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# Fourier Transform: Some Progress of a Chameleonic Math-Object

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# FOURIER TRANSFORM: SOME PROGRESS OF A CHAMELEONIC MATH-OBJECT

EDOARDO NICCOLAI

ABSTRACT. The Fourier transform is investigated in some of its multiple forms, inherently to the initial question, which acts as a fuse to the entire investigation.

KEYWORDS: Furier transform — Fourier transform in 4-Space, fractional Fourier expansion, Langevin–Einstein equation, Mittag-Leffler function, probability measure, random function, stochastic gravitation.

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## Thierry's Question

A new writing that takes its spark from a puzzle de mon ami français.

Autre question pour toi: comment se ferait la transformation de Fourier généralisée dans tes espaces fractoïdes [22]? Car pour Laurent [Nottale] déjà en relativité d'échelle restreinte on a des super résultats...

$\Lambda$ LDBB sert bien sûr à la transition classique-quantique tout comme la  $L$  de Compton dans le cas relativiste; quand on fait des changements d'échelles, il faut leur appliquer aussi (comme pour les vitesses, mais avec une autre constant qui est l'équivalent de  $c$ ) une transformation de type Lorentz en relativité restreinte, ici vue en relativité d'échelle, ce qui nous permet de résoudre le problème du rayon du proton, voir par ex. [25].

Ce qu'il y a de plus profond là dessous est que (et utile aussi en mécanique quantique): l'espace des  $x$  et l'espace des  $p$  ne sont pas reliés par une simple transformation de Fourier usuelle ici, mais dans le cas de  $(x, t)$  fractal (qui sous tend la mécanique quantique selon la relativité d'échelle) par une transformation de Fourier généralisée [...].

[Voir le livre] [24] [où se trouve] un exposé détaillé sur le calcul de la constant cosmologique à partir des seules fluctuations gravitationnelles du vide, et sa comparaison aux mesures.

Autre joli problème abordé avec cette transformation de Fourier généralisée récemment par Laurent est ici [26].

Une autre conséquence d'un espace-temps fractal est qu'il est nécessaire d'introduire une transformée de Fourier généralisée pour passer de la position  $x$  à l'impulsion  $p$  (ou de  $v$  à  $p_v$ ) ce qui s'écrit sous la forme suivante en relativité d'échelle en relativité restreinte:

$$F(p) = f\left(\frac{p}{p_0} a\right), \text{ avec } a = 1/(1 + \ln^2(p/p_0)/C^2)^{\frac{1}{2}}. \quad (1)$$

$C$  étant une constant = rapport d'échelle de longueur =  $\log(L/\lambda g)$ ,  $\lambda g$  est une longueur caractéristique du système considéré, et  $L$  est telle que  $\Lambda = 1/L^2$ , ou  $\Lambda$  est la constant cosmologique.

Il convient de rechercher la généralisation possible en relativité d'échelle-relativité générale de ce type de transformation...

— T. LEHNER, via email to me

## 1. Behind Us

Let us begin *ex abrupto*.

### 1.1. Theoretical Background Framework

The aforementioned problem is closely linked to a focal issue: dealing with fractal fluctuations (and with a locally determinable irreversibility, where it exists and is practicable). This brings us to what we can call *fractal patterns (scilicet: geometry) of the flow paths* infinitely countable, with the variables of fluid dynamics or thermodynamics. We are thus inside a stochastic mechanism, which is a niche of probability theory. That is, we are inside the world where the calculus (from Born onwards) is done with the probability amplitude.

If  $H_{\mu}$  represent the entropy with reference to probability measures, and  $\text{vol}$  the volume, the Fourier transform<sup>a</sup>  $\widehat{\psi}$  of a function  $\psi(H_{\mu}, \text{vol})$ , relates to the squared modulus, which translates into the definition of the probability density distribution  $\mathcal{P}(H_{\mu}, \text{vol})$ , or  $\mathcal{P}(x_1, \dots, x_j, x_{j+1}, \dots, x_j)_{H_{\mu}}$  in path integrals. This gives a solution to an equation à la Schrödinger — for a solution of the Schrödinger equation through the Fourier transform method [8] with proof, see my [20, prop. 14.2.1].

NB. The basic Eq. I considered in that book is

$$\psi(x, t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \widehat{\psi}_0(k) e^{ik\left(x - \frac{\omega(k)}{k}t\right)}, \quad (2)$$

where  $\psi_0 \in L^2(\mathbb{R})$  is a Schwartz function, where  $L^2$  is a Hilbert space.

### 1.2. Status Quo for the Pseudo-Euclidean Space

The Fourier transform for the special scale relativity, acting on a flat Minkowski (or Lorentz–Minkowski) space-time  $\mathbb{M}^4 = \mathbb{R}_{1,3}^4$ , is described by Eq. (1), or by other equations of that type, e.g. Nottale's own one [24, p. 539]:

$$\ln \frac{\frac{p_0}{p}}{\sqrt{1 + \ln^2 \frac{p_0}{p} / C^2}} = \ln \left( \frac{p_0}{p} \right) \exp \left\{ \frac{1}{\sqrt{1 + \ln^2 \frac{p_0}{p} / C^2}} \right\}, \quad (3)$$

where  $C = \ln C_s$  is a dimensionless scale constant, which is the logarithm of a scale ratio with a second scale constant  $C_s$ .

<sup>a</sup> Abbreviated as FT.

## 2. In Front of Us

To reply to the dilemma, I will move the argument from the physical plane to the geometric and analytical one.

### 2.1. Fourier Transform via Stochastic Processes Onto a Probability Measure

#### 2.1.1. Fourier Transform of a Probability Measure

Let  $\mu$  be a probability measure. To be precise, we choose a Borel-like measure [1] established on the  $\sigma$ -algebra  $\mathcal{B}_\sigma(\mathcal{X})$  of Borel sets, where  $\mathcal{X}$  can be considered here as a Lindelöf space [13].

