Projections in enlargements of filtrations under Jacod's absolute continuity hypothesis for marked point processes *

Pavel V. Gapeev, Monique Jeanblanc, Dongli Wu

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Abstract

We consider the initial and progressive enlargements of a filtration (called the reference filtration) generated by a marked point process with a strictly positive random time. We assume Jacod's absolute continuity hypothesis, that is, the existence of a nonnegative conditional density for the random time with respect to the reference filtration. Then, starting with the predictable integral representation of a martingale in the initial enlargement of the reference filtration, we derive explicit expressions for the coefficients which appear in the predictable integral representations for the optional projections of the martingale on the progressively enlarged filtration and on the reference filtration. We also provide similar results for the optional projection of a martingale (in the progressively enlarged filtration) on the reference filtration. This paper also extends the results obtained in our previous paper [14] in the Brownian motion setting to the case of absolute continuity hypothesis.

1 Introduction

In this paper, we consider the initial (resp. progressive) enlargement of a filtration $\mathbb{F}$ (called hereafter the reference filtration) with a strictly positive random variable τ (called hereafter the random time), denoted by $\mathbb{F}^{(\tau)}$ (resp. $\mathbb{G}$). We study the case in which $\mathbb{F}$ is generated by a marked point process (MPP for short. The reason why we are working with such processes is that a marked point process in $\mathbb{F}$ remains a marked point process. in any larger filtration, in particular is a semi-martingale in any enlargement of $\mathbb{F}$ with possibly a different compensator and under some conditions admits the weak predictable representation property (WPRP for short).

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We assume that the law of τ has no atoms and that Jacod's absolute continuity hypothesis introduced in [3] and [15] holds (see Section 3 below for details). We recall that, under some hypotheses on τ (see Hypothesis 3.4 below), the weak predictable representation property holds in the filtration $\mathbb{F}^{(\tau)}$ and (adding a pure jump martingale) in the filtration $\mathbb{G}$. We study the relationship between the representation of martingales in the initially (resp. progressively) enlarged filtration and the various optional projections. The paper is an extension of our previous paper [14] to the case of models driven by marked point processes. We refer the reader to the monograph [1] for results on enlargements of filtrations. Our results can be useful to compare the optimal strategies of investors having different information flows (see, e.g. [3] and [15]). Note that, without any difficulties, one can study models driven by independent Brownian motions and marked point processes, but this would simply lead to longer formulae. Detailed studies of the weak predictable representation property when the process also has a continuous martingale part were provided in [9] (see also [10]-[11]).

The paper is organised as follows. In Section 2, we recall standard results of stochastic analysis that we use in the paper. In Section 3, we give some basic definitions and results related to the initial and progressive enlargements of a filtration $\mathbb{F}$ generated by a marked point process with a random time τ, denoted by $\mathbb{F}^{(\tau)}$ and $\mathbb{G}$, respectively, under Jacod's hypothesis. In Section 4, we recall that the weak predictable representation property holds in the reference filtration with respect to the compensated random measure and prove that the weak predictable representation property holds with respect to an explicit martingale and a compensated random measure in the enlargements of filtration involved. In Section 5, we consider the optional projections of an $\mathbb{F}^{(\tau)}$-martingale on the filtrations $\mathbb{G}$ and $\mathbb{F}$. We derive explicit expressions for the coefficients in the integral representations of these optional projections in terms of the original $\mathbb{F}^{(\tau)}$-martingale and the components in its representation as a stochastic integral and give analogous results in the case of $\mathbb{F}$-optional projections of a $\mathbb{G}$-martingale. In Section 6, we consider the optional projections of a strictly positive $\mathbb{F}^{(\tau)}$-martingale on $\mathbb{G}$ and $\mathbb{F}$ and the $\mathbb{F}$-optional projection of a strictly positive $\mathbb{G}$-martingale. We describe the set of equivalent martingale measures in the associated extension of the exponential model driven by a marked point process and enhanced with the random time τ. In particular, we show that the set of equivalent martingale measures in the model with the progressively enlarged filtration $\mathbb{G}$ is essentially larger than the one obtained by means of the optional projections on $\mathbb{G}$ of the Radon-Nikodym densities in the model with the initially enlarged filtration $\mathbb{F}^{(\tau)}$. Some technical proofs are presented in Appendix.

2 Preliminary definitions and results

We work on a standard complete probability space $(\Omega, \mathcal{G}, \mathbb{P})$, on which there exists a sequence $(T_n, Z_n)_{n\geq 1}$, where $(T_n)_{n\geq 1}$ is a strictly increasing sequence of finite strictly positive random variables with no accumulation points, and $(Z_n)_{n\geq 1}$ a sequence of real-valued random variables. We shall say that the sequence $(T_n, Z_n)_{n\geq 1}$ is a *marked point process* (MPP) on $\mathbb{R}$ with no accumulation points (see Def. 1.1.6 in [25], Section 1.2, pages 3-4 in [22], Chapter 4 in [6], and Chapter VIII in [7]).

We denote by $\mathcal{B}(\mathbb{R}_+)$ the σ-algebra of the Borel sets on $\mathbb{R}_+ \equiv [0, \infty)$ and introduce the associated random measure μ on $\mathcal{G} \times \mathcal{B}(\mathbb{R}_+) \times \mathcal{B}(\mathbb{R})$ which is defined, for any set $A \in \mathcal{B}(\mathbb{R})$ and

any $t \geq 0$, by

$$\mu(\omega; (0,t], A) = \sum_{n \geq 1} \mathbb{1}_{\{T_n(\omega) \leq t\}} \, \mathbb{1}_{\{Z_n(\omega) \in A\}} \, ,$$

which is called the *jump measure* of the marked point process. We denote by $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$ the natural filtration of the MPP given by

$$\mathcal{F}_t = \sigma\big(\mu((a,b], A), 0 \leq a < b \leq t, A \in \mathcal{B}(\mathbb{R})\big), \, \forall t \geq 0 \, ,$$

which is a right-continuous filtration (see Proposition 3.39 in [18]). We call $\mathbb{F}$ hereafter the *reference filtration*, and note that all T_n, for $n \geq 1$, are $\mathbb{F}$-stopping times. We define the compensator ν of the jump measure μ with respect to $\mathbb{F}$ as the unique random measure

$$\nu(\omega; (0,t], A) = \int_0^t \int_A \nu(\omega; ds, dz), \, \forall t \geq 0 \, ,$$

is $\mathbb{F}$-predictable and we introduce

$$\widetilde{\mu}((0,t], A) := \mu((0,t], A) - \nu((0,t], A), \, \forall t \geq 0 \, ,$$

which is an $\mathbb{F}$-martingale. We shall say that $\widetilde{\mu}$ is the $\mathbb{F}$-*compensated* martingale of the marked point process, and, by abuse of language, that ν is the compensator of μ. More generally, if $\mathbb{K}$ is a filtration larger that $\mathbb{F}$, we say that $\nu^{\mathbb{K}}$ is the $\mathbb{K}$-compensator of μ if, for any $A \in \mathcal{B}(\mathbb{R})$, the process

$$\widetilde{\mu}^{\mathbb{K}}((0,t], A) := \mu((0,t], A) - \nu^{\mathbb{K}}((0,t], A), \, \forall t \geq 0 \, ,$$

is a $\mathbb{K}$-martingale, and the process $\nu^{\mathbb{K}}((0,\cdot], A)$ is $\mathbb{K}$-predictable. With an abuse of language, we shall write that $\widetilde{\mu}^{\mathbb{K}}$ is a $\mathbb{K}$-martingale.

Hypothesis 2.1 *We assume, as in Chapter VIII, Definition D5, page 236 of [7] and [27], that the $\mathbb{F}$-compensator ν admits the representation*

$$\nu(\omega; dt, dz) = \eta_t(\omega; dz) \, dt, \, \forall t \geq 0 \, , \tag{1}$$

where $\eta(dz)$ is a transition kernel.

As usual, $\mathcal{P}(\mathbb{F})$ (resp. $\mathcal{O}(\mathbb{F})$) is the predictable (resp. optional) σ-algebra on $\mathbb{F}$. For a family of processes $\xi(z) = (\xi_t(z))_{t \geq 0}$ parameterized by $z \in \mathbb{R}$, we shall say that ξ is $\mathcal{P}(\mathbb{F}) \otimes \mathcal{B}(\mathbb{R})$-measurable, if the map $(t, \omega, z) \to \xi_t(\omega; z)$ is $\mathcal{P}(\mathbb{F}) \otimes \mathcal{B}(\mathbb{R})$-measurable, and we define $\mathcal{O}(\mathbb{F}) \otimes \mathcal{B}(\mathbb{R})$-measurable processes in a similar way.

Recall that, under Hypothesis 2.1, if ξ is a $\mathcal{P}(\mathbb{F}) \otimes \mathcal{B}(\mathbb{R})$-measurable process such that

$$\int_0^t \int_{\mathbb{R}} |\xi_s(z)| \, \eta_s(dz) \, ds < \infty, \, \forall t \geq 0 \, , \tag{2}$$

the process $Y = (Y_t)_{t \geq 0}$ defined as

$$Y_t = Y_0 + \int_0^t \int_{\mathbb{R}} \xi_s(z) \, \widetilde{\mu}(ds, dz), \, \forall t \geq 0 \, , \tag{3}$$

is an $\mathbb{F}$-martingale (see Chapter VIII, Corollary C4, page 235 in [7]).

Furthermore, any $\mathbb{F}$-martingale Y admits a representation as in (3) with ξ satisfying (2) (see Chapter VIII, Theorem T8, page 239 in [7] and Theorem 2.2 in [27]). This property is referred to as the *weak predictable representation property* (WPRP) of the marked point process μ in the filtration $\mathbb{F}$ with respect to the *compensated jump measure* $\mu - \nu$ (see also Theorem 13.19 in [16], or Th. 1.13.2 in [22], or Theorem 1.1.21 in [25]). Such a representation is essentially unique ($\mathbb{P} \times \eta_t(dz) \times dt$-a.s.).

Let $X = (X_t)_{t\geq 0}$ be a measurable process and $\mathbb{H}$ be a filtration satisfying the usual hypotheses of completeness and right continuity. We denote by ${}^{p,\mathbb{F}}X = ({}^{p,\mathbb{F}}X_t)_{t\geq 0}$ (resp. ${}^{o,\mathbb{F}}X = ({}^{o,\mathbb{F}}X_t)_{t\geq 0}$) its $\mathbb{F}$-predictable (resp. optional) projection when they exist (see Chapter V, Th. 5.1 (resp. 5.2) in [16] or Section 1.3.1, page 15 in [1]).

3 Jacod's absolute continuity hypothesis

In the whole paper, we work on a complete probability space $(\Omega, \mathcal{G}, \mathbb{P})$ which supports a marked point process with a right-continuous and completed natural filtration $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ and a strictly positive random variable τ. Note that the inclusion $\mathcal{F}_\infty \subset \mathcal{G}$ holds and, in general, this inclusion is strict. We recall that any $\mathbb{F}$-martingale admits a càdlàg modification, see Corollary 2.48 in [16]. In what follows, τ is a given random time defined on $(\Omega, \mathcal{G})$, i.e. a strictly positive random variable.

Hypothesis 3.1 *We assume in the whole paper that Jacod's absolute continuity hypothesis (in short Jacod's hypothesis) holds, that is, the regular conditional distributions of τ given $\mathcal{F}_t$ are absolutely continuous with respect to ρ, the unconditional law of the random variable τ is of the form*

$$\mathbb{P}(\tau \in \cdot \,|\, \mathcal{F}_t) \ll \mathbb{P}(\tau \in \cdot), \forall t \geq 0 \,(\mathbb{P}\text{-}a.s.)\,.$$

Note that Hypothesis 3.1 was not assumed in [9]. We assume that Hypothesis 3.1 holds in order to be able to obtain explicit expressions for the coefficients of the process in the weak predictable representations given considered initially and progressively enlarged filtrations $\mathbb{F}^{(\tau)}$ and $\mathbb{G}$. This assumption implies (see Lemma 2.3 in [13]) that there exists a family of nonnegative processes $p(u) = (p_t(u))_{t\geq 0}$ such that the function $(\omega, t, u) \mapsto p_t(u;\omega)$ is $\mathcal{O}(\mathbb{F}) \otimes \mathcal{B}(\mathbb{R}_+)$-measurable, and, for each $u \geq 0$, the process $p(u)$ is a càdlàg $\mathbb{F}$-martingale. Moreover, for any Borel bounded function f, the following equality holds

$$\mathbb{E}\big[f(\tau) \,\big|\, \mathcal{F}_t\big] = \int_0^\infty f(u)\, p_t(u)\, \rho(du),\ \forall t \geq 0 \,(\mathbb{P}\text{-a.s.})\,. \tag{4}$$

The expression in (4) implies that

$$\mathbb{P}(\tau > s \,|\, \mathcal{F}_t) = \int_s^\infty p_t(u)\, \rho(du),\ \forall t, s \geq 0 \,(\mathbb{P}\text{-a.s.})\,,$$

so that the equality

$$\int_0^\infty p_t(u)\, \rho(du) = 1,\ (\mathbb{P}\text{-a.s.}),$$

is satisfied, and $p_0(u) = 1$, for each $u \geq 0$.

We shall call the family of $\mathbb{F}$-optional processes $p(u)$, for each $u \geq 0$, the $\mathbb{F}$-conditional density family with respect to $\rho(du)$. Note that, even if p is not strictly positive, $p(\tau)$ is strictly positive (see formula (4.10) in [1]). □

The following proposition is proved as a consequence of the weak predictable representation property in [24, Pro. 2.1] (see also Chapter II, Definition 1.27 in [19] and Chapter III, Theorem 4.24 in [19]).

