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# $k$ -anti-power uniform morphisms

Francis Wlazinski

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## Abstract

Words whose every  $k(\geq 3)$  successive factors of the same length are all different, i.e.,  $k$ -anti-power words are natural extensions of square-free words (every two successive factors of the same length are different). We give a way to verify whether a uniform morphism preserves  $k$ -anti-power words, i.e., the image by this morphism of all  $k$ -anti-power words are  $k$ -anti-power words.

A consequence of the existence of such morphisms is the possibility of generating an infinite  $k$ -anti-power word.

## 1 Preliminaries

Let us recall some basic notions of Combinatorics of words.

### 1.1 Words

An *alphabet*  $A$  is a finite set of symbols called *letters*. A *word* over  $A$  is a finite sequence of letters from  $A$ . The *empty word*  $\varepsilon$  is the empty sequence of letters. Equipped with the concatenation operation, the set  $A^*$  of words over  $A$  is a free monoid with  $\varepsilon$  as neutral element and  $A$  as set of generators. Since an alphabet with one element is limited interest to us, we always assume that the cardinality of considered alphabets is at least two. Given a non-empty word  $u = a_1 \dots a_n$  with  $a_i \in A$  for any integer  $i$  from 1 to  $n$ , the *length* of  $u$  denoted by  $|u|$  is the integer  $n$  that is the number of letters of  $u$ . By convention, we have  $|\varepsilon| = 0$ . We denote by  $A^+$  the set of words of positive length over  $A$ , i.e.,  $A^+ = A^* \setminus \{\varepsilon\}$ .

Given two words  $w$  and  $u$ , we denote by  $|w|_u$  the number of different words  $p$  such that  $pu$  is a prefix of  $w$ . For instance, if  $w = abaababa$ , we have  $|w|_a = 5$ ,  $|w|_{aba} = 3$ .

An infinite word over  $A$  is a map from  $\mathbb{N}$  to  $A$  that is an infinite sequence of letters  $a_1 \dots a_n \dots$  with  $a_i \in A$ . And  $A^{\mathbb{N}}$  is the set of all infinite words over  $A$ .

The *mirror image* of  $u$ , denoted by  $\tilde{u}$ , is the word  $a_n \dots a_2 a_1$ . In the particular case of the empty word,  $\tilde{\varepsilon} = \varepsilon$ .

A word  $u$  is a *factor* of a word  $v$  if there exist two (possibly empty) words  $p$  and  $s$  such that  $v = pus$ . We denote  $\text{Fcts}(v)$  the set of all factors of  $v$ . If  $u \in \text{Fcts}(v)$ , we also say that  $v$

contains the word  $u$  (as a factor). If  $p = \varepsilon$ ,  $u$  is a *prefix* of  $v$ . If  $s = \varepsilon$ ,  $u$  is a *suffix* of  $v$ . If  $u \neq v$ ,  $u$  is a *proper* factor of  $v$ . If  $u$ ,  $p$  and  $s$  are non-empty,  $u$  is an *internal* factor of  $v$ . Furthermore,  $\text{alph}(v) = \{u \in \text{Fcts}(v) \mid |u| = 1\}$  is the set of all the letters in  $v$ .

Let  $w$  be a non-empty word and let  $i, j$  be two integers such that  $0 \leq i - 1 \leq j \leq |w|$ . We denote by  $w[i..j]$  the factor of  $w$  such that  $|w[i..j]| = j - i + 1$  and  $w = pw[i..j]s$  for two words  $s$  and  $p$  verifying  $|p| = i - 1$ . When  $j > i$ ,  $w[i..j]$  is simply the factor of  $w$  that starts at the  $i^{\text{th}}$  letter and ends at the  $j^{\text{th}}$ . Note that, when  $j = i - 1$ , we have  $w[i..j] = \varepsilon$ . When  $i = j$ , we also denote by  $w[i]$  the factor  $w[i..i]$  which is the  $i^{\text{th}}$  letter of  $w$ . In particular,  $w[1]$  and  $w[|w|]$  are respectively the first and the last letter of  $w$ .

Powers of a word are defined inductively by  $u^0 = \varepsilon$ , and for any integer  $n \geq 1$ ,  $u^n = uu^{n-1}$ . Given an integer  $k \geq 2$ , since the case  $\varepsilon^k$  is of little interest, we call a  $k$ -*power* any word  $u^k$  with  $u \neq \varepsilon$ . And a word is  $k$ -*power-free* if it does not contain any  $k$ -power as factor. A *primitive* word is a word which is not a  $\ell$ -power of another word whatever the integer  $\ell \geq 2$ .

Given two integers  $p > q \geq 1$  and two words  $x$  and  $y$  with  $xy \neq \varepsilon$ , a word of the form  $(xy)^\alpha x$  with  $\frac{|(xy)^\alpha x|}{|xy|} = \alpha + \frac{|x|}{|xy|} = \frac{p}{q}$  is called a  $\frac{p}{q}$ -*power*. For instance, the word *anchorman* is a  $(1 + \frac{2}{7} =) \frac{9}{7}$ -power and the word *abaabaa* is a  $(2 + \frac{1}{3} =) \frac{7}{3}$ -power. In particular, a  $\frac{3}{2}$ -power is a word of the form  $xyx$  with  $|x| = |y| \neq \varepsilon$ . For instance, the word *antman* is a  $\frac{3}{2}$ -power. A word is  $\frac{p}{q}$ -*power-free* if it does not contain any  $\ell$ -power as factor with  $\ell \geq \frac{p}{q}$ . The word *abcaba* is not  $\frac{8}{5}$ -power-free. Indeed, it contains the word *abcab* which is a  $\frac{5}{3}$ -power.