A quick reminder:

(1) a Borel measure on  $\mathcal{X}$  is nothing more than a measure in which the Borel sets are measurable, while the latter are the members of the smallest collection with open or closed sets formed by countable unions and countable intersections (if the Borel sets are in a Hausdorff  $\sigma$ -compact space [10], one has the members of the smallest  $\sigma$ -ring involving compact sets);

(2) the Borel  $\sigma$ -algebra is the collection of all Borel sets on  $\mathcal{X}$  (and it is the smallest  $\sigma$ -algebra that contains all the open or closed Borel sets, and it is closed under the operations of countable union and countable intersection);

(3) a Lindelöf space is

(i) a topological space, provided that it is possible to extract a countable subcover from any open cover of  $\mathcal{X}$ ,

(ii) a (complete and separable) metric space  $(\mathcal{X}, \rho)$ , provided that any open cover of  $\mathcal{X}$  has a subcover with a countable number of members, where  $\rho$  is the distance between two points, say  $x$  and  $y$ , so

$$\rho(x, y) \stackrel{\text{eqv}}{=} |x - y|_{\mathcal{X}},$$

(iii) a  $\sigma$ -compact space.

The *Fourier transform of the probability measure*  $\mu$  can be conceived in this way:

$$\widehat{\mu}(t) = \int_{\mathbb{R}^n} e^{-2\pi i \langle t, x \rangle} d\mu(x), \quad (4)$$

where  $t \in \mathbb{R}^n$ , and  $\langle \cdot, \cdot \rangle$  denotes the (dot) scalar product, which is a case of the inner product when one is dealing with the  $\mathbb{R}$ -field.

*Marginalia 2.1 (Imagining a  $\mathbb{S}^{n-1}$ -space).* Let  $\mathbb{S}^{n-1} = \{x \in \mathbb{R}^n \mid \|x\| = 1\}$  be a unit sphere the surface measure of which is declared by  $S^{n-1}$ , and let  $\xi = (\xi_1, \dots, \xi_n) \in \mathbb{R}^n$  be the spatial frequency (transform variable), putting  $\varphi \in \mathcal{C}_0^\infty$  (see Section 2.3), and  $r \in \mathbb{R}^n$ . The Fourier transform of  $\mu$  in the  $\mathbb{S}^{n-1}$ -space is but

$$\widehat{\mu}(\xi) = \int_{\mathbb{S}^{n-1}} \int e^{-2\pi i \xi(r+te)} \varphi(t) dt dS^{n-1}. \quad (5)$$

#### 2.1.2. The Grafting of Stochastic Processuality, ou seja a Random Dynamical System for the Fourier Transform

Let  $(\widehat{\Omega}, \mathcal{B}_\sigma, \mu)$  be a probability space, with a Borel  $\sigma$ -algebra on  $\widehat{\Omega}$ , where  $\widehat{\Omega}$  is a non-empty set, and  $\mathbb{E}$  a random set, that is, a measurable subset  $\mathbb{E} \subset \widehat{\Omega} \times \mathcal{X}$  in relation to the product  $\sigma$ -algebra on  $\widehat{\Omega} \times \mathcal{X}$ , or a family  $\mathbb{E} = \{E_{\widehat{\omega}}\}_{\widehat{\omega} \in \widehat{\Omega}}$  of subsets of  $\mathcal{X}$ , where  $E$  is a non-empty bounded subset of  $\mathcal{X}$ , once it has been fixed that the mapping

$$\widehat{\omega} \mapsto \rho(x, E_{\widehat{\omega}})$$

is  $\mathcal{B}_\sigma$ -measurable, for  $\widehat{\omega} \in \widehat{\Omega}$  and  $x \in \mathcal{X}$ .

NB. The set  $\mathbb{E}$  is

- closed on condition that  $E_{\widehat{\omega}}$  serves as a closed  $\widehat{\omega}$ -fiber, for  $\widehat{\omega} \in \widehat{\Omega}$ ,
- compact on condition that  $E_{\widehat{\omega}}$  serves as a compact  $\widehat{\omega}$ -fiber, for  $\widehat{\omega} \in \widehat{\Omega}$ ,
- tempered on condition that

$$E_{\widehat{\omega}} \subset \{x \in \mathcal{X} \mid \rho(x, x_0) \leq f(\widehat{\omega})\},$$

for  $x_0 \in \mathcal{X}$ , placing a random variable  $f(\widehat{\omega})$ .

This allows us to define the map

$$\mathbb{P}: \mathcal{B}_\sigma \xrightarrow{\text{probability measure}} \mathbb{E}$$

as a *general stochastic measure* on the Borel  $\sigma$ -algebra in connection with a *fractal Brownian motion* [2] (see below, Sections 2.2, 2.2.1). Note the painless transition from the concept of *probability measure* to that of *stochastic measure*, which come to blend together.

We are therefore able to determine the *Fourier transform of a stochastic measure*  $\mathbb{P}$ , understood in terms of *random function*, with the presence of  $(\hat{\omega})$ -random variables:

$$\begin{cases} \widehat{\mathbb{P}}(t) = \int_{\mathbb{R}^n} e^{-2\pi i \langle t, x \rangle} d\mathbb{P}(x) = \int_{\mathbb{R}^n} \cos 2\pi \langle t, x \rangle d\mathbb{P}(x) - i \int_{\mathbb{R}^n} \sin 2\pi \langle t, x \rangle d\mathbb{P}(x), & (6a) \\ \mathcal{B}_\sigma \xrightarrow{\widehat{\mathbb{P}}_t(\hat{\omega})} \mathbb{E} \subset \hat{\Omega} \times \mathcal{X}. & (6b) \end{cases}$$

### 2.1.3. The Inversion Formula

It may be useful to see what form the *Fourier inversion formula* takes. Let us take the most affordable case. First of all we must state that  $\mathcal{C}^\infty(\mathbb{R}^n)$  is the set of infinitely, i.e.  $\infty$ -times, continuously differentiable  $\mathbb{R}$ -functions  $\varphi_\mathbb{P}$ . We shall introduce probability function  $\mathcal{P}$  along with a probability limit. For all

$$\varphi_\mathbb{P} \in \mathbb{R} \rightarrow \mathcal{C}^\infty(\mathbb{R}^n),$$

one obtains

$$\int_{\mathbb{R}^n} \varphi_\mathbb{P}(x) d\mathbb{P}(x) = \mathcal{P} \lim_{\epsilon \rightarrow 0^+} \int_{\mathbb{R}^n} \varphi_\mathbb{P}(x) dx \int_{\mathbb{R}^n} e^{-4\pi^2 \epsilon \|t\|^2} e^{2\pi i \langle x, t \rangle} \widehat{\mathbb{P}}(t) dt. \quad (7)$$

