (1.5) is null controllable. Moreover, we can choose v satisfying

$$\mu_3 v\in H^1(0,T;L^2(\omega))\cap L^2(0,T;H^2(\omega))\tag{2.19}$$

and

$$\|\mu_3 v_t\|^2_{L^2(\omega_T)}+\|v(\cdot,0)\|^2_{H^1(\omega)}\le C\left(\|\mu F\|^2_{L^2(0,T;\mathbb{L}^2)}+\|\Psi_0\|^2_{\mathbb{L}^2}\right).\tag{2.20}$$

Proof. The proof of this result is inspired by the method of Fursikov and Imanuvilov [22]. Let us consider the following space:

$$\mathbf{P}:=\left\{Y=(y,y_\Gamma)\ :\ y\in C^2(\overline{\Omega_T})\ \text{ and }\ y_{|\Gamma}(\cdot,t)=y_\Gamma(\cdot,t),\ \ t\in[0,T]\right\}$$

Note that $\mathbf{P}$ can be identified with $C^2(\overline{\Omega_T})$ and is dense in $\mathbb{E}$. We define the bilinear form $\mathbf{B}:\mathbf{P}\times\mathbf{P}\longrightarrow\mathbb{R}$ by

$$\mathbf{B}(Y,Z):=\int_{\Omega_T}\mu_0^{-2}\mathbf{L}_1^\star Y\mathbf{L}_1^\star Z\,\mathrm{d}x\,\mathrm{d}t+\int_{\Gamma_T}\mu_0^{-2}\mathbf{L}_2^\star Y\mathbf{L}_2^\star Z\,\mathrm{d}S\,\mathrm{d}t+\int_{\Omega_T}\mu_1^{-2}\chi yz\,\mathrm{d}x\,\mathrm{d}t,$$

where $\chi \in C_0^\infty(\omega)$ (the space of test functions with compact support in ω) is given such that $0 \leq \chi \leq 1$ and $\chi_{|\omega'} = 1$ ($\mathbf{L}_1^\star$ and $\mathbf{L}_2^\star$ are defined in (2.4)). We also define the linear form $\mathbf{F} : \mathbf{P} \longrightarrow \mathbb{R}$ by

$$\mathbf{F}(Z) := \langle F, Z\rangle_{L^2(0,T;\mathbb{L}^2)} + \langle \Psi_0, Z(\cdot,0)\rangle_{\mathbb{L}^2}.$$

We claim that $\mathbf{B}$ is an inner product in $\mathbf{P}$ and $\mathbf{F}$ is continuous for the norm $\|\cdot\|_{\mathbf{B}}$ associated with the scalar product $\mathbf{B}$. Indeed, due to Carleman estimate (2.11), there exists a constant $C := C(s,\lambda,T) > 0$ such that for all $Z := (z, z_\Gamma) \in \mathbf{P}$, one has

$$\begin{aligned}\int_\Omega |z(\cdot,0)|^2 \,\mathrm{d}x + \int_\Gamma |z_\Gamma(\cdot,0)|^2 \,\mathrm{d}S + \int_{\Omega_T} \mu^{-2}|z|^2 \,\mathrm{d}x\,\mathrm{d}t + \int_{\Gamma_T} \mu^{-2}|z_\Gamma|^2 \,\mathrm{d}S\,\mathrm{d}t \\ \leq C\, \mathbf{B}(Z,Z). \end{aligned} \tag{2.21}$$

In particular, $\mathbf{B}$ is a scalar product in $\mathbf{P}$. To ensure the continuity of $\mathbf{F}$, using Cauchy-Schwarz inequality and (2.21), we get

$$|\mathbf{F}(Z)| \leq C\left(\|\mu F\|_{L^2(0,T;\mathbb{L}^2)} + \|\Psi_0\|_{\mathbb{L}^2}\right)\|Z\|_{\mathbf{B}}. \tag{2.22}$$

In the sequel, we will denote by $\overline{\mathbf{P}}$ the completion of $\mathbf{P}$ for the norm $\|\cdot\|_{\mathbf{B}}$ and we will still denote $\mathbf{B}$ and $\mathbf{F}$ the corresponding continuous extensions. From the Riesz Representation theorem, there exists a unique $\Phi = (\phi, \phi_\Gamma) \in \overline{\mathbf{P}}$ such that

$$\mathbf{B}(\Phi, Z) = \mathbf{F}(Z) \quad \forall Z \in \overline{\mathbf{P}}. \tag{2.23}$$

Using (2.22) and (2.23), we obtain

$$\|\Phi\|_{\mathbf{B}} \leq C\left(\|\mu F\|_{L^2(0,T;\mathbb{L}^2)} + \|\Psi_0\|_{\mathbb{L}^2}\right). \tag{2.24}$$

Let us introduce (Ψ, v) with

$$\Psi := \mu_0^{-2}(\mathbf{L}_1^*\Phi, \mathbf{L}_2^*\Phi) \quad \text{and} \quad v := -\chi\mu_1^{-2}\phi|_{\omega_T}. \tag{2.25}$$

According to (2.25) and the definition of $\mathbf{B}$ and $\chi^2 \leq \chi$, we obtain

$$\int_{\Omega_T} \mu_0^2|\psi|^2 \,\mathrm{d}x\,\mathrm{d}t + \int_{\Gamma_T} \mu_0^2|\psi_\Gamma|^2 \,\mathrm{d}S\,\mathrm{d}t + \int_{\omega_T} \mu_1^2|v|^2 \,\mathrm{d}x\,\mathrm{d}t \leq \mathbf{B}(\Phi,\Phi).$$

Using this equation and estimate (2.24), we deduce estimate (2.18). As a consequence $\Psi \in L^2(0,T;\mathbb{L}^2)$, $v \in L^2(\omega_T)$ and from (2.23), Ψ is the unique distributional solution of (1.2) with the control v. Let $W := -\chi\mu_1^{-2}\Phi$ and $g := \mathbf{L}_1^\star(\mu_3 W)$. Then,

$$\begin{aligned} g &= -\chi\mu_3\mu_1^{-2}\mathbf{L}_1^\star\Phi + \chi(\mu_3\mu_1^{-2})_t\phi + 2\sigma(0)\mu_3\mu_1^{-2}\nabla\chi\cdot\nabla\phi \\ &\quad +\sigma(0)\mu_3\mu_1^{-2}\Delta\chi\,\phi := g_1 + g_2 + g_3 + g_4. \end{aligned} \tag{2.26}$$

Using the definition of ψ in (2.25) and Remark 4, we obtain

$$\begin{cases} |g_1| \leq C\mu_0|\psi|, \quad |g_2| \leq C\mu^{-1}|\phi|, \\ |g_3| \leq C\mu^{-1}\ell|\nabla\phi|, \quad |g_4| \leq C\mu^{-1}|\phi|. \end{cases} \tag{2.27}$$

From the Carleman estimate (2.11) and (2.24), we get

$$\int_{\Omega_T} \mu^{-2}\left[\ell^2|\nabla\phi|^2 + |\phi|^2\right] \,\mathrm{d}x\,\mathrm{d}t \leq C\left(\|\mu F\|^2_{L^2(0,T;\mathbb{L}^2)} + \|\Psi_0\|^2_{\mathbb{L}^2}\right). \tag{2.28}$$

Taking into account (2.18), (2.26), (2.27) and (2.28), we obtain

$$g \in L^2(\Omega_T) \quad \text{and} \quad \|g\|^2_{L^2(\Omega_T)} \leq C\left(\|\mu F\|^2_{L^2(0,T;\mathbb{L}^2)} + \|\Psi_0\|^2_{\mathbb{L}^2}\right). \tag{2.29}$$

Since $\chi \in C_0^\infty(\omega)$ and $\omega \Subset \Omega$, then $\mu_3 W$ is the strong solution of

$$\begin{cases} \mathbf{L}_1^\star(\mu_3 W) = g & \text{in } \Omega_T, \\ \mathbf{L}_2^\star(\mu_3 W) = 0 & \text{on } \Gamma_T, \\ (\mu_3 w)_\Gamma = (\mu_3 w)_{|\Gamma} & \text{on } \Gamma_T, \\ (\mu_3 w(\cdot,T), \mu_3 w_\Gamma(\cdot,T)) = (0,0) & \text{in } \Omega\times\Gamma. \end{cases}$$

Using estimates (2.8) and (2.29), we obtain

$$\|\mu_3 W\|^2_{\mathbb{E}} \leq C\left(\|\mu F\|^2_{L^2(0,T;\mathbb{L}^2)} + \|\Psi_0\|^2_{\mathbb{L}^2}\right). \tag{2.30}$$

Since $\mu_3 w|_\omega = \mu_3 v$, we deduce (2.19). From the following continuous embedding (see Theorem 4 in [15])

$$H^1(0,T;L^2(\omega)) \cap L^2(0,T;H^2(\omega)) \hookrightarrow C([0,T];H^1(\omega))$$

and estimate (2.30), we obtain

$$v(\cdot,0) \in H^1(\omega) \quad \text{and} \quad \|v(\cdot,0)\|^2_{H^1(\omega)} \leq C\left(\|\mu F\|^2_{L^2(0,T;\mathbb{L}^2)} + \|\Psi_0\|^2_{\mathbb{L}^2}\right). \tag{2.31}$$

Since $\mu_3 v_t = (\mu_3 v)_t - \mu_{3,t} v$ and $|\mu_{3,t}| \leq C\mu_1$, then

$$\|\mu_3 v_t\|^2_{L^2(\omega_T)} \leq C\left(\|\mu F\|^2_{L^2(0,T;\mathbb{L}^2)} + \|\Psi_0\|^2_{\mathbb{L}^2}\right). \tag{2.32}$$

Finally, from (2.31) and (2.32), we obtain (2.20). □

Proposition 4. *Under the assumptions of Proposition 3 and if (Ψ, v) is a state-control for (1.5) provided by Proposition 3 associated with data $\Psi_0 = (\psi_0, \psi_{0,\Gamma})$ and $F = (f, f_\Gamma)$ such that $\mu F \in L^2(0,T;\mathbb{L}^2)$ and $\mu_4 F_t \in L^2(0,T;\mathbb{L}^2)$. The following estimates are satisfied*

(1) For any $\Psi_0 \in \mathbb{L}^2$, one has

$$\sup_{0\leq t\leq T} \mu_2^2(t)\|\Psi(\cdot,t)\|^2_{\mathbb{L}^2} + \|\mu_2 \nabla\Psi\|^2_{L^2(0,T;\mathbb{L}^2)} \leq C\left(\|\mu F\|^2_{L^2(0,T;\mathbb{L}^2)} + \|\Psi_0\|^2_{\mathbb{L}^2}\right). \tag{2.33}$$

(2) If we assume that $\Psi_0 \in \mathbb{H}^1$, one has

$$\begin{aligned}&\sup_{0\leq t\leq T} \mu_3^2(t)\|\nabla\Psi(\cdot,t)\|^2_{\mathbb{L}^2} + \|\mu_3\Psi_t\|^2_{L^2(0,T;\mathbb{L}^2)} + \|\mu_3\Delta\Psi\|^2_{L^2(0,T;\mathbb{L}^2)}\\ &\qquad \leq C\left(\|\mu F\|^2_{L^2(0,T;\mathbb{L}^2)} + \|\Psi_0\|^2_{\mathbb{H}^1}\right).\end{aligned} \tag{2.34}$$

(3) If we assume that $\Psi_0 \in \mathbb{H}^2$, one has

$$\begin{aligned}&\sup_{0\leq t\leq T} \mu_4^2(t)\|\Psi_t(\cdot,t)\|^2_{\mathbb{L}^2} + \|\mu_4\nabla\Psi_t\|^2_{L^2(0,T;\mathbb{L}^2)}\\ &\qquad \leq C\left(\|\mu F\|^2_{L^2(0,T;\mathbb{L}^2)} + \|\mu_4 F_t\|^2_{L^2(0,T;\mathbb{L}^2)} + \|\Psi_0\|^2_{\mathbb{H}^2}\right).\end{aligned} \tag{2.35}$$

(4) If we assume that $\Psi_0 \in \mathbb{H}^3$ and $F(\cdot,0) \in \mathbb{H}^1$, one has

$$\begin{aligned}&\sup_{0\leq t\leq T} \mu_5^2(t)\|\nabla\Psi_t(\cdot,t)\|^2_{\mathbb{L}^2} + \|\mu_5\Psi_{tt}\|^2_{L^2(0,T;\mathbb{L}^2)} + \|\mu_5\Delta\Psi_t\|^2_{L^2(0,T;\mathbb{L}^2)}\\ &+ \sup_{0\leq t\leq T} \mu_5^2(t)\|\Delta\Psi(\cdot,t)\|^2_{\mathbb{L}^2} + \|\mu_5\nabla\Psi_t\|^2_{L^2(0,T;\mathbb{L}^2)}\\ &\qquad \leq C\left(\|\mu F\|^2_{L^2(0,T;\mathbb{L}^2)} + \|\mu_4 F_t\|^2_{L^2(0,T;\mathbb{L}^2)} + \|F(\cdot,0)\|^2_{\mathbb{H}^1} + \|\Psi_0\|^2_{\mathbb{H}^3}\right).\end{aligned} \tag{2.36}$$

Proof. (1) Firstly, multiplying the first equation of (1.5) by $\mu_2^2\psi$ and integrating in Ω, one has

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \mu_2^2|\psi|^2\,\mathrm{d}x + \sigma(0)\int_\Omega \mu_2^2|\nabla\psi|^2\,\mathrm{d}x = \int_\Omega \mu_2\mu_{2,t}|\psi|^2\,\mathrm{d}x\\ &+\sigma(0)\int_\Gamma \mu_2^2\psi\partial_\nu\psi\,\mathrm{d}S - a'(0)\int_\Omega \mu_2^2|\psi|^2\,\mathrm{d}x + \int_\omega \mu_2^2\psi v\,\mathrm{d}x + \int_\Omega \mu_2^2\psi f\,\mathrm{d}x.\end{aligned}$$

Secondly, multiplying the second equation of (1.5) by $\mu_2^2\psi_\Gamma$ and integrating on Γ, using the Stokes divergence formula (2.3), we obtain

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Gamma \mu_2^2|\psi_\Gamma|^2\,\mathrm{d}S + \delta(0)\int_\Gamma \mu_2^2|\nabla_\Gamma\psi_\Gamma|^2\,\mathrm{d}S = \int_\Gamma \mu_2\mu_{2,t}|\psi_\Gamma|^2\,\mathrm{d}S\\ &-\sigma(0)\int_\Gamma \mu_2^2\psi_\Gamma\partial_\nu\psi\,\mathrm{d}S - b'(0)\int_\Gamma \mu_2^2|\psi_\Gamma|^2\,\mathrm{d}S + \int_\Gamma \mu_2^2\psi_\Gamma f_\Gamma\,\mathrm{d}S.\end{aligned}$$

Next, we add these identities, using the fact that $|\mu_2\mu_{2,t}| \leq C\mu_0^2$, $|\mu_2^2| \leq C\mu_0^2$ and Young's inequality, we get

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\mu_2\Psi\|^2_{\mathbb{L}^2} + \sigma(0)\int_\Omega \mu_2^2|\nabla\psi|^2\,\mathrm{d}x + \delta(0)\int_\Gamma \mu_2^2|\nabla_\Gamma\psi_\Gamma|^2\,\mathrm{d}S\\ &\leq C\|\mu_0\Psi\|^2_{\mathbb{L}^2} + C\left(\int_\omega \mu_2^2|v|^2\,\mathrm{d}x + \int_\Omega \mu_2^2|f|^2\,\mathrm{d}x + \int_\Gamma \mu_2^2|f_\Gamma|^2\,\mathrm{d}S\right).\end{aligned}$$

Integrating over $(0,t)$, one has

$$\sup_{0\le t\le T}\mu_2^2(t)\|\Psi(\cdot,t)\|_{\mathbb{L}^2}^2+\|\mu_2\nabla\Psi\|_{L^2(0,T;\mathbb{L}^2)}^2 \le C\left(\|\mu_0\Psi\|_{\mathbb{L}^2}^2+\|\mu_2 v\|_{L^2(\omega_T)}^2+\|\mu_2 F\|_{L^2(0,T;\mathbb{L}^2)}^2+\|\Psi_0\|_{\mathbb{L}^2}^2\right).$$

Using the fact that $\mu_2\le C\mu_1$, $\mu_2\le C\mu$ and estimate (2.18), we deduce (2.33).