Proposition 3.2 *For each $u \geq 0$, the $\mathbb{F}$-martingale $p(u)$ admits the representation*

$$dp_t(u) = \int_{\mathbb{R}} f_t(u,z)\,\widetilde{\mu}(dt,dz), \forall t \geq 0, \quad p_0(u) = 1\,, \tag{5}$$

for a $\mathcal{P}(\mathbb{F}) \otimes \mathcal{B}(\mathbb{R}_+) \otimes \mathcal{B}(\mathbb{R})$-measurable process f, and

$$\int_0^t \int_{\mathbb{R}} |f_s(u,z)|\,\eta_s(dz)\,ds < \infty, \forall t \geq 0\,.$$

Proof: The existence of f is due to the weak predictable representation property in the filtration $\mathbb{F}$. Note that at the jump times of the process we have $\Delta p_t(u) = f_t(u,z)$, and hence, using the non-negativity of p, we see that $f_t(u,z) \geq -p_{t-}(u)$, $\forall t \geq 0$, $\forall z \in \mathbb{R}$. We shall prove latter on that this inequality holds everywhere. □

Let us denote by $H = (H_t)_{t\geq 0}$ with $H_t = \mathbb{1}_{\{\tau \leq t\}}$, for all $t \geq 0$, the *indicator default process*, where τ denotes the time at which a default occurs. Moreover, since H is a $\mathbb{G}$-adapted càdlàg process, we can introduce the $\mathbb{F}$-supermartingale $G = (G_t)_{t\geq 0}$ defined by $G = {}^{o,\mathbb{F}}(1-H)$, that is, the $\mathbb{F}$-optional projection of $1 - H$ satisfying the property

$$G_t = \mathbb{P}(\tau > t \,|\, \mathcal{F}_t),\ \forall t \geq 0\,(\mathbb{P}\text{-a.s.})\,, \tag{6}$$

which, according to the equality (4), can be represented in the form

$$G_t = \int_t^\infty p_t(u)\,\rho(du),\ \forall t \geq 0\,(\mathbb{P}\text{-a.s.})\,.$$

Note that $G_0 = 1$.

Hypothesis 3.3 *We assume that G, G_- are strictly positive.*

The $\mathbb{F}$-supermartingale G is called the *Azéma supermartingale* of the random time τ.

Hypothesis 3.4 *We assume that the distribution law ρ of the strictly positive random variable τ avoids $\mathbb{F}$-stopping times (in particular, ρ is non-atomic).*

4 Enlargements of filtrations and martingales

We will consider two enlarged filtrations: the initial enlargement of $\mathbb{F}$ obtained by adding the σ-field $\sigma(\tau)$ at time 0 and denoted $\mathbb{F}^{(\tau)}$, and the progressive enlargement of $\mathbb{F}$ obtained by progressively adding information of $\sigma(\tau \wedge t)$ at time $t \geq 0$, or, more precisely, the smallest right-continuous filtration $\mathbb{G}$ containing $\mathbb{F}$ and turning out τ into a stopping time.

The aim of the paper is to explicitly compute the components in the integral representations of the optional projections of the $\mathbb{F}^{(\tau)}$-martingales and of the $\mathbb{G}$-martingales. In this section, we recall some well known results. We give the form of the $\mathbb{F}^{(\tau)}$-semimartingale decomposition and $\mathbb{G}$-semimartingale decomposition of $\widetilde{\mu}$ defined in (2) as well as the $\mathbb{G}$-semimartingale decomposition of H. We underline that the martingale part $\widetilde{\mu}^{(\tau)}$ of the $\mathbb{F}^{(\tau)}$-semimartingale decomposition of $\widetilde{\mu}$ enjoys the $\mathbb{F}^{(\tau)}$-predictable representation property, while the pair $(\widetilde{\mu}^{\mathbb{G}}, M^{\mathbb{G}})$ of the martingale parts of the $\mathbb{G}$-semimartingale decompositions of $\widetilde{\mu}$ and H enjoys the $\mathbb{G}$-predictable representation property, where the integral with respect to the pair is understood componentwise.

4.1 The initially enlarged filtration

As in the introduction, let us denote by $\mathbb{F}^{(\tau)} = (\mathcal{F}_t^{(\tau)})_{t\geq 0} = (\mathcal{F}_t \vee \sigma(\tau))_{t\geq 0}$ the initial enlargement of the filtration $\mathbb{F}$ with the random time τ. Note that $\mathcal{F}_0^{(\tau)} = \sigma(\tau)$. We recall that, under Jacod's hypothesis, any $\mathbb{F}$-local martingale is an $\mathbb{F}^{(\tau)}$-special semimartingale (see, e.g., Theorem 2.1 in [17] or Proposition 5.30, page 116 in [1]). Note that, according to Proposition 4.20 in [1], the filtration $\mathbb{F}^{(\tau)}$ is right-continuous.

We further denote $\mathbb{F}^{(\tau)}$-optional processes with the superscript (τ) as in $Y^{(\tau)}$. We denote $\mathbb{F}$-adapted processes by capital letters as X, or lower case x, or φ, or even x^0.

We also recall that, for any $t \geq 0$ fixed, any $\mathcal{F}_t^{(\tau)}$-measurable random variable $Y_t^{(\tau)}$ is of the form $Y_t(\omega, \tau(\omega))$, for some $\mathcal{F}_t \otimes \mathcal{B}(\mathbb{R}_+)$-measurable function $(\omega, u) \mapsto Y_t(\omega, u)$ (see, e.g., Proposition 2.7, part (i) in [8]). In particular, any $\mathcal{F}_0^{(\tau)}$-measurable random variable is a Borel function of τ. Recall that any $\mathbb{F}^{(\tau)}$-predictable process can be represented in the form $Y_t(\omega, \tau(\omega))$, for all $t \geq 0$, where the mapping $(\omega, t, u) \mapsto Y_t(\omega, u)$ defined on $\Omega \times \mathbb{R}_+ \times \mathbb{R}_+$ and valued in $\mathbb{R}$ is $\mathcal{P}(\mathbb{F}) \otimes \mathcal{B}(\mathbb{R}_+)$-measurable. Moreover, under Jacod's hypothesis, any $\mathbb{F}^{(\tau)}$-optional process $Y^{(\tau)} = (Y_t^{(\tau)})_{t\geq 0}$ can be written as $Y_t^{(\tau)} = Y_t(\tau)$, for all $t \geq 0$, where the process $Y = (Y_t(u), t \geq 0, u \in \mathbb{R}_+)$ is $\mathcal{O}(\mathbb{F}) \otimes \mathcal{B}(\mathbb{R}_+)$-measurable (see Theorem 6.9 in [28]).

As an immediate consequence of Jacod's hypothesis, we observe that, for each $t \geq 0$, if the $\mathcal{F}_t^{(\tau)}$-measurable random variable $Y_t(\tau)$ is integrable, then the following representation holds

$$\mathbb{E}\big[Y_t(\tau) \,\big|\, \mathcal{F}_t\big] = \int_0^\infty Y_t(u)\, p_t(u)\, \rho(du),\ \forall t \geq 0$$

(see, e.g., Proposition 4.18 (b), page 85 in [1]).

In the following proposition, we give the $\mathbb{F}^{(\tau)}$-semimartingale decomposition of $\widetilde{\mu}$, defined in (2).

Proposition 4.1 *The $\mathbb{F}^{(\tau)}$-semimartingale decomposition of the $\mathbb{F}$-martingale $\widetilde{\mu}$ is given by*

$$\widetilde{\mu}((0,t], A) = \widetilde{\mu}^{(\tau)}((0,t], A) + \int_0^t \int_A \frac{f_s(\tau, z)}{p_{s-}(\tau)}\, \eta_s(dz)\, ds,\ \forall t \geq 0\,, \forall A \in \mathcal{B}(\mathbb{R})\,,$$

where $\widetilde{\mu}^{(\tau)}$ is an $\mathbb{F}^{(\tau)}$-martingale and f is given in (5). We assume that $\nu^{\tau)}$ has a finite mass. In other terms, the process $(T_n, Z_n)_{n\geq 1}$ is a marked point process with $\mathbb{F}^{(\tau)}$-compensator $\nu^{(\tau)}$, where we have

$$\nu^{(\tau)}(dt, dz) = \left(1 + \frac{f_t(\tau, z)}{p_{t-}(\tau)}\right) \eta_t(dz)\, dt,\ \forall t \geq 0\,, \forall z \in \mathbb{R}\,. \tag{7}$$

In particular, we have $1 + f_t(\tau, z)/p_{t-}(\tau) \geq 0$, $\forall t \geq 0$, $\forall z \in \mathbb{R}$.

Proof: From the results of initial enlargement[1], the process $\widetilde{\mu}^{(\tau)}$ defined by

$$\widetilde{\mu}^{(\tau)}((0,t], A) = \widetilde{\mu}((0,t], A] - \int_0^t \frac{d\langle \widetilde{\mu}((0,\cdot], A), p(u)\rangle_s^{\mathbb{F}}}{p_{s-}(u)}\bigg|_{u=\tau},\ \forall t \geq 0\,, \forall A \in \mathcal{B}(\mathbb{R})\,,$$

is an $\mathbb{F}^{(\tau)}$-martingale. In order to compute the predictable covariation, we start by computing the quadratic covariation of the processes $\widetilde{\mu}$ and $p(u)$, for each $u \geq 0$. Obviously, we have

$$\left[\widetilde{\mu}, p(u)\right]_t = \int_0^t \int_{\mathbb{R}} f_s(u, z)\, \mu(ds, dz),\ \forall t, u \geq 0\,,$$

and hence, from Theorem 6.28, part 2 in [16],

$$\left\langle \widetilde{\mu}, p(u)\right\rangle_t^{\mathbb{F}} = \int_0^t \int_{\mathbb{R}} f_s(u, z)\, \eta_s(dz)\, ds,\ \forall t, u \geq 0\,.$$

It follows that

$$\widetilde{\mu}^{(\tau)}((0,t], \mathbb{R}) = \mu((0,t], \mathbb{R}) - \int_0^t \int_{\mathbb{R}} \left(\frac{f_s(\tau, z)}{p_{s-}(\tau)} + 1\right) \eta_s(dz)\, ds,\ \forall t \geq 0\,,$$

is an $\mathbb{F}^{(\tau)}$-martingale and the $\mathbb{F}^{(\tau)}$-compensator of μ is

$$\nu^{(\tau)}(dt, dz) = \left(\frac{f_t(\tau, z)}{p_{t-}(\tau)} + 1\right) \eta_t(dz)\, dt,\ \forall t \geq 0\,, \forall z \in \mathbb{R}\,,$$

since the process

$$\widetilde{\mu}^{(\tau)} = \mu - \nu - \int_0^{\cdot} \int_{\mathbb{R}} \frac{f_s(\tau, z)}{p_{s-}(\tau)}\, \eta_s(dz)\, ds \equiv \mu - \nu^{(\tau)}$$

is an $\mathbb{F}^{(\tau)}$-martingale. This completes the proof. □

Note that the weak predictable representation property for the marked point process $\widetilde{\mu}^{(\tau)}$ holds in $\mathbb{F}^{(\tau)}$ (See Proposition 4.6 or [4]).

[1] One applies Theorem 2.1 in [17] which states that, under Jacod's hypothesis, for any $\mathbb{F}$-martingale $X = (X_t)_{t\geq 0}$, the process $X(\tau) = (X_t(\tau))_{t\geq 0}$ defined by

$$X_t(\tau) = X_t - \int_0^t \frac{d\langle X, p(u)\rangle_s^{\mathbb{F}}}{p_{s-}(u)}\bigg|_{u=\tau},\ \forall t \geq 0\,,$$

is an $\mathbb{F}^{(\tau)}$-martingale. Note that $p(\tau)$ does not vanish.

Proposition 4.2 *Each $(\mathbb{P}, \mathbb{F}^{(\tau)})$-martingale $Y^{(\tau)} = (Y_t(\tau))_{t\geq 0}$ admits a representation of the form*

$$Y_t(\tau) = Y_0(\tau) + \int_0^t \int_{\mathbb{R}} \psi_s(\tau, z)\, \widetilde{\mu}^{(\tau)}(ds, dz),\ \forall t \geq 0\,, \tag{8}$$

for some $\mathcal{P}(\mathbb{F}) \otimes \mathcal{B}(\mathbb{R}_+) \otimes \mathcal{B}(\mathbb{R})$-measurable process ψ satisfying

$$\int_0^t \int_{\mathbb{R}} |\psi_s(\tau, z)|\, \nu^{(\tau)}(ds, dz) < \infty,\ \forall t \geq 0\,,$$

where $\nu^{(\tau)}$ is defined in (7).

4.2 The progressively enlarged filtration

We denote by $\mathbb{G} = (\mathcal{G}_t)_{t\geq 0}$ the progressive enlargement of $\mathbb{F}$ with τ, that is, the right-continuous version of $\mathcal{G}^0$ where

$$\mathcal{G}_t^0 = \bigcap_{s>t} \big(\mathcal{F}_s \vee \sigma(\tau \wedge s)\big),\ \forall t \geq 0\,. \tag{9}$$

Note that τ is a $\mathbb{G}$-stopping time and that, according to the hypothesis that the random variable τ is strictly positive, the σ-algebra $\mathcal{G}_0$ is trivial, so that the initial value of a $\mathbb{G}$-adapted process is a deterministic one. Observe that, under Jacod's hypothesis, any $\mathbb{F}$-martingale is a $\mathbb{G}$-semimartingale (see, e.g., Proposition 5.30, page 116 in [1] or Theorem 3.1 in [20]), and thus, a special semimartingale according to Chapter VI, Theorem 4, page 367 in [26].

We observe that the completion of the two enlargements $\mathbb{G}$ and $\mathbb{F}^{(\tau)}$ follows from $\mathcal{F}_\infty \subset \mathcal{G}_\infty \subset \mathcal{F}_\infty^{(\tau)} \subset \mathcal{A}$.