A *border* of a word  $w$  is a factor  $v \notin \{\varepsilon, w\}$  of  $w$  that is both a prefix and a suffix of  $w$ . When  $|w| > 2|v|$ , we say that the *bordered* word  $w$  is a *sesquipower* or a *Zimin word*.

The two following propositions are well known. The first one is about bordered words.

**Proposition 1.1** [8] *Let  $A$  be an alphabet and  $u, v, w$  three non-empty words over  $A$ .*

*If  $vt = uv$  then there exist two words  $r$  and  $s$  over  $A$  and an integer  $n$  such that  $v = (rs)^n r$ ,  $u = rs$  and  $t = sr$ .*

**Lemma 1.2** [6, 7] *If a non-empty word  $v$  is an internal factor of  $vv$ , i.e., if there exist two non-empty words  $x$  and  $y$  such that  $vv = xvy$ , then there exist a non-empty word  $t$  and two integers  $i, j \geq 1$  such that  $x = t^i$ ,  $y = t^j$ , and  $v = t^{i+j}$ .*

Given an integer  $k \geq 2$  and an integer  $j \geq 1$ , a  $(k, j)$ -*anti-power sequence* or simply a  $k$ -*anti-power* [5] is a concatenation of  $k$  consecutive pairwise different words of the same length  $j$ .

For instance, if  $A = \{a, b\}$ , the words  $u = aababab$  and  $v = ababababaaa$  are respectively  $(4, 2)$ -anti-power and  $(4, 3)$ -anti-power sequences. But the prefix *abababab* of  $v$  is not a 4-anti-power sequence: it is even a 4-power.

Given an integer  $k \geq 1$ , if  $\text{Card}(A) = \alpha \geq 2$  then there exist  $\alpha^n$  different words in  $A^*$  of length  $n \geq 1$ . Among the words of length  $k \times n$ , there are  $\alpha^n$  different  $k$ -powers (of length  $k \times n$ ) and  $A_{\alpha^n}^k = \frac{(\alpha^n)!}{(\alpha^n - k)!}$  different  $(k, n)$ -anti-power sequences if  $\alpha^n \geq k$  and 0 otherwise.

It particularly means that there exists an integer  $k_0$  such that there are no  $(k', n)$ -anti-power sequences over  $A$  for any  $k' \geq k_0$ .

For any alphabet  $A$  with  $\text{Card}(A) = \alpha \geq 2$  and for any integer  $k \geq 2$ , there exists a smallest integer  $p_0$  such that  $\alpha^{p_0} \geq k$ . And, if  $p \geq p_0$ , the set of  $(k, p)$ -anti-power sequences is greater than the set of  $k$ -powers of length  $p \times k$ .

A *2-anti-power word* is simply a square-free word. Given an integer  $k \geq 3$ , a word  $w$  is a *k-anti-power word* if it is a  $(k-1)$ -anti-power word and, when  $|w| \geq k$ , if any factor of  $w$  of length  $k \times j$  for every integer  $1 \leq j \leq \lfloor \frac{|w|}{k} \rfloor$  is a  $(k, j)$ -anti-power sequence. As precised,  $w$  can contain a factor of length  $k$  only if  $|w| \geq k$ . But, by the choosen definition, a word of length  $n$  with  $2 \leq n < k$  is a *k-anti-power word* if and only if it is a *n-anti-power word*. Of course, such a word exists only if  $\text{Card}(A) = \alpha \geq n$ .

In other words, for any integer  $2 \leq \ell \leq k$ , if we choose  $\ell$  successive factors of the same length of a *k-anti-power word*, they are all different.

A word  $w$  is not a *k-anti-power word* ( $k \geq 2$ ) if and only if it contains a factor of the form  $U = u_1 u_2 \dots u_\ell$  for an integer  $2 \leq \ell \leq k$  and for some non-empty words  $u_1, u_2, \dots, u_\ell$  of the same length and such that  $u_1 = u_\ell$ . The word  $U$  is a  $\frac{\ell}{\ell-1}$ -power. So, a word  $w$  that is not a *k-anti-power word* contains at least a  $\frac{k}{k-1}$ -power (when  $\ell = k$ ). The converse is false. For instance, the word  $abcdeabc$  is a  $\frac{8}{5} (> \frac{4}{3})$  power. But it is a 4-anti-power word over a five-letter alphabet.

Let  $\mathcal{A}_{A,k}$ , or  $\mathcal{A}_k$  if no doubt exists for the alphabet  $A$ , be the set of all *k-anti-power words* over  $A$ . By definition, we have  $\mathcal{A}_k \subset \mathcal{A}_{k-1} \dots \subset \mathcal{A}_2$ . So let  $\mathcal{B}_{A,k}$ , or  $\mathcal{B}_k$ , be the set  $\mathcal{A}_{k-1} \setminus \mathcal{A}_k$ .

**Remark 1.3** *We have  $w \in \mathcal{A}_k$  if and only if  $\tilde{u} \in \mathcal{A}_k$ .*

**Lemma 1.4** [5]

*Let  $v$  be a border of a word  $w$  and let  $u$  be the word such that  $w = uv$ .*

*If  $|w| > \ell|u|$  for an integer  $\ell$  then  $u^\ell$  is a prefix of  $w$ .*

More precisely, by Proposition 1.1, if  $v$  is a border of a word  $w = uv (\neq \varepsilon)$ , there exist two words  $r$  and  $s$  over  $A$  and an integer  $n \geq 0$  such that  $u = rs$  and  $w = (rs)^{n+1}r$ . When  $|w| \geq 2|u|$ , we get that  $w$  is a  $\frac{p}{q}$ -power with  $\frac{p}{q} \geq 2$ . So, Proposition 1.1 and Lemma 1.4 mean that a bordered word  $w$  that contains a border of length at least  $\frac{|w|}{2}$  is not a *k-anti-power word* for all integers  $k \geq 2$ . So, we will only be interested in Zimin words, i.e.,  $\frac{p}{q}$ -powers with  $1 < \frac{p}{q} < 2$ .