(2) Firstly, multiplying the first equation of (1.5) by $\mu_3^2\psi_t$ and integrating over Ω, we obtain

$$\int_\Omega \mu_3^2|\psi_t|^2\,\mathrm{d}x+\frac{\sigma(0)}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega\mu_3^2|\nabla\psi|^2\,\mathrm{d}x=\sigma(0)\int_\Omega\mu_3\mu_{3,t}|\nabla\psi|^2\,\mathrm{d}x$$
$$+\sigma(0)\int_\Gamma\mu_3^2\psi_t\partial_\nu\psi\,\mathrm{d}S-a'(0)\int_\Omega\mu_3^2\psi\psi_t\,\mathrm{d}x+\int_\omega\mu_3^2\psi_t v\,\mathrm{d}x+\int_\Omega\mu_3^2\psi_t f\,\mathrm{d}x.$$

Using Young's inequality and the following elementary estimates

$$|\mu_3\mu_{3,t}|\le C\mu_2^2,\qquad \mu_3\le C\mu_0, \tag{2.37}$$

we get

$$\frac12\int_\Omega\mu_3^2|\psi_t|^2\,\mathrm{d}x+\frac{\sigma(0)}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega\mu_3^2|\nabla\psi|^2\,\mathrm{d}x\le C\left(\int_\Omega\mu_2^2|\nabla\psi|^2\,\mathrm{d}x\right. \tag{2.38}$$
$$\left.+\int_\Omega\mu_0^2|\psi|^2\,\mathrm{d}x\right)+\sigma(0)\int_\Gamma\mu_3^2\psi_t\partial_\nu\psi\,\mathrm{d}S+C\left(\int_\omega\mu_3^2|v|^2\,\mathrm{d}x+\int_\Omega\mu_3^2|f|^2\,\mathrm{d}x\right).$$

On the other hand, multiplying the second equation of (1.5) by $\mu_3^2\psi_{\Gamma,t}$ and integrating over Γ, by the Stokes divergence formula (2.3), we find

$$\int_\Gamma\mu_3^2|\psi_{\Gamma,t}|^2\,\mathrm{d}S+\frac{\delta(0)}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Gamma\mu_3^2|\nabla_\Gamma\psi_\Gamma|^2\,\mathrm{d}S=\delta(0)\int_\Gamma\mu_3\mu_{3,t}|\nabla_\Gamma\psi_\Gamma|^2\,\mathrm{d}S$$
$$-\sigma(0)\int_\Gamma\mu_3^2\psi_{\Gamma,t}\partial_\nu\psi\,\mathrm{d}S-b'(0)\int_\Gamma\mu_3^2\psi_\Gamma\psi_{\Gamma,t}\,\mathrm{d}S+\int_\Gamma\mu_3^2\psi_{\Gamma,t}f_\Gamma\,\mathrm{d}S.$$

Using Young's inequality and (2.37), we have

$$\frac12\int_\Gamma\mu_3^2|\psi_{\Gamma,t}|^2\,\mathrm{d}S+\frac{\delta(0)}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Gamma\mu_3^2|\nabla_\Gamma\psi_\Gamma|^2\,\mathrm{d}S\le C\left(\int_\Gamma\mu_2^2|\nabla_\Gamma\psi_\Gamma|^2\,\mathrm{d}S\right.$$
$$\left.+\int_\Gamma\mu_0^2|\psi_\Gamma|^2\,\mathrm{d}S\right)-\sigma(0)\int_\Gamma\mu_3^2\psi_{\Gamma,t}\partial_\nu\psi\,\mathrm{d}S+C\int_\Gamma\mu_3^2|f_\Gamma|^2\,\mathrm{d}S. \tag{2.39}$$

By summing (2.38) and (2.39), integrating over $(0,t)$, using $\mu_3\le C\mu_1$, $\mu_3\le C\mu$ and estimates (2.18) and (2.33), one has

$$\|\mu_3\Psi_t\|_{L^2(0,T;\mathbb{L}^2)}^2+\sup_{0\le t\le T}\mu_3^2(t)\|\nabla\Psi(\cdot,t)\|_{\mathbb{L}^2}^2\le C\left(\|\mu F\|_{L^2(0,T;\mathbb{L}^2)}^2+\|\Psi_0\|_{\mathbb{H}^1}^2\right). \tag{2.40}$$

Now, multiplying the first equation of (1.5) by $-\mu_3^2\Delta\psi$ and integrating over Ω, we obtain

$$\frac12\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega\mu_3^2|\nabla\psi|^2\,\mathrm{d}x+\sigma(0)\int_\Omega\mu_3^2|\Delta\psi|^2\,\mathrm{d}x=\int_\Omega\mu_3\mu_{3,t}|\nabla\psi|^2\,\mathrm{d}x$$
$$+\int_\Gamma\mu_3^2\psi_t\partial_\nu\psi\,\mathrm{d}S+a'(0)\int_\Omega\mu_3^2\Delta\psi\psi\,\mathrm{d}x-\int_\omega\mu_3^2\Delta\psi v\,\mathrm{d}x-\int_\Omega\mu_3^2\Delta\psi f\,\mathrm{d}x.$$

Using $|\mu_3\mu_{3,t}|\le C\mu_2$, Young's inequality and the continuity of the normal derivative from $H^2(\Omega)$ to $L^2(\Gamma)$, we find

$$\frac12\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega\mu_3^2|\nabla\psi|^2\,\mathrm{d}x+\frac{\sigma(0)}{2}\int_\Omega\mu_3^2|\Delta\psi|^2\,\mathrm{d}x\le C\left(\int_\Omega\mu_2^2|\nabla\psi|^2\,\mathrm{d}x\right.$$
$$\left.+\int_\Gamma\mu_3^2|\psi_{\Gamma,t}|^2\,\mathrm{d}S+\int_\Omega\mu_0^2|\psi|^2\,\mathrm{d}x+\int_\omega\mu_1^2|v|^2\,\mathrm{d}x+\int_\Omega\mu^2|f|^2\,\mathrm{d}x\right).$$

Integrating over $(0,t)$ and using estimates (2.18), (2.33) and (2.40), we obtain

$$\sup_{0\le t\le T}\mu_3^2(t)\|\nabla\psi(\cdot,t)\|_{L^2(\Omega)}^2+\|\mu_3\Delta\psi\|_{L^2(\Omega_T)}^2\le C\left(\|\mu F\|_{L^2(0,T;\mathbb{L}^2)}^2+\|\Psi_0\|_{\mathbb{H}^1}^2\right). \tag{2.41}$$

On the other hand, multiplying the second equation of (1.5) by $-\mu_3^2\Delta_\Gamma\psi_\Gamma$ and integrating over Γ, we get

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Gamma \mu_3^2|\nabla_\Gamma\psi_\Gamma|^2\,\mathrm{d}S+\delta(0)\int_\Gamma \mu_3^2|\Delta_\Gamma\psi_\Gamma|^2\,\mathrm{d}S=\int_\Gamma \mu_3\mu_{3,t}|\nabla_\Gamma\psi_\Gamma|^2\,\mathrm{d}S$$
$$+\sigma(0)\int_\Gamma \mu_3^2\Delta_\Gamma\psi_\Gamma\partial_\nu\psi\,\mathrm{d}S+b'(0)\int_\Gamma \mu_3^2\Delta_\Gamma\psi_\Gamma\psi_\Gamma\,\mathrm{d}S-\int_\Gamma \mu_3^2\Delta_\Gamma\psi_\Gamma f_\Gamma\,\mathrm{d}S.$$

Using Young's inequality and the continuity of the normal derivative from $H^2(\Omega)$ to $L^2(\Gamma)$, one has

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Gamma \mu_3^2|\nabla_\Gamma\psi_\Gamma|^2\,\mathrm{d}S+\frac{\delta(0)}{2}\int_\Gamma \mu_3^2|\Delta_\Gamma\psi_\Gamma|^2\,\mathrm{d}S\le C\left(\int_\Gamma \mu_2^2|\nabla_\Gamma\psi_\Gamma|^2\,\mathrm{d}S\right.$$
$$\left.+\int_\Omega \mu_3^2|\Delta\psi|^2\,\mathrm{d}x+\int_\Omega \mu_3^2|\psi|^2\,\mathrm{d}x+\int_\Gamma \mu_3^2|\psi_\Gamma|^2\,\mathrm{d}S+\int_\Gamma \mu_3^2|f_\Gamma|^2\,\mathrm{d}S\right).$$

Integrating over $(0,t)$ and using estimates (2.33) and (2.41), we get

$$\sup_{0\le t\le T}\mu_3^2(t)\|\nabla_\Gamma\psi_\Gamma(\cdot,t)\|^2_{L^2(\Gamma)}+\|\mu_3\Delta_\Gamma\psi_\Gamma\|^2_{L^2(\Gamma_T)}\le C\left(\|\mu F\|^2_{L^2(0,T;\mathbb{L}^2)}+\|\Psi_0\|^2_{\mathbb{H}^1}\right). \tag{2.42}$$

Finally, from (2.40)-(2.42), we deduce (2.34).

(3) Differentiating with respect to time the equations in (1.5), one has

$$\begin{cases}\psi_{tt}-\sigma(0)\Delta\psi_t+a'(0)\psi_t=\mathbb{1}_\omega v_t+f_t & \text{in } \Omega_T\\ \psi_{\Gamma,tt}-\delta(0)\Delta_\Gamma\psi_{\Gamma,t}+\sigma(0)\partial_\nu\psi_t+b'(0)\psi_{\Gamma,t}=f_{\Gamma,t} & \text{on } \Gamma_T\\ \psi_{\Gamma,t}=\psi_{t\,|\Gamma} & \text{on } \Gamma_T.\end{cases} \tag{2.43}$$

Multiplying the first equation of (2.43) by $\mu_4^2\psi_t$ and integrating over Ω, we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \mu_4^2|\psi_t|^2\,\mathrm{d}x+\sigma(0)\int_\Omega \mu_4^2|\nabla\psi_t|^2\,\mathrm{d}x=\int_\Omega \mu_4\mu_{4,t}|\psi_t|^2\,\mathrm{d}x$$
$$+\sigma(0)\int_\Gamma \mu_4^2\psi_t\partial_\nu\psi_t\,\mathrm{d}S-a'(0)\int_\Omega \mu_4^2|\psi_t|^2\,\mathrm{d}x+\int_\omega \mu_4^2\psi_t v_t\,\mathrm{d}x+\int_\Omega \mu_4^2\psi_t f_t\,\mathrm{d}x.$$

Using Young's inequality, $|\mu_4\mu_{4,t}|\le C\mu_3^2$ and $\mu_4^2\le C\mu_3^2$, we get

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \mu_4^2|\psi_t|^2\,\mathrm{d}x+\sigma(0)\int_\Omega \mu_4^2|\nabla\psi_t|^2\,\mathrm{d}x\le C\int_\Omega \mu_3^2|\psi_t|^2\,\mathrm{d}x$$
$$+\sigma(0)\int_\Gamma \mu_4^2\psi_t\partial_\nu\psi_t\,\mathrm{d}S+C\int_\omega \mu_3^2|v_t|^2\,\mathrm{d}x+C\int_\Omega \mu_4^2|f_t|^2\,\mathrm{d}x. \tag{2.44}$$

Multiplying the second equation of (2.43) by $\mu_4^2\psi_{\Gamma,t}$ and integrating over Γ, we get

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Gamma \mu_4^2|\psi_{\Gamma,t}|^2\,\mathrm{d}S+\delta(0)\int_\Gamma \mu_4^2|\nabla_\Gamma\psi_{\Gamma,t}|^2\,\mathrm{d}S=\int_\Gamma \mu_4\mu_{4,t}|\psi_{\Gamma,t}|^2\,\mathrm{d}S$$
$$-\sigma(0)\int_\Gamma \mu_4^2\psi_{\Gamma,t}\partial_\nu\psi_t\,\mathrm{d}S-b'(0)\int_\Gamma \mu_4^2|\psi_{\Gamma,t}|^2\,\mathrm{d}S+\int_\Gamma \mu_4^2\psi_{\Gamma,t}f_{\Gamma,t}\,\mathrm{d}S.$$
$$\le C\int_\Gamma \mu_3^2|\psi_{\Gamma,t}|^2\,\mathrm{d}S-\sigma(0)\int_\Gamma \mu_4^2\psi_{\Gamma,t}\partial_\nu\psi_t\,\mathrm{d}S+C\int_\Gamma \mu_4^2|f_{\Gamma,t}|^2\,\mathrm{d}S. \tag{2.45}$$

By summing (2.44) and (2.45) and integrating the estimate obtained over $(0,t)$, we find

$$\sup_{0\le t\le T}\mu_4^2(t)\|\Psi_t(\cdot,t)\|^2_{\mathbb{L}^2}+\|\mu_4\nabla\Psi_t\|^2_{L^2(0,T;\mathbb{L}^2)}$$
$$\le C\left(\|\mu_3\Psi_t\|^2_{L^2(0,T;\mathbb{L}^2)}+\|\mu_3 v_t\|^2_{L^2(\omega_T)}+\|\mu_4 F_t\|^2_{L^2(0,T;\mathbb{L}^2)}+\|\Psi_t(\cdot,0)\|^2_{\mathbb{L}^2}\right).$$

On the other hand, we have

$$\|\Psi_t(\cdot,0)\|^2_{\mathbb{L}^2}\le C\left(\|\Psi_0\|^2_{\mathbb{H}^2}+\|v(\cdot,0)\|^2_{L^2(\omega)}+\|F(\cdot,0)\|^2_{\mathbb{L}^2}\right). \tag{2.46}$$

Since $\mu_4 F\in H^1(0,T;\mathbb{L}^2)$ and $H^1(0,T;\mathbb{L}^2)\hookrightarrow C([0,T];\mathbb{L}^2)$ with continuous embedding, then

$$\|F(\cdot,0)\|^2_{\mathbb{L}^2}\le C\|\mu F\|^2_{L^2(0,T;\mathbb{L}^2)}+\|\mu_4 F_t\|^2_{L^2(0,T;\mathbb{L}^2)}. \tag{2.47}$$

Using estimates (2.46)-(2.47), (2.34) and (2.20), we obtain (2.35).