We further indicate with the superscript $\mathbb{G}$ the processes which are $\mathbb{G}$-adapted, as $Y^{\mathbb{G}}$, as we shall do now for the $\mathbb{G}$-adapted process $\mathbb{1}_{\{\tau\leq t\}}$ which will be denoted $H^{\mathbb{G}}$. We recall that, under Jacod's hypothesis, any $\mathbb{G}$-*optional* process $Y^{\mathbb{G}}$ can be written as

$$Y_t^{\mathbb{G}} = \mathbb{1}_{\{\tau>t\}}\, Y_t^0 + \mathbb{1}_{\{\tau\leq t\}}\, Y_t^1(\tau),\ \forall t \geq 0\,,$$

where Y^0 is $\mathbb{F}$-optional and Y^1 is $\mathcal{O}(\mathbb{F}) \otimes \mathcal{B}(\mathbb{R}_+)$-measurable (see Theorem 6.9 in [28]). A particular case occurs when $Y^{\mathbb{G}}$ is the optional projection of a process $Y^{(\tau)}$. In that case, one has

$$Y_t^0 = \frac{1}{G_t} \int_t^\infty Y_t(u)\, p_t(u)\, \rho(du),\ \forall t \geq 0\,, \quad \text{and} \quad Y_t^1(u) = Y_t(u),\ \forall t, u \geq 0 \,\text{with}\, t \geq u\,,$$

where the process G is defined in (6). Here, Y^0 is called the $\mathbb{F}$-optional reduction of $Y^{\mathbb{G}}$. We also recall that any $\mathbb{G}$-*predictable* process $Y_t^{\mathbb{G}} = (Y_t^{\mathbb{G}})_{t\geq 0}$ can be written as

$$Y_t^{\mathbb{G}} = \mathbb{1}_{\{\tau\geq t\}}\, Y_t^0 + \mathbb{1}_{\{\tau<t\}}\, Y_t^1(\tau),\ \forall t \geq 0\,,$$

where the process Y^0 is $\mathbb{F}$-predictable and Y^1 is $\mathcal{P}(\mathbb{F}) \otimes \mathcal{B}(\mathbb{R}_+)$-measurable (see, e.g., Proposition 2.11, page 36 in [1]). In this case, Y^0 is called the $\mathbb{F}$-predictable reduction of $Y^{\mathbb{G}}$. Note that the càg process $(\mathbb{1}_{\{\tau<t\}})_{t\geq 0}$ is $\mathbb{G}$-predictable.

As it follows from the Doob-Meyer decomposition of the supermartingale $H^{\mathbb{G}}$ and the fact that any $\mathbb{G}$-predictable process is equal, on the set $\{\tau \geq t\}$ to an $\mathbb{F}$-predictable process, there exists an $\mathbb{F}$-predictable increasing process $\Lambda = (\Lambda_t)_{t\geq 0}$ such that the process $M^{\mathbb{G}} = (M_t^{\mathbb{G}})_{t\geq 0}$ defined by

$$M_t^{\mathbb{G}} = H_t^{\mathbb{G}} - \Lambda_{t\wedge\tau},\ \forall t \geq 0\,, \tag{10}$$

is a $\mathbb{G}$-martingale. It is known that, under Jacod's hypothesis, the process Λ admits the representation (we also use the fact that ρ has no atoms)

$$\Lambda_t = \int_0^t \frac{p_s(s)}{G_s}\,\rho(ds) = \int_0^t \frac{p_{s-}(s)}{G_{s-}}\,\rho(ds),\ \forall t \geq 0\,,$$

(see Proposition 4.4 in [12] or Corollary 5.27 (b), page 114 in [1]). In this respect, the process $\lambda = (\lambda_t)_{t\geq 0}$ defined by $\lambda_t = p_{t-}(t)/G_{t-}$, for $t \geq 0$, is the intensity rate of τ with respect to the measure ρ (see Proposition 2.15, page 37 in [1]).

The Doob-Meyer decomposition of the Azéma supermartingale can be given explicitly and its multiplicative decomposition is as follows.

Proposition 4.3 *Suppose that Jacod's hypothesis holds. The Doob-Meyer decomposition of the Azéma supermartingale G is*

$$G_t = 1 - \int_0^t G_s\,\lambda_s\,\rho(ds) + \int_0^t \int_{\mathbb{R}} \int_s^\infty f_s(u,z)\,\widetilde{\mu}(ds,dz)\,\rho(du),\ \forall t \geq 0\,, \tag{11}$$

or in a simplified form

$$G_t = 1 - \int_0^t G_s\,\lambda_s\,\rho(ds) + \int_0^t \int_{\mathbb{R}} \varphi(s,z)\,\widetilde{\mu}(ds,dz),\ \forall t \geq 0\,,$$

where

$$\varphi(t,z) = \int_t^\infty f_t(u,z)\,\rho(du),\ \forall t \geq 0\,, \forall z \in \mathbb{R}\,. \tag{12}$$

Proof: The Doob-Meyer decomposition of G is obtained using Itô-Ventzell formula as developed in Theorem 3.1 in [23] to the process

$$\begin{aligned} G_t(x) = \mathbb{P}(\tau > x \,|\, \mathcal{F}_t) &= \int_x^\infty p_t(u)\,\rho(du) \\ &= \int_x^\infty p_0(u)\rho(du) + \int_{s=0}^t \int_{\mathbb{R}} \int_{u=x}^\infty f_s(u,z)\,\widetilde{\mu}(ds,dz)\,\rho(du),\ \forall t \geq 0\,, \end{aligned}$$

with a parameter x, where the forward integral (with respect to the compensated measure) in [23] is the usual stochastic integral in our setting since we integrate predictable processes. Then (with the notation of [23] $\gamma = 0$, $H(s,x) = \int_x^\infty f_s(u,z)\rho(du)$),

$$G_t = G_t(t) = 1 - \int_0^t \lambda_s\,G_s\,\rho(ds) + \int_0^t \int_s^\infty \int_{\mathbb{R}} f_s(u,z)\,\rho(du)\,\widetilde{\mu}(ds,dz),\ \forall t \geq 0\,.$$

□

In the following proposition, we give the semimartingale decomposition of the process $\widetilde{\mu}$ defined in (12) in the filtration $\mathbb{G}$.

Proposition 4.4 *The $\mathbb{G}$-semimartingale decomposition of the $\mathbb{F}$-martingale $\widetilde{\mu}([0,\cdot],A)$ is given by for any $A \in \mathcal{B}(\mathbb{R})$*

$$\widetilde{\mu}((0,t],A) = \widetilde{\mu}^{\mathbb{G}}((0,t],A) + \int_0^{t\wedge\tau}\int_A \frac{\varphi_s(z)}{G_{s-}}\,\eta_s(dz)\,ds + \int_{t\wedge\tau}^{t}\int_A \frac{f_s(\tau,z)}{p_{s-}(\tau)}\,\eta_s(dz)\,ds,\ \forall t\geq 0\,, \quad (13)$$

where $\widetilde{\mu}^{\mathbb{G}}((0,\cdot],A)$ is a $\mathbb{G}$-martingale, φ is defined in (12)*, and f is defined in* (5)*. The predictable random measure*

$$\nu^{\mathbb{G}}(dt,dz) = \left[\mathbb{1}_{\{\tau\geq t\}}\left(\frac{\varphi_t(z)}{G_t}+1\right) + \mathbb{1}_{\{\tau<t\}}\left(\frac{f_t(\tau,z)}{p_{t-}(\tau)}+1\right)\right]\eta_t(dz)\,dt,\ \forall t\geq 0,\ \forall z\in\mathbb{R}\,, \quad (14)$$

is the $\mathbb{G}$-compensator of the random jump measure μ.

Proof: Recall that the process G admits a Doob-Meyer decomposition as $G = m - (H^{\mathbb{G}})^{p,\mathbb{F}}$ with a martingale $m = (m_t)_{t\geq 0}$ (see [2]) where, from (11) and (12), we get

$$m_t = \int_0^t\int_{\mathbb{R}} \varphi_s(z)\,\widetilde{\mu}(ds,dz)\,,\forall t\geq 0\,.$$

The $\mathbb{G}$-semimartingale decomposition[3] of the $\mathbb{F}$-martingale $\widetilde{\mu}$ is given by

$$\begin{aligned}\widetilde{\mu}((0,t],A) &= \widetilde{\mu}^{\mathbb{G}}((0,t],A) + \int_0^{t\wedge\tau}\frac{d\langle\widetilde{\mu}(A),m\rangle_s^{\mathbb{F}}}{G_{s-}} + \int_{t\wedge\tau}^{t}\left.\frac{d\langle\widetilde{\mu}(A),p(u)\rangle_s^{\mathbb{F}}}{p_{s-}(u)}\right|_{u=\tau}\\ &= \widetilde{\mu}^{\mathbb{G}}((0,t],A) + \int_0^{t\wedge\tau}\int_A \frac{\varphi_s(z)}{G_{s-}}\,\eta_s(dz)\,ds + \int_{t\wedge\tau}^{t}\int_A \frac{f_s(\tau,z)}{p_{s-}(\tau)}\,\eta_s(dz)\,ds\,,\ \forall t\geq 0,\ \forall A\in\mathcal{B}(\mathbb{R})\,,\end{aligned}$$

where $(\widetilde{\mu}^{\mathbb{G}}((0,t],A))_{t\geq 0}$ is a $\mathbb{G}$-martingale. It thus follows that the $\mathbb{G}$-compensator of μ is given by (14). □

Remark 4.5 The compensator $\nu^{\mathbb{G}}$ of μ in (14) being increasing, the process $f_t(\tau,z)/p_{t-}(\tau)+1$, $\forall t\geq 0$, $\forall z\in\mathbb{R}$, is nonnegative, as well as the process $\varphi_t(z)/G_t+1$, $\forall t\geq 0$, $\forall z\in\mathbb{R}$.

Proposition 4.6 *Each $(\mathbb{P},\mathbb{G})$-martingale $Y^{\mathbb{G}} = (Y_t^{\mathbb{G}})_{t\geq 0}$ can be represented as*

$$Y_t^{\mathbb{G}} = Y_0^{\mathbb{G}} + \int_0^t\int_{\mathbb{R}} \alpha_s^{\mathbb{G}}(z)\,\widetilde{\mu}^{\mathbb{G}}(ds,dz) + \int_0^t \beta_s^0\,dM_s^{\mathbb{G}},\ \forall t\geq 0\,, \quad (15)$$

[2] It is known that, under the assumption that the random time τ avoids all $\mathbb{F}$-stopping times, the dual optional projection of H is continuous and equal to the dual predictable projection of H, denoted by H^p (see Proposition 1.48 (a), page 22 in [1]). Therefore the martingale m which appears in the general formulae of the semimartingale decomposition (see Proposition 5.30, page 116 in [1]) is equal to the martingale part of the Doob-Meyer decomposition of G, that is, one has $G = m - H^p$ with an $\mathbb{F}$-martingale $m = (m_t)_{t\geq 0}$. In particular, the predictable projection of G is ${}^pG = {}^pm - H^p = m_- - H^p = G_-$.

[3] One can use Remark 2 and Theorem 5.30, page 116 in [1] to deduce that, for any $\mathbb{F}$-martingale X, the process $X^{\mathbb{G}} = (X_t^{\mathbb{G}})_{t\geq 0}$ defined by

$$X_t^{\mathbb{G}} = X_t - \int_0^{t\wedge\tau}\frac{d\langle X,m\rangle_s^{\mathbb{F}}}{G_{s-}} - \int_{t\wedge\tau}^{t}\left.\frac{d\langle X,p(u)\rangle_s^{\mathbb{F}}}{p_{s-}(u)}\right|_{u=\tau},\ \forall t\geq 0\,,$$

is a $\mathbb{G}$-martingale.

for some $\mathcal{P}(\mathbb{G}) \otimes \mathcal{B}(\mathbb{R})$-measurable process $\alpha^{\mathbb{G}}$ satisfying

$$\int_0^t \int_{\mathbb{R}} \left|\alpha_s^{\mathbb{G}}(z)\right| \nu^{\mathbb{G}}(ds, dz) < \infty, \, \forall t \geq 0 \, ,$$

where $\nu^{\mathbb{G}}$ is defined in (14)*. Here, the process $\alpha^{\mathbb{G}}$ is of the form*

$$\alpha_t^{\mathbb{G}}(z) = \mathbb{1}_{\{\tau \geq t\}} \, \alpha_t^0(z) + \mathbb{1}_{\{\tau < t\}} \, \alpha_t(\tau, z), \, \forall t \geq 0, \, \forall z \in \mathbb{R} \, , \tag{16}$$

where α^0 is a $\mathcal{P}(\mathbb{F}) \otimes \mathcal{B}(\mathbb{R})$-measurable process, α is a $\mathcal{P}(\mathbb{F}) \otimes \mathcal{B}(\mathbb{R}_+) \otimes \mathcal{B}(\mathbb{R})$-measurable process, while β^0 is an $\mathbb{F}$-predictable process.

Proof: Due to the avoidance property, the process $\mu((0, \cdot], A) + H^{\mathbb{G}}$ is a $\mathbb{G}$-measurable marked point process, with sequence of jumps $(\cup_n T_n) \cup \tau$ and jumps $(Z_n, n \geq 1)$ (after ordering the jump times). The result follows from the definition of $M^{\mathbb{G}}$ in (10) above. See also [4]. □

Remark 4.7 Note that, if the process $\beta^{\mathbb{G}}$ admits the representation

$$\beta_t^{\mathbb{G}} = \mathbb{1}_{\{\tau \geq t\}} \, \beta_t^0 + \mathbb{1}_{\{\tau < t\}} \, \beta_t^1(\tau), \, \forall t \geq 0 \, ,$$

then the equality

$$\int_0^t \beta_s^{\mathbb{G}} \, dM_s^{\mathbb{G}} = \int_0^t \beta_s^0 \, dM_s^{\mathbb{G}}, \, \forall t \geq 0 \, ,$$

holds, for any choice of the $\mathcal{P}(\mathbb{G}) \otimes \mathcal{B}(\mathbb{R})$-measurable process β^1, since $M^{\mathbb{G}}$ is flat after τ (i.e., $M_t^{\mathbb{G}} = M_{t \wedge \tau}^{\mathbb{G}}$, for all $t \geq 0$).