**Lemma 1.5** *Let  $A$  be an alphabet, let  $d \geq 1$  be an integer and let  $X, Y \in A^*$  such that  $XY \neq \varepsilon$ . If  $d|XY|$  and  $|Y| \leq (d-1)|X|$  then  $XYX \notin \mathcal{A}_k$  for all integers  $k > d$ .*

**Proof.**

In fact, we will show that  $XYX \notin \mathcal{A}_{d+1}$ .

If  $d = 1$  then  $Y = \varepsilon$ ,  $X \neq \varepsilon$  and  $XYX = X^2 \notin \mathcal{A}_2$ .

If  $d \geq 2$ , we have  $|XY| > \frac{d-1}{d}|XY| \geq |Y|$ . Let  $X_2$  be the suffix of  $X$  such that  $|X_2Y| = \frac{d-1}{d}|XY|$  and let  $X_1$  be the non-empty word such that  $X = X_1X_2$ . Since  $|X_1| = |XY| - |X_2Y| = \frac{1}{d}|XY|$ , the word  $[X_1]X_2Y[X_1] \notin \mathcal{A}_{d+1}$ .  $\square$

Taking  $d = \gcd(|XY|; k)$ , we get:

**Corollary 1.6** *Let  $A$  be an alphabet and let  $X, Y \in A^*$  such that  $XY \neq \varepsilon$ .*

*For any integer  $k \geq 2$ , if  $\gcd(|XY|; k) \neq k$  and if  $|Y| \leq (\gcd(|XY|; k) - 1)|X|$ , then  $XYX \notin \mathcal{A}_k$ .*

**Remark 1.7** *When  $d = 2$  in Lemma 1.5, we get that if  $|XY|$  is even and if  $|Y| \leq |X|$  then  $XYX \notin \mathcal{A}_k$  for all integers  $k \geq 3$ .*

**Lemma 1.8** *Let  $A$  be the alphabet  $\{a_1, a_2, \dots, a_k, \dots\}$  and let  $k \geq 3$  be an integer. The word  $[a_1a_2 \dots a_{k-1}]a_k[a_1a_2 \dots a_{k-1}]$  is a  $k$ -anti-power word if and only if  $k$  is prime.*

**Proof.**

Let  $w_{sp,k}$  be the word  $[a_1a_2 \dots a_{k-1}]a_k[a_1a_2 \dots a_{k-1}]$ .

Since  $w_{sp,k}$  is a  $\ell$ -anti-power word for an integer  $2 \leq \ell < k$  only if  $\gcd(k; \ell) = 1$ , and, since  $\mathcal{A}_k \subset \mathcal{A}_\ell$ , the word  $w_{sp,k}$  is a  $k$ -anti-power word only if  $k$  is prime.

Let us now assume that  $k$  is prime and, by contradiction, let us assume that  $w_{sp,k} \notin \mathcal{A}_k$ . It means that  $w_{sp,k}$  contains a factor of the form  $U = u_1u_2 \dots u_\ell$  for an integer  $2 \leq \ell \leq k$  and for some words  $u_1, u_2, \dots, u_\ell$  of the same length  $\Lambda \geq 1$  with  $u_1 = u_\ell$ . Let  $i_0$  be the integer such that  $a_{i_0}$  is the first letter of  $u_1$  and let  $u'_1$  be the word  $u_1 = a_{i_0}u'_1$ . So  $a_{i_0}u'_1u_2 \dots u_{\ell-1}a_{i_0}u'_1$  is a factor of  $w_{sp,k}$ . Since  $a_{i_0}$  appears only twice in  $w_{sp,k}$ , we get that  $k = |a_{i_0}u'_1u_2 \dots u_{\ell-1}| = (\ell - 1) \times \Lambda$ . Since  $k$  is a prime number, we get that  $\ell = 2$  and  $\Lambda = k$ . It follows that  $|u_1u_\ell| = 2k > |w_{sp,k}|$ : this is impossible.  $\square$

**Lemma 1.9** *Let  $A$  be an alphabet and let  $X, Y \in A^*$  such that  $|X|_a \times |Y|_a = 0$  for all letters  $a \in A$ . If  $X \in \mathcal{A}_k$  and  $Y \in \mathcal{A}_k$  and  $|Y| > (k - 2)|X|$  then  $XYX \in \mathcal{A}_k$  for any integer  $k \geq 2$ .*

**Proof.**

By contradiction, let us assume that  $XYX \notin \mathcal{A}_k$  with  $X \neq \varepsilon$ . It means that  $XYX$  contains a factor of the form  $U = u_1u_2 \dots u_\ell$  for an integer  $2 \leq \ell \leq k$  and for some words  $u_1, u_2, \dots, u_\ell$  of the same length  $\Lambda \geq 1$  with  $u_1 = u_\ell$ .

Let  $a$  be the first letter of  $u_1$  and let  $u'_1$  be the word such that  $u_1 = au'_1$ .

If  $|X|_a = 0$  then  $au'_1u_2 \dots a \in \text{Fcts}(Y)$ . We get that  $U \in \text{Fcts}(Y)$ : a contradiction with the hypotheses. Thus,  $|Y|_a = 0$  and it implies that  $au'_1 = u_\ell \in \text{Fcts}(X)$ . Since  $U \notin \text{Fcts}(X)$ , it follows that  $|Y| \leq |u_2 \dots u_{\ell-1}| = (\ell - 2) \times \Lambda \leq (k - 2) \times |X|$ : a contradiction with the hypotheses.  $\square$

An *infinite  $k$ -anti-power word* is an infinite word whose all finite factors are  $k$ -anti-power words. Obviously, the first question is whether such a word exists.