(4) Firstly, multiplying the first equation of (2.43) by $\mu_5^2\psi_{tt}$ and integrating over Ω, we obtain

$$\int_\Omega \mu_5^2|\psi_{tt}|^2\,\mathrm{d}x+\frac{\sigma(0)}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \mu_5^2|\nabla\psi_t|^2\,\mathrm{d}x=\sigma(0)\int_\Omega \mu_5\mu_{5,t}|\nabla\psi_t|^2\,\mathrm{d}x$$
$$+\sigma(0)\int_\Gamma \mu_5^2\psi_{tt}\partial_\nu\psi_t\,\mathrm{d}S-a'(0)\int_\Omega \mu_5^2\psi_t\psi_{tt}\,\mathrm{d}x+\int_\omega \mu_5^2\psi_{tt}v_t\,\mathrm{d}x+\int_\Omega \mu_5^2\psi_{tt}f_t\,\mathrm{d}x.$$

Using Young's inequality and the following elementary estimates

$$|\mu_5\mu_{5,t}|\le C\mu_4^2,\qquad \mu_{k+1}\le C\mu_k,$$

we get

$$\frac12\int_\Omega \mu_5^2|\psi_{tt}|^2\,\mathrm{d}x+\frac{\sigma(0)}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \mu_5^2|\nabla\psi_t|^2\,\mathrm{d}x\le C\int_\Omega \mu_4^2|\nabla\psi_t|^2\,\mathrm{d}x+\sigma(0)\int_\Gamma \mu_5^2\psi_{tt}\partial_\nu\psi_t\,\mathrm{d}S$$
$$+C\int_\Omega \mu_4^2|\psi_t|^2\,\mathrm{d}x+C\int_\omega \mu_3^2|v_t|^2\,\mathrm{d}x+\int_\Omega \mu_4^2|f_t|^2\,\mathrm{d}x. \tag{2.48}$$

Secondly, multiplying the second equation of (2.43) by $\mu_5^2\psi_{\Gamma,tt}$ and integrating over Γ, we find

$$\int_\Gamma \mu_5^2|\psi_{\Gamma,tt}|^2\,\mathrm{d}S+\frac{\delta(0)}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Gamma \mu_5^2|\nabla_\Gamma\psi_{\Gamma,t}|^2\,\mathrm{d}S=\delta(0)\int_\Gamma \mu_5\mu_{5,t}|\nabla_\Gamma\psi_{\Gamma,t}|^2\,\mathrm{d}S$$
$$-\sigma(0)\int_\Gamma \mu_5^2\psi_{\Gamma,tt}\partial_\nu\psi_t\,\mathrm{d}S-b'(0)\int_\Gamma \mu_5^2\psi_{\Gamma,t}\psi_{\Gamma,tt}\,\mathrm{d}S+\int_\Gamma \mu_5^2\psi_{\Gamma,tt}f_{\Gamma,t}\,\mathrm{d}S.$$

Using Young's inequality, we have

$$\frac12\int_\Gamma \mu_5^2|\psi_{\Gamma,tt}|^2\,\mathrm{d}S+\frac{\delta(0)}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Gamma \mu_5^2|\nabla_\Gamma\psi_{\Gamma,t}|^2\,\mathrm{d}S\le C\int_\Gamma \mu_4^2|\nabla_\Gamma\psi_{\Gamma,t}|^2\,\mathrm{d}S$$
$$-\sigma(0)\int_\Gamma \mu_5^2\psi_{\Gamma,tt}\partial_\nu\psi_t\,\mathrm{d}S+C\int_\Gamma \mu_4^2|\psi_{\Gamma,t}|^2\,\mathrm{d}S+C\int_\Gamma \mu_4^2|f_{\Gamma,t}|^2\,\mathrm{d}S. \tag{2.49}$$

By summing (2.48) and (2.49) and integrating over $(0,t)$, one has

$$\|\mu_5\Psi_{tt}\|^2_{L^2(0,T;\mathbb{L}^2)}+\sup_{0\le t\le T}\mu_5^2(t)\|\nabla\Psi_t(\cdot,t)\|^2_{\mathbb{L}^2}\le C\Big(\sup_{0\le t\le T}\mu_4^2(t)\|\Psi_t(\cdot,t)\|^2_{\mathbb{L}^2}$$
$$+\|\mu_4\nabla\Psi_t\|^2_{L^2(0,T;\mathbb{L}^2)}+\|\mu_3 v_t\|^2_{L^2(\omega_T)}+\|\mu F\|^2_{L^2(0,T;\mathbb{L}^2)}+\|\mu_4F_t\|^2_{L^2(0,T;\mathbb{L}^2)}$$
$$+\|\Psi_0\|^2_{\mathbb{H}^2}+\|\nabla\Psi_t(\cdot,0)\|^2_{\mathbb{L}^2}\Big). \tag{2.50}$$

On the other hand, we can easily get

$$\|\nabla\Psi_t(\cdot,0)\|^2_{\mathbb{L}^2}\le C\left(\|\Psi_0\|^2_{\mathbb{H}^3}+\|v(\cdot,0)\|^2_{H^1(\omega)}+\|F(\cdot,0)\|^2_{\mathbb{H}^1}\right). \tag{2.51}$$

From (2.50)-(2.51) and estimates (2.35), (2.20), we obtain

$$\|\mu_5\Psi_{tt}\|^2_{L^2(0,T;\mathbb{L}^2)}+\sup_{0\le t\le T}\mu_5^2(t)\|\nabla\Psi_t(\cdot,t)\|^2_{\mathbb{L}^2}$$
$$\le C\left(\|\mu F\|^2_{L^2(0,T;\mathbb{L}^2)}+\|\mu_4F_t\|^2_{L^2(0,T;\mathbb{L}^2)}+\|F(\cdot,0)\|^2_{\mathbb{H}^1}+\|\Psi_0\|^2_{\mathbb{H}^3}\right). \tag{2.52}$$

Multiplying the first equation of (2.43) by $-\mu_5^2\Delta\psi_t$ and integrating over Ω, we obtain

$$\frac12\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \mu_5^2|\nabla\psi_t|^2\,\mathrm{d}x+\sigma(0)\int_\Omega \mu_5^2|\Delta\psi_t|^2\,\mathrm{d}x=\int_\Omega \mu_5\mu_{5,t}|\nabla\psi_t|^2\,\mathrm{d}x$$
$$+\int_\Gamma \mu_5^2\psi_{tt}\partial_\nu\psi_t\,\mathrm{d}S+a'(0)\int_\Omega \mu_5^2\Delta\psi_t\psi_t\,\mathrm{d}x-\int_\omega \mu_5^2\Delta\psi_t v_t\,\mathrm{d}x-\int_\Omega \mu_5^2\Delta\psi_t f_t\,\mathrm{d}x.$$

Using Young's inequality and the continuity of the normal derivative from $H^2(\Omega)$ to $L^2(\Gamma)$, we find

$$\frac12\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \mu_5^2|\nabla\psi_t|^2\,\mathrm{d}x+\frac{\sigma(0)}{2}\int_\Omega \mu_5^2|\Delta\psi_t|^2\,\mathrm{d}x\le C\left(\int_\Omega \mu_4^2|\nabla\psi_t|^2\,\mathrm{d}x\right.$$
$$\left.+\int_\Gamma \mu_5^2|\psi_{\Gamma,tt}|^2\,\mathrm{d}S+\int_\Omega \mu_4^2|\psi_t|^2\,\mathrm{d}x+\int_\omega \mu_3^2|v_t|^2\,\mathrm{d}x+\int_\Omega \mu_4^2|f_t|^2\,\mathrm{d}x\right).$$

Integrating over $(0,t)$, we obtain

$$\begin{aligned}&\sup_{0\le t\le T}\int_\Omega \mu_5^2|\nabla\psi_t(\cdot,t)|^2\,\mathrm{d}x+\int_{\Omega_T}\mu_5^2|\Delta\psi_t|^2\,\mathrm{d}x\,\mathrm{d}t\\&\le C\left(\int_{\Omega_T}\mu_4^2|\nabla\psi_t|^2\,\mathrm{d}x\,\mathrm{d}t+\sup_{0\le t\le T}\int_\Omega\mu_4^2|\psi_t|^2\,\mathrm{d}x+\int_{\Gamma_T}\mu_5^2|\psi_{\Gamma,tt}|^2\,\mathrm{d}S\,\mathrm{d}t\right.\\&\left.+\int_{\omega_T}\mu_3^2|v_t|^2\,\mathrm{d}x\,\mathrm{d}t+\int_{\Omega_T}\mu_4^2|f_t|^2\,\mathrm{d}x\,\mathrm{d}t+\int_\Omega|\nabla\psi_t(\cdot,0)|^2\,\mathrm{d}x\right).\end{aligned}$$

Using estimates (2.35),(2.51) and (2.52), we get

$$\begin{aligned}&\sup_{0\le t\le T}\mu_5^2(t)\|\nabla\psi_t(\cdot,t)\|^2_{L^2(\Omega)}+\|\mu_5\Delta\psi_t\|^2_{L^2(\Omega_T)}\\&\le C\left(\|\mu F\|^2_{L^2(0,T;\mathbb{L}^2)}+\|\mu_4F_t\|^2_{L^2(0,T;\mathbb{L}^2)}+\|F(\cdot,0)\|^2_{\mathbb{H}^1}+\|\Psi_0\|^2_{\mathbb{H}^3}\right).\end{aligned}\tag{2.53}$$

Now, multiplying the second equation of (2.43) by $-\mu_5^2\Delta_\Gamma\psi_{\Gamma,t}$ and integrating over Γ, we get

$$\begin{aligned}&\frac12\frac{\mathrm{d}}{\mathrm{d}t}\int_\Gamma\mu_5^2|\nabla_\Gamma\psi_{\Gamma,t}|^2\,\mathrm{d}S+\delta(0)\int_\Gamma\mu_5^2|\Delta_\Gamma\psi_{\Gamma,t}|^2\,\mathrm{d}S=\int_\Gamma\mu_5\mu_{5,t}|\nabla_\Gamma\psi_{\Gamma,t}|^2\,\mathrm{d}S\\&+\sigma(0)\int_\Gamma\mu_5^2\Delta_\Gamma\psi_{\Gamma,t}\partial_\nu\psi_t\,\mathrm{d}S+b'(0)\int_\Gamma\mu_5^2\Delta_\Gamma\psi_{\Gamma,t}\psi_{\Gamma,t}\,\mathrm{d}S-\int_\Gamma\mu_5^2\Delta_\Gamma\psi_{\Gamma,t}f_{\Gamma,t}\,\mathrm{d}S.\end{aligned}$$

Similarly as above, one has

$$\begin{aligned}&\frac12\frac{\mathrm{d}}{\mathrm{d}t}\int_\Gamma\mu_5^2|\nabla_\Gamma\psi_{\Gamma,t}|^2\,\mathrm{d}S+\frac{\delta(0)}{2}\int_\Gamma\mu_5^2|\Delta_\Gamma\psi_{\Gamma,t}|^2\,\mathrm{d}S\le C\left(\int_\Gamma\mu_4^2|\nabla_\Gamma\psi_{\Gamma,t}|^2\,\mathrm{d}S\right.\\&\left.+\int_\Omega\mu_5^2|\psi_t|^2\,\mathrm{d}x+\int_\Omega\mu_5^2|\Delta\psi_t|^2\,\mathrm{d}x+\int_\Gamma\mu_5^2|\psi_{\Gamma,t}|^2\,\mathrm{d}S+\int_\Gamma\mu_4^2|f_{\Gamma,t}|^2\,\mathrm{d}S\right).\end{aligned}$$

Hence

$$\begin{aligned}&\sup_{0\le t\le T}\mu_5^2(t)\|\nabla_\Gamma\psi_{\Gamma,t}(\cdot,t)\|^2_{L^2(\Gamma)}+\|\mu_5\Delta_\Gamma\psi_{\Gamma,t}\|^2_{L^2(\Gamma_T)}\\&\le C\left(\|\mu F\|^2_{L^2(0,T;\mathbb{L}^2)}+\|\mu_4F_t\|^2_{L^2(0,T;\mathbb{L}^2)}+\|F(\cdot,0)\|^2_{\mathbb{H}^1}+\|\Psi_0\|^2_{\mathbb{H}^3}\right).\end{aligned}\tag{2.54}$$