5 Optional projections of martingales

Let $Y^{(\tau)}$ be an $\mathbb{F}^{(\tau)}$-martingale. Then, $Y^{(\tau)}$ admits the integral representation given by (8). We study the $\mathbb{G}$-optional projection $Y^{\mathbb{G}}$ (a $\mathbb{G}$-martingale) of the process $Y^{(\tau)}$ and the $\mathbb{F}$-optional projection Y (an $\mathbb{F}$-martingale) of $Y^{(\tau)}$. The $\mathbb{G}$-martingale $Y^{\mathbb{G}}$ admits the integral representation given by (15), with some process $\alpha^{\mathbb{G}}$ in the form of (16), which has a $\mathcal{P}(\mathbb{F}) \otimes \mathcal{B}(\mathbb{R})$-measurable process α^0, a $\mathcal{P}(\mathbb{F}) \otimes \mathcal{B}(\mathbb{R}^+) \otimes \mathcal{B}(\mathbb{R})$-measurable process α, as well as an $\mathbb{F}$-predictable process β^0.

Observe that any *square integrable $\mathbb{F}$-martingale Y* admits the representation (3) with some $\mathcal{P}(\mathbb{F}) \otimes \mathcal{B}(\mathbb{R})$-measurable process ξ satisfying

$$\mathbb{E}\left[\int_0^t \int_{\mathbb{R}} \xi_s^2(z) \, \eta_s(dz) \, ds \right] < \infty, \, \forall t \geq 0 \, ,$$

(see Chapter VIII, Theorem T8, page 239 in [7]).

Similarly, we observe that any *square integrable $\mathbb{F}^{(\tau)}$-martingale $Y(\tau)$* admits the representation (8) with some $\mathcal{P}(\mathbb{F}) \otimes \mathcal{B}(\mathbb{R}_+) \otimes \mathcal{B}(\mathbb{R})$-measurable process ψ satisfying

$$\mathbb{E}\left[\int_0^t \int_{\mathbb{R}} \psi_s^2(\tau, z) \, \nu^{(\tau)}(ds, dz) \right] < \infty, \, \forall t \geq 0 \, , \tag{17}$$

where $\nu^{(\tau)}$ is defined in (7) (see Chapter VIII, Theorem T8, page 239 in [7]).

Finally, we observe that any *square integrable* $\mathbb{G}$*-martingale* $Y^{\mathbb{G}}$ admits the representation (15) with some $\mathcal{P}(\mathbb{G}) \otimes \mathcal{B}(\mathbb{R})$-measurable process $\alpha^{\mathbb{G}}$ satisfying

$$\mathbb{E}\left[\int_0^t \int_{\mathbb{R}} \left(\alpha_s^{\mathbb{G}}(z)\right)^2 \nu^{\mathbb{G}}(ds, dz)\right] < \infty, \, \forall t \geq 0\,, \tag{18}$$

and $\mathbb{F}$-predictable process β^0, where $\nu^{\mathbb{G}}$ is defined in (14) (see Chapter VIII, Theorem T8, page 239 in [7]).

5.1 The projections of $\mathbb{F}^{(\tau)}$-martingales on $\mathbb{G}$

Proposition 5.1 *Let $Y^{(\tau)}$ be an $\mathbb{F}^{(\tau)}$-martingale with the representation (8) above:*

$$Y_t(\tau) = Y_0(\tau) + \int_0^t \int_{\mathbb{R}} \psi_s(\tau, z)\, \widetilde{\mu}^{(\tau)}(ds, dz)$$

for some $\mathcal{P}(\mathbb{F}) \otimes \mathcal{B}(\mathbb{R}_+) \otimes \mathcal{B}(\mathbb{R})$-measurable process ψ, We study the $\mathbb{G}$-optional projection $Y^{\mathbb{G}}$ of the process $Y^{(\tau)}$. Note that $Y^{\mathbb{G}}$ is a $\mathbb{G}$-martingale. The $\mathbb{G}$-martingale $Y^{\mathbb{G}}$ admits the integral representation given by (15), with some process $\alpha^{\mathbb{G}}$ in the form of (16), which has a $\mathcal{P}(\mathbb{F}) \otimes \mathcal{B}(\mathbb{R})$-measurable process α^0, a $\mathcal{P}(\mathbb{F}) \otimes \mathcal{B}(\mathbb{R}^+) \otimes \mathcal{B}(\mathbb{R})$-measurable process α, as well as an $\mathbb{F}$-predictable process β^0. Then $Y_0^{\mathbb{G}} = \mathbb{E}[Y_t(\tau)]$. The $\mathcal{P}(\mathbb{F}) \otimes \mathcal{B}(\mathbb{R})$-measurable process α^0, the $\mathcal{P}(\mathbb{F}) \otimes \mathcal{B}(\mathbb{R}_+) \otimes \mathcal{B}(\mathbb{R})$-measurable process α and the $\mathbb{F}$-predictable process β^0 are of the form

$$\begin{aligned}
&\alpha_t^0(z) = \frac{1}{\varphi_t(z) + G_{t-}} \qquad (19)\\
&\times \int_t^\infty \left(\left(\psi_t(u,z) + Y_{t-}(u)\right) \left(f_t(u,z) + p_{t-}(u)\right) - Y_{t-}(u) \left(\frac{\varphi_t(z)}{G_{t-}} + 1\right) p_{t-}(u) \right) \rho(du),\\
&\forall t \geq 0\,, \forall z \in \mathbb{R}\,,\\
&\alpha_t(u,z) = \psi_t(u,z), \, \forall u \geq t \geq 0\,, \forall z \in \mathbb{R}\,, \qquad (20)\\
&\beta_t^0 = {}^{p,\mathbb{F}}\big(Y_{t-}(t)\big) - Y_{t-}^0, \, \forall t \geq 0\,, \qquad (21)
\end{aligned}$$

where $Y^0 = (Y_t^0)_{t \geq 0}$ is the $\mathbb{F}$-predictable reduction of $Y^{\mathbb{G}}$ given by

$$Y_{t-}^0 = \frac{1}{G_{t-}} \int_t^\infty Y_{t-}(u)\, p_{t-}(u)\, \rho(du), \, \forall t \geq 0\,, \tag{22}$$

and Y_-^0 is its left limit.

Proof: In the first part of the proof (the first and the second step), we assume that the $\mathbb{F}^{(\tau)}$-martingale $Y^{(\tau)}$ is square integrable, so that the $\mathbb{G}$-martingale $Y^{\mathbb{G}}$ is square integrable too. In the first step, we determine $\alpha^{\mathbb{G}}(z)$, for each $z \in \mathbb{R}$, and, in the second step, we determine β^0. We generalize the result to any $\mathbb{F}^{(\tau)}$-martingale by localisation in the second part of the proof (third step).

We introduce the sign $\overset{\text{TP}}{=}$ to indicate that the tower property for conditional expectations is applied.

First step: We assume that the $\mathbb{F}^{(\tau)}$-martingale $Y^{(\tau)}$ is square integrable, so that the $\mathbb{G}$-martingale $Y^{\mathbb{G}}$ is square integrable too. In particular, $Y_0(\tau)$ is square integrable and the $\mathcal{P}(\mathbb{F})\otimes\mathcal{B}(\mathbb{R}_+)\otimes\mathcal{B}(\mathbb{R})$-measurable process ψ satisfies (17) as well as the $\mathcal{P}(\mathbb{G})\otimes\mathcal{B}(\mathbb{R})$-measurable process $\alpha^{\mathbb{G}}$ satisfies (18). Then, consider a bounded $\mathcal{P}(\mathbb{G})\otimes\mathcal{B}(\mathbb{R})$-measurable process $\gamma^{\mathbb{G}}$ such that $\gamma^{\mathbb{G}}(z)=(\gamma_t^{\mathbb{G}}(z))_{t\geq 0}$, for each $z\in\mathbb{R}$, as well as a bounded $\mathbb{F}$-predictable process $\theta^0=(\theta_t^0)_{t\geq 0}$, and define the process $K^{\mathbb{G}}=(K_t^{\mathbb{G}})_{t\geq 0}$ by

$$K_t^{\mathbb{G}} = K_0^{\mathbb{G}} + \int_0^t\int_{\mathbb{R}} \gamma_s^{\mathbb{G}}(z)\,\widetilde{\mu}^{\mathbb{G}}(ds,dz) + \int_0^t \theta_s^0\,dM_s^{\mathbb{G}},\ \forall t\geq 0\,, \tag{23}$$

where $\nu^{\mathbb{G}}$ is defined in (14). It is seen that the process $K^{\mathbb{G}}$ is a square integrable $\mathbb{G}$-martingale, since $\gamma^{\mathbb{G}}$ satisfies the condition

$$\mathbb{E}\bigg[\int_0^t\int_{\mathbb{R}} \big(\gamma_s^{\mathbb{G}}(z)\big)^2\,\nu^{\mathbb{G}}(ds,dz)\bigg] < \infty,\ \forall t\geq 0\,, \tag{24}$$

and the process θ^0 is $\mathbb{F}$-predictable and bounded. In this case, the square integrable random variable $Y_t^{\mathbb{G}}=\mathbb{E}[Y_t(\tau)\,|\,\mathcal{G}_t]$ is the only $\mathcal{G}_t$-measurable random variable such that

$$\mathbb{E}\big[Y_t(\tau)\,K_t^{\mathbb{G}}\big] = \mathbb{E}\big[Y_t^{\mathbb{G}}\,K_t^{\mathbb{G}}\big],\ \forall t\geq 0\,, \tag{25}$$

holds. Thus, since one has

$$\mathbb{E}\big[Y_t(\tau)\,K_0^{\mathbb{G}}\big] = \mathbb{E}\big[Y_t^{\mathbb{G}}\,K_0^{\mathbb{G}}\big],\ \forall t\geq 0\,,$$

the equality (25) is equivalent to the system of two following equalities

$$\mathbb{E}\bigg[Y_t(\tau)\int_0^t\int_{\mathbb{R}} \gamma_s^{\mathbb{G}}(z)\,\widetilde{\mu}^{\mathbb{G}}(ds,dz)\bigg] = \mathbb{E}\bigg[Y_t^{\mathbb{G}}\int_0^t\int_{\mathbb{R}} \gamma_s^{\mathbb{G}}(z)\,\widetilde{\mu}^{\mathbb{G}}(ds,dz)\bigg],\ \forall t\geq 0\,, \tag{26}$$

and

$$\mathbb{E}\bigg[Y_t(\tau)\int_0^t \theta_s^0\,dM_s^{\mathbb{G}}\bigg] = \mathbb{E}\bigg[Y_t^{\mathbb{G}}\int_0^t \theta_s^0\,dM_s^{\mathbb{G}}\bigg],\ \forall t\geq 0\,. \tag{27}$$

We now determine the processes α^0 and α from the equality (26). On the one hand, one has

$$\begin{aligned}
&\mathbb{E}\bigg[Y_t(\tau)\int_0^t\int_{\mathbb{R}} \gamma_s^{\mathbb{G}}(z)\,\big(\mu(ds,dz)-\nu^{\mathbb{G}}(ds,dz)\big)\bigg]\\
&=\mathbb{E}\bigg[Y_t(\tau)\bigg(\int_0^t\int_{\mathbb{R}} \gamma_s^{\mathbb{G}}(z)\,\big(\mu(ds,dz)-\nu^{(\tau)}(ds,dz)\big)\\
&\qquad+\int_0^t\int_{\mathbb{R}} \gamma_s^{\mathbb{G}}(z)\,\big(\nu^{(\tau)}(ds,dz)-\nu^{\mathbb{G}}(ds,dz)\big)\bigg)\bigg],\ \forall t\geq 0\,,
\end{aligned}$$

where $\nu^{(\tau)}$ is defined in (7). Integrating by parts on the time interval $[0,t]$ the product the two $\mathbb{F}^{(\tau)}$-martingales $Y^{(\tau)}$ and $\Upsilon=(\Upsilon_t)_{t\geq 0}$ defined by

$$\Upsilon_t = \int_0^t\int_{\mathbb{R}} \gamma_s^{\mathbb{G}}(z)\,\widetilde{\mu}^{(\tau)}(ds,dz),\ \forall t\geq 0\,, \tag{28}$$

and taking into account the fact that $\Upsilon_{t-}dY_t(\tau)$ and $Y_{t-}(\tau)d\Upsilon_t$ correspond to true martingales, as we shall prove in Appendix below, one has

$$\mathbb{E}\Big[Y_t(\tau)\int_0^t\int_{\mathbb{R}}\gamma_s^{\mathbb{G}}(z)\,\widetilde{\mu}^{(\tau)}(ds,dz)\Big]=\mathbb{E}\Big[\int_0^t\int_{\mathbb{R}}\gamma_s^{\mathbb{G}}(z)\,\psi_s(\tau,z)\,\nu^{(\tau)}(ds,dz)\Big],\ \forall t\geq 0\,.$$