**Theorem 2.1.** *Eq. (7) is a statement that can be proved.*

*Proof.* Let  $[\![ \cdots ]\!]$ ,  $[\![ \cdots ]\!]$ , and  $[\![ \cdots ]\!]$ , indicate the left and right repeat signs, over the influence of the musical symbols:

$$\begin{aligned} & [\![ \mathcal{P} \lim_{\epsilon \rightarrow 0^+} \int_{\mathbb{R}^n} \varphi_\mathbb{P}(x) dx \int_{\mathbb{R}^n} e^{-4\pi^2 \epsilon \|t\|^2} e^{2\pi i \langle x, t \rangle} ]\!] \widehat{\mathbb{P}}(t) dt \\ &= [\![ \cdots ]\!] dt \int_{\mathbb{R}^n} e^{-2\pi i \langle y, t \rangle} d\mathbb{P}(y) \\ &= [\![ \mathcal{P} \lim_{\epsilon \rightarrow 0^+} \int_{\mathbb{R}^n} d\mathbb{P}(y) \int_{\mathbb{R}^n} \varphi_\mathbb{P}(x) ]\!] dx \int_{\mathbb{R}^n} e^{-4\pi^2 \epsilon \|t\|^2} e^{2\pi i \langle x, t \rangle} e^{-2\pi i \langle y, t \rangle} dt \\ &= [\![ \cdots ]\!] [(4\pi\epsilon)^{-\frac{n}{2}} e^{-\frac{\|x-y\|^2}{4\epsilon}} dx:] \\ &= \int_{\mathbb{R}^n} d\mathbb{P}(y) \lim_{\epsilon \rightarrow 0^+} \int_{\mathbb{R}^n} \varphi_\mathbb{P}(x) [\![ \cdots ]\!] = \int_{\mathbb{R}^n} \varphi_\mathbb{P}(y) d\mathbb{P}(y). \end{aligned} \quad (8)$$

□

### 2.1.4. Equality betwixt Probability Measure and Random Function (FT)

To reveal the relationship between the probability measure  $\mathbb{P}$ —which we have superimposed above on a stochastic one—and the Fourier transform of  $\widehat{\mathbb{P}}$ , just take a bounded function, e.g.  $\varphi_\epsilon \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ , belonging to a Lebesgue space  $L^p$ , so that

$$\int_{\mathbb{R}^n} \varphi_\epsilon(x) \widehat{\mathbb{P}}(x) dx = \int_{\mathbb{R}^n} \widehat{\varphi}_\epsilon(x) d\mathbb{P}(x). \quad (9)$$

The *proof* of this depends directly on the integrability of  $\widehat{\varphi}_\epsilon$  in regard to  $\mathbb{P}$ , i.e.

$$\int_{\mathbb{R}^n} \widehat{\varphi}_\epsilon(t) d\mathbb{P}(t) = \int_{\mathbb{R}^n} \varphi_\epsilon(x) dx \int_{\mathbb{R}^n} e^{-2\pi i \langle t, x \rangle} d\mathbb{P}(t) = \int_{\mathbb{R}^n} \varphi_\epsilon(x) \widehat{\mathbb{P}}(x) dx. \quad (10)$$

### 2.1.5. The Convergence of the Commensurate Probability Distributions

The above gives the opportunity to rapidly show that the probability measure  $\mu$  has a classical consequence, which is associated with a stochastic process

$$\phi_\tau : \hat{\Omega} \rightarrow \mathcal{M} \stackrel{\text{equiv}}{\cong} \mathcal{M}^4 \text{ viz. } \mathbb{M}^4 = \mathbb{R}_{1,3}^4, \text{ }^a$$

with each  $f(\hat{\omega})$ , that we assume takes place in a generic Riemannian space, in order to cover every relativistic possibility. We are talking about a triple  $(\hat{\Omega}, \mathcal{B}_\sigma, \mu)$  of  $\phi_{(t)}$  here: see my [22]. The consequence is this, cf. e.g. [28, p. 172]: since every characteristic function, say  $\chi$ , and the distribution function are in one-to-one correspondence, the convergence of  $\chi$  involves a weak convergence of the commensurate probability distributions.

If  $\mathcal{C}^0$  illustrates the set of all continuous functions  $\varphi \in \mathcal{C}^0$ , then

$$\int_{\mathbb{R}^n} \varphi(t) \widehat{\mu}_{n \rightarrow \infty}(t) dt \xrightarrow{\mathcal{B}_\sigma \text{ on } \mu} \mathcal{P} \int_{\mathbb{R}^n} \varphi(t) \widehat{\mu}(t) dt, \quad (11)$$

ergo

$$\int_{\mathbb{R}^n} \widehat{\varphi}(x) d\mu_n(x) \xrightarrow{\text{via } \mathcal{P}\text{-probability density distribution}} 0. \quad (12)$$

This expression makes it possible to connect Eq. (9) with Eq. (11).

### 2.2. Fourier Transform of a $\mathbb{R}$ -valued Random Brownian Measure

We appoint

$$\mathbb{B}_{\mu_{(t)}} = \sum_{k \geq 1} \epsilon_k \sigma_k \frac{ds}{(2\pi)^{\frac{n}{2}}} \quad (13)$$

as the  $\mathbb{R}$ -valued Brownian random measure, where  $(\epsilon_k)_{k \geq 1} = \epsilon_1, \dots, \epsilon_k$  is a succession of variables,  $(\sigma_k)_{k \in \mathbb{N}} = \{\sigma_1, \dots, \sigma_k\}$  is an orthonormal basis (a local frame field) of  $\mathbb{R}^n$ , and  $\frac{ds}{(2\pi)^{\frac{n}{2}}} = \mu(ds)$ .

The integral transform of  $\mathbb{B}_{\mu_{(t)}}$  has a punctilious definition. We should start from here:

$$\int_{\mathbb{R}^n} \varphi(\xi) \widehat{\mathbb{B}}_{\mu_{(t)}} d\xi = \int_{\mathbb{R}^n} \widehat{\varphi}(s) \mathbb{B}_{\mu_{(t)}} ds = \sum_{k \geq 1} \int_{\mathbb{R}\text{-valued}} \varphi(\xi) \widehat{\sigma}_k(\xi) \frac{d\xi}{(2\pi)^{\frac{n}{2}}} \epsilon_k, \quad (14)$$

and the *Fourier transform of a  $\mathbb{R}$ -valued Brownian random measure* is

$$\widehat{\mathbb{B}}_{\mu_{(t)}} d\xi = \sum_{k \geq 1} \epsilon_k \widehat{\sigma}_k(\xi) \frac{d\xi}{(2\pi)^{\frac{n}{2}}}. \quad (15)$$

#### 2.2.1. Fourier-harmonized Picture of a Brownian Fractal Structure

Is it possible to *harmonize* (in the sense of Fourier analysis) a Brownian fractal structure? The answer is yes.