If  $A = \{a, b\}$ , the only 2-anti-power words are  $aba$ ,  $bab$  and their factors. But, for any  $k \geq 3$ , the only  $k$ -anti-power words are  $a$ ,  $b$ ,  $ab$  and  $ba$ .

If  $\text{Card}(A) \geq 3$ , there exist infinite 2-anti-power (square-free) words [1, 10, 11].

If  $k = 3$  and  $A = \{a, b, c\}$ , the only 3-anti-power words are  $abcab$ , the exchange of letters of this word and their factors. Let us note that the word  $abcab$  is not  $\frac{3}{2}$ -power-free.

A  $\frac{3}{2}$ -power-free word contain neither a factor of the form  $xyx$  with  $|x| = |y|$ , nor a factor of the form  $xx$ . Thus a  $\frac{3}{2}$ -power-free word is a 3-anti-power word (but the converse does not hold). Thus a Dejean's word [4, 3, 9] over a four-letter alphabet, which does not contain any  $\ell$ -power with  $\ell > \frac{7}{5}$ -power-free, is a 3-anti-power word.

More generally, if  $k \geq 3$  is a prime number and if  $A$  is a  $k$ -letter alphabet  $\{a_1, a_2, \dots, a_k\}$ , the word  $a_1 a_2 \dots a_{k-1} a_k a_1 a_2 \dots a_{k-1}$  is a  $k$ -anti-power word. But it contains a  $\frac{2k-1}{k}$  ( $> \frac{k}{k-1}$ ) power. Furthermore, a Dejean's word over a  $(k+1)$ -letter alphabet that avoids any power greater than  $\frac{k+1}{k}$  ( $\leq \frac{k}{k-1}$ ) is a  $k$ -anti-power word.

**Remark 1.10** *If we had chosen not to add that a  $k$ -anti-power word must be a  $(k-1)$ -anti-power word, we would have, for instance, that, for  $A = \{a; b; c\}$ , the word  $abcabcab$  would have been a 3-anti-power word but not a 2-anti-power word.*

*More precisely, without the condition that a  $k$ -anti-power word  $w$  must be a  $(k-1)$ -anti-power word, we only could say that all prefixes and all suffixes of  $w$  of length between  $k-1$  and  $\left\lfloor \frac{(k-1)|w|}{k} \right\rfloor$  are  $(k-1)$ -anti-power words.*

*For an infinite word, it does not change anything to add the condition that a  $k$ -anti-power word  $w$  must be a  $(k-1)$ -anti-power word. Indeed, every factor of  $w$  whose length is a multiple of  $k-1$  can be extended to a factor of  $w$  whose length is a multiple of  $k$ . Obviously, if these  $k$  factors are different, the same holds for  $k-1$  ones.*

**Corollary 1.11** *Let  $k \geq 2$  be an integer, let  $A$  be the alphabet  $\{a_0, a_1, \dots, a_{k+3}, \dots\}$ , and let  $w$  be a Dejean's word over  $\{a_0, a_1, \dots, a_k\}$ . The word  $a_{k+1} a_{k+2} w a_{k+1} a_{k+2}$  is a  $k$ -anti-power word if at least one of these conditions holds :*

- i.  $|w| \geq 2k - 3$ .*
- ii.  $|w| \geq k - 2$  and  $|w|$  is not a multiple of 2*

*The word  $a_{k+1} a_{k+2} a_{k+3} w a_{k+1} a_{k+2} a_{k+3}$  is a  $k$ -anti-power word if at least one of these conditions holds :*

- i.  $|w| \geq 3k - 5$ .*
- ii.  $|w| \geq 2k - 4$  and  $|w|$  is not a multiple of 3*
- iii.  $|w| \geq k - 3$ ,  $|a_{k+3} w|$  is not a multiple of 2 and  $|w|$  is not a multiple of 3*

More generally, for any integer  $p \geq 2$ , let  $w_p$  be the word  $a_{k+1} \dots a_{k+p} w a_{k+1} \dots a_{k+p}$ . If  $|w| + i$  is not a multiple of  $p - i$  for every integer  $0 \leq i \leq p - 2$  and if  $|w| \geq k - p$  then  $w_p$  is a  $k$ -anti-power word.

**Proof.** It directly comes from Lemma 1.9 and the fact that the word  $Sw$  is a  $k$ -anti-power word for any suffix  $S$  of  $a_{k+1} \dots a_{k+p}$ .  $\square$

## 1.2 Morphisms

Let  $A$  and  $B$  be two alphabets. A *morphism*  $f$  from  $A^*$  to  $B^*$  is a mapping from  $A^*$  to  $B^*$  such that  $f(uv) = f(u)f(v)$  for all words  $u, v$  over  $A$ . When  $B$  has no importance, we say that  $f$  is a morphism on  $A$  or that  $f$  is defined on  $A$ .

Given an integer  $L \geq 1$ ,  $f$  is  $L$ -uniform if  $|f(a)| = L$  for every letter  $a$  in  $A$ . A morphism  $f$  is *uniform* if it is  $L$ -uniform for some integer  $L \geq 1$ .

Let  $k \geq 2$  be an integer and let  $A$  and  $B$  be two alphabets. A morphism  $f$  from  $A^*$  to  $B^*$  is a  $k$ -anti-power morphism if and only if  $f(w)$  is a  $k$ -anti-power word over  $B$  for every  $k$ -anti-power word  $w$  over  $A$ , i.e.,  $f(\mathcal{A}_{A,k}) \subset \mathcal{A}_{B,k}$ . For instance, the *identity endomorphism*  $Id$  ( $\forall a \in A, Id(a) = a$ ) is a  $k$ -anti-power morphism.