Now, integrating by parts on the time interval $[0,t]$ the product of the martingale $Y^{(\tau)}$ and the bounded variation process $\Gamma(\tau)=(\Gamma_t(\tau))_{t\geq 0}$ defined by

$$\Gamma_t(\tau)=\int_0^t\int_{\mathbb{R}}\gamma_s^{\mathbb{G}}(z)\,\big(\nu^{(\tau)}(ds,dz)-\nu^{\mathbb{G}}(ds,dz)\big),\ \forall t\geq 0\,,$$

one obtains

$$\mathbb{E}\big[Y_t(\tau)\,\Gamma_t(\tau)\big]=\mathbb{E}\Big[\int_0^t\int_{\mathbb{R}}Y_{s-}(\tau)\,\gamma_s^{\mathbb{G}}(z)\,\big(\nu^{(\tau)}(ds,dz)-\nu^{\mathbb{G}}(ds,dz)\big)\Big],\ \forall t\geq 0\,.$$

On the other hand, one has by integration by parts

$$\mathbb{E}\Big[Y_t^{\mathbb{G}}\int_0^t\int_{\mathbb{R}}\gamma_s^{\mathbb{G}}(z)\,\widetilde{\mu}^{(\tau)}(ds,dz)\Big]=\mathbb{E}\Big[\int_0^t\int_{\mathbb{R}}\gamma_s^{\mathbb{G}}(z)\,\alpha_s^{\mathbb{G}}(z)\,\nu^{\mathbb{G}}(ds,dz)\Big],\ \forall t\geq 0\,.$$

Finally, (26) is equivalent to, for any $\gamma^{\mathbb{G}}$ satisfying (24), we have

$$\begin{aligned}&\mathbb{E}\Big[\int_0^t\int_{\mathbb{R}}\gamma_s^{\mathbb{G}}(z)\,\Big(\psi_s(\tau,z)\,\nu^{(\tau)}(ds,dz)+Y_{s-}(\tau)\,\big(\nu^{(\tau)}(ds,dz)-\nu^{\mathbb{G}}(ds,dz)\big)\Big)\Big]\\&=\mathbb{E}\Big[\int_0^t\int_{\mathbb{R}}\gamma_s^{\mathbb{G}}(z)\,\alpha_s^{\mathbb{G}}(z)\,\nu^{\mathbb{G}}(ds,dz)\Big],\ \forall t\geq 0\,.\end{aligned}\tag{29}$$

For $\gamma^{\mathbb{G}}(z)$ such that $\gamma_t^{\mathbb{G}}(z)=\mathbb{1}_{\{\tau\geq t\}}\gamma_t^0(z)$, $\forall t>0$, $\forall z\in\mathbb{R}$, where γ^0 is $\mathcal{P}(\mathbb{F})\otimes\mathcal{B}(\mathbb{R})$-measurable, using the identities (7) and (14), we have

$$\begin{aligned}&\mathbb{E}\Big[\int_0^t\int_{\mathbb{R}}\gamma_s^0(z)\,\mathbb{1}_{\{\tau\geq s\}}\left[\big(\psi_s(\tau,z)+Y_{s-}(\tau)\big)\left(\frac{f_s(\tau,z)}{p_{s-}(\tau)}+1\right)-Y_{s-}(\tau)\left(\frac{\varphi_s(z)}{G_{s-}}+1\right)\right]\eta_s(dz)\,ds\Big]\\&=\mathbb{E}\Big[\int_0^t\int_{\mathbb{R}}\gamma_s^0(z)\,\mathbb{1}_{\{\tau\geq s\}}\,\alpha_s^0(z)\left(\frac{\varphi_s(z)}{G_{s-}}+1\right)\eta_s(dz)\,ds\Big],\ \forall t\geq 0\,,\end{aligned}\tag{30}$$

and, introducing by tower property a conditioning with respect to $\mathcal{F}_s$ and using the existence of the conditional density, setting $F_t(u,z)=f_t(u,z)/p_{s-}(u)+1$ and $\Phi_t(z)=\varphi_t(z)/G_{t-}+1$, $\forall t,u\geq 0$, $\forall z\in\mathbb{R}$, the left-hand side of (30) is equal to

$$\begin{aligned}&\mathbb{E}\Big[\int_0^t\int_{\mathbb{R}}\gamma_s^0(z)\,\mathbb{1}_{\{\tau\geq s\}}\Big(\big(\psi_s(\tau,z)+Y_{s-}(\tau)\big)\,F_s(\tau,z)-Y_{s-}(\tau)\,\Phi_s(z)\Big)\,\eta_s(dz)\,ds\Big]\\&\overset{\text{TP}}{=}\mathbb{E}\Big[\int_0^t\int_{\mathbb{R}}\gamma_s^0(z)\int_s^\infty\Big(\big(\psi_s(u,z)+Y_{s-}(u)\big)\,F_s(\tau,z)-Y_{s-}(u)\,\Phi_s(z)\Big)\,p_{s-}(u)\,\rho(du)\,\eta_s(dz)\,ds\Big],\\&\forall t\geq 0\,,\end{aligned}\tag{31}$$

where, in the last equality, we have used the fact that the $\mathbb{F}$-predictable projection of $p(u)$ is $p_-(u)$, the process $p(u)$ being a martingale, for each $u \geq 0$ fixed.

We note also that, using the fact that G_- is the $\mathbb{F}$-predictable projection of G (see Remark 2), the right-hand side of (30) is

$$\begin{aligned}
&\mathbb{E}\bigg[\int_0^t \int_{\mathbb{R}} \gamma_s^0(z)\, \mathbb{1}_{\{\tau \geq s\}}\, \alpha_s^0(z)\, \Phi_s(z)\, \eta_s(dz)\, ds\bigg] \\
&= \mathbb{E}\bigg[\int_0^t \int_{\mathbb{R}} \gamma_s^0(z)\, G_s\, \alpha_s^0(z)\, \Phi_s(z)\, \eta_s(dz)\, ds\bigg] \\
&= \mathbb{E}\bigg[\int_0^t \int_{\mathbb{R}} \gamma_s^0(z)\, G_{s-}\, \alpha_s^0(z)\, \Phi_s(z)\, \eta_s(dz)\, ds\bigg],\ \forall t \geq 0\,. \qquad (32)
\end{aligned}$$

It follows from (30) that the right-hand sides of (31) and (32) are equal, for any γ^0, and hence,

$$\begin{aligned}
&\int_t^\infty \Big(\big(\psi_t(u,z) + Y_{t-}(u)\big)\, F_t(u,z) - Y_{t-}(u)\, \Phi_t(z)\Big)\, p_{t-}(u)\, \rho(du) \\
&= \int_t^\infty \left(\big(\psi_t(u,z) + Y_{t-}(u)\big)\, \big(f_t(u,z) + p_t(u)\big) - Y_{t-}(u)\left(\frac{\varphi_t(z)}{G_{t-}} + 1\right)\right) p_{t-}(u)\, \rho(du) \\
&= G_{t-}\, \alpha_t^0(z)\, \Phi_t(z) = \alpha_t^0(z)\, \big(\varphi_t(z) + G_{t-}\big),\ \forall t \geq 0,\ \forall z \in \mathbb{R}\,,
\end{aligned}$$

and expression (19) holds.

Using the identities (7) and (14), for $\gamma^{\mathbb{G}}$ of the form $\gamma_t^{\mathbb{G}} = \gamma_t(\tau, z)\mathbb{1}_{\{\tau < t\}}$, $\forall s > 0$, for $\gamma \in \mathbb{P}(\mathbb{F}) \otimes \mathcal{B}(\mathbb{R}_+) \otimes \mathcal{B}(\mathbb{R})$, equality (29) leads to

$$\begin{aligned}
&\mathbb{E}\bigg[\int_0^t \int_{\mathbb{R}} \gamma_s(\tau, z)\, \mathbb{1}_{\{\tau < s\}}\, \psi_s(\tau, z)\, F_s(\tau, z)\, \eta_s(dz)\, ds\bigg] \\
&= \mathbb{E}\bigg[\int_0^t \int_{\mathbb{R}} \gamma_s(\tau, z)\, \mathbb{1}_{\{\tau < s\}}\, \alpha_s(\tau, z)\, F_s(\tau, z)\, \eta_s(dz)\, ds\bigg],\ \forall t \geq 0\,,
\end{aligned}$$

and we can choose $\alpha = \psi$ on the event $\{\tau < t\}$, so that expression (20) holds.

Second step: In the second step, we compute the value of β^0, from the expression (27). It is straightforward to see that

$$\mathbb{E}\bigg[Y_t^{\mathbb{G}} \int_0^t \theta_s^0\, dM_s^{\mathbb{G}}\bigg] = \mathbb{E}\bigg[\int_0^t \beta_s^0\, \theta_s^0\, \lambda_s\, \mathbb{1}_{\{\tau > s\}}\, \rho(ds)\bigg] \overset{\text{TP}}{=} \mathbb{E}\bigg[\int_0^t \beta_s^0\, \theta_s^0\, \lambda_s\, G_s\, \rho(ds)\bigg],\ \forall t \geq 0\,.$$

From the definition of $M^{\mathbb{G}}$, it follows that

$$\begin{aligned}
&\mathbb{E}\bigg[Y_t(\tau) \int_0^t \theta_s^0\, dM_s^{\mathbb{G}}\bigg] = \mathbb{E}\bigg[Y_t(\tau)\left(\mathbb{1}_{\{\tau \leq t\}}\, \theta_\tau^0 - \int_0^t \mathbb{1}_{\{\tau > s\}}\, \theta_s^0\, \lambda_s\, \rho(ds)\right)\bigg] \\
&\overset{\text{TP}}{=} \mathbb{E}\bigg[\int_0^t Y_t(s)\, \theta_s^0\, p_t(s)\, \rho(ds) - \int_0^t \theta_s^0\, \lambda_s\, \mathbb{E}\big[Y_s^{(\tau)}\, \mathbb{1}_{\{\tau > s\}} \,\big|\, \mathcal{F}_s\big]\, \rho(ds)\bigg] \\
&= \mathbb{E}\bigg[\int_0^t Y_{s-}(s)\, p_{s-}(s)\, \theta_s^0\, \rho(ds) - \int_0^t \theta_s^0\, \lambda_s \left(\int_s^\infty Y_{s-}(u)\, p_{s-}(u)\, \rho(du)\right) \rho(ds)\bigg] \\
&= \mathbb{E}\bigg[\int_0^t {}^{p,\mathbb{F}}\Sigma_s\, p_{s-}(s)\, \theta_s^0\, \rho(ds) - \int_0^t \theta_s^0\, \lambda_s \left(\int_s^\infty Y_{s-}(u)\, p_{s-}(u)\, \rho(du)\right) \rho(ds)\bigg],\ \forall t \geq 0\,,
\end{aligned}$$

where we have used in the third equality that $Y(u)p(u)$ is an $\mathbb{F}$-martingale [1, Pro 4.33] with predictable projection $Y_-(u)p_-(u)$, for each $u \geq 0$, and defined $\Sigma = (\Sigma_t)_{t\geq 0}$ by $\Sigma_t = Y_{t-}(t)$, for all $t \geq 0$. We are not able to give conditions so that Σ is predictable, since we do not have regularity of the process $Y_{t-}(u)$ with respect to u, for each $u \geq 0$, this is why we have to take its predictable projection.

It follows that

$$\begin{aligned}\beta_t^0 &= \frac{1}{\lambda_t G_{t-}}\left({}^{p,\mathbb{F}}\Sigma_t\, p_{t-}(t) - \lambda_t \int_t^\infty Y_{t-}(u)\, p_{t-}(u)\, \rho(du)\right) \\ &= {}^{p,\mathbb{F}}\Sigma_t - \frac{1}{G_{t-}} \int_t^\infty Y_{t-}(u)\, p_{t-}(u)\, \rho(du),\ \forall t \geq 0\,,\end{aligned} \tag{33}$$

where we have used the fact that $\lambda_t = p_{t-}(t)/G_{t-}$, for $t \geq 0$. The expression in (33) implies the fact means that (21) holds with (22).

Third step: The extension to $\mathbb{F}^{(\tau)}$-martingales is done using usual localisation procedure (see Third step of Proof of Proposition 5.1 in [14]). □

5.2 The projections of $\mathbb{F}^{(\tau)}$-martingales on $\mathbb{F}$

Proposition 5.2 *Let $Y(\tau)$ be an $\mathbb{F}^{(\tau)}$-martingale with the representation given by equality* (8)*. Then, its $\mathbb{F}$-optional projection $Y = (Y_t)_{t\geq 0}$ admits the representation* (3)*, with $\mathcal{P}(\mathbb{F}) \otimes \mathcal{B}(\mathbb{R})$-measurable process ξ, given by*

$$\xi_t(z) = \int_0^\infty \Big(\psi_t(u,z)\,\big(f_t(u,z) + p_{t-}(u)\big) + Y_{t-}(u)\, f_t(u,z)\Big)\, \rho(du),\ \forall t \geq 0\,.$$

Proof: As before, we assume that $Y(\tau)$ is square integrable. Then, consider a bounded $\mathcal{P}(\mathbb{F}) \otimes \mathcal{B}(\mathbb{R})$-measurable process ζ such that $\zeta(z) = (\zeta_t(z))_{t\geq 0}$, for each $z \in \mathbb{R}$, and define the process $K = (K_t)_{t\geq 0}$ by

$$K_t = K_0 + \int_0^t \int_{\mathbb{R}} \zeta_s(z)\, \widetilde{\mu}(ds, dz),\ \forall t \geq 0\,.$$

It is seen that the process K is a square integrable $\mathbb{G}$-martingale, since the process ζ satisfies the condition

$$\mathbb{E}\left[\int_0^t \int_{\mathbb{R}} \zeta_s^2(z)\, \eta_s(dz)\, ds\right] < \infty,\ \forall t \geq 0\,.$$