Let  $\phi_t = \phi_0 + \int_0^t \beta_s ds + \epsilon_{\mathbb{R}} W_t$  be a stochastic process, the generalization of which is a *random field*, where  $\beta_s$  is a process almost surely with bounded variation of some path,  $\epsilon_{\mathbb{R}} > 0$  is a real constant, and  $W_t$  is the Wiener stochastic process, which is almost surely continuous in  $t$ , and square-integrable martingale regarding a non-decreasing family  $\mathcal{U}_t$ ,  $t \in [0, \infty)$  of  $\sigma$ -subalgebras of the  $\sigma$ -algebra  $\mathcal{B}$ .

Compare all this with [22, secc. 3.1, 7.3.1].

The harmonized math-picture is quickly done:

$$\phi_t(x) \int_{\mathbb{R}^n} \frac{-e^{-ix \cdot \xi} (-1 + e^{ix \cdot \xi})}{C_{\mathfrak{B}}^{\frac{1}{2}} \|\xi\|^{\mathfrak{B} + \frac{n}{2}}} \widehat{\mathbb{B}}_{\mu_{(t)}} d\xi, \quad x \in \mathbb{R}^n, \quad (16)$$

stabilizing a constant

$$C_{\mathfrak{B}} = \int_{\mathbb{R}^n} \frac{-2(\cos \xi - 1)}{\|\xi\|^{n+2\mathfrak{B}}} \left( \frac{d\xi}{(2\pi)^{\frac{n}{2}}} \right) \quad (17)$$

depending on a vector space  $\mathfrak{B}$ , which is a closed subspace of  $L^2(\hat{\Omega}, \mathcal{B}_\sigma, \mu)$ .

<sup>a</sup> Cf. [21].

### 2.3. Fourier Transform Distribution on a Set of Bump Functions

Let  $\lambda_{\mathfrak{p}} \stackrel{\text{viz}}{=} (\lambda_{\mathfrak{p}})_{\mathcal{D}} \in \mathcal{D}(\mathbb{R})$  be a (compactly supported) bump function whose set is  $\mathcal{C}_0^\infty$  (having continuous derivatives of any order) on a Euclidean vector  $\mathbb{R}$ -space, where  $\mathcal{D}(\cdot)$  is the space of  $\lambda_{\mathfrak{p}}$ . Let it be assumed that the Fourier transform  $\widehat{\varphi}$  coincides with a  $k$ -dimensional  $\mathcal{E}$ -distribution on a set of bump functions  $\widehat{\lambda}_{\mathfrak{p}}(\omega)e^{i\omega\tau}$ .

Since  $\lambda_{\mathfrak{p}}$  is also a  $\mathfrak{p}$ -integrable function, we will characterize the Fourier transform distribution on the bump function  $\lambda_{\mathfrak{p}}(\tau - t)$  in such a way that

$$\widehat{\varphi}\left(\widehat{\lambda}_{\mathfrak{p}}(\omega)e^{i\omega\tau}\right) = \frac{\mathcal{E}}{2\pi}\left(\lambda_{\mathfrak{p}}(\tau - t)\right). \quad (18)$$

I wanted to present this case because it is advantageous in many fields, even if it is not applicable to fractal structures, because is rapidly decreasing, ergo infinitely differentiable.

### 2.4. Fourier Transform in Fractal Space thru Mittag-Leffler Function

Next we will take a look at the transform in a fractal dimension (with the famous changes in scale). There are various analytical techniques. I will use the one that relies on the Mittag-Leffler function [18] = [19], which originally (in the first part of the equation below) has this form:

$$E_\alpha(x) = 1 + \frac{x}{\Gamma(1 + \alpha.1)} + \frac{x^2}{\Gamma(1 + \alpha.2)} + \frac{x^3}{\Gamma(1 + \alpha.3)} + \cdots \stackrel{\text{eqv}}{=} \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(\alpha k + 1)}, \quad (19)$$

where  $\alpha$  is a positive constant, and  $\Gamma(\cdot)$  is the gamma function.

#### 2.4.1. Fractional Fourier Expansion of a Periodic Function and its Coefficients

Let

$$\varphi(\tau) = \sum_{k=-\infty}^{\infty} \varphi_k^\alpha E_\alpha\left(\frac{\pi i(k\tau)}{I_t}\right)^\alpha, \quad -I_t \leq \tau \leq +I_t, \tau \in \mathbb{R}, k \in \mathbb{Z}_*, \quad (20)$$

designate a *Fourier-type periodic and bounded function*, having the peculiarity of being *locally integrable on a fractal space*, which we make coincide with a certain interval  $I_t$ . The coefficients of Eq. (20) are consequently governed by this formula:

$$\varphi_k^\alpha = 1/[-I_t, +I_t]^\alpha \int_{-I_t}^{+I_t} \varphi(\tau) E_\alpha - \left(\frac{\pi i(k\tau)}{I_t}\right)^\alpha [d\tau]^\alpha. \quad (21)$$

By writing the limit for the two equations above, we arrive at

$$\begin{aligned} \lim_{I_t \rightarrow +\infty} \varphi(\tau) &= \lim_{I_t \rightarrow +\infty} \left\{ \frac{1}{[-I_t, +I_t]^\alpha} \sum_{k=-\infty}^{\infty} \left[ \frac{1}{\Gamma(1+\alpha)} \int_{-I_t}^{+I_t} \varphi(\tau) E_\alpha - \left(\frac{\pi i(k\tau)}{I_t}\right)^\alpha [d\tau]^\alpha \right] E_\alpha\left(i\tau \frac{\pi k}{I_t}\right)^\alpha \right\} \\ &= \frac{1}{(2\pi)^\alpha} \int_{-\infty}^{+\infty} \left[ \cdots \right] E_\alpha\left(i\tau \frac{\pi k}{I_t}\right)^\alpha \left(d \frac{\pi k}{I_t}\right)^\alpha, \end{aligned} \quad (22)$$

and

$$\begin{aligned} \lim_{I_t \rightarrow +\infty} \left[ \cdots \right] &= \lim_{I_t \rightarrow +\infty} \left( \frac{1}{\Gamma(1 + \alpha)} \int_{-I_t}^{+I_t} \varphi(\tau) E_\alpha\left(-i\tau \frac{\pi k}{I_t}\right)^\alpha [d\tau]^\alpha \right) \\ &= \lim_{I_t \rightarrow +\infty} \left( \frac{1}{\Gamma(1 + \alpha)} \int_{-\infty}^{+\infty} \varphi(\tau) E_\alpha\left(i\tau \frac{\pi k}{I_t}\right)^\alpha [d\tau]^\alpha \right). \end{aligned} \quad (23)$$