In particular, a 2-anti-power morphism is a square-free morphism. These last morphisms have been characterized in [2].

Given a morphism  $f$  on  $A$ , the *mirror morphism*  $\tilde{f}$  of  $f$  is defined for all words  $w$  over  $A$ , by  $\tilde{f}(w) = f(\tilde{w})$ . In particular,  $\tilde{f}(a) = \tilde{f(a)}$  for every letter  $a$  in  $A$ .

**Remark 1.12** *By remark 1.3, we get that  $f$  is a  $k$ -anti-power morphism if and only if  $\tilde{f}$  is a  $k$ -anti-power morphism.*

A morphism on  $A$  is called *prefix* (resp. *suffix*) if, for all different letters  $a$  and  $b$  in  $A$ , the word  $f(a)$  is not a prefix (resp. not a suffix) of  $f(b)$ . A prefix (resp. suffix) morphism is non-erasing. A morphism is *bifix* if it is prefix and suffix.

Proofs of the following lemmas are left to the reader.

**Lemma 1.13** *Let  $f$  be a bifix morphism on an alphabet  $A$  and let  $u, v, w$ , and  $t$  be words over  $A$ .*

*The equality  $f(u) = f(v)p$  with  $p$  be a prefix of  $f(w)$  implies  $u = vw'$  for a prefix  $w'$  of  $w$  such that  $f(w') = p$ .*

*And the equality  $f(u) = sf(v)$  with  $s$  a suffix of  $f(t)$  implies  $u = t'v$  for a suffix  $t'$  of  $t$  such that  $f(t') = s$ .*

**Lemma 1.14** *Let  $f$  be a prefix morphism on an alphabet  $A$ , let  $u$  and  $v$  be words over  $A$ , and let  $a$  and  $b$  be letters in  $A$ . Furthermore, let  $p_1$  (resp.  $p_2$ ) be a prefix of  $f(a)$  (resp. of  $f(b)$ ). If  $(p_1; p_2) \neq (\varepsilon; f(b))$  and if  $(p_1; p_2) \neq (f(a); \varepsilon)$  then the equality  $f(u)p_1 = f(v)p_2$  implies  $u = v$  and  $p_1 = p_2$ .*

**Lemma 1.15** *Let  $f$  be a suffix morphism on an alphabet  $A$ , let  $u$  and  $v$  be words over  $A$ , and let  $a$  and  $b$  be letters in  $A$ . Furthermore, let  $s_1$  (resp.  $s_2$ ) be a suffix of  $f(a)$  (resp. of  $f(b)$ ). If  $(s_1; s_2) \neq (\varepsilon; f(b))$  and if  $(s_1; s_2) \neq (f(a); \varepsilon)$  then the equality  $s_1 f(u) = s_2 f(v)$  implies  $u = v$  and  $s_1 = s_2$ .*

Taking  $p_1 = p_2 = \varepsilon$  (resp.  $s_1 = s_2 = \varepsilon$ ) in Lemma 1.14 (resp Lemma 1.15), we get that a prefix (resp. suffix) morphism is injective.

**Definition 1.16** *A morphism  $f$  from  $A^*$  to  $B^*$  is a ps-morphism (Keränen [6] called  $f$  a ps-code) if and only if the equalities*

$$f(a) = ps, f(b) = ps' \text{ and } f(c) = p's$$

*with  $a, b, c \in A$  (possibly  $c = b$ ) and  $p, s, p', s' \in B^*$  imply  $b = a$  or  $c = a$ .*

Obviously, taking  $c = b$ , and  $s = \varepsilon$  in a first time and  $p = \varepsilon$  in a second time, we obtain that a ps-morphism is a bifix morphism.

**Lemma 1.17** [6, 7] *If  $f$  is not a ps-morphism then  $f$  is not a  $k$ -power-free morphism for every integer  $k \geq 2$ .*

**Proposition 1.18** *Let  $k \geq 2$  be an integer, let  $A$  and  $B$  be two alphabets with  $\text{Card}(A) = p \geq k$  and let  $f$  be a  $L$ -uniform morphism from  $A^*$  to  $B^*$  (with  $L \geq 1$ ). If there exist five letters  $a, b, c, d$  and  $x$  (possibly equal) and four words  $p, s, \pi$  and  $\sigma$  such that  $s$  is a suffix of  $f(a)$ ,  $p$  is a prefix of  $f(b)$ ,  $\sigma$  is a suffix of  $f(c)$ ,  $\pi$  is a prefix of  $f(d)$ ,  $(\pi; p) \neq (\varepsilon; f(b))$ ,  $(\sigma; s) \neq (\varepsilon; f(a))$  and  $sp = \sigma f(x)\pi$  then  $f$  is not a  $k$ -anti-power morphism.*

**Proof.**

In fact, we will show that there exists a square-free word of length at most two whose image by  $f$  contains a square. Therefore, this word will be a  $k$ -anti-power-word for any  $k \geq 2$  but its image will not be. This will mean that  $f$  is not a  $k$ -anti-power morphism.

Since  $(\pi; p) \neq (\varepsilon; f(b))$ ,  $(\sigma; s) \neq (\varepsilon; f(a))$  and  $|sp| = L + |\sigma\pi|$ , we get  $|s| > |\sigma|$  and  $|p| > |\pi|$ .

Let  $s'$  be the non empty prefix of  $f(x)$  such that  $s = \sigma s'$  and let  $p'$  be the non empty suffix of  $f(x)$  such that  $p = p'\pi$ .

If  $x = a$  and  $x = b$ , then  $f(x)$  is an internal factor of  $f(xx)$ . By Lemma 1.2,  $f(x)$  is not primitive, i.e.,  $f$  is not a  $k$ -anti-power morphism.