In this case, the square integrable random variable $Y_t = \mathbb{E}[Y_t(\tau)\,|\,\mathcal{F}_t]$ is the only $\mathcal{F}_t$-measurable random variable such that

$$\mathbb{E}\big[Y_t(\tau)\, K_t\big] = \mathbb{E}\big[Y_t\, K_t\big],\ \forall t \geq 0\,, \tag{34}$$

holds. Thus, the equality (34) is equivalent to the following equality

$$\mathbb{E}\left[Y_t(\tau)\, \int_0^t \int_{\mathbb{R}} \zeta_s(z)\, \widetilde{\mu}(ds, dz)\right] = \mathbb{E}\left[Y_t\, \int_0^t \int_{\mathbb{R}} \zeta_s(z)\, \widetilde{\mu}(ds, dz)\right],\ \forall t \geq 0\,.$$

On the one hand, one has

$$\mathbb{E}\Big[Y_t(\tau)\int_0^t\int_{\mathbb{R}}\zeta_s(z)\,\widetilde{\mu}(ds,dz)\Big]$$
$$=\mathbb{E}\Big[Y_t(\tau)\int_0^t\int_{\mathbb{R}}\zeta_s(z)\,\widetilde{\mu}^{(\tau)}(ds,dz)+Y_t(\tau)\int_0^t\int_{\mathbb{R}}\zeta_s(z)\,\big(\nu^{(\tau)}(ds,dz)-\eta_s(dz)\,ds\big)\Big],\ \forall t\geq 0\,,$$

where $\nu^{(\tau)}$ is defined in (7). Integrating by parts on the time interval $[0,t]$ the product of the two $\mathbb{F}^{(\tau)}$-martingales $Y(\tau)$ and $\Phi(\tau)=(\Phi_t(\tau))_{t\geq 0}$ defined by

$$\Phi_t(\tau)=\int_0^t\int_{\mathbb{R}}\zeta_s(z)\,\widetilde{\mu}^{(\tau)}(ds,dz),\ \forall t\geq 0\,,$$

one has, using the square integrability assumption, that

$$\mathbb{E}\big[Y_t(\tau)\,\Phi_t(\tau)\big]=\mathbb{E}\Big[\int_0^t\int_{\mathbb{R}}\zeta_s(z)\,\psi_s(\tau,z)\,\nu^{(\tau)}(ds,dz)\Big]$$
$$=\mathbb{E}\Big[\int_0^t\int_{\mathbb{R}}\zeta_s(z)\,\psi_s(\tau,z)\,F_s(\tau,z)\,\eta_s(dz)\,ds\Big],\forall t\geq 0\,.$$

By integrating by parts the product of $Y(\tau)$ and the process $\Delta(\tau)=(\Delta_t(\tau))_{t\geq 0}$ of bounded variation defined by

$$\Delta_t(\tau)=\int_0^t\int_{\mathbb{R}}\zeta_s(z)\,\big(\nu^{(\tau)}(ds,dz)-\eta_s(dz)\,ds\big),\ \forall t\geq 0\,,$$

one obtains, using the equality (7), that

$$\mathbb{E}\big[Y_t(\tau)\,\Delta_t(\tau)\big]=\mathbb{E}\Big[\int_0^t\int_{\mathbb{R}}\zeta_s(z)\,Y_{s-}(\tau)\,\frac{f_s(\tau,z)}{p_s(\tau)}\,\eta_s(dz)\,ds\Big],\ \forall t\geq 0\,.$$

Hence, we have

$$\mathbb{E}\Big[Y_t(\tau)\int_0^t\int_{\mathbb{R}}\zeta_s(z)\,\widetilde{\mu}(ds,dz)\Big]=\mathbb{E}\big[Y_t(\tau)\,\Phi_t(\tau)\big]+\mathbb{E}\big[Y_t(\tau)\,\Delta_t(\tau)\big],\ \forall t\geq 0\,.$$

On the other hand, one has

$$\mathbb{E}\Big[Y_t\int_0^t\int_{\mathbb{R}}\zeta_s(z)\,\widetilde{\mu}(ds,dz)\Big]=\mathbb{E}\Big[\int_0^t\int_{\mathbb{R}}\zeta_s(z)\,\xi_s(z)\,\eta_s(dz)\,ds\Big],\ \forall t\geq 0\,.$$

Finally, the expression (34) implies

$$\mathbb{E}\Big[\int_0^t\int_{\mathbb{R}}\zeta_s(z)\int_0^\infty\Big(\zeta_s(u,z)\,f_s(u,z)+Y_{s-}(u)\,\big(f_s(u,z)-1\big)\Big)\,p_{s-}(u)\,\rho(du)\,\eta_s(dz)\,ds\Big]$$
$$=\mathbb{E}\Big[\int_0^t\int_{\mathbb{R}}\zeta_s(z)\,\xi_s(z)\,\eta_s(dz)\,ds\Big],\ \forall t\geq 0\,,$$

then, we obtain the expression (5.2). □

5.3 The projections of $\mathbb{G}$-martingales on $\mathbb{F}$

Proposition 5.3 *Let $Y^{\mathbb{G}}$ be a $\mathbb{G}$-martingale with the representation given by equality* (15)

$$Y_t^{\mathbb{G}} = Y_0^{\mathbb{G}} + \int_0^t \int_{\mathbb{R}} \alpha_s^{\mathbb{G}}(z)\, \widetilde{\mu}^{\mathbb{G}}(ds, dz) + \int_0^t \beta_s^0\, dM_s^{\mathbb{G}},\ \forall t \geq 0\,,$$

and the decomposition given in (4.2) *above.*

Then, its $\mathbb{F}$-optional projection Y is given by (3) above,

$$Y_t = Y_0 + \int_0^t \int_{\mathbb{R}} \xi_s(z)\, \widetilde{\mu}(ds, dz),\ \forall t \geq 0$$

where the $\mathcal{P}(\mathbb{F}) \otimes \mathcal{B}(\mathbb{R})$-measurable process ξ is given by

$$\begin{aligned}\xi_t(z) &= \alpha_t^0(z)\left(\varphi_t(z) + G_{t-}\right) + Y_{t-}^0\, \varphi_t(z) \\ &\quad + \int_0^t \Big(\alpha_t(u,z)\left(f_t(u,z) + p_{t-}(u)\right) + Y_{t-}(u)\, f_t(u,z)\, p_{t-}(u)\Big)\, \rho(du),\ \forall t \geq 0,\ \forall z \in \mathbb{R}\,,\end{aligned}$$

with the supermartingale G given by the equality (6).

Proof: As before, for any $\mathbb{G}$-adapted bounded process $\theta^{\mathbb{G}}$, we consider the equality satisfied by Y such that

$$\mathbb{E}\Big[Y_t \int_0^t \int_{\mathbb{R}} \theta_s^{\mathbb{G}}\, \widetilde{\mu}(ds, dz)\Big] = \mathbb{E}\Big[Y_t^{\mathbb{G}} \int_0^t \int_{\mathbb{R}} \theta_s^{\mathbb{G}}\, \widetilde{\mu}(ds, dz)\Big],\ \forall t \geq 0\,.$$

The left-hand side is equal to

$$\mathbb{E}\Big[\int_0^t \int_{\mathbb{R}} \xi_s(z)\, \theta_s^{\mathbb{G}}\, \eta_s(dz)\, ds\Big],\ \forall t \geq 0\,.$$

The right-hand side is equal to

$$\begin{aligned}&\mathbb{E}\Big[Y_t^{\mathbb{G}} \int_0^t \int_{\mathbb{R}} \theta_s^{\mathbb{G}}\left(\widetilde{\mu}^{\mathbb{G}}(ds, dz) + \nu_s^{\mathbb{G}}(ds, dz) - \eta_s(dz)\, ds\right)\Big] \\ &= \mathbb{E}\Big[\int_0^t \int_{\mathbb{R}} \alpha_s^{\mathbb{G}}\, \theta_s^{\mathbb{G}}\, \nu^{\mathbb{G}}(ds, dz) + \int_0^t \int_{\mathbb{R}} \theta_s^{\mathbb{G}}\, Y_{s-}^{\mathbb{G}}\left(\nu^{\mathbb{G}}(ds, dz) - \eta_s(dz)\, ds\right)\Big] \\ &= \mathbb{E}\Big[\int_0^t \int_{\mathbb{R}} \theta_s^{\mathbb{G}}\left(\alpha_s^0(z)\left(\frac{\varphi_s(z)}{G_{s-}} + 1\right) + Y_{s-}^0\, \frac{\varphi_s(z)}{G_{s-}}\right) \eta_s(dz)\, \mathbb{1}_{\{\tau > s\}}\, ds\Big] \\ &\quad + \mathbb{E}\Big[\int_0^t \int_{\mathbb{R}} \theta_s^{\mathbb{G}}\left(\int_0^s \Big(\alpha_s(u,z)\left(f_s(u,z) + p_{s-}(u)\right) + Y_{s-}(u)\, f_s(u,z)\Big)\, \rho(du)\right) \eta_s(dz)\, ds\Big], \\ &= \mathbb{E}\Big[\int_0^t \int_{\mathbb{R}} \theta_s^{\mathbb{G}}\left(\alpha_s^0(z)\left(\varphi_s(z) + G_s\right) + Y_{s-}^0\, \varphi_s(z)\right) \eta_s(dz)\, ds\Big] \\ &\quad + \mathbb{E}\Big[\int_0^t \int_{\mathbb{R}} \theta_s^{\mathbb{G}}\left(\int_0^s \Big(\alpha_s(u,z)\left(f_s(u,z) + p_s(u)\right) + Y_{s-}(u)\, f_s(u,z)\Big)\, \rho(du)\right) \eta_s(dz)\, ds\Big], \\ &\forall t \geq 0\,,\end{aligned}$$

where $\nu^{\mathbb{G}}$ is defined in (14). Hence, this equality being true for any $\theta^{\mathbb{G}}$, the proof is complete. □

Remark 5.4 Using the same methodology, wee can extend the result in the Brownian case, established in [14] under equivalence Jacod's hypothesis to the case of absolute continuity hypothesis. We do not give details,the results are the same as the ones in [14].

6 Changes of probability measures and applications

In this section, as an example of application of the results from the previous section, we consider the relationships between strictly positive $\mathbb{F}^{(\tau)}$-martingales (or $\mathbb{G}$-martingales) and their optional projections. We then apply the results in a financial market framework to study the set of equivalent martingale measures in different filtrations.

A probability measure $\mathbb{Q}$ is said to be *locally equivalent* to $\mathbb{P}$ on the filtration $\mathbb{H}$ if there exists a strictly positive $\mathbb{H}$-martingale $L = (L_t)_{t\geq 0}$ such that

$$\left.\frac{d\mathbb{Q}}{d\mathbb{P}}\right|_{\mathcal{H}_t} = L_t,\ \forall t \geq 0\,.$$

The martingale L is called the *Radon-Nikodym* density of $\mathbb{Q}$ with respect to $\mathbb{P}$. The "locally" terminology is needed, since as in [3], we cannot define the new probability measure $\mathbb{Q}$ on $\mathcal{H}_\infty$, because the density process L is not necessarily an uniformly integrable martingale on $\mathbb{H}$.

6.1 The projections of strictly positive $\mathbb{F}^{(\tau)}$-martingales on $\mathbb{G}$

Let $L(\tau)$ be a strictly positive $\mathbb{F}^{(\tau)}$-martingale. Then, in particular, we have $L_0(\tau) > 0$ ($\mathbb{P}$-a.s.). Moreover, applying Proposition 4.2 with $Y(\tau) = L(\tau)$, we can write $L(\tau)$ in the form of

$$L_t(\tau) = L_0(\tau) + \int_0^t L_{s-}(\tau) \int_{\mathbb{R}^*} \big(\Theta_s(\tau, z) - 1\big)\, \widetilde{\mu}^{(\tau)}(ds, dz)\, \forall t \geq 0\,, \tag{35}$$

where $L_t(\tau)(\Theta_t(\tau, z) - 1) = \psi_t(\tau, z)$, $\forall t \geq 0$, $\forall z \in \mathbb{R}$. Note that, since at jumps times $L_{T_n} = L_{T_n-}\Theta_{T_n}$, $\forall n \geq 1$, one has $\Theta > 0$. Note that, if $\mathbb{E}[L_0(\tau)] = 1$, then we can associate to the strictly positive $\mathbb{F}^{(\tau)}$-martingale $L(\tau)$ the probability measure $\widetilde{\mathbb{P}}$ locally equivalent to $\mathbb{P}$ on the filtration $\mathbb{F}^{(\tau)}$ defined by

$$\left.\frac{d\widetilde{\mathbb{P}}}{d\mathbb{P}}\right|_{\mathcal{F}_t^{(\tau)}} = L_t(\tau),\ \forall t \geq 0\,.$$

Remark 6.1 As in Remark 6.1 in [14], the particular choice of $L_0(\tau) = 1$ ($\mathbb{P}$-a.s.) is equivalent to the property $\widetilde{\mathbb{P}}(\tau > u) = \mathbb{P}(\tau > u)$, for each $u \geq 0$.

We now consider the $\mathbb{G}$-optional projection $L^{\mathbb{G}} = (L_t^{\mathbb{G}})_{t\geq 0}$ of the strictly positive martingale $L(\tau)$. In this case, applying Proposition 4.6 with $Y^{\mathbb{G}} = L^{\mathbb{G}}$, and setting $L(\varkappa - 1) = \alpha^{\mathbb{G}}$, so that $\varkappa > 0$, we see that $L^{\mathbb{G}}$ admits the representation

$$L_t^{\mathbb{G}} = L_0^{\mathbb{G}} + \int_0^t L_{s-}^{\mathbb{G}} \int_{\mathbb{R}} \big(\varkappa_s^{\mathbb{G}}(z) - 1\big)\, \widetilde{\mu}^{\mathbb{G}}(ds, dz) + \int_0^t L_{s-}^{\mathbb{G}}\, \xi_s^0\, dM_s^{\mathbb{G}},\ \forall t \geq 0\,. \tag{36}$$

with a strictly positive and $\mathcal{P}(\mathbb{G}) \otimes \mathcal{B}(\mathbb{R})$-measurable process $\varkappa^{\mathbb{G}}$ satisfying

$$\int_0^t \int_{\mathbb{R}} \left| \varkappa_s^{\mathbb{G}}(z) - 1 \right| \nu^{\mathbb{G}}(ds, dz) < \infty, \, \forall t \geq 0 \, ,$$

where $\nu^{\mathbb{G}}$ is defined in (14).