#### 2.4.2. The Case of a Continuously Non-differentiable Transform

Now we have enough to build what we need. The crux of our interest is the *non-differentiable Fourier fractal function*. Assembling all the pieces of the jigsaw, this expression can be explained as follows:

$$\frac{1}{2}(\varphi(\tau + 0) + \varphi(\tau - 0)) = \left[ \frac{1}{(2\pi)^\alpha} \int_{-\infty}^{+\infty} \left( \frac{1}{\Gamma(1 + \alpha)} \int_{-\infty}^{+\infty} \varphi(\tau) E_\alpha(-i\tau\hat{\omega} \cdot d\tau)^\alpha E_\alpha(i\tau\hat{\omega})^\alpha \right) [d\hat{\omega}]^\alpha \right], \quad (24)$$



id est

$$\widehat{\varphi}(\tau) = [[: \cdots :]]. \quad (25)$$

Eqq. (24) (25) can also be easily linked again to a stochastic process (Section 2.1), see e.g. [11] [27].

### 2.4.3. An Algebraic Paradigm

Let  $\varphi_n(t, \tau) \stackrel{\text{viz}}{=} \ker(\varphi_n(t, \tau))$  specify the homomorphic kernel, or the impulse response, so as to have a universal application for some linear map. If  $\alpha = n$  in boundary condition

$$\varphi_n(t, \tau) \begin{cases} \delta(\tau - t), & \text{w/ } \varphi_{n=0}, \\ e^{-i2\pi(t\tau)}, & \text{w/ } \varphi_{n=1}, \\ \delta(\tau + t), & \text{w/ } \varphi_{n=2}, \\ e^{i2\pi(t\tau)}, & \text{w/ } \varphi_{n=3}, \end{cases} \quad \begin{array}{l} (26a) \\ (26b) \\ (26c) \\ (26d) \end{array}$$

the *fractional Fourier transform into generalized version* has this formulistic synopsis,

$$\widehat{\varphi}_\alpha(x_t) = \int_{-\infty}^{+\infty} x(\tau) \varphi_\alpha(t, \tau) d\tau, \quad (27)$$

and its solution is 4-periodic in  $\alpha$ -value.

## 2.5. Fourier Transform in Fractal Space via Probability Weights and Contractive Similarities

A fractal measure on  $\mathbb{R}^n$  can be viewed from a different perspective, returning to the probability Borel-like measure (and its stochastic nature) but with this specific value, by the system of contractive similarities  $\Pi_k$ ,

$$\mu = \sum_{k=1}^r \mathbb{P}_k \mu \circ \Pi_k^{-1} \quad (28)$$

where

$\mathbb{P}_k$  are the probability weights,  $0 < \mathbb{P}_k < 1$ , with  $\sum_{k=1}^r \mathbb{P}_k = 1$ , and  
 $\Pi_k = \Pi_1, \dots, \Pi_k$  are contractions on a (compact) metric space such that

$$\Pi_k(x) = \theta_k \Lambda(x) + w_k,$$

in which

$\theta_k$  is a fixed value,  $0 < \theta_k < 1$ ,

$\Lambda = [\Lambda^\mu_\nu]$  is a matrix of an orthogonal transformation (cf. e.g. the general, proper, orthochronous Lorentz transformation), and

$w_k$  is a  $\mathbb{R}$ -vector.

Let

$$D_f = \frac{\sum_{k=1}^r \mathbb{P}_k \log \mathbb{P}_k}{\sum_{k=1}^r \mathbb{P}_k \log \theta_k}. \quad (29)$$

be the fractal dimension, or the Hausdorff (Hausdorff–Besicovitch) dimension.

### 2.5.1. The Emergence of an Asymptotic Periodicity of the $\mu$ -Fourier transform

The  $\mu$ -Fourier transform presents, within this scheme, an asymptotic periodicity,<sup>a</sup>

$$\int_{|x| \leq \Lambda} |\widehat{\mu}(x)|^{2k} dx = \Lambda^{n - (D_C)k}, \quad \Lambda \rightarrow \infty, \quad (30)$$

where  $D_C = \frac{\log 2}{\log 3}$  is the dimension of the *poussière cantorienne* [3].

<sup>a</sup> This coincides with a fractal Plancherel formula for the FT of distributions.

If  $c_1$  and  $c_2$  are two constants, and create the condition that  $\theta^{D_C} = \sum_{k=1}^r \mathbb{P}_k^2$ , the limit inferior and limit superior of this collection are

$$\left\{ \begin{array}{l} \limsup_{\Lambda \rightarrow \infty} \frac{1}{\Lambda^{n-D_C}} \int_{|x| \leq \Lambda} |\widehat{\mu}(x)|^2 dx \leq c_1, \\ \limsup_{\Lambda \rightarrow \infty} \frac{1}{\Lambda^{n-D_C}} \int_{|x| \leq \Lambda} |\widehat{\mu}(x)|^2 dx \geq c_2. \end{array} \right. \quad (31a)$$

$$\left\{ \begin{array}{l} \limsup_{\Lambda \rightarrow \infty} \frac{1}{\Lambda^{n-D_C}} \int_{|x| \leq \Lambda} |\widehat{\mu}(x)|^2 dx \leq c_1, \\ \limsup_{\Lambda \rightarrow \infty} \frac{1}{\Lambda^{n-D_C}} \int_{|x| \leq \Lambda} |\widehat{\mu}(x)|^2 dx \geq c_2. \end{array} \right. \quad (31b)$$

An accurate mathematical orientation of this kind, as it is exposed in the two previous Sections, can be found in [31].