From (2.53) and (2.54), we obtain

$$\begin{aligned}&\sup_{0\le t\le T}\mu_5^2(t)\|\nabla\Psi_t(\cdot,t)|^2_{\mathbb{L}^2}+\|\mu_5\Delta\Psi_t\|^2_{L^2(0,T;\mathbb{L}^2)}\\&\le C\left(\|\mu F\|^2_{L^2(0,T;\mathbb{L}^2)}+\|\mu_4F_t\|^2_{L^2(0,T;\mathbb{L}^2)}+\|F(\cdot,0)\|^2_{\mathbb{H}^1}+\|\Psi_0\|^2_{\mathbb{H}^3}\right).\end{aligned}\tag{2.55}$$

Multiplying the first equation of (1.5) by $-\mu_5^2\Delta\psi_t$ and integrating over Ω, we obtain

$$\begin{aligned}&\int_\Omega\mu_5^2|\nabla\psi_t|^2\,\mathrm{d}x+\frac{\sigma(0)}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega\mu_5^2|\Delta\psi|^2\,\mathrm{d}x=\int_\Gamma\mu_5^2\psi_t\partial_\nu\psi_t\,\mathrm{d}S\\&+\sigma(0)\int_\Omega\mu_5\mu_{5,t}|\Delta\psi|^2\,\mathrm{d}x+a'(0)\int_\Omega\mu_5^2\psi\Delta\psi_t\,\mathrm{d}x\\&-\int_\omega\mu_5^2\Delta\psi_tv\,\mathrm{d}x-\int_\Omega\mu_5^2\Delta\psi_tf\,\mathrm{d}x.\end{aligned}$$

Using Young's inequality and the trace theorem for the normal derivative, we have

$$\begin{aligned}&\int_\Omega\mu_5^2|\nabla\psi_t|^2\,\mathrm{d}x+\frac{\sigma(0)}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega\mu_5^2|\Delta\psi|^2\,\mathrm{d}x\le C\left(\int_\Gamma\mu_3^2|\psi_t|^2\,\mathrm{d}S+\int_\Omega\mu_3^2|\psi_t|^2\,\mathrm{d}x\right.\\&+\int_\Omega\mu_5^2|\Delta\psi_t|^2\,\mathrm{d}x+\int_\Omega\mu_3^2|\Delta\psi|^2\,\mathrm{d}x+\int_\Omega\mu_0^2|\psi|^2\,\mathrm{d}x+\int_\omega\mu_1^2|v|^2\,\mathrm{d}x\\&\left.+\int_\Omega\mu^2|f|^2\,\mathrm{d}x\right).\end{aligned}\tag{2.56}$$

Multiplying the second equation of (1.5) by $-\mu_5^2\Delta_\Gamma\psi_{\Gamma,t}$ and integrating over Γ, we obtain

$$\int_\Gamma\mu_5^2|\nabla_\Gamma\psi_{\Gamma,t}|^2\,\mathrm{d}S+\frac{\delta(0)}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Gamma\mu_5^2|\Delta_\Gamma\psi_\Gamma|^2\,\mathrm{d}S=\delta(0)\int_\Gamma\mu_5\mu_{5,t}|\Delta_\Gamma\psi_\Gamma|^2\,\mathrm{d}S$$

$$+\sigma(0)\int_\Gamma \mu_5^2\Delta_\Gamma\psi_{\Gamma,t}\partial_\nu\psi\ \mathrm{d}S + b'(0)\int_\Gamma \mu_5^2\psi_\Gamma\Delta_\Gamma\psi_{\Gamma,t}\ \mathrm{d}S - \int_\Gamma \mu_5^2\Delta_\Gamma\psi_{\Gamma,t}f_\Gamma\ \mathrm{d}S.$$

Hence,

$$\begin{aligned}\int_\Gamma \mu_5^2|\nabla_\Gamma\psi_{\Gamma,t}|^2\ \mathrm{d}S + \frac{\delta(0)}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Gamma \mu_5^2|\Delta_\Gamma\psi_\Gamma|^2\ \mathrm{d}S \le C\Big(&\int_\Gamma \mu_5^2|\Delta_\Gamma\psi_\Gamma|^2\ \mathrm{d}S\\ &+\int_\Gamma \mu_5^2|\Delta_\Gamma\psi_{\Gamma,t}|^2\ \mathrm{d}S + \int_\Omega \mu_0^2|\psi|^2\ \mathrm{d}x + \int_\Omega \mu_3^2|\Delta\psi|^2\ \mathrm{d}x\\ &+\int_\Gamma \mu_0^2|\psi_\Gamma|^2\ \mathrm{d}S + \int_\Gamma \mu^2|f_\Gamma|^2\ \mathrm{d}S\Big).\end{aligned} \tag{2.57}$$

Summing (2.56) and (2.57) and integrating over $(0,t)$, one has

$$\begin{aligned}&\|\mu_5\nabla\Psi_t\|^2_{L^2(0,T;\mathbb{L}^2)} + \sup_{0\le t\le T}\mu_5^2(t)\|\Delta\Psi(\cdot,t)\|^2_{\mathbb{L}^2} \le C\Big(\|\mu_3\Psi_t\|^2_{L^2(0,T;\mathbb{L}^2)}\\ &+\|\mu_3\Delta\Psi\|^2_{L^2(0,T;\mathbb{L}^2)} + \|\mu_5\Delta\Psi_t\|^2_{L^2(0,T;\mathbb{L}^2)} + \|\mu_0\Psi\|^2_{L^2(0,T;\mathbb{L}^2)} + \|\mu_1 v\|^2_{L^2(\omega_T)}\\ &+\|\mu F\|^2_{L^2(0,T;\mathbb{L}^2)} + \|\Psi_0\|^2_{\mathbb{H}^2}\Big).\end{aligned}$$

Using (2.34), (2.18) and (2.55), we obtain

$$\begin{aligned}&\|\mu_5\nabla\Psi_t\|^2_{L^2(0,T;\mathbb{L}^2)} + \sup_{0\le t\le T}\mu_5^2(t)\|\Delta\Psi(\cdot,t)\|^2_{\mathbb{L}^2}\\ &\qquad\le C\Big(\|\mu F\|^2_{L^2(0,T;\mathbb{L}^2)} + \|\mu_4 F_t\|^2_{L^2(0,T;\mathbb{L}^2)} + \|F(\cdot,0)\|^2_{\mathbb{H}^1} + \|\Psi_0\|^2_{\mathbb{H}^3}\Big).\end{aligned} \tag{2.58}$$

Finally, from (2.52), (2.55) and (2.58), we deduce (2.36). □

3. Controllability of the quasi-linear equation

We will establish the local controllability of equation (1.2) using Lyusternik-Graves' Theorem 1.3. Throughout this section, we assume that $d \le 3$.

3.1. **Study of the mapping Λ.** We introduce suitable spaces so that the mapping Λ verifies the conditions of Theorem 1.3.

$$\begin{aligned}\mathbb{X} := \Big\{(\Psi, v)\ :\ &\mu_0\Psi,\ \mu_4\Psi_t,\ \Delta\Psi,\ \mu_5\Delta\Psi_t \in L^2(0,T;\mathbb{L}^2),\ \ \mu_1 v,\ \mu_3 v_t \in L^2(\omega_T),\\ &\mu(\mathbf{L}_1\Psi - \mathbb{1}_\omega v, \mathbf{L}_2\Psi),\ \mu_4(\mathbf{L}_1\Psi - \mathbb{1}_\omega v, \mathbf{L}_2\Psi)_t \in L^2(0,T;\mathbb{L}^2),\ \Psi(\cdot,0)\in\mathbb{H}^3,\\ &v(\cdot,0)\in H^1(\omega),\ \sup_{0\le t\le T}\mu_5^2(t)\|\Psi_t(\cdot,t)\|^2_{\mathbb{H}^1} < \infty,\ \sup_{0\le t\le T}\mu_5^2(t)\|\Psi(\cdot,t)\|^2_{\mathbb{H}^2} < \infty\Big\},\\ \mathbb{G} := \{F\ :\ &\mu F,\ \mu_4 F_t \in L^2(0,T;\mathbb{L}^2),\quad F(\cdot,0)\in\mathbb{H}^1\},\\ \mathbb{Y} := \mathbb{G}\times\mathbb{H}^3,&\end{aligned}$$

where $\mathbf{L}_1$ and $\mathbf{L}_2$ are defined in (2.4). These spaces are naturally equipped with the following norms

$$\begin{aligned}\|(\Psi,v)\|_{\mathbb{X}} := \Big(&\|\mu_0\Psi\|^2_{L^2(0,T;\mathbb{L}^2)} + \|\mu_4\Psi_t\|^2_{L^2(0,T;\mathbb{L}^2)} + \|\mu_1 v\|^2_{L^2(\omega_T)} + \|\mu_3 v_t\|^2_{L^2(\omega_T)}\\ &+\|\Psi(\cdot,0)\|^2_{\mathbb{H}^3} + \|v(\cdot,0)\|^2_{H^1(\omega)} + \|\mu(\mathbf{L}_1\Psi - \mathbb{1}_\omega v, \mathbf{L}_2\Psi)\|^2_{L^2(0,T;\mathbb{L}^2)}\\ &+\|\mu_4(\mathbf{L}_1\Psi - \mathbb{1}_\omega v, \mathbf{L}_2\Psi)_t\|^2_{L^2(0,T;\mathbb{L}^2)} + \|\Delta\Psi\|^2_{L^2(0,T;\mathbb{L}^2)} + \|\mu_5\Delta\Psi_t\|^2_{L^2(0,T;\mathbb{L}^2)}\\ &+\sup_{0\le t\le T}\mu_5^2(t)\|\Psi_t(\cdot,t)\|^2_{\mathbb{H}^1} + \sup_{0\le t\le T}\mu_5^2(t)\|\Psi(\cdot,t)\|^2_{\mathbb{H}^2}\Big)^{1/2},\\ \|F\|_{\mathbb{G}} := \Big(&\|\mu F\|^2_{L^2(0,T;\mathbb{L}^2)} + \|\mu_4 F_t\|^2_{L^2(0,T;\mathbb{L}^2)} + \|F(\cdot,0)\|^2_{\mathbb{H}^1}\Big)^{1/2}.\end{aligned}$$

One can easily prove that these are Hilbert spaces.

Remark 5. For any $(\Psi,v)\in\mathbb{X}$ and $F\in\mathbb{G}$, we have:

$$\Psi\in H^1(0,T;\mathbb{L}^2)\cap L^2(0,T;\mathbb{H}^2), v\in H^1(0,T;L^2(\omega)) \quad\text{and}\quad F\in H^1(0,T;\mathbb{L}^2).$$

Consequently,

$$\Psi\in C([0,T];\mathbb{H}^1), v\in C([0,T];L^2(\omega)) \quad\text{and}\quad F\in C([0,T];\mathbb{L}^2).$$

In particular, $\Psi(\cdot,0), v(\cdot,0)$ and $F(\cdot,0)$ are well defined.

Remark 6. It is simple to note that there is a constant $C>0$ such that

$$\int_0^T \mu_5^2(t)\|\Psi_t(\cdot,t)\|_{\mathbb{H}^2}^2\ \mathrm{d}t \le C\|(\Psi,v)\|_{\mathbb{X}}^2 \quad \forall (\Psi,v)\in\mathbb{X}.$$

Consider the mapping $\Lambda:\mathbb{X}\longrightarrow\mathbb{Y}$ given by

$$\Lambda(\Psi,v) := ((\Lambda_1(\Psi,v),\Lambda_2(\Psi,v)),\Psi(\cdot,0)),$$

where

$$\begin{aligned}\Lambda_1(\Psi,v) &:= \psi_t - \nabla\cdot(\sigma(\psi)\nabla\psi) + a(\psi) - \mathbb{1}_\omega v,\\ \Lambda_2(\Psi,v) &:= \psi_{\Gamma,t} - \nabla_\Gamma\cdot(\delta(\psi_\Gamma)\nabla_\Gamma\psi_\Gamma) + \sigma(\psi_\Gamma)\partial_\nu\psi + b(\psi_\Gamma).\end{aligned}$$

The following lemma shows that the mapping Λ is well defined.

Lemma 3.1. *Let $r>0$. There is a positive constant $C:=C(r)>0$ such that the mapping $\Lambda:\mathbb{X}\to\mathbb{Y}$ defined above verifies*

$$\|\Lambda(\Psi,v)\|_{\mathbb{Y}}^2 \le C\left(\|(\Psi,v)\|_{\mathbb{X}}^2 + \|(\Psi,v)\|_{\mathbb{X}}^4 + \|(\Psi,v)\|_{\mathbb{X}}^6\right), \qquad (\Psi,v)\in\overline{B}_{\mathbb{X}}((0,0),r). \tag{3.1}$$

In particular, $\Lambda:\mathbb{X}\to\mathbb{Y}$ is well defined.