Here, the process $\varkappa^{\mathbb{G}}$ is of the form

$$\varkappa_t^{\mathbb{G}}(z) = \mathbb{1}_{\{\tau \geq t\}} \, \varkappa_t^0(z) + \mathbb{1}_{\{\tau < t\}} \, \varkappa_t(\tau, z), \, \forall t \geq 0, \, \forall z \in \mathbb{R} \, ,$$

where $\varkappa^0$ is a $\mathcal{P}(\mathbb{F}) \otimes \mathcal{B}(\mathbb{R})$-measurable process, $\varkappa$ is a $\mathcal{P}(\mathbb{F}) \otimes \mathcal{B}(\mathbb{R}_+) \otimes \mathcal{B}(\mathbb{R})$-measurable process, while $\xi^0 > -1$ is an $\mathbb{F}$-predictable process.

Proposition 6.2 *Let $L(\tau) = (L_t(\tau))_{t \geq 0}$ be a strictly positive martingale of the form (35). Then, its $\mathbb{G}$-optional projection $L^{\mathbb{G}}$ satisfies (36) with the $\mathbb{G}$-predictable processes $\varkappa^{\mathbb{G}}$ and the $\mathbb{F}$-predictable process ξ^0 given by*

$$\begin{aligned}
&\varkappa_t^{\mathbb{G}}(z) - 1 = \frac{\mathbb{1}_{\{\tau \geq t\}}}{L_{t-}^0(\varphi_t(z) + G_{t-})} \qquad (37) \\
&\times \int_t^\infty L_{t-}(u) \left(\Theta_t(u,z) \left(f_t(u,z) + p_{t-}(u) \right) - \left(\frac{\varphi_t(z)}{G_{t-}} + 1 \right) p_{t-}(u) \right) \rho(du) \\
&+ \mathbb{1}_{\{\tau < t\}} \left(\Theta_t(\tau, z) - 1 \right), \, \forall t \geq 0 \, , \, \forall z \in \mathbb{R} \, , \\
&\xi_t^0 = \frac{^{p,\mathbb{F}}(L_{t-}(t))}{L_{t-}^0} - 1, \, \forall t \geq 0 \, , \qquad (38) \\
&L_{t-}^0 = \frac{1}{G_{t-}} \int_t^\infty L_{t-}(u) \, p_{t-}(u) \, \rho(du), \, \forall t \geq 0 \, ,
\end{aligned}$$

where $L^0 = (L_t^0)_{t \geq 0}$ is the $\mathbb{F}$-predictable reduction of $L^{\mathbb{G}}$ and L_-^0 is its left-hand limit.

Proof: Consider the $\mathbb{F}^{(\tau)}$-martingale $L(\tau)$ given by equality (35). In this case, its $\mathbb{G}$-optional projection $L^{\mathbb{G}}$ has the form of (36). Then, Proposition 5.1 applies with $Y(\tau) = L(\tau)$ and $\psi(u,z) = L_-(u)(\Theta(u,z) - 1)$, for all $u \geq 0$ and $z \in \mathbb{R}$, and therefore, equalities $\psi = L_t^{\mathbb{G}}(\varkappa_t^{\mathbb{G}}(z) - 1) = \alpha_t^{\mathbb{G}}(z)$. That is, $L_{t-}^0(\varkappa_t^0(z) - 1) = \alpha_t^0(z)$ and $L_{t-}(u)(\varkappa_t(u,z) - 1) = \alpha_t(u,z)$, for all $u \geq 0$ and $z \in \mathbb{R}$, and $\xi_t^0 L_{t-}^0 = \beta_t^0$ hold, for all $t \geq 0$ and $z \in \mathbb{R}$, and

$$\begin{aligned}
&L_{t-}(u) \left(\varkappa_t(u,z) - 1 \right) = \frac{1}{\varphi_t(z) + G_{t-}} \\
&\times \int_t L_{t-}(u) \left(\Theta_t(u,z) \big(f_t(u,z) + p_{t-}(u) \big) - \left(\frac{\varphi_t(z)}{G_{t-}} + 1 \right) p_{t-}(u) \right) \rho(du), \, \forall t \geq 0 \, .
\end{aligned}$$

□

6.2 The projections of strictly positive $\mathbb{F}^{(\tau)}$-martingales on $\mathbb{F}$

Let $L(\tau)$ be a strictly positive $\mathbb{F}^{(\tau)}$-martingale of the form (35). Then, applying Proposition 5.2 with $Y(\tau) = L(\tau)$, we obtain that the $\mathbb{F}$-optional projection L of $L(\tau)$ admits the integral representation

$$L_t = \mathbb{E}\big[L_0(\tau)\big] + \int_0^t L_{s-} \int_{\mathbb{R}} \chi_s(z)\, \widetilde{\mu}(ds, dz),\ \forall t \geq 0\,,$$

where the $\mathcal{P}(\mathbb{F}) \otimes \mathcal{B}(\mathbb{R})$-measurable process χ is given by

$$\chi_t(z) = \frac{1}{L_{t-}} \int_0^\infty L_{t-}(u) \Big(\Theta_t(u,z)\,\big(f_t(u,z) + p_{t-}(u)\big) + f_t(u,z)\Big)\, \rho(du),\ \forall t \geq 0\,, \forall z \in \mathbb{R}\,.$$

6.3 The projections of strictly positive $\mathbb{G}$-martingales on $\mathbb{F}$

It follows from Proposition 5.3 that any strictly positive $\mathbb{G}$-martingale $L^{\mathbb{G}} = (L^{\mathbb{G}}_t)_{t\geq 0}$ admits the equivalent representation (36) and, being a $\mathbb{G}$-optional process, it admits the decomposition

$$L^{\mathbb{G}}_t = \mathbb{1}_{\{\tau > t\}}\, L^0_t + \mathbb{1}_{\{\tau \leq t\}}\, L^1_t(\tau),\ \forall t \geq 0\,,$$

where the process L^0 is $\mathbb{F}$-optional and the process L^1 is $\mathcal{O}(\mathbb{F}) \otimes \mathcal{B}(\mathbb{R}_+)$-measurable. By similar arguments, it follows that its $\mathbb{F}$-optional projection $L = (L_t)_{t\geq 0}$ admits the integral representation

$$L_t = L^{\mathbb{G}}_0 + \int_0^t L_{s-} \int_{\mathbb{R}} \sigma_s(z)\, \widetilde{\mu}(ds, dz),\ \forall t \geq 0\,,$$

where $\sigma(z) = (\sigma_t(z))_{t\geq 0}$ is an $\mathbb{F}$-predictable process. In order to derive σ, it suffices to apply Proposition 5.3 with $Y^{\mathbb{G}} = L^{\mathbb{G}}$, $(L_-(\varkappa(z) - 1))^0 = \alpha^0(z)$, $L_-(\varkappa(z) - 1) = \alpha(z)$ and $L_-\sigma(z) = \xi(z)$, for all $z \in \mathbb{R}$, so that $Y^0 = L^0$ and $Y = L$. The equality $(L_-(\varkappa(z) - 1))^0 = L^0_-(\varkappa^0(z) - 1)$, for all $z \in \mathbb{R}$, follows from the definition of predictable reduction. Therefore, we conclude, after easy simplifications, that

$$\begin{aligned}\sigma_t(z) = \frac{1}{L_{t-}} \Bigg(& L^0_{t-} \Big(\varkappa^0_t(z)\,\big(\varphi_t(z) + G_{t-}\big) - G_t\Big) \\ & + \int_0^t L^1_t(u) \Big(\big(\varkappa_t(u,z) - 1\big)\,\big(f_t(u,z) + p_{t-}(u)\big) + f_t(u,z)\, p_{t-}(u)\Big)\, \rho(du)\Bigg),\ \forall t \geq 0\,, \forall z \in \mathbb{R}\,.\end{aligned}$$

6.4 The equivalent martingale measures

Let us now consider a model of a financial market in which the risky asset price process $S = (S_t)_{t\geq 0}$ follows the stochastic differential equation

$$dS_t = S_t\, \delta_t\, dt + S_{t-} \int_{\mathbb{R}} h_t(z)\, \widetilde{\mu}(dt, dz)\,,$$

where h belongs to $\mathcal{P}(\mathbb{F}) \otimes \mathcal{B}(\mathbb{R})$ and is greater than -1 (to satisfy the positivity of S). We also assume that the riskless asset has a zero interest rate and δ is $\mathbb{F}$-adapted.

Change of probability in $\mathbb{F}$: A change of the probability measure in $\mathbb{F}$ has a Radon-Nikodym density process $L = (L_t)_{t\geq 0}$ satisfying the stochastic differential equation given from the weak predictable representation property [24, Pro. 2.1] by

$$dL_t = L_{t-} \int_{\mathbb{R}} \big(\alpha_t(z) - 1\big)\, \widetilde{\mu}(dt, dz), \quad L_0 = 1\,, \tag{39}$$

for a strictly positive $\mathcal{P}(\mathbb{F}) \otimes \mathcal{B}(\mathbb{R})$-measurable process α.

This Radon-Nikodym density process will correspond to an equivalent martingale measure if SL is an $\mathbb{F}$-martingale. From the integration-by-parts formula, we have

$$S_t\, L_t = S_0 + \int_0^t L_{s-}\, dS_s + \int_0^t S_{s-}\, dL_s + [S, L]_t,\ \forall t \geq 0\,,$$

where the quadratic variation[4] is given by

$$\begin{aligned}
\big[S, L\big]_t &= \sum_{T_n \leq t} S_{T_n-}\, L_{T_n-}\, h_{T_n}(Z_n)\, (\alpha_{T_n}(Z_n) - 1) \\
&= \int_0^t S_{s-}\, L_{s-} \int_{\mathbb{R}} h_s(z)\, \big(\alpha_s(z) - 1\big)\, \widetilde{\mu}(ds, dz) \\
&\quad + \int_0^t S_{s-}\, L_{s-} \int_{\mathbb{R}} h_s(z)\, \big(\alpha_s(z) - 1\big)\, \eta_s(dz)\, ds,\ \forall t \geq 0\,,
\end{aligned}$$

and

$$\begin{aligned}
S_t\, L_t &= S_0 + \int_0^t L_{s-}\, dS_s + \int_0^t S_{s-}\, dL_s + [S, L]_t, \\
&= S_0 + \int_0^t L_{s-} S_{s-} \left(\delta_s + \int_{\mathbb{R}} h_s(z)\, \frac{f_s(\tau, z)}{p_{s-}(\tau)}\, \eta_s(dz)\right) ds \\
&\quad + \int_0^t S_{s-}\, L_{s-} \int_{\mathbb{R}} h_s(z)\, \big(\alpha_s(z) - 1\big)\ \eta_s(dz)\, ds + \overline{M}_t,\ \forall t \geq 0\,,
\end{aligned}$$

where the process $\overline{M} = (\overline{M}_t)_{t\geq 0}$ is an $\mathbb{F}$-martingale.

We now define

$$\theta_t = \delta_t + \int_{\mathbb{R}} h_t(z)\, \frac{f_t(\tau, z)}{p_{t-}(\tau)}\, \eta_t(dz)$$

and we obtain that the process SL is an $\mathbb{F}$-martingale if and only if α (see (39)) satisfies the equality

$$\theta_t + \int_{\mathbb{R}} h_t(z)\, \big(\alpha_t(z) - 1\big)\, \eta_t(dz) = 0,\ \forall t \geq 0\,.$$

[4]For any two semimartingales $X = (X_t)_{t\geq 0}$ and $Y = (Y_t)_{t\geq 0}$, one has

$$[X, Y]_t = \langle X^c, Y^c \rangle_t + \sum_{0\leq s\leq t} \Delta X_s\, \Delta Y_s,\ \forall t \geq 0\,,$$

where X^c and Y^c are continuous (local) martingale parts of X and Y, while $\Delta X_t = X_t - X_{t-}$ and $\Delta Y_t = Y_t - Y_{t-}$ (see Chapter VIII, Definition 8.2, page 209 in [16]).

(see Section 2.1 in [24] for similar results). This equivalent martingale measure is unique in that case.