## 2.6. Fourier Transform in 4-Space

### 2.6.1. Quantum Probability Amplitude for a Particle

How does a particle behave if one has to treat it with the Fourier transform and one has to calculate the probability amplitude according to Feynman's [7] recipe? Write his path integral in this way [20, sec. 26.1.3]:

$$\mathcal{U}_C(\mathbb{k}) = \lim_{\epsilon \rightarrow 0} \int_{\mathbb{k} \subset \mathbb{R}_{1,3}^4} \mathcal{Y}_C(x_1, \dots, x_j, x_{j+1}, \dots, x_j) dx_1, \dots, dx_j dx_{j+1}, \dots, dx_j, \quad (32)$$

where  $\mathcal{U}_C$  appoints a probability amplitude for  $\mathbb{k} \subset \mathbb{R}_{1,3}^4$ , and  $\mathcal{Y}_C$  is the probability amplitude functional of  $x_j$  demarcating the entire path  $x_\tau$ .

Well, the *spatial Fourier 4-transform* can be outlined with an integral of this kind:

$$\int_{\mathbb{k} \subset \mathbb{R}_{1,3}^4} \mathcal{U}_C(\mathbb{k}) e^{ip_j x^j} d^4 x = \frac{e^{-\frac{i\pi}{2}}}{16\pi^2} \int_0^\infty \frac{ds}{s^2} e^{-i(m^2-1\epsilon)s} \int d^4 e^{-\frac{ix^2}{4s} + ip_j x^j} = \frac{1}{\epsilon + i(m-p)(m+p)}. \quad (33)$$

The inverse Fourier transform formula is

$$\mathcal{U}_C(\mathbb{k}) = \int \frac{d^4}{16\pi^4} \left( \frac{ie^{-ip_j x^j}}{p^2 - m^2 + i\epsilon} \right) = \int_0^\infty ds \int \frac{d^4}{16\pi^4} e^{-ip_j x^j - is(m^2 - p^2 - i\epsilon)}. \quad (34)$$

### 2.6.2. Space-Time Fourier Transform (General Definition)

Given a *spatial Fourier transform*

$$\left\{ \begin{array}{l} \varphi(x) \mapsto \widehat{\varphi}(\xi), \\ \widehat{\varphi}(\xi) = \int_{\mathbb{R}^n} e^{-ix \cdot \xi}, \end{array} \right. \quad (35a)$$

$$\left\{ \begin{array}{l} \widehat{\varphi}(\xi) = \int_{\mathbb{R}^n} e^{-ix \cdot \xi}, \end{array} \right. \quad (35b)$$

where  $x$  is the independent variable, and stated a complex scalar field

$$F: \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{C}, \quad (36)$$

a general description of the *space-time Fourier transform* is this:

$$\tilde{F}(\tau, \xi) = \int_{\mathbb{R}} \int_{\mathbb{R}^n} F(t, x) e^{-i(t\tau + x \cdot \xi)} dt dx, \quad (37)$$

whose inverse formula will be

$$F(t, x) = \frac{1}{(2\pi)^{n+1}} \int_{\mathbb{R}} \int_{\mathbb{R}^n} e^{i(t\tau + x \cdot \xi)} \tilde{F}(\tau, \xi) d\tau d\xi. \quad (38)$$

NB. The norm of this transform can be made more captivating by adding the Sobolev space  $W^{a,b}(\mathbb{R}^{1,n-1})$  in  $L^2_{\tau,\xi}$ -space,  $a, b \in \mathbb{R}$ , where  $\mathbb{R}^{1,n-1}$  is the Minkowski space, so that

$$\|F\|_{W^{a,b}}^2 = \int_{\mathbb{R}^{1,n-1}} (1 + |\xi|^2 + |\tau|^2)^a (1 + |\tau|^2 - |\xi|^2)^b |\tilde{F}|^2(\tau, \xi) d\tau d\xi, \quad (39)$$

so that  $W^{a,b}$  is the completion of the Schwartz space [9] [6] [30]  $\mathcal{S}_i(\mathbb{R}^{1,n-1})$ . A similar result occurs if we insert the Minkowski space-time  $\mathbb{M}^4 = \mathbb{R}_{1,3}^4$  or  $\mathbb{R}_{3,1}^4$ .

*Marginalia 2.2.* The scalar can be (re)writable in the following manner:

$$f = \int_{\mathbb{R}} \|f_z\|_{L_i^\infty W_x^b} dz \lesssim \|f\|_{W^{a,b}} \quad (40)$$

with  $\lesssim$  being the less-than or similar symbol. This is because (*sketch proof*)

$$\tilde{f}(\tau, \xi) = \int_{\mathbb{R}} \tilde{f}(z + |\xi|, \xi) \delta(a - \tau + |\xi|). \quad (41)$$

Via Cauchy–Schwarz inequality [4] [29], one gets to

$$\int_{\mathbb{R}} \|\tilde{f}(|\xi| + z, \xi)\|_{W^b} dz \lesssim \|f\|_{W^{a,b}}. \quad (42)$$

### 2.6.3. 4D Space-Time Fourier Transform

We come to a more (classical) deterministic context, to approach the concept of 4-dimensionality, for the benefit of the theory of relativity.

We can determine the space-time Fourier transform with two procedures. The first is constructed with the transform variable  $\xi$ , or better, with

$$\varphi(x) = \frac{1}{16\pi^4} \int \widehat{\varphi}(\xi) e^{i\xi \cdot x} d^4x, \quad (43)$$

where the denominator of the fraction is equal to  $(2\pi)^4$ , and  $\xi \cdot x = \eta_{\mu\nu} \xi^\mu x^\nu$  ( $\eta_{\mu\nu}$  is obviously the Minkowskian tensor). Hence

$$\widehat{\varphi}(\xi) = \int \varphi(x) e^{-i\xi \cdot x} d^4x. \quad (44)$$

The second one explicitly starts from the pseudo-Euclidean 4-vectoriality. Let  $k \cdot x = \dot{k} \eta x$ , where  $\eta \stackrel{\text{viz}}{=} \eta_{\mu\nu}^{(1,3)\pm}$  is the Minkowski metric tensor with signatures  $(+, -, -, -)$  or  $(-, +, +, +)$ . Imposing a space-time function

$$\varphi : \{x = (x, t) \in \mathbb{R}^3 \times \mathbb{R}^+\} \rightarrow \mathbb{C}, \quad (45)$$

and symbolizing, classically, the angular frequency with  $\omega$ , a more remarkable denotation, for the Fourier transform in 4-space, is achieved in two manner, which are mathematically superimposable, even if physically (*juste pour dire...*) they have their own distinction:

$$\widehat{\varphi}(k) = \int \varphi(x) e^{-ik \cdot x} dx = \int \varphi(x, t) e^{i(\omega t - k \cdot x)} dt d^3x, \quad (46)$$

$$\widehat{\varphi}(k, \omega) = 2\pi^{-\frac{n}{2}} \int \varphi(x, t) e^{i(\omega t - k \cdot x)} dt d^3x. \quad (47)$$