Proof. Let $r>0$ and $(\Psi,v)\in\overline{B}_{\mathbb{X}}((0,0),r)$. From the definition of the norm of $\mathbb{X}$ and the continuity of Sobolev embedding $\mathbb{H}^2\hookrightarrow\mathbb{L}^\infty$, we can easily obtain that

$$\|\Psi\|_{\mathbb{L}^\infty} \le C_0\|(\Psi,v)\|_{\mathbb{X}} \le C_0 r \quad \text{for all}\,(\Psi,v)\in\overline{B}_{\mathbb{X}}((0,0),r). \tag{3.2}$$

Throughout this proof, we will use the fact that the i^{th} derivative of σ and δ, the j^{th} derivative of a and b are Lipschitz-continuous on the interval $J_0:=[-C_0r,C_0r]$ for $i=0,1,2$ and $j=0,1$. One has

$$\begin{aligned}\|\Lambda(\Psi,v)\|_{\mathbb{Y}}^2 &= \|\mu(\Lambda_1(\Psi,v),\Lambda_2(\Psi,v))\|_{L^2(0,T;\mathbb{L}^2)}^2 + \|\mu_4(\Lambda_1(\Psi,v),\Lambda_2(\Psi,v))_t\|_{L^2(0,T;\mathbb{L}^2)}^2\\ &\quad + \|(\Lambda_1(\Psi,v),\Lambda_2(\Psi,v))(\cdot,0)\|_{\mathbb{H}^1}^2 + \|\Psi(\cdot,0)\|_{\mathbb{H}^3}^2.\end{aligned} \tag{3.3}$$

Then, it suffices to show that each term of (3.3) is bounded as in estimate (3.1). Firstly, we claim that

$$\|\mu(\Lambda_1(\Psi,v),\Lambda_2(\Psi,v))\|_{L^2(0,T;\mathbb{L}^2)}^2 \le C\left(\|(\Psi,v)\|_{\mathbb{X}}^2 + \|(\Psi,v)\|_{\mathbb{X}}^4\right). \tag{3.4}$$

We have

$$\begin{aligned}&\|\mu(\Lambda_1(\Psi,v),\Lambda_2(\Psi,v))\|_{L^2(0,T;\mathbb{L}^2)}^2 = \|\mu\Lambda_1(\Psi,v)\|_{L^2(\Omega_T)}^2 + \|\mu\Lambda_2(\Psi,v)\|_{L^2(\Gamma_T)}^2\\ &\le C\left(\int_{\Omega_T}\mu^2|\mathbf{L}_1\Psi - \mathbb{1}_\omega v|^2\ \mathrm{d}x\ \mathrm{d}t + \int_{\Omega_T}\mu^2|\nabla\cdot((\sigma(\psi)-\sigma(0))\nabla\psi)\,|^2\ \mathrm{d}x\ \mathrm{d}t\right.\\ &+\int_{\Omega_T}\mu^2|a(\psi)-a'(0)\psi|^2\ \mathrm{d}x\ \mathrm{d}t + \int_{\Gamma_T}\mu^2|\mathbf{L}_2\Psi|^2\ \mathrm{d}S\ \mathrm{d}t\\ &+\int_{\Gamma_T}\mu^2|\nabla_\Gamma\cdot((\delta(\psi_\Gamma)-\delta(0))\nabla_\Gamma\psi_\Gamma)\,|^2\ \mathrm{d}S\ \mathrm{d}t\\ &\left.+\int_{\Gamma_T}\mu^2|b(\psi_\Gamma)-b'(0)\psi_\Gamma|^2\ \mathrm{d}S\ \mathrm{d}t + \int_{\Gamma_T}\mu^2|\psi_\Gamma|^2|\partial_\nu\psi|^2\ \mathrm{d}S\ \mathrm{d}t\right) := C\left(\sum_{j=1}^7 I_j\right).\end{aligned} \tag{3.5}$$

According to the definition of the norm of $\mathbb{X}$, note that

$$I_1 + I_4 = \|\mu(\mathbf{L}_1\Psi - \mathbb{1}_\omega v, \mathbf{L}_2\Psi)\|_{L^2(0,T;\mathbb{L}^2)}^2 \le \|(\Psi,v)\|_{\mathbb{X}}^2. \tag{3.6}$$

Using $a(0)=b(0)=0$ and Taylor's Inequality, we obtain

$$\begin{aligned}I_3+I_6 &\le \frac{M^2}{4}\int_{\Omega_T}\mu^2|\psi|^4\ \mathrm{d}x\ \mathrm{d}t + \frac{M_\Gamma^2}{4}\int_{\Gamma_T}\mu^2|\psi_\Gamma|^4\ \mathrm{d}S\ \mathrm{d}t\\ &\le C\left(\int_{\Omega_T}\mu_5^4|\psi|^4\ \mathrm{d}x\ \mathrm{d}t + \int_{\Gamma_T}\mu_5^4|\psi_\Gamma|^4\ \mathrm{d}S\ \mathrm{d}t\right)\\ &\le C\sup_{0\le t\le T}\mu_5^4(t)\|\Psi(\cdot,t)\|_{L^2(0,T;\mathbb{L}^2)}^4\end{aligned}$$

$$\leq C\|(\Psi,v)\|_{\mathbb{X}}^4, \tag{3.7}$$

where $M=\sup_{x\in J_0}|a''(x)|$ and $M_\Gamma=\sup_{x\in J_0}|b''(x)|$. Let us now analyze I_2. Since σ is Lipschitz-continuous on J_0 and σ' is bounded on J_0, one has

$$\begin{aligned}
I_2 &= \int_{\Omega_T}\mu^2|\sigma'(\psi)|^2|\nabla\psi|^4\,\mathrm{d}x\,\mathrm{d}t+\int_{\Omega_T}\mu^2|(\sigma(\psi)-\sigma(0))\Delta\psi|^2\,\mathrm{d}x\,\mathrm{d}t\\
&\leq C\left(\int_{\Omega_T}\mu^2|\nabla\psi|^4\,\mathrm{d}x\,\mathrm{d}t+\int_{\Omega_T}\mu^2|\psi|^2|\Delta\psi|^2\,\mathrm{d}x\,\mathrm{d}t\right)\\
&\leq C\left(\int_0^T\mu^2(t)\int_\Omega|\nabla\psi(\cdot,t)|^4\,\mathrm{d}x\,\mathrm{d}t+\int_0^T\mu^2(t)\|\psi(\cdot,t)\|_{L^\infty(\Omega)}^2\int_\Omega|\Delta\psi(\cdot,t)|^2\,\mathrm{d}x\,\mathrm{d}t\right).
\end{aligned}$$

The fact that $d\leq 3$ implies that $H^2(\Omega)\hookrightarrow W^{1,4}(\Omega)$ and $H^2(\Omega)\hookrightarrow L^\infty(\Omega)$ with continuous embeddings. Hence

$$\begin{aligned}
I_2 &\leq C\int_0^T\mu^2(t)\|\psi(\cdot,t)\|_{H^2(\Omega)}^4\,\mathrm{d}t\\
&\leq C\sup_{0\leq t\leq T}\mu_5^4(t)\|\psi(\cdot,t)\|_{H^2(\Omega)}^4\\
&\leq C\|(\Psi,v)\|_{\mathbb{X}}^4.
\end{aligned}\tag{3.8}$$

Similarly, we obtain

$$I_5 \leq C\|(\Psi,v)\|_{\mathbb{X}}^4. \tag{3.9}$$

For the last term I_7, thanks to the continuity of the Sobolev embedding $H^2(\Gamma)\hookrightarrow L^\infty(\Gamma)$ and the continuity of the normal derivative from $H^2(\Omega)$ to $L^2(\Gamma)$, we have

$$\begin{aligned}
I_7 &\leq C\int_0^T\mu^2(t)\|\psi_\Gamma(\cdot,t)\|_{L^\infty(\Gamma)}^2\|\partial_\nu\psi(\cdot,t)\|_{L^2(\Gamma)}^2\,\mathrm{d}t\\
&\leq C\int_0^T\mu^2(t)\|\psi_\Gamma(\cdot,t)\|_{H^2(\Gamma)}^2\|\psi(\cdot,t)\|_{H^2(\Omega)}^2\,\mathrm{d}t\\
&\leq C\int_0^T\mu^2(t)\|\Psi(\cdot,t)\|_{\mathbb{H}^2}^4\,\mathrm{d}t\\
&\leq C\sup_{0\leq t\leq T}\mu_5^4(t)\|\Psi(\cdot,t)\|_{\mathbb{H}^2}^4\\
&\leq C\|(\Psi,v)\|_{\mathbb{X}}^4.
\end{aligned}\tag{3.10}$$

From estimates (3.5)-(3.10), we obtain (3.4). Secondly, we prove that

$$\|\mu_4(\Lambda_1(\Psi,v),\Lambda_2(\Psi,v))_t\|_{L^2(0,T;\mathbb{L}^2)}^2\leq C\left(\|(\Psi,v)\|_{\mathbb{X}}^2+\|(\Psi,v)\|_{\mathbb{X}}^4+\|(\Psi,v)\|_{\mathbb{X}}^6\right). \tag{3.11}$$

Using the fact that a' and b' are bounded on J_0, we find

$$\begin{aligned}
&\|\mu_4(\Lambda_1(\Psi,v),\Lambda_2(\Psi,v))_t\|_{L^2(0,T;\mathbb{L}^2)}^2=\|\mu_4(\Lambda_1(\Psi,v))_t\|_{L^2(\Omega_T)}^2+\|\mu_4(\Lambda_2(\Psi,v))_t\|_{L^2(\Gamma_T)}^2\\
&\leq C\left(\int_{\Omega_T}\mu_4^2\left|(\mathbf{L}_1\Psi-\mathbb{1}_\omega v)_t\right|^2\,\mathrm{d}x\,\mathrm{d}t+\int_{\Omega_T}\mu_4^2\left|(\nabla\cdot((\sigma(\psi)-\sigma(0))\nabla\psi))_t\right|^2\,\mathrm{d}x\,\mathrm{d}t\right.\\
&+\int_{\Omega_T}\mu_4^2|\psi_t|^2\,\mathrm{d}x\,\mathrm{d}t+\int_{\Gamma_T}\mu_4^2\left|(\mathbf{L}_2\Psi)_t\right|^2\,\mathrm{d}S\,\mathrm{d}t+\int_{\Gamma_T}\mu_4^2\left|(\nabla_\Gamma\cdot((\delta(\psi_\Gamma)-\delta(0))\nabla_\Gamma\psi_\Gamma))_t\right|^2\,\mathrm{d}S\,\mathrm{d}t\\
&\left.+\int_{\Gamma_T}\mu_4^2|\psi_{\Gamma,t}|^2\,\mathrm{d}S\,\mathrm{d}t+\int_{\Gamma_T}\mu_4^2\left|((\sigma(\psi_\Gamma)-\sigma(0))\,\partial_\nu\psi)_t\right|^2\,\mathrm{d}S\,\mathrm{d}t\right)\\
&:=C\left(\sum_{j=1}^7J_j\right).
\end{aligned}\tag{3.12}$$

It is clear that

$$J_1+J_4=\|\mu_4(\mathbf{L}_1\Psi-\mathbb{1}_\omega v,\mathbf{L}_1\Psi)_t\|_{L^2(0,T;\mathbb{L}^2)}^2\leq\|(\Psi,v)\|_{\mathbb{X}}^2, \tag{3.13}$$

$$J_3+J_6=\|\mu_4\Psi_t\|_{L^2(0,T;\mathbb{L}^2)}^2\leq\|(\Psi,v)\|_{\mathbb{X}}^2. \tag{3.14}$$

Let us show the estimate for J_2. One has

$$\begin{aligned} J_2 &\leq C\left(\int_{\Omega_T}\mu_4^2|\sigma''(\psi)\psi_t|^2|\nabla\psi|^4\,\mathrm{d}x\,\mathrm{d}t+\int_{\Omega_T}\mu_4^2|\sigma'(\psi)\nabla\psi_t\cdot\nabla\psi|^2\,\mathrm{d}x\,\mathrm{d}t\right.\\ &\quad\left.+\int_{\Omega_T}\mu_4^2|\sigma'(\psi)\psi_t\Delta\psi|^2\,\mathrm{d}x\,\mathrm{d}t+\int_{\Omega_T}\mu_4^2|(\sigma(\psi)-\sigma(0))\Delta\psi_t|^2\,\mathrm{d}x\,\mathrm{d}t\right)\\ &\leq C\left(\int_{\Omega_T}\mu_4^2|\psi_t|^2|\nabla\psi|^4\,\mathrm{d}x\,\mathrm{d}t+\int_{\Omega_T}\mu_4^2|\nabla\psi_t\cdot\nabla\psi|^2\,\mathrm{d}x\,\mathrm{d}t\right.\\ &\quad\left.+\int_{\Omega_T}\mu_4^2|\psi_t|^2|\Delta\psi|^2\,\mathrm{d}x\,\mathrm{d}t+\int_{\Omega_T}\mu_4^2|\psi|^2|\Delta\psi_t|^2\,\mathrm{d}x\,\mathrm{d}t\right):=C\left(\sum_{j=1}^{4}J_{2j}\right). \end{aligned}$$

Using Sobolev embeddings as above and Remark 6, we get

$$\begin{aligned} J_{21} &\leq \int_0^T\mu_4^2(t)\|\psi_t(\cdot,t)\|^2_{L^\infty(\Omega)}\left(\int_\Omega|\nabla\psi(\cdot,t)|^4\,\mathrm{d}x\right)\,\mathrm{d}t\\ &\leq C\int_0^T\mu_4^2(t)\|\psi_t(\cdot,t)\|^2_{H^2(\Omega)}\|\psi(\cdot,t)\|^4_{H^2(\Omega)}\,\mathrm{d}t\\ &\leq C\left(\sup_{0\leq t\leq T}\mu_5^4(t)\|\psi(\cdot,t)\|^4_{H^2(\Omega)}\right)\int_0^T\mu_5^2(t)\|\psi_t(\cdot,t)\|^2_{H^2(\Omega)}\,\mathrm{d}t\\ &\leq C\|(\Psi,v)\|^6_{\mathbb{X}}. \end{aligned}$$

Applying Cauchy-Schwarz inequality, we obtain

$$\begin{aligned} J_{22} &\leq \int_0^T\mu_4^2(t)\left(\int_\Omega|\nabla\psi_t(\cdot,t)|^2|\nabla\psi(\cdot,t)|^2\,\mathrm{d}x\right)\,\mathrm{d}t\\ &\leq \int_0^T\mu_4^2(t)\left(\int_\Omega|\nabla\psi_t(\cdot,t)|^4\,\mathrm{d}x\right)^{1/2}\left(\int_\Omega|\nabla\psi(\cdot,t)|^4\,\mathrm{d}x\right)^{1/2}\,\mathrm{d}t\\ &\leq \int_0^T\mu_4^2(t)\|\psi_t(\cdot,t)\|^2_{H^2(\Omega)}\|\psi(\cdot,t)\|^2_{H^2(\Omega)}\,\mathrm{d}t\\ &\leq C\left(\sup_{0\leq t\leq T}\mu_5^2(t)\|\psi(\cdot,t)\|^2_{H^2(\Omega)}\right)\int_0^T\mu_5^2(t)\|\psi_t(\cdot,t)\|^2_{H^2(\Omega)}\,\mathrm{d}t\\ &\leq C\|(\Psi,v)\|^4_{\mathbb{X}}. \end{aligned}$$

Using the fact that $H^2(\Omega)\hookrightarrow L^\infty(\Omega)$ is continuous and Remark 6, one has

$$\begin{aligned} J_{23} &\leq \int_0^T\mu_4^2(t)\|\psi_t(\cdot,t)\|^2_{L^\infty(\Omega)}\left(\int_\Omega|\Delta\psi(\cdot,t)|^2\,\mathrm{d}x\right)\,\mathrm{d}t\\ &\leq C\int_0^T\mu_4^2(t)\|\psi_t(\cdot,t)\|^2_{H^2(\Omega)}\|\Delta\psi(\cdot,t)\|^2_{L^2(\Omega)}\,\mathrm{d}t\\ &\leq C\left(\sup_{0\leq t\leq T}\mu_5^2(t)\|\Delta\psi(\cdot,t)\|^2_{L^2(\Omega)}\right)\int_0^T\mu_5^2(t)\|\psi_t(\cdot,t)\|^2_{H^2(\Omega)}\,\mathrm{d}t\\ &\leq C\|(\Psi,v)\|^4_{\mathbb{X}}. \end{aligned}$$