Therefore, at least one of the word  $ax$  or  $bx$  is not a square. But  $f(ax)$  contains the square  $s's'$  and  $f(xb)$  contains the square  $p'p'$ .  $\square$

**Proposition 1.19** *Let  $k \geq 2$  be an integer. Let  $A$  and  $B$  be two alphabets. Let  $f$  be a  $L$ -uniform morphism from  $A^*$  to  $B^*$  (with  $L \geq 2$ ). Let us assume that there exists a prime number  $p$  such that  $\text{Card}(A) \geq p \geq k$ .*

*If there exists an integer  $2 \leq d < k$  such that  $d|L$  then  $f$  is not a  $k$ -anti-power morphism.*



**Proof.**

Let us assume that  $A = \{a_1, a_2 \dots a_p, \dots\}$ .

The word  $w_{(A,p)} = a_1 a_2 \dots a_{p-1} a_p a_1 a_2 \dots a_{p-1} \in \mathcal{A}_p \subset \mathcal{A}_k \subset \mathcal{A}_{d+1}$ . But, by Lemma 1.5, the word  $[f(a_1)(a_2) \dots f(a_{p-1})] f(a_p) [f(a_1)(a_2) \dots f(a_{p-1})] \notin \mathcal{A}_{d+1}$ , i.e.,  $f(w_{(A,p)}) \notin \mathcal{A}_k$ .  $\square$

**Proposition 1.20** *Let  $k \geq 3$  and  $L \geq 2$  be two integers. Let  $A$  and  $B$  be two alphabets with  $\text{Card}(A) \geq k$  and let  $f$  be  $L$ -uniform morphism from  $A^*$  to  $B^*$ . Let us assume that there exist three letters  $a, b$  and  $c$  such that  $a \notin \{b, c\}$  and  $f(bac)$  contains a square  $T^2 (\neq \varepsilon)$ .*

*If there exists an integer  $2 \leq d < k$  such that  $d \mid |T|$  then  $f$  is not a  $k$ -anti-power morphism.*

**Proof.**

If  $b \neq c$  then  $abc$  is a  $k$ -anti-power word and  $f$  is not a  $k$ -anti-power morphism.

From now, let us assume that  $f(bab) = \pi T^2 \sigma$  (with  $|T| \geq 1$ ).

If  $|\pi| > |f(b)|$  (resp.  $|\sigma| > |f(b)|$ ) then  $T^2$  is factor of  $f(ab)$  (resp.  $f(ba)$ ) with  $ab$  (resp.  $ba$ ) a  $k$ -anti-power word:  $f$  is not a  $k$ -anti-power morphism.

Thus there exist two words  $p$  and  $s$  such that  $f(a) = ps$ ,  $\pi T = f(b)p$  and  $T\sigma = sf(b)$ . Without loss of generality (using mirror image for instance), we may assume that  $|p| \geq |s|$ .

If  $|T| = |L|$  then  $T = sp$ ,  $f(b) = ps$  and  $f(ba)$  contains the square  $(ps)^2$ :  $f$  is not a  $k$ -anti-power morphism.

If  $|T| > |L|$ , let  $Y$  be non-empty word such that  $T = sYp$ . It follows that  $f(b) = \pi sY = Yp\sigma$ . By Lemma 1.1, there exist two words  $u$  and  $v$  and an integer  $n$  such that  $Y = u(vu)^n$ ,  $p\sigma = vu$  and  $\pi s = uv$ . Moreover, since  $Y \neq \varepsilon$ , we get that  $uv \neq \varepsilon$ . If  $n \geq 1$  then  $f(b)$  contains  $(uv)^2$  that is  $f$  is not a  $k$ -anti-power morphism. From now,  $u = Y$  and  $f(b) = YvY$ . If  $|p| \geq |v|$  (resp.  $|s| \geq |v|$ ) then  $p$  starts with  $v$  (resp.  $s$  ends with  $v$ ) and  $f(ba)$  (resp.  $fab$ ) contains the square  $(Yv)^2$  (resp.  $(vY)^2$ ):  $f$  is not a  $k$ -anti-power morphism. When  $|p| < |v|$  and  $|s| < |v|$ , the word  $v$  starts with  $p$  and ends with  $s$ . But  $|v| < |f(b)| = |f(a)| = |ps|$ . Thus there exist three words  $p'$ ,  $s'$  and  $Z \neq \varepsilon$  such that  $v = p'Zs'$ ,  $p = p'Z$  and  $s = Zs'$ . It means that  $f(a)$  contains  $Z^2$ :  $f$  is not a  $k$ -anti-power morphism.

If  $0 < |T| < |L| (= |sp|)$ , since  $T$  starts with  $s$  and ends with  $p$ , let  $p'$ ,  $s'$  and  $Y \neq \varepsilon$  be the words such that  $p = Yp'$ ,  $s = s'Y$  and  $T = s'Yp'$ . It follows that  $f(a) = Yp's'Y$ . Moreover, from the equality  $f(bab) = \pi T^2 \sigma$ , we get that  $f(b)$  starts with  $p'$  and ends with  $s'$ . Thus, there exist a word  $B$  such that  $|B| = 2|Y|$  and  $f(b) = p'Bs'$ .

We have  $f(ab) = Y[p'][s'Y][p']Bs'$  and  $f(ba) = p'B[s'Y][p'][s'Y]$ . If there exist an integer  $2 \leq d < k$  such that  $d \mid |T|$ , i.e.,  $d \mid |p's'Y|$ , since either  $|s'Y| \leq |p'| \leq (d-1)|p'|$  or  $|p'| \leq |s'Y| \leq (d-1)|s'Y|$ , by Lemma 1.5,  $f(ab) \notin \mathcal{A}_{B,k}$  or  $f(ba) \notin \mathcal{A}_{B,k}$ :  $f$  is not a  $k$ -anti-power morphism.