Change of probability in $\mathbb{G}$: In the filtration $\mathbb{G}$, using the equality (14), we have

$$\begin{aligned} dS_t &= S_t\,\delta_t\,dt + S_{t-}\int_{\mathbb{R}} h_t(z)\left(\nu^{\mathbb{G}}(dt,dz)-\nu(dt,dz)\right) + S_{t-}\int_{\mathbb{R}} h_t(z)\,\widetilde{\mu}^{\mathbb{G})}(dt,dz) \\ &= S_{t-}\left(\delta_t + \int_{\mathbb{R}} h_t(z)\left(\mathbb{1}_{\{\tau\geq t\}}\,\frac{\varphi_t(z)}{G_{t-}} + \mathbb{1}_{\{\tau<t\}}\,\frac{f_t(\tau,z)}{p_{t-}(\tau)}\right)\eta_t(dz)\right)dt + S_{t-}\int_{\mathbb{R}} h_t(z)\,\widetilde{\mu}^{\mathbb{G}}(dt,dz) \\ &= S_{t-}\,\theta_t^{\mathbb{G}}\,dt + S_{t-}\int_{\mathbb{R}} h_t(z)\,\widetilde{\mu}^{\mathbb{G}}(dt,dz)\,, \end{aligned}$$

where the last term is an $\mathbb{G}$-martingale and

$$\theta_t^{\mathbb{G}} = \delta_t + \int_{\mathbb{R}} h_t(z)\left(\mathbb{1}_{\{\tau\geq t\}}\,\frac{\varphi_t(z)}{G_{t-}} + \mathbb{1}_{\{\tau<t\}}\,\frac{f_t(\tau,z)}{p_{t-}(\tau)}\right)\eta_t(dz)\,.$$

Using the fact that any positive $\mathbb{G}$-martingale has the form

$$dL_t^{\mathbb{G}} = L_{t-}^{\mathbb{G}}\int_{\mathbb{R}}\left(\varkappa^{\mathbb{G}}(z)-1\right)\widetilde{\mu}^{\mathbb{G}}(dt,dz) + \xi_t^0\,dM_t^{\mathbb{G}}$$

for some strictly positive $\mathcal{P}(\mathbb{G})\otimes\mathcal{B}(\mathbb{R})$-measurable process $\varkappa^{\mathbb{G}}$ (see Proposition 4.6 and that

$$\left[S,L^{\mathbb{G}}\right]_t^{\mathbb{G}} = \int_0^t\int_{\mathbb{R}} h_s(z)\left(\varkappa_t^{\mathbb{G}}(z)-1\right)\eta_t^{\mathbb{G}}(dz),\ \forall t\geq 0\,,$$

the set $\mathcal{P}(\mathbb{G})$ of (locally) equivalent martingale measures on $\mathbb{G}$ corresponds to the set of Radon-Nikodym density processes of the form

$$dL_t^{\mathbb{G}} = L_{t-}^{\mathbb{G}}\left(\int_{\mathbb{R}}\left(\varkappa_t^{\mathbb{G}}(z)-1\right)\widetilde{\mu}^{\mathbb{G}}(dt,dz) + \xi_s^0\,dM_t^{\mathbb{G}}\right)$$

with $\varkappa^{\mathbb{G}}$ such that

$$\theta_t^{\mathbb{G}} + \int_{\mathbb{R}} h_t(z)\left(\varkappa_t^{\mathbb{G}}(z)-1\right)\nu^{\mathbb{G}}(dt,dz) = 0,\ \forall t\geq 0\,.$$

This change of probability is not unique, ξ^0 is to be chosen.

Change of probability in $\mathbb{F}^{(\tau)}$: In the filtration $\mathbb{F}^{(\tau)}$, using the representation (7), we have

$$\begin{aligned} dS_t &= S_t\,\delta_t\,dt + S_{t-}\int_{\mathbb{R}} h_t(z)\left(\nu^{(\tau)}(dt,dz)-\nu(dt,dz)\right) + S_{t-}\int_{\mathbb{R}} h_t(z)\,\widetilde{\mu}^{\tau}(dt,dz) \\ &= S_{t-}\left(\delta_t + \int_{\mathbb{R}} h_t(z)\,\frac{f_t(\tau,z)}{p_{t-}(\tau)}\,\eta_t(dz)\right)dt + S_{t-}\int_{\mathbb{R}} h_t(z)\,\widetilde{\mu}^{(\tau)}(dt,dz)\,, \end{aligned}$$

and the last term is an $\mathbb{F}^{(\tau)}$-martingale.

As it is seen in Proposition 4.2, a change of probability in $\mathbb{F}^{(\tau)}$ has the Radon-Nikodym density process $L(\tau) = (L_t(\tau))_{t\geq 0}$ satisfying the stochastic differential equation given in (35) which can be written in the form

$$dL_t(\tau) = L_{t-}(\tau) \int_{\mathbb{R}} \big(\Theta_t(\tau, z) - 1\big)\, \widetilde{\mu}^{(\tau)}(dt, dz), \quad L_0(\tau) = \ell(\tau)\,,$$

for a strictly positive $\mathcal{P}(\mathbb{F}^{(\tau)}) \otimes \mathcal{B}(\mathbb{R})$-measurable process Θ and a strictly positive Borel function $\ell(u)$, for $u \geq 0$. This Radon-Nikodym density process will correspond to an equivalent martingale measure if $SL(\tau)$ is an $\mathbb{F}^{(\tau)}$-martingale. Using the fact that

$$\begin{aligned}\big[S, L(\tau)\big]_t = &\int_0^t S_{s-}\, L_{s-}(\tau) \int_{\mathbb{R}} h_s(z)\, \big(\Theta_s(\tau, z) - 1\big)\, \widetilde{\mu}^{(\tau)}(ds, dz) \\ &+ \int_0^t S_{s-}\, L_{s-}(\tau) \int_{\mathbb{R}} h_s(z)\, \big(\Theta_s(\tau, z) - 1\big)\, \frac{f_s(\tau, z)}{p_{s-}(\tau)}\, \eta_s(dz)\, ds,\ \forall t \geq 0\,,\end{aligned}$$

and the $\mathbb{F}^{(\tau)}$-martingale property of the first term above, we obtain that the process $SL(\tau)$ is an $\mathbb{F}^{(\tau)}$-martingale if and only if Θ satisfies the equality

$$\delta_t + \int_t^\infty h_t(z)\, \frac{f_t(\tau, z)}{p_{t-}(\tau)}\, \eta_t(dz) + \int_t^\infty h_t(z)\, \big(\Theta_t(\tau, z)) - 1\big)\, \frac{f_t(\tau, z)}{p_{t-}(\tau)}\, \eta_t(dz) = 0,\ \forall t \geq 0\,,$$

which can be simplified to

$$\delta_t + \int_t^\infty h_t(z)\, \Theta_t(\tau, z)\, \frac{f_t(\tau, z)}{p_{t-}(\tau)}\, \eta_t(dz) = 0,\ \forall t \geq 0\,.$$

This change of probability is not unique, ℓ being to be chosen.

Let $\mathcal{P}^*$ be the set of $\mathbb{G}$-optional projections $L^{*,\mathbb{G}}$ of $L^*(\tau)$, which satisfies (36) where the processes $\varkappa^{\mathbb{G}}$ and ξ^0 given by equalities (37) and (38). More precisely, one has

$$\begin{aligned}&\varkappa^{\mathbb{G}}_t(z) - 1 = \mathbb{1}_{\{\tau \geq t\}}\, \frac{1}{L^{*,0}_{t-}(\varphi_t(z) + G_{t-})} \qquad (40)\\
&\times \int_t^\infty L^*_{t-}(u) \left(\Theta_t(u, z)\, \big(f_t(u, z) + p_{t-}(u)\big) - \left(\frac{\varphi_t(z)}{G_{t-}} + 1\right) p_{t-}(u)\right) \rho(du)\\
&+ \mathbb{1}_{\{\tau < t\}}\, \big(\Theta_t(\tau, z) - 1\big),\ \forall t \geq 0\,,\ \forall z \in \mathbb{R}\,,\\
&\xi^0_t = \frac{{}^{p,\mathbb{F}}(L^*_{t-}(t))}{L^{*,0}_{t-}} - 1,\ \forall t \geq 0\,, \qquad (41)\\
&L^{*,0}_{t-} = \frac{1}{G_{t-}} \int_t^\infty L^*_{t-}(u)\, p_{t-}(u)\, \rho(du),\ \forall t \geq 0\,, \qquad (42)\end{aligned}$$

where $L^{*,0}$ is the $\mathbb{F}$-predictable reduction of $L^{*,\mathbb{G}}$. Here, each element of $\mathcal{P}^*$ is a (locally) equivalent martingale measure on $\mathbb{G}$. Note that $\varkappa^{\mathbb{G}}$ does not depend on the choice of L^*_0 (see (40)), whereas ξ^0 depends on it.

Since in (6.4), there are no constraints on ξ^0, the set $\mathcal{P}(\mathbb{G})$ is strictly greater than $\mathcal{P}^*$.

7 Appendix

Using the same methodology as in [14] we prove the martingale property of the two local martingales used in the proof of Proposition 5.1.

• We first prove that the $\mathbb{F}^{(\tau)}$-local martingale $\widetilde{M}(\tau) = (\widetilde{M}_t(\tau))_{t\geq 0}$ defined by

$$\widetilde{M}_t(\tau) = \int_0^t \Upsilon_{s-}\, dY_s(\tau),\, \forall t \geq 0\,,$$

is a true martingale. This will be the case when, for any $T > 0$ fixed, the property

$$\mathbb{E}\Big[\sup_{0\leq t\leq T} \big|\widetilde{M}_t(\tau)\big|\Big] < \infty$$

holds (see Chapter I, Theorem 51, page 38 in [26]). By Burkholder-Davis-Gundy's inequality[5], this condition is satisfied if

$$\mathbb{E}\Big[\big(\langle\widetilde{M}(\tau)\rangle_T^{\mathbb{F}^{(\tau)}}\big)^{1/2}\Big] < \infty\,.$$

Note that we have

$$\begin{aligned}
\mathbb{E}\big[\big(\langle\widetilde{M}(\tau)\rangle_T^{\mathbb{F}^{(\tau)}}\big)^{1/2}\big] &= \mathbb{E}\bigg[\bigg(\int_0^T \Upsilon_{s-}^2\, \psi_s^2(\tau,z)\, \nu^{(\tau)}(ds,dz)\bigg)^{1/2}\bigg] \\
&\leq \mathbb{E}\bigg[\sup_{0\leq s\leq T}\big|\Upsilon_s\big|\bigg(\int_0^T\int_{\mathbb{R}} \psi_s^2(\tau,z)\, \nu^{(\tau)}(ds,dz)\bigg)^{1/2}\bigg] \\
&\leq \mathbb{E}\bigg[\sup_{0\leq s\leq T}\big|\Upsilon_s\big|^2\bigg] + \mathbb{E}\bigg[\int_0^T\int_{\mathbb{R}} \psi_s^2(\tau,z)\, \nu^{(\tau)}(ds,dz)\bigg]\,,
\end{aligned}$$

where we have used the fact that $|ab| \leq (a^2 + b^2)$, for any $a, b \in \mathbb{R}$. Using again Burkholder-Davis-Gundy's inequality Υ defined in (28) being a martingale, we obtain that

$$\mathbb{E}\Big[\sup_{0\leq s\leq T}\big|\Upsilon_s\big|^2\Big] \leq \widetilde{C}\,\mathbb{E}\bigg[\int_0^T (\gamma_s^{\mathbb{G}})^2\, \nu^{(\tau)}(ds,dz)\bigg] < \infty\,,$$

for some constant $\widetilde{C} > 0$. Moreover, by the assumption of square integrability of the $\mathbb{F}^{(\tau)}$-martingale $Y(\tau)$, we have

$$\mathbb{E}\bigg[\int_0^T\int_{\mathbb{R}} \psi_s^2(\tau,z)\, \nu^{(\tau)}(ds,dz)\bigg] < \infty\,,$$

so that the process $\widetilde{M}(\tau)$ is a martingale.

• We now prove that the $\mathbb{F}^{(\tau)}$-local martingale $\widehat{M}(\tau) = (\widehat{M}_t(\tau))_{t\geq 0}$ defined by

$$\widehat{M}_t(\tau) = \int_0^t Y_s(\tau)\, d\Upsilon_s,\, \forall t \geq 0\,,$$

[5]Burkholder-Davis-Gundy's inequality states that, if M is a local martingale, for any $p \geq 1$, then the expression

$$\mathbb{E}\Big[\sup_{0\leq t\leq T}\big|M_t\big|^p\Big] \leq C_p\,\mathbb{E}\big[(\langle M\rangle_T)^{p/2}\big]$$

holds, for some $C_p > 0$ depending on p only (see, e.g., Chapter IV, Section 4, Theorem 48, page 195 in [26]).

is a true martingale. As above, by Burkholder-Davis-Gundy's inequality, this will be the case when, for any $T > 0$ fixed

$$\mathbb{E}\Big[\big(\langle\widehat{M}(\tau)\rangle_T^{\mathbb{F}^{(\tau)}}\big)^{1/2}\Big] < \infty\,.$$

Note that we have

$$\begin{aligned}
\mathbb{E}\big[\big(\langle\widehat{M}(\tau)\rangle_T^{\mathbb{F}^{(\tau)}}\big)^{1/2}\big] &= \mathbb{E}\bigg[\bigg(\int_0^T Y_s^2(\tau)\,\big(\gamma_s^{\mathbb{G}}(z)\big)^2\,\nu^{(\tau)}(ds,dz)\bigg)^{1/2}\bigg] \\
&\leq \mathbb{E}\bigg[\sup_{0\leq s\leq T}\big|Y_s(\tau)\big|\,\bigg(\int_0^T\int_{\mathbb{R}}\big(\gamma_s^{\mathbb{G}}(z)\big)^2\,\nu^{(\tau)}(ds,dz)\bigg)^{1/2}\bigg] \\
&\leq \mathbb{E}\bigg[\sup_{0\leq s\leq T}\big|Y_s(\tau)\big|^2\bigg] + \mathbb{E}\bigg[\int_0^T\int_{\mathbb{R}}\big(\gamma_s^{\mathbb{G}}(z)\big)^2\,\nu^{(\tau)}(ds,dz)\bigg]\,.
\end{aligned}$$

It follows, using again Burkholder-Davis-Gundy's inequality, that

$$\mathbb{E}\Big[\sup_{0\leq s\leq T}\big|Y_s(\tau)\big|^2\Big] \leq \widehat{C}\,\mathbb{E}\bigg[\int_0^T \psi_s^2(\tau,z)\,\nu^{(\tau)}(ds,dz)\bigg] < \infty\,,$$

for some constant $\widehat{C} > 0$. Moreover, by the assumption of square integrability of the $\mathbb{F}^{(\tau)}$-martingale Υ, we have

$$\mathbb{E}\bigg[\int_0^T\int_{\mathbb{R}}\big(\gamma_s^{\mathbb{G}}\big)^2\,\nu^{(\tau)}(ds,dz)\bigg] < \infty\,,$$

so that the process $\widehat{M}(\tau)$ is a martingale.

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