We also have

$$\begin{aligned} J_{24} &\leq \int_0^T\mu_4^2(t)\|\psi(\cdot,t)\|^2_{L^\infty(\Omega)}\left(\int_\Omega|\Delta\psi_t(\cdot,t)|^2\,\mathrm{d}x\right)\,\mathrm{d}t\\ &\leq C\int_0^T\mu_4^2(t)\|\psi(\cdot,t)\|^2_{H^2(\Omega)}\|\Delta\psi_t(\cdot,t)\|^2_{L^2(\Omega)}\,\mathrm{d}t\\ &\leq C\left(\sup_{0\leq t\leq T}\mu_5^2(t)\|\psi(\cdot,t)\|^2_{H^2(\Omega)}\right)\int_0^T\mu_5^2(t)\|\Delta\psi_t(\cdot,t)\|^2_{L^2(\Omega)}\,\mathrm{d}t\\ &\leq C\|(\Psi,v)\|^4_{\mathbb{X}}. \end{aligned}$$

Then

$$J_2 \leq C\left(\|(\Psi,v)\|_{\mathbb{X}}^6 + \|(\Psi,v)\|_{\mathbb{X}}^4\right). \tag{3.15}$$

Analogously, we obtain

$$J_5 \leq C\left(\|(\Psi,v)\|_{\mathbb{X}}^6 + \|(\Psi,v)\|_{\mathbb{X}}^4\right). \tag{3.16}$$

It remains to analyze the last term J_7. Using σ' is bounded on J_0, σ is Lipscitz-continuous on J_0, $H^2(\Gamma) \hookrightarrow L^\infty(\Gamma)$ is continuous, the normal derivative is continuous from $H^2(\Omega)$ to $L^2(\Gamma)$ and Remark 6, we find

$$\begin{aligned}
J_7 &\leq C\left(\int_{\Gamma_T} \mu_4^2|\psi_{\Gamma,t}|^2|\partial_\nu\psi|^2 \,\mathrm{d}S\,\mathrm{d}t + \int_{\Gamma_T} \mu_4^2|\psi_\Gamma|^2|\partial_\nu\psi_t|^2 \,\mathrm{d}S\,\mathrm{d}t\right)\\
&\leq C\left(\int_0^T \mu_4^2(t)\|\psi_{\Gamma,t}(\cdot,t)\|_{L^\infty(\Gamma)}^2 \int_\Gamma |\partial_\nu\psi(\cdot,t)|^2 \,\mathrm{d}S\,\mathrm{d}t\right.\\
&\quad \left. + \int_0^T \mu_4^2(t)\|\psi_\Gamma(\cdot,t)\|_{L^\infty(\Gamma)}^2 \int_\Gamma |\partial_\nu\psi_t(\cdot,t)|^2 \,\mathrm{d}S\,\mathrm{d}t\right)\\
&\leq C\left(\int_0^T \mu_4^2(t)\|\psi_{\Gamma,t}(\cdot,t)\|_{H^2(\Gamma)}^2\|\psi(\cdot,t)\|_{H^2(\Omega)}^2 \,\mathrm{d}t\right.\\
&\quad \left. + \int_0^T \mu_4^2(t)\|\psi_\Gamma(\cdot,t)\|_{H^2(\Gamma)}^2\|\psi_t(\cdot,t)\|_{H^2(\Omega)}^2 \,\mathrm{d}t\right)\\
&\leq C\left(\sup_{0\leq t\leq T}\mu_5^2(t)\|\psi(\cdot,t)\|_{H^2(\Omega)}^2 \int_0^T \mu_5^2(t)\|\psi_{\Gamma,t}(\cdot,t)\|_{H^2(\Gamma)}^2 \,\mathrm{d}t\right.\\
&\quad \left. + \sup_{0\leq t\leq T}\mu_5^2(t)\|\psi_\Gamma(\cdot,t)\|_{H^2(\Gamma)}^2 \int_0^T \mu_5^2(t)\|\psi_t(\cdot,t)\|_{H^2(\Omega)}^2 \,\mathrm{d}t\right)\\
&\leq C\|(\Psi,v)\|_{\mathbb{X}}^4.
\end{aligned} \tag{3.17}$$

From estimates (3.12)-(3.17), we obtain (3.11). Lastly, we show that

$$\|(\Lambda_1(\Psi,v)(\cdot,0),\Lambda_2(\Psi,v)(\cdot,0))\|_{\mathbb{H}^1} \leq C\left(\|(\Psi,v)\|_{\mathbb{X}}^2 + \|(\Psi,v)\|_{\mathbb{X}}^4 + \|(\Psi,v)\|_{\mathbb{X}}^6\right). \tag{3.18}$$

We have,

$$\begin{aligned}
&\|(\Lambda_1(\Psi,v)(\cdot,0),\Lambda_2(\Psi,v)(\cdot,0))\|_{\mathbb{H}^1}^2 = \|\Lambda_1(\Psi,v)(\cdot,0)\|_{H^1(\Omega)}^2 + \|\Lambda_2(\Psi,v)(\cdot,0)\|_{H^1(\Gamma)}^2\\
&\leq C\left(\|(\psi_t - \sigma(0)\Delta\psi - \mathbb{1}_\omega v)(\cdot,0)\|_{H^1(\Omega)}^2 + \|a(\psi)(\cdot,0)\|_{H^1(\Omega)}^2\right.\\
&+\|(\nabla\cdot((\sigma(\psi)-\sigma(0))\nabla\psi))(\cdot,0)\|_{H^1(\Omega)}^2 + \|(\psi_{\Gamma,t} - \delta(0)\Delta_\Gamma\psi_\Gamma)(\cdot,0)\|_{H^1(\Gamma)}^2\\
&+\|(\sigma(\psi_\Gamma)\partial_\nu\psi)(\cdot,0)\|_{H^1(\Gamma)}^2 + \|b(\psi_\Gamma)(\cdot,0)\|_{H^1(\Gamma)}^2\\
&\left.+\|(\nabla_\Gamma\cdot((\delta(\psi_\Gamma)-\delta(0))\nabla_\Gamma\psi_\Gamma))(\cdot,0)\|_{H^1(\Gamma)}^2\right) := C\left(\sum_{j=1}^7 K_j\right).
\end{aligned} \tag{3.19}$$

We obtain

$$\begin{aligned}
K_1 + K_4 &\leq C\left(\|\Psi_t(\cdot,0)\|_{\mathbb{H}^1}^2 + \|\Delta\Psi(\cdot,0)\|_{\mathbb{H}^1}^2 + \|v(\cdot,0)\|_{H^1(\omega)}^2\right)\\
&\leq C\left(\sup_{0\leq t\leq T}\mu_5^2(t)\|\Psi_t(\cdot,t)\|_{\mathbb{H}^1}^2 + \|\Psi(\cdot,0)\|_{\mathbb{H}^3}^2 + \|v(\cdot,0)\|_{H^1(\omega)}^2\right)\\
&\leq C\|(\Psi,v)\|_{\mathbb{X}}^2.
\end{aligned} \tag{3.20}$$

Since a and b are Lipschitz-continuous on J_0 with $a(0) = b(0) = 0$, then

$$\begin{aligned}
K_2 + K_6 &\leq C\|\Psi(\cdot,0)\|_{\mathbb{H}^1}^2\\
&\leq C\|(\Psi,v)\|_{\mathbb{X}}^2.
\end{aligned} \tag{3.21}$$

Let us treat the term K_3. Using the fact that σ, σ' and σ'' are bounded on J_0, Sobolev embeddings as above and $\nabla\psi(\cdot,0)\in H^2(\Omega)^d$, one has

$$\begin{aligned}
K_3 &\leq C\left(\|(\sigma'(\psi(\cdot,0))|\nabla\psi(\cdot,0)|^2\|^2_{H^1(\Omega)}+\|(\sigma(\psi(\cdot,0))-\sigma(0))\Delta\psi(\cdot,0)\|^2_{H^1(\Omega)}\right)\\
&\leq C\left(\||\nabla\psi(\cdot,0)|^2\|^2_{H^1(\Omega)}+\||\nabla\psi(\cdot,0)|^2\nabla\psi(\cdot,0)\|^2_{L^2(\Omega)}+\|\Delta\psi(\cdot,0)\|^2_{H^1(\Omega)}\|\Delta\psi(\cdot,0)\nabla\psi(\cdot,0)\|^2_{L^2(\Omega)}\right)\\
&\leq C\left(\|\psi(\cdot,0)\|^4_{H^3(\Omega)}+\|\psi(\cdot,0)\|^6_{H^3(\Omega)}+\|\psi(\cdot,0)\|^2_{H^3(\Omega)}\right)\\
&\leq C\left(\|(\Psi,v)\|^2_{\mathbb{X}}+\|(\Psi,v)\|^4_{\mathbb{X}}+\|(\Psi,v)\|^6_{\mathbb{X}}\right). \qquad (3.22)
\end{aligned}$$

A similar argument shows that

$$K_7\leq C\left(\|(\Psi,v)\|^2_{\mathbb{X}}+\|(\Psi,v)\|^4_{\mathbb{X}}+\|(\Psi,v)\|^6_{\mathbb{X}}\right). \qquad (3.23)$$

It remains to analyze the term K_5. Since σ and σ' are bounded on J_0, the normal derivative $\partial_\nu : H^s(\Omega)\longrightarrow H^{s-3/2}(\Gamma)$ is continuous for $s>3/2$ and Sobolev embeddings, we obtain

$$\begin{aligned}
K_5 &\leq C\left(\|\partial_\nu\psi(\cdot,0)\|^2_{H^1(\Gamma)}+\|\partial_\nu\psi(\cdot,0)\nabla_\Gamma\psi_\Gamma(\cdot,0)\|^2_{L^2(\Gamma)}\right)\\
&\leq C\left(\|\psi(\cdot,0)\|^2_{H^3(\Omega)}+\|\nabla_\Gamma\psi_\Gamma(\cdot,0)\|^2_{L^\infty(\Gamma)}\|\partial_\nu\psi(\cdot,0)\|^2_{L^2(\Gamma)}\right)\\
&\leq C\left(\|\psi(\cdot,0)\|^2_{H^3(\Omega)}+\|\psi_\Gamma(\cdot,0)\|^2_{H^3(\Gamma)}\|\psi(\cdot,0)\|^2_{H^2(\Omega)}\right)\\
&\leq C\left(\|(\Psi,v)\|^2_{\mathbb{X}}+\|(\Psi,v)\|^4_{\mathbb{X}}\right). \qquad (3.24)
\end{aligned}$$

From estimates (3.19)-(3.24), we obtain (3.18). □

Lemma 3.2. *The mapping $\Lambda:\mathbb{X}\to\mathbb{Y}$ is continuously differentiable.*

Proof. To simplify, isolating the linear part of Λ, let us put

$$\begin{aligned}
\Lambda_3(\Psi,v) &:= (\psi_t-\mathbb{1}_\omega v,\psi_{\Gamma,t},\Psi(\cdot,0))\\
\Lambda_4(\Psi,v) &:= (\nabla\cdot(\sigma(\psi)\nabla\psi)-a(\psi),\nabla_\Gamma\cdot(\delta(\psi_\Gamma)\nabla_\Gamma\psi_\Gamma)-\sigma(\psi_\Gamma)\partial_\nu\psi-b(\psi_\Gamma),0)\\
&:= (\Lambda_{41}(\Psi,v),\Lambda_{42}(\Psi,v),0).
\end{aligned}$$

As $\Lambda=\Lambda_3-\Lambda_4$ and Λ_3 is linear continuous, then it is continuously differentiable and $D\Lambda_3(\Psi,v)=\Lambda_3$. Let us show that Λ_4 is continuously differentiable. We will start by proving that Λ_4 is Gateaux-differentiable at any $(\Psi,v)\in\mathbb{X}$ and determine the G-derivative $\Lambda'(\Psi,v)$. Let $\varepsilon\in(-1,1)\setminus\{0\}$ and $(\Psi,v),(\Phi,u)\in\mathbb{X}$. One has

$$\begin{aligned}
&\frac{\Lambda_{41}((\Psi,v)+\varepsilon(\Phi,u))-\Lambda_{41}(\Psi,v)}{\varepsilon}=\frac{\sigma(\psi+\varepsilon\phi)-\sigma(\psi)}{\varepsilon}\Delta\psi\\
&+\left[\frac{\sigma'(\psi+\varepsilon\phi)-\sigma'(\psi)}{\varepsilon}\right]|\nabla\psi|^2+\sigma(\psi+\varepsilon\phi)\Delta\phi+2\sigma'(\psi+\varepsilon\phi)\nabla\psi\cdot\nabla\phi\\
&-\frac{a(\psi+\varepsilon\phi)-a(\psi)}{\varepsilon}+\varepsilon\sigma'(\psi+\varepsilon\phi)|\nabla\phi|^2:=\sum_{j=1}^{6}L^\varepsilon_j
\end{aligned}$$

and

$$\begin{aligned}
&\frac{\Lambda_{42}((\Psi,v)+\varepsilon(\Phi,u))-\Lambda_{42}(\Psi,v)}{\varepsilon}=\frac{\delta(\psi_\Gamma+\varepsilon\phi_\Gamma)-\delta(\psi_\Gamma)}{\varepsilon}\Delta_\Gamma\psi_\Gamma\\
&+\left[\frac{\delta'(\psi_\Gamma+\varepsilon\phi_\Gamma)-\delta'(\psi_\Gamma)}{\varepsilon}\right]|\nabla_\Gamma\psi_\Gamma|^2+\delta(\psi_\Gamma+\varepsilon\phi_\Gamma)\Delta_\Gamma\phi_\Gamma\\
&+2\delta'(\psi_\Gamma+\varepsilon\phi_\Gamma)\nabla_\Gamma\psi_\Gamma\cdot\nabla_\Gamma\phi_\Gamma-\frac{\sigma(\psi_\Gamma+\varepsilon\phi_\Gamma)-\sigma(\psi_\Gamma)}{\varepsilon}\partial_\nu\psi-\sigma(\psi_\Gamma+\varepsilon\phi_\Gamma)\partial_\nu\phi\\
&-\frac{b(\psi_\Gamma+\varepsilon\phi_\Gamma)-b(\psi_\Gamma)}{\varepsilon}+\varepsilon\delta'(\psi_\Gamma+\varepsilon\phi_\Gamma)|\nabla_\Gamma\phi_\Gamma|^2:=\sum_{j=1}^{8}M^\varepsilon_j.
\end{aligned}$$