$\square$

**Remark 1.21** *Let  $B_0$  be the alphabet  $\{a, b, c, d, e, f, g, h, i, j, k, l, m\}^*$ .*

*Let  $f$  be a uniform morphism from  $\{a, b\}^*$  to  $B_0^*$  such that  $f(a) = abcdefghiab$ ,  $f(b) = cdefjklmghi$ .*

Then  $f(bab)$  contains the square  $(ghiabcdef)^2$ . Therefore,  $f$  is not a 2-anti-power morphism (is not a square-free morphism).

For any integer  $k \geq 3$ , the only  $k$ -anti power words over  $\{a, b\}$  are  $a$ ,  $b$ ,  $ab$ , and  $ba$ .

Moreover,  $f(ab)$  and  $f(ba)$  are 3-anti-power words:  $f$  is a 3-anti-power morphism.

But  $f(ba)$  contains  $ghiabcdefghi$ . Thus  $f$  is not a  $k$ -anti-power morphism for any integer  $k \geq 4$ .

Let  $A = \{a_1, a_2, \dots, a_\alpha\}$  be an alphabet. A *trivial* morphism from  $A^*$  to  $B^*$  is an injective 1-uniform morphism. That is,  $f(a_i) = b_i$  for some letter  $b_i \in B$  and  $b_i \neq b_j$  when  $i \neq j$ .

**Proposition 1.22** *Let  $k \geq 2$  be an integer and let  $A$  be an alphabet. Let  $f$  be a non-trivial  $L$ -uniform morphism defined on  $A$ . If  $(\text{alph}(f(a)))_{a \in A}$  are not pairwise disjoint sets and if  $L < k$  then  $f$  is not a  $k$ -anti-power morphism.*

**Proof.**

Let assume that  $\text{Card}(A) = \alpha$  and  $A = \{a_1, a_2, \dots, a_\alpha\}$ .

Since  $(\text{alph}(f(a)))_{a \in A}$  are not pairwise disjoint sets, let  $x$  be a letter and let  $\ell \neq m$  two integers such that  $x \in \text{alph}(f(a_\ell)) \cap \text{alph}(f(a_m))$ .

We have  $f(a_\ell) = A'_\ell x A''_\ell$  and  $f(a_m) = A'_m x A''_m$  for four words  $A'_\ell$ ,  $A''_\ell$ ,  $A'_m$  and  $A''_m$  such that  $A'_\ell + A''_\ell = A'_m + A''_m = L - 1$ .

So  $f(a_\ell a_m) = A'_\ell x A''_\ell A'_m x A''_m$  and  $f(a_m a_\ell) = A'_m x A''_m A'_\ell x A''_\ell$  with either  $|xA''_\ell A'_m| \leq L \leq k - 1$  or  $|xA''_m A'_\ell| \leq L \leq k - 1$ . Since  $a_\ell a_m$  and  $a_m a_\ell$  are  $k$ -anti-power words, it means that  $f$  is not a  $k$ -anti-power morphism.  $\square$

A morphism  $f$  on  $A$  is  $k$ -anti-power up to  $\ell$  ( $k, \ell \geq 2$ ) if and only if  $f(w)$  is a  $k$ -anti-power word for every  $k$ -anti-power word  $w$  over  $A$  of length at most  $\ell$ .

**Remark 1.23** *Let  $A$  and  $B$  be two alphabets with  $\text{Card}(A) \geq k \geq 3$  and let  $f$  be a 1-uniform morphism. Then  $f$  is a  $k$ -anti-power morphism if and only if  $f$  is trivial. So  $f$  is a  $k$ -anti-power morphism if and only if it is a  $k$ -anti-power morphism up to 2.*

**Proposition 1.24** *Let  $k \geq 3$  be an integer and let  $A$  and  $B$  be two alphabets. Let us assume that there exists a prime number  $p$  such that  $\text{Card}(A) \geq p \geq k$ . Let  $f$  be a square-free  $L$ -uniform morphism from  $A^*$  to  $B^*$ .*

*Then  $f$  is a  $k$ -anti-power morphism if and only if it is a  $k$ -anti-power morphism up to  $2p - 1$ .*

**Proof.**

By definition of  $k$ -anti-power morphisms, we only have to prove the "if" part of Proposition 1.24.

By Lemma 1.17,  $f$  (square-free) is a ps-morphism and so injective.

We may assume that, if there exists an integer  $d \geq 2$  such that  $d|L$  then  $d \geq k$ . Indeed (see the Proof of Proposition 1.19), if  $d|L$  and  $d < k$  then there exists a word  $w_{A,p} \in \mathcal{A}_{A,k}$  such that  $|w_{A,p}| = 2p - 1$  and  $f(w_{A,p}) \notin \mathcal{A}_{B,k}$ :  $f$  is not a  $k$ -anti-power morphism. It particularly means that if  $2 \leq \ell \leq k$ , then  $\gcd(L, \ell - 1) = 1$ .

By contradiction, we assume that a shortest  $k$ -anti-power word  $w$  (not necessarily unique) such that  $f(w)$  contains a non- $k$ -anti-power satisfies  $|w| \geq 2p$ . We will show that this assumption leads to contradictions.

Since the length of  $w$  is minimal, we may assume that there exist an integer  $2 \leq \ell \leq k$  and some words  $p, s, u_1, u_2, \dots, u_\ell$  such that  $f(w) = pu_1u_2 \dots u_\ell s = p \left( \prod_{q=1}^{\ell} u_q \right) s$  where  $p$  is a prefix of  $f(w[1])$  different from  $f(w[1])$  and  $s$  is a suffix of  $f(w[|w|])$  different from  $f(w[|w|])$ . Moreover, the words  $(u_j)_{j=1, \dots, \ell}$  have the same length  $\Lambda (\geq 1)$  and  $u_1 = u_\ell$ .