### 2.6.4. Fourier Transform on $\mathbb{R}^4$ with Lorentz Metric

As the title suggests, the 4D transform  $\varphi \mapsto \widehat{\varphi}$  in Lorentz(–Minkowski) space-time  $\mathbb{L}^4 = \mathbb{R}_{1,3}^4$  or  $\mathbb{R}_{3,1}^4$ , i.e.  $\mathbb{M}^4$ , is effortlessly describable with the following equality:

$$\widehat{\varphi}(p) = \int_{\mathbb{L}^4} e^{ip_\mu x^\mu} \varphi(x) d^4x, \quad (48)$$

where  $x$  and  $p$  are the position and momentum spaces. Its inverse is

$$\varphi(x) = \int_{\mathbb{L}^4} e^{-ip_\mu x^\mu} \widehat{\varphi}(p) \frac{d^4x}{16\pi^4}. \quad (49)$$

### 2.6.5. Space-Time Fourier Transform with the Energy-Momentum Tensor

The space-time Fourier transform with the (stress) energy-momentum tensor formalism looks like this:

$$\tilde{\mathbb{T}}^{\mu\nu}(\omega, \omega x) = \int_{\mathbb{R}^3} \widehat{\mathbb{T}}^{\mu\nu}(\omega, \dot{x}) e^{-i\omega x \cdot \dot{x}} d^3\dot{x}, \quad (50)$$

recalling that the transform as regards the time is simply

$$\mathbb{T}^{\mu\nu}(t, x) = \int_{\mathbb{R}} \widehat{\mathbb{T}}^{\mu\nu}(\omega, x) e^{-i\omega t} d\omega. \quad (51)$$

### 3. Stochastic Space-time au Moyen de Langevin–Einstein Equation

The easiest climb to reach the top of the resolution is the one known as the *col alpin* of Langevin equation for *gravitational purposes*, with *stochastic perturbations* — compare with the work of P. Langevin on Brownian motion [12]:

$$\begin{aligned} & \frac{1}{8\pi G_N} G^{\mu\nu}(x) - 2C_{s_1}^\kappa \mathcal{M}_{G_1}^{\mu\nu}(x) - 2C_{s_2}^\kappa \mathcal{M}_{G_2}^{\mu\nu}(x) \\ & + \frac{1}{4} \int \left( \ker(\Theta)_{\mathcal{M}_{G_1}}(x-y) \mathcal{M}_{G_1}^{\mu\nu}(y) + \ker(\Theta)_{\mathcal{M}_{G_2}}(x-y) \mathcal{M}_{G_2}^{\mu\nu}(y) \right) d^4y \stackrel{\text{eqv}}{=} \mathcal{J}^{\mu\nu}(x), \end{aligned} \quad (52)$$

where

- $G_N$  is the Newtonian constant of gravitation,
- $C_{s_1}^\kappa$  and  $C_{s_2}^\kappa$  are two dimensionless coupling constants,
- $\mathcal{M}_{G_1}$  and  $\mathcal{M}_{G_2}$  are two tensors affiliated with gravity,
- $\ker(\Theta)$  is the noise kernel, and
- $\mathcal{J}^{\mu\nu}$  is the (Gaussian) stochastic component appertaining to this equation.

#### 3.1. Fourier Transform inside the Stochastic Gravitation

From the above we figure out the *transform*, in the kingdom of *Fourerian analysis*, of the Einstein–Langevin Eq. (52),

$$F^{\mu\nu}{}_{\rho\varpi}(p) \tilde{G}^{\rho\varpi}(p) = 8\pi G_N \tilde{\mathcal{J}}^{\mu\nu}(p), \quad (53)$$

expecting that  $F^{\mu\nu}{}_{\rho\varpi}$  is a 4-dimensional (field) bitensor.

#### Suggestions for Reading

For historical information, the first article that addresses the topic of the *Fourier transform of fractional order*, aka *fractional or angular Fourier transforms*, is that of V. Namias [23]; details in [15].

There are also a series of articles [16] [17] [14] [5] with a preponderance of engineering, so they only partially stimulate my interests.

NOTE. The use of the double formalism,  $\widehat{\mathbb{1}}$  (**widehat**) and  $\widetilde{\mathbb{1}}$  (**tilde**), may appear ambiguous (and it is indeed, to a profane eye); but it serves to make distinctions on the type of the (Fourier) transform, or rather, on the contextual mathematical category.