We consider the linear mapping $\mathbf{D}\Lambda_4(\Psi,v):\mathbb{X}\to\mathbb{Y}$ defined by

$$\begin{aligned}
&\mathbf{D}\Lambda_4(\Psi,v)(\Phi,u)=\big(\sigma'(\psi)\phi\Delta\psi+\sigma''(\psi)\phi|\nabla\psi|^2+\sigma(\psi)\Delta\phi+2\sigma'(\psi)\nabla\psi\cdot\nabla\phi\\
&-a'(\psi)\phi,\delta'(\psi_\Gamma)\phi_\Gamma\Delta_\Gamma\psi_\Gamma+\delta''(\psi_\Gamma)\phi_\Gamma|\nabla_\Gamma\psi_\Gamma|^2+\delta(\psi_\Gamma)\Delta_\Gamma\phi_\Gamma
\end{aligned}$$

$$+2\delta'(\psi_\Gamma)\nabla_\Gamma\psi_\Gamma\cdot\nabla_\Gamma\phi_\Gamma-\sigma'(\psi_\Gamma)\phi_\Gamma\partial_\nu\psi-\sigma(\psi_\Gamma)\partial_\nu\phi-b'(\psi_\Gamma)\phi_\Gamma,0)$$
$$:=\left(\sum_{j=1}^{6}L_j,\sum_{j=1}^{8}M_j,0\right)\quad\text{with } L_6=M_8=0.$$

Using the dominated convergence theorem, we can show that

$$\frac{\Lambda_4((\Psi,v)+\varepsilon(\Phi,u))-\Lambda_4(\Psi,v)}{\varepsilon}\longrightarrow\mathbf{D}\Lambda_4(\Psi,u)(\Phi,v)\text{ in }\mathbb{Y}\text{ as }\varepsilon\longrightarrow 0.$$

Indeed, we have

$$\left\|\frac{\Lambda_4((\Psi,v)+\varepsilon(\Phi,u))-\Lambda_4(\Psi,v)}{\varepsilon}-\mathbf{D}\Lambda_4(\Psi,v)(\Phi,u)\right\|_{\mathbb{Y}}^2$$
$$\leq C\left(\sum_{j=1}^{6}\left[\|\mu(L_j^\varepsilon-L_j)\|^2_{L^2(\Omega_T)}+\|\mu_4(L_j^\varepsilon-L_j)_t\|^2_{L^2(\Omega_T)}+\|(L_j^\varepsilon-L_j)(\cdot,0)\|^2_{H^1(\Omega)}\right]\right.$$
$$\left.+\sum_{j=1}^{8}\left[\|\mu(M_j^\varepsilon-M_j)\|^2_{L^2(\Gamma_T)}+\|\mu_4(M_j^\varepsilon-M_j)_t\|^2_{L^2(\Gamma_T)}+\|(M_j^\varepsilon-M_j)(\cdot,0)\|^2_{H^1(\Gamma)}\right]\right).$$

Let us start with the term $\|\mu(L_1^\varepsilon-L_1)\|^2_{L^2(\Omega_T)}$.

$$\|\mu(L_1^\varepsilon-L_1)\|^2_{L^2(\Omega_T)}=\int_{\Omega_T}\mu^2\left|\frac{\sigma(\psi+\varepsilon\phi)-\sigma(\psi)-\varepsilon\phi\sigma'(\psi)}{\varepsilon}\right|^2|\Delta\psi|^2\,\mathrm{d}x\,\mathrm{d}t.$$

Since σ is of class C^2, by Taylor-Lagrange inequality, we have

$$\mu^2\left|\frac{\sigma(\psi+\varepsilon\phi)-\sigma(\psi)-\sigma'(\psi)\varepsilon\phi}{\varepsilon}\right|^2|\Delta\psi|^2\leq\frac{M^2}{4}\mu^2|\phi|^4|\Delta\psi|^2,$$

where

$$M:=\sup_{r\in K_0}|\sigma''(r)|,\quad K_0:=[-C_0(\|(\Psi,v)\|_{\mathbb{X}}+\|(\Phi,u)\|_{\mathbb{X}}),C_0(\|(\Psi,v)\|_{\mathbb{X}}+\|(\Phi,u)\|_{\mathbb{X}})]$$

and the constant C_0 is defined in (3.2). Moreover

$$\begin{aligned}\int_{\Omega_T}\mu^2|\phi|^4|\Delta\psi|^2\,\mathrm{d}x\,\mathrm{d}t&\leq C\sup_{0\leq t\leq T}\mu_5^4(t)\|\phi(\cdot,t)\|^4_{H^2(\Omega)}\int_{\Omega_T}|\Delta\psi|^2\,\mathrm{d}x\,\mathrm{d}t\\&\leq C\|(\Phi,u)\|^4_{\mathbb{X}}\|(\Psi,v)\|^2_{\mathbb{X}}.\end{aligned}$$

Then, using the dominated convergence theorem, we obtain $\|\mu(L_1^\varepsilon-L_1)\|^2_{L^2(\Omega_T)}\longrightarrow 0$ as $\varepsilon\longrightarrow 0$. For the term $\|\mu_4(L_1^\varepsilon-L_1)_t\|^2_{L^2(\Omega_T)}$, one has

$$\begin{aligned}\|\mu_4(L_1^\varepsilon-L_1)_t\|^2_{L^2(\Omega_T)}&\leq C\left(\int_{\Omega_T}\mu_4^2\left|\frac{\sigma'(\psi+\varepsilon\phi)-\sigma'(\psi)-\varepsilon\phi\sigma''(\psi)}{\varepsilon}\right|^2|\psi_t|^2|\Delta\psi|^2\,\mathrm{d}x\,\mathrm{d}t\right.\\&+\int_{\Omega_T}\mu_4^2|\sigma'(\psi+\varepsilon\phi)-\sigma'(\psi)|^2|\phi_t|^2|\Delta\psi|^2\,\mathrm{d}x\,\mathrm{d}t\\&\left.+\int_{\Omega_T}\mu_4^2\left|\frac{\sigma(\psi+\varepsilon\phi)-\sigma(\psi)-\varepsilon\phi\sigma'(\psi)}{\varepsilon}\right|^2|\Delta\psi_t|^2\,\mathrm{d}x\,\mathrm{d}t\right).\end{aligned}$$

Using the dominated convergence theorem as above, we obtain $\|\mu_4(L_1^\varepsilon-L_1)_t\|^2_{L^2(\Omega_T)}\longrightarrow 0$ as $\varepsilon\longrightarrow 0$. We have

$$\|(L_1^\varepsilon-L_1)(\cdot,0)\|^2_{H^1(\Omega)}=\|(L_1^\varepsilon-L_1)(\cdot,0)\|^2_{L^2(\Omega)}+\|\nabla(L_1^\varepsilon-L_1)(\cdot,0)\|^2_{L^2(\Omega)}$$
$$\leq C\left(\int_{\Omega_T}\left|\frac{\sigma(\psi(\cdot,0)+\varepsilon\phi(\cdot,0))-\sigma(\psi(\cdot,0))-\varepsilon\phi(\cdot,0)\sigma'(\psi(\cdot,0))}{\varepsilon}\right|^2|\Delta\psi(\cdot,0)|^2\,\mathrm{d}x\,\mathrm{d}t\right.$$
$$\left.+\int_{\Omega_T}\left|\frac{\sigma'(\psi(\cdot,0)+\varepsilon\phi(\cdot,0))-\sigma'(\psi(\cdot,0))-\varepsilon\phi(\cdot,0)\sigma''(\psi(\cdot,0))}{\varepsilon}\right|^2|\nabla\psi(\cdot,0)|^2\,\mathrm{d}x\,\mathrm{d}t\right.$$

$$+\int_{\Omega_T}|\sigma'(\psi(\cdot,0)+\varepsilon\phi(\cdot,0))-\sigma'(\psi(\cdot,0))|^2|\nabla\psi(\cdot,0)|^2\,\mathrm{d}x\,\mathrm{d}t$$
$$+\int_{\Omega_T}\left|\frac{\sigma(\psi(\cdot,0)+\varepsilon\phi(\cdot,0))-\sigma(\psi(\cdot,0))-\varepsilon\phi(\cdot,0)\sigma'(\psi(\cdot,0))}{\varepsilon}\right|^2|\nabla\Delta\psi(\cdot,0)|^2\,\mathrm{d}x\,\mathrm{d}t\Bigg).$$

In the same way, one has $\|(L_1^\varepsilon-L_1)(\cdot,0)\|^2_{H^1(\Omega)}\longrightarrow 0$ as $\varepsilon\longrightarrow 0$. By the same arguments, we treat the other terms, thus concluding that Λ_4 is Gateaux-differentiable at (Ψ,v) and $\Lambda_4'(\Psi,v)=\mathbf{D}\Lambda_4(\Psi,v)$. To conclude, it is sufficient to show $\Lambda_4':\mathbb{X}\to\mathcal{L}(\mathbb{X},\mathbb{Y})$ is continuous. Let $((\Psi^n,v^n)),(\Psi,v),(\Phi,u)\in\mathbb{X}$ such that $((\Psi^n,v^n))$ converges to (Ψ,v) in $\mathbb{X}$ and we will prove that

$$\|(\Lambda_4'(\Psi^n,v^n)(\Phi,u)-\Lambda_4'(\Psi,v)(\Phi,u))\|_{\mathbb{Y}}\leq\varepsilon_n\|(\Phi,u)\|_{\mathbb{X}}\tag{3.25}$$

for some (ε_n) converging to 0. One has

$$\Lambda_4'(\Psi^n,v^n)(\Phi,u)-\Lambda_4'(\Psi,v)(\Phi,u)=(\Lambda_5^n,\Lambda_6^n,0),$$

where

$$\begin{aligned}\Lambda_5^n &= (\sigma''(\psi^n)-\sigma''(\psi))|\nabla\psi^n|^2\phi+(\sigma''(\psi)\phi\nabla\psi^n+\sigma'(\psi)\nabla\phi)\cdot(\nabla\psi^n-\nabla\psi)\\ &\quad+(\sigma'(\psi^n)-\sigma'(\psi))(\Delta\psi^n\phi+2\nabla\psi^n\cdot\nabla\phi)+(\sigma(\psi^n)-\sigma(\psi))\Delta\phi\\ &\quad-(a'(\psi^n)-a'(\psi))\phi:=\sum_{j=1}^5\Lambda_{5j}^n\end{aligned}$$

and

$$\begin{aligned}\Lambda_6^n &= (\delta''(\psi_\Gamma^n)-\delta''(\psi_\Gamma))|\nabla_\Gamma\psi_\Gamma^n|^2\phi_\Gamma+(\delta''(\psi_\Gamma)\phi_\Gamma\nabla_\Gamma\psi_\Gamma^n+\delta'(\psi_\Gamma)\nabla_\Gamma\phi_\Gamma)\cdot(\nabla_\Gamma\psi_\Gamma^n-\nabla_\Gamma\psi_\Gamma)\\ &\quad+(\delta'(\psi_\Gamma^n)-\delta'(\psi_\Gamma))(\Delta_\Gamma\psi_\Gamma^n\phi_\Gamma+2\nabla_\Gamma\psi_\Gamma^n\cdot\nabla_\Gamma\phi_\Gamma)+(\delta(\psi_\Gamma^n)-\delta(\psi_\Gamma))\Delta_\Gamma\phi_\Gamma\\ &\quad-(b'(\psi_\Gamma^n)-b'(\psi_\Gamma))\phi_\Gamma-(\sigma'(\psi_\Gamma^n)-\sigma'(\psi_\Gamma))\phi_\Gamma\partial_\nu\psi^n-\sigma'(\psi_\Gamma)\phi_\Gamma\partial_\nu(\psi^n-\psi)\\ &\quad-(\sigma(\psi_\Gamma^n)-\sigma(\psi_\Gamma))\partial_\nu\phi:=\sum_{j=1}^8\Lambda_{6j}^n.\end{aligned}$$

Then

$$\begin{aligned}\|\Lambda_4'(\Psi^n,u^n)(\Phi,v)-\Lambda_4'(\Psi,u)(\Phi,v)\|_{\mathbb{Y}}^2 &= \|\mu(\Lambda_5^n,\Lambda_6^n)\|^2_{L^2(0,T;\mathbb{L}^2)}+\|\mu_4(\Lambda_5^n,\Lambda_6^n)_t\|^2_{L^2(0,T;\mathbb{L}^2)}\\ &\quad+\|(\Lambda_5^n,\Lambda_6^n)(\cdot,0)\|^2_{\mathbb{H}^1}.\end{aligned}$$

Firstly, let us show that $\|\mu(\Lambda_5^n,\Lambda_6^n)\|^2_{L^2(0,T;\mathbb{L}^2)}\leq C\|(\Psi^n,v^n)-(\Psi,v)\|^2_{\mathbb{X}}\|(\Phi,u)\|^2_{\mathbb{X}}$. We can easily see that

$$\|\mu(\Lambda_5^n,\Lambda_6^n)\|^2_{L^2(0,T;\mathbb{L}^2)}\leq C\left(\sum_{j=1}^5\|\mu\Lambda_{5j}^n\|^2_{L^2(\Omega_T)}+\sum_{j=1}^8\|\mu\Lambda_{6j}^n\|^2_{L^2(\Gamma_T)}\right).$$

Let us analyze the first term. Using the fact that the embeddings $H^2(\Omega)\hookrightarrow L^\infty(\Omega)$ and $H^2(\Omega)\hookrightarrow W^{1,4}(\Omega)$ are continuous and σ'' is locally Lipschitz-continuous, we obtain

$$\begin{aligned}\|\mu\Lambda_{51}^n\|^2_{L^2(\Omega_T)} &= \int_{\Omega_T}\mu^2|\sigma''(\psi^n)-\sigma''(\psi)|^2|\nabla\psi^n|^4|\phi|^2\,\mathrm{d}x\,\mathrm{d}t\\ &\leq C\int_{\Omega_T}\mu^2|\psi^n-\psi|^2|\nabla\psi^n|^4|\phi|^2\,\mathrm{d}x\,\mathrm{d}t\\ &\leq C\int_0^T\mu^2(t)\|(\psi^n-\psi)(\cdot,t)\|^2_{H^2(\Omega)}\|\phi(\cdot,t)\|^2_{H^2(\Omega)}\int_\Omega|\nabla\psi^n(x,t)|^4\,\mathrm{d}x\,\mathrm{d}t\\ &\leq C\sup_{0\leq t\leq T}\mu_5^2(t)\|(\psi^n-\psi)(\cdot,t)\|^2_{H^2(\Omega)}\sup_{0\leq t\leq T}\mu_5^2(t)\|\phi(\cdot,t)\|^2_{H^2(\Omega)}\int_0^T\int_\Omega|\nabla\psi^n(x,t)|^4\,\mathrm{d}x\,\mathrm{d}t\\ &\leq C\sup_{0\leq t\leq T}\mu_5^2(t)\|(\psi^n-\psi)(\cdot,t)\|^2_{H^2(\Omega)}\sup_{0\leq t\leq T}\mu_5^2(t)\|\phi(\cdot,t)\|^2_{H^2(\Omega)}\int_0^T\|\psi^n(\cdot,t)\|^4_{H^2(\Omega)}\,\mathrm{d}t\\ &\leq C\sup_{0\leq t\leq T}\mu_5^2(t)\|(\psi^n-\psi)(\cdot,t)\|^2_{H^2(\Omega)}\sup_{0\leq t\leq T}\mu_5^2(t)\|\phi(\cdot,t)\|^2_{H^2(\Omega)}\sup_{0\leq t\leq T}\mu_5^4(t)\|\psi^n(\cdot,t)\|^4_{H^2(\Omega)}\\ &\leq C\|(\Psi^n,v^n)-(\Psi,v)\|^2_{\mathbb{X}}\|(\Psi^n,v^n)\|^4_{\mathbb{X}}\|(\Phi,u)\|^2_{\mathbb{X}}\end{aligned}$$

$$\leq C\|(\Psi^n, v^n) - (\Psi, v)\|_{\mathbb{X}}^2 \|(\Phi, u)\|_{\mathbb{X}}^2.$$

Similar arguments lead to the same conclusions

$$\|\mu \Lambda_{5j}^n\|_{L^2(\Omega_T)}^2 \leq C\|(\Psi^n, v^n) - (\Psi, v)\|_{\mathbb{X}}^2 \|(\Phi, u)\|_{\mathbb{X}}^2, \quad j = 2, \cdots, 5$$

and

$$\|\mu \Lambda_{6j}^n\|_{L^2(\Gamma_T)}^2 \leq C\|(\Psi^n, v^n) - (\Psi, v)\|_{\mathbb{X}}^2 \|(\Phi, u)\|_{\mathbb{X}}^2, \quad j = 1, \cdots, 8.$$

All this implies that

$$\|\mu(\Lambda_5^n, \Lambda_6^n)\|_{L^2(0,T;\mathbb{L}^2)}^2 \leq C\|(\Psi^n, v^n) - (\Psi, v)\|_{\mathbb{X}}^2 \|(\Phi, u)\|_{\mathbb{X}}^2. \tag{3.26}$$

Secondly, let us show that $\|\mu_4(\Lambda_5^n, \Lambda_6^n)_t\|_{L^2(0,T;\mathbb{L}^2)}^2 \leq \varepsilon_1^n \|(\Phi, u)\|_{\mathbb{X}}^2$ for some (ε_1^n) converging to 0. We can easily see that

$$\|\mu_4(\Lambda_5^n, \Lambda_6^n)_t\|_{L^2(0,T;\mathbb{L}^2)}^2 \leq C\left(\sum_{j=1}^5 \|\mu_4(\Lambda_{5j}^n)_t\|_{L^2(\Omega_T)}^2 + \sum_{j=1}^8 \|\mu_4(\Lambda_{6j}^n)_t\|_{L^2(\Gamma_T)}^2\right).$$

Let us treat the first term. Since σ'' is locally Lipschitz-continuous and the continuity of embeddings $H^2(\Omega) \hookrightarrow L^\infty(\Omega)$ and $H^2(\Omega) \hookrightarrow W^{1,4}(\Omega)$. Then

$$\begin{aligned}
\|\mu_4(\Lambda_{51}^n)_t\|_{L^2(\Omega_T)}^2 \quad &\leq \quad C\left(\int_{\Omega_T} \mu_4^2 |\sigma'''(\psi^n) - \sigma'''(\psi)|^2 |\psi_t|^2 |\nabla\psi^n|^4 |\phi|^2 \, \mathrm{d}x \, \mathrm{d}t\right. \\
&\quad + \int_{\Omega_T} \mu_4^2 |\sigma'''(\psi^n)|^2 |\psi_t^n - \psi_t|^2 |\nabla\psi^n|^4 |\phi|^2 \, \mathrm{d}x \, \mathrm{d}t \\
&\quad + \int_{\Omega_T} \mu_4^2 |(\sigma''(\psi^n) - \sigma''(\psi))|^2 |\nabla\psi^n|^2 |\nabla\psi_t^n|^2 |\phi|^2 \, \mathrm{d}x \, \mathrm{d}t \\
&\quad \left. + \int_{\Omega_T} \mu_4^2 |(\sigma''(\psi^n) - \sigma''(\psi))|^2 |\nabla\psi^n|^4 |\phi_t|^2 \, \mathrm{d}x \, \mathrm{d}t\right) \\
&\leq \quad \varepsilon_{11}^n \|(\Phi, u)\|_{\mathbb{X}}^2,
\end{aligned}$$

where $\varepsilon_{11}^n = C\left(\sup_{(x,t)\in\Omega_T} |\sigma'''(\psi^n(x,t)) - \sigma'''(\psi(x,t))|^2 + \|(\Psi^n, v^n) - (\Psi, v)\|_{\mathbb{X}}^2\right)$.
Since $\|\psi^n - \psi\|_{L^\infty(\Omega_T)} \leq C\|(\Psi^n, v^n) - (\Psi, v)\|_{\mathbb{X}}$ and σ''' is locally uniformly continuous, thus (ε_{11}^n) converges to 0. Similar arguments lead to the same conclusions:

$$\|\mu_4(\Lambda_{5j}^n)_t\|_{L^2(\Omega_T)}^2 \leq \varepsilon_{j1}^n \|(\Phi, u)\|_{\mathbb{X}}^2, \quad j = 2, \cdots, 5$$

and

$$\|\mu_4(\Lambda_{6j}^n)_t\|_{L^2(\Omega_T)}^2 \leq \varepsilon_{\Gamma,j1}^n \|(\Phi, u)\|_{\mathbb{X}}^2, \quad j = 1, \cdots, 8,$$

where (ε_{j1}^n) and $(\varepsilon_{\Gamma,j1}^n)$ converge to 0. All this implies that

$$\|\mu_4(\Lambda_5^n, \Lambda_6^n)_t\|_{L^2(0,T;\mathbb{L}^2)}^2 \leq C\left(\sum_{j=1}^5 \varepsilon_{j1}^n + \sum_{j=1}^8 \varepsilon_{\Gamma,j1}^n\right) \|(\Phi, u)\|_{\mathbb{X}}^2. \tag{3.27}$$

Lastly, let us prove that $\|(\Lambda_5^n, \Lambda_6^n)(\cdot, 0)\|_{\mathbb{H}^1}^2 \leq \varepsilon_2^n \|(\Phi, u)\|_{\mathbb{X}}^2$ for some (ε_2^n) converging to 0. We can easily see that

$$\|(\Lambda_5^n, \Lambda_6^n)(\cdot, 0)\|_{\mathbb{H}^1}^2 \leq C\left(\sum_{j=1}^5 \|\Lambda_{5j}^n(\cdot, 0)\|_{H^1(\Omega)}^2 + \sum_{j=1}^8 \|\Lambda_{6j}^n(\cdot, 0)\|_{H^1(\Gamma)}^2\right).$$

Let us analyze the first term. Using the continuity of the Sobolev embedding $H^2(\Omega) \hookrightarrow L^\infty(\Omega)$, σ'' is locally Lipschitz-continuous, σ''' is locally uniformly continuous and bounded and $\|\psi^n - \psi\|_{L^\infty(\Omega_T)} \leq C\|(\Psi^n, v^n) - (\Psi, v)\|_{\mathbb{X}}$, one has

$$\begin{aligned}
\|\Lambda_{51}^n(\cdot, 0)\|_{H^1(\Omega)}^2 &= \int_\Omega |\Lambda_{51}^n(x, 0)|^2 \, \mathrm{d}x + \int_\Omega |\nabla \Lambda_{51}^n(x, 0)|^2 \, \mathrm{d}x \\
&\leq C\left(\int_\Omega |\sigma''(\psi^n(x, 0)) - \sigma''(\psi(x, 0))|^2 |\nabla\psi^n(x, 0)|^4 |\phi(x, 0)|^2 \, \mathrm{d}x\right.
\end{aligned}$$

$$
\begin{aligned}
&+\int_\Omega |\sigma'''(\psi^n(x,0))-\sigma'''(\psi(x,0))|^2|\nabla\psi^n(x,0)|^6|\phi(x,0)|^2\,\mathrm{d}x\\
&+\int_\Omega |\sigma'''(\psi(x,0))|^2|\nabla(\psi^n-\psi)(x,0)|^2|\nabla\psi^n(x,0)|^4|\phi(x,0)|^2\,\mathrm{d}x\\
&+\int_\Omega |\sigma''(\psi^n(x,0))-\sigma''(\psi(x,0))|^2|\nabla(|\nabla\psi^n(x,0)|^2)|^2|\phi(x,0)|^2\,\mathrm{d}x\\
&+\int_\Omega |\sigma''(\psi^n(x,0))-\sigma''(\psi(x,0))|^2|\nabla\psi^n(x,0)|^4|\nabla\phi(x,0)|^2\,\mathrm{d}x\Big)\\
&\leq \varepsilon_{12}^n\|(\Phi,u)\|_{\mathbb{X}}^2,
\end{aligned}
$$

where $\varepsilon_{12}^n = C\left(\sup_{x\in\Omega}|\sigma'''(\psi^n(x,0))-\sigma'''(\psi(x,0))|^2+\|(\Psi^n,v^n)-(\Psi,v)\|_{\mathbb{X}}^2\right)$, which converges to 0. Similar arguments lead to the same conclusions:

$$\|\Lambda_{5j}^n(\cdot,0)\|_{H^1(\Omega)}^2\leq\varepsilon_{j2}^n\|(\Phi,u)\|_{\mathbb{X}}^2,\quad j=2,\cdots,5$$

and

$$\|\Lambda_{6j}^n(\cdot,0)\|_{H^1(\Gamma)}^2\leq\varepsilon_{\Gamma,j2}^n\|(\Phi,u)\|_{\mathbb{X}}^2,\quad j=1,\cdots,8,$$

where (ε_{j2}^n) and $(\varepsilon_{\Gamma,j2}^n)$ converge to 0. All this implies that

$$\|(\Lambda_5^n,\Lambda_6^n)(\cdot,0)\|_{\mathbb{H}^1}^2\leq C\left(\sum_{j=1}^5\varepsilon_{j2}^n+\sum_{j=1}^8\varepsilon_{\Gamma,j2}^n\right)\|(\Phi,u)\|_{\mathbb{X}}^2. \tag{3.28}$$

Finally, from (3.26), (3.27) and (3.28), we obtain (3.25). □

3.2. **Proof of Theorem 1.2.** Due to Lemmas 3.1 and 3.2, the mapping Λ is well-defined and continuously differentiable. Moreover

$$\Lambda'(0,0)(\Psi,v)=(\mathbf{L}_1\Psi-\mathbb{1}_\omega v,\mathbf{L}_2\Psi,\Psi(\cdot,0)).$$

Null controllability results of inhomogenuous linearized equation (1.5) obtained in Propositions 3 and 4, show that $\Lambda'(0,0):\mathbb{X}\longrightarrow\mathbb{Y}$ is onto. We conclude that Lyusternik-Graves' Theorem 1.3 can be applied to the operator Λ, in particular for any sufficiently small $(F,\Psi_0)\in\mathbb{Y}$, there exists $(\Psi,v)\in\mathbb{X}$ such that $\Lambda(\Psi,v)=(F,\Psi_0)$. Consequently, $\Psi=(\psi,\psi_\Gamma)$ is the solution of the system (1.2) associated to the control v, and the exponential growth of the weight μ_0 as $t\to T^-$ and $\mu_0\Psi\in L^2(0,T,\mathbb{L}^2)$, ensures that

$$(\psi(\cdot,T),\psi_\Gamma(\cdot,T))=(0,0)\quad\text{on }\Omega\times\Gamma,$$

which leads to the proof of Theorem (1.2). □

4. Conclusion and final comments

In this work, we have studied the local null controllability of a quasi-linear reaction-diffusion equation of volume-surface type subject to dynamic boundary conditions. The strategy we have adopted relies on the null controllability of the inhomogeneous linearized equation and a local inversion theorem. Although we have considered reaction and diffusion coefficients that depend only on the state variable, our strategy can be adapted to more general diffusion and reaction coefficients depending on both the state and its gradients.

To the best of the authors knowledge, the null controllability of quasi-linear parabolic equations with general boundary conditions has not been considered before in the literature. Most of the previous works have dealt only with Dirichlet and Neumann boundary conditions, as far as we know.

From a numerical perspective, it would be of much interest to investigate the approximation of null controls numerically based on the theoretical results we have obtained. This will eventually be done in a forthcoming paper.

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Lab-SIV, Faculty of Sciences-Agadir, Ibnou Zohr University, B.P. 8106, Agadir, Morocco
Email address: fouad.et-tahri@edu.uiz.ac.ma

LMDP, UMMISCO (IRD-UPMC), Cadi Ayyad University, Faculty of Sciences Semlalia, B.P. 2390, Marrakesh, Morocco
Email address: s.chorfi@uca.ac.ma; maniar@uca.ma

Lab-SIV, Multidisciplinary Faculty-Ouarzazate, Ibnou Zohr University, BP 638, Ouarzazate 45000, Morocco
Email address: dsboutaayamou@gmail.com