## Bibliography · List of Works Mentioned in the Article

- [1] É. Borel, *Un théorème sur les ensembles mesurables*, C. R. Acad. Sci. Paris, 1903, 2<sup>e</sup> Semestre, Tome CXXXVII, N<sup>o</sup> 23, Séance du 7 décembre 1903, pp. 966-967.
- [2] R. Brown, *A Brief Account of Microscopical Observations made in the Months of June, July, and August, 1827, on the Particles contained in the Pollen of Plants; and on the General Existence of Active Molecules in Organic and Inorganic Bodies*, in *The Miscellaneous Botanical Works of Robert Brown, Vol. I, containing I. Geographico-Botanical, and II. Structural and Physiological Memoirs*, R. Hardwicke, London, MDCCCLXVI (1866), pp. 463-486.
- [3] G. Cantor, *Ueber unendliche, lineare Punktmannichfaltigkeiten*, 5, Math. Ann., Vol. 21, N<sup>o</sup> 4, 1883, pp. 545-591.
- [4] A.-L. Cauchy, *Sur les Formules qui résultent de l'emploi du signe  $>$  ou  $<$ , et sur les Moyennes entre plusieurs quantités*, in [?], Notes sur l'analyse algébrique · Note II, pp. 438-459].
- [5] G. Cariolaro, T. Erseghe, P. Kraniuskas, N. Laurenti, *Multiplicity of fractional Fourier transforms and their relationships*, IEEE Trans. Signal Process, Vol. 48, N<sup>o</sup> 1, 2000, pp. 227-241.
- [6] J. Dieudonné, L. Schwartz, *La dualité dans les espaces  $(\mathcal{F})$  et  $(\mathcal{L}\mathcal{F})$* , Ann. Inst. Fourier, Vol. 1, 1949, pp. 61-101.
- [7] R.P. Feynman, *Space-Time Approach to Non-Relativistic Quantum Mechanics*, Rev. Mod. Phys., Vol. 20, N<sup>o</sup> 2, 1948, pp. 367-387.
- [8] J.B.J. Fourier, *Théorie analytique de la chaleur*, F. Didot Père et Fils, Paris, 1822.
- [9] A. Grothendieck, *Sur les espaces  $(\mathcal{F})$  et  $(\mathcal{D}\mathcal{F})$* , Summa Bras. Math., Vol. 3, N<sup>o</sup> 6, 1954, pp. 57-123. Fr. version is not found; version consulted: Ru. transl. by D.A. Raikov, Matematika, Vol. 2, N<sup>o</sup> 3, 1958, pp. 81-127.
- [10] F. Hausdorff, *Grundzüge der Mengenlehre*, Verlag von Veit & Comp., Leipzig, 1914.
- [11] T. Kawata, *Fourier Analysis of Nonstationary Stochastic Processes*, Trans. Amer. Math. Soc., Vol. 118, 1965, pp. 276-302.
- [12] P. Langevin, *Sur la théorie du mouvement brownien*, C. R. Acad. Sci. Paris, 1908, Tome CXLVI, Janvier-Juin 1908, Séance du 9 Mars 1908, pp. 530-533.
- [13] [E.]L. Lindelöf, *Sur quelques points de la théorie des ensembles*, C. R. Acad. Sci. Paris, 1903, Tome CXXXVII, N<sup>o</sup> 18, Séance du 2 Novembre 1903, pp. 697-700.
- [14] A.W. Lohmann, *Image rotation, Wigner rotation, and the fractional Fourier transform*, J. Opt. Soc. Am. A, Vol. 10, N<sup>o</sup> 10, 1993, pp. 2181-2186.
- [15] A.C. McBride, F.H. Kerr, *On Namias's Fractional Fourier Transforms Get access Arrow*, IMA J. Appl. Math., Vol. 39, N<sup>o</sup> 2, 1987, pp. 159-175.
- [16] D. Mendlovic and H.M. Ozaktas, *Fractional Fourier transforms and their optical implementation: I*, J. Opt. Soc. Am. A, Vol. 9, N<sup>o</sup> 10, 1993, pp. 1875-1881.
- [17] —, *Fractional Fourier transforms and their optical implementation: II*, J. Opt. Soc. Am. A, Vol. 10, N<sup>o</sup> 12, 1993, pp. 2522-2531.
- [18] [M.]G. Mittag-Leffler, *Sur la Nouvelle Fonction  $E_\alpha(x)$* , C. R. Acad. Sci. Paris, 1903, Second Semestre, Tome CXXXVII, N<sup>o</sup> 15 (12 Octobre 1903), Séance du 12 Octobre 1903, pp. 554-558.
- [19] —, *Sopra la funzione  $E_\alpha(x)$* , Atti R. Accad. Lincei, Rend. Cl. Sci. fis. mat. e natur., Ser. Quinta, Vol. XIII, 1<sup>o</sup> Sem., 1904 (Seduta del 3 gennaio 1904), pp. 3-5.
- [20] E. Niccolai, *Notes in Pure Mathematics & Mathematical Structures in Physics*, the latest revision is available here: *Download page at* <https://www.edoardoniccolai.com>; public repositories: doi:10.5281.10427921, 2023-, or hal-04362941 (HAL ID), 2023-. Outdated versions: arXiv:2105.14863 [math-ph] (please do not download the book from the arXiv repository, 2021-2023).
- [21] —, *Spin & Torsion Tensors on Gauge Gravity: a Re-examination of the Einstein–Cartan Spatio-Temporal Theory*, doi:10.5281.7568818, 2023-, or hal-03948127 (HAL ID), 2023-.
- [22] —, *Stochastic Covariant Derivatives in a (Curved) Space-Time: a Glimpse into the Fractoid Spaces*, hal-04081660 (HAL ID), 2023-, or doi:10.5281.8413143, 2023-. Outdated version: arXiv:2303.10177 [math.PR].
- [23] V. Namias, *The Fractional Order Fourier Transform and its Application to Quantum Mechanics*, J. Inst. Maths Applies, Vol. 25, N<sup>o</sup> 3, 1980, pp. 241-265.
- [24] L. Nottale, *Scale Relativity and Fractal Space-Time. A New Approach to Unifying Relativity and Quantum Mechanics*, World Scientific, Singapore, 2011.
- [25] —, *Scale relativity of the proton radius: solving the puzzle*, arXiv:1904.05772 [physics.gen-ph], 2019 [v1].
- [26] —, *Scale-relativistic correction to the muon  $g - 2$  and its hadronic contribution*, 3 October 2023 (this is an update to a publicly available article from 2021).

- [27] S. Ogawa, *On a stochastic Fourier transformation*, Stochastics, Vol. 85, № 2, 2013, pp. 286-294.
- [28] Yu.V. Prokhorov, A.N. Shiryaev (Eds.), *Probability Theory III: Stochastic Calculus*, through the contribution of S.V. Anulova, A.Yu. Veretennikov, N.V. Krylov, R.Sh. Liptser, A.N. Shiryaev, Transl. from the Ru. by P.B. Slater, Springer-Verlag, Berlin, Heidelberg, 1998.
- [29] H.A. Schwarz, *Über ein Flächen kleinsten Flächeninhalts betreffendes Problem der Variationsrechnung*, Acta Soc. Sci. Fenn., Tomus XV, MDCCCLXXXVIII (1888), pp. 318-361.
- [30] L. Schwartz, *Produits tensoriels topologiques d'espaces vectoriels topologiques. Espaces vectoriels topologiques nucléaires*, Séminaire Schwartz, Tome 1, 1953-1954, exp. № 1-24.
- [31] R.S. Strichartz, *Self-Similar Measures and Their Fourier Transforms I*, Indiana Univ. Math. J., Vol. 39, № 3, 1990, pp. 797-817.

DEDICATION

To Marsili-Alissi Gálakta,  
'cos we were, and are, two lives inextricably intertwined.  
(Just the two of us we can understand this).  
Death has not taken you away from me.

I remain a mirror of your eyes on which nature reflected its absence of corruption.