Let  $i_0 = 1$  and, for all integers  $1 \leq j \leq \ell$ , let  $i_j$  be the shortest integer such that  $p \prod_{q=1}^j u_q$  is the prefix of  $f(w[1..i_j])$ . By definition, we have  $i_\ell = |w|$ .

If there exist two different integers  $j$  and  $j'$  such that  $i_j = i_{j'}$  then we get that  $\Lambda \leq L$ . It implies that  $f(w) = |p| + |s| + \ell \times \Lambda \leq (k + 2)L$ , i.e.,  $|w| \leq k + 2 < 2p$ : a contradiction.

Thus  $1 = i_0 < i_1 < \dots < i_\ell = |w|$ . If we denote  $a_j$  the letter  $w[i_j]$  and  $x_j$  the word  $w[i_{j-1} + 1..i_j - 1]$ , we have  $w = a_0 x_1 a_1 x_2 a_2 \dots x_\ell a_\ell$ . And there exist some words  $p_j$  and  $s_j$  ( $0 \leq j \leq \ell$ ) such that  $f(a_j) = p_j s_j$  with  $p_0 = p$  and  $s_\ell = s$ . By definition, the words  $s_0, p_1, p_2, \dots, p_\ell$  are non empty and  $u_j = s_{j-1} f(x_j) p_j$  for all  $1 \leq j \leq \ell$ .

Let us note that, for all integers  $1 \leq j_1, j_2 \leq \ell$ , we have  $||x_{j_1}| - |x_{j_2}|| \leq 1$ . Indeed, in the contrary, for instance if  $|x_{j_1}| \geq |x_{j_2}| + 2$  (obviously with  $j_1 \neq j_2$ ), we get that  $\Lambda = |u_{j_1}| = |s_{j_1-1} f(x_{j_1}) p_{j_1}| > |f(x_{j_1})| \geq |f(x_{j_2})| + 2L$ . Moreover  $\Lambda = |u_{j_2}| = |s_{j_2-1} f(x_{j_2}) p_{j_2}| < |f(x_{j_2})| + 2L$ : this is impossible. Since  $\sum_{q=1}^{\ell} x_q \geq 2p - \ell - 1 \geq \ell - 1$ , it also implies that  $\text{card}\{q \mid 1 \leq q \leq \ell \text{ and } x_q = \varepsilon\} \leq 1$ .

If  $j_1 < j_2$  and  $u_{j_1} = u_{j_2}$  with  $(j_1, j_2) \neq (1, \ell)$  then  $w[i_{j_1-1}..i_{j_2}]$  is a proper factor of  $w$  and  $f(w[i_{j_1-1}..i_{j_2}])$  contains the non- $(j_2 - j_1 + 1)$ -anti-power word  $\prod_{q=j_1}^{j_2} u_q$  that is a non- $k$ -anti-power word: a contradiction with the hypothesis of the minimal length of  $w$ .

In particular, we have  $s_0 f(x_1) p_1 = u_1 = u_\ell = s_{\ell-1} f(x_\ell) p_\ell$ .

- *Case 1* :  $|s_0| \neq |s_{\ell-1}|$  and  $(s_0, s_{\ell-1}) \neq (f(a_0), \varepsilon)$

Since  $\text{card}\{q \mid 1 \leq q \leq \ell \text{ and } x_q = \varepsilon\} \leq 1$ , we have either  $x_1 \neq \varepsilon$  or  $x_\ell \neq \varepsilon$ .

If  $s_{\ell-1} \neq \varepsilon$  and  $(0 <) |s_0| < |s_{\ell-1}|$ , by a length criterion, we necessarily have  $x_1 \neq \varepsilon$ . Let  $\chi$  be the first letter of  $x_1$  and let  $x'_1$  be the word such that  $x_1 = \chi x'_1$ . If  $x_\ell = \varepsilon$ , let  $P = p_\ell$  and if  $x_\ell \neq \varepsilon$ , let  $P = f(\gamma)$  where  $\gamma$  is the first letter of  $x_\ell$ . In particular, we have  $P$  non-empty. Let  $\pi$  be the non empty prefix of  $f(x'_1) p_1$  such that  $s_0 f(\chi) \pi = s_{\ell-1} P$ . By Proposition 1.18, this last equation implies that  $f$  is not a square-free morphism: a contradiction with the hypotheses.

The case  $s_{\ell-1} \neq \varepsilon$  with  $|s_0| > |s_{\ell-1}|$  and the case  $s_{\ell-1} = \varepsilon$  with  $s_0 \neq f(a_0)$  are solved as previous one using Proposition 1.18.

- *Case 2* :  $(s_0, s_{\ell-1}) = (f(a_0), \varepsilon)$

We get that  $f(a_0x_1)p_1 = f(x_\ell)p_\ell$ . By Lemma 1.14, we get  $a_0x_1 = x_\ell$  and  $p_1 = p_\ell$ . It implies that  $|s_1| = |s_\ell|$ .

It follows that  $(\ell - 1) \times \Lambda = \left| \prod_{q=2}^{\ell} u_q \right| = |s_1 f(x_2 a_2 x_3 a_3 \dots x_\ell) p_\ell| = |f(x_2 a_2 x_3 a_3 \dots x_\ell a_\ell)| = L \times |x_2 a_2 x_3 a_3 \dots x_\ell a_\ell| = L \times (\ell - 1) + L \times \sum_{q=2}^{\ell} |x_q|$ . Since  $\gcd(L, \ell - 1) = 1$ , we get that  $\sum_{q=2}^{\ell} |x_q|$  ( $\geq 1$ ) is a multiple of  $\ell - 1$ . But  $||x_{j_1}| - |x_{j_2}|| \leq 1$  for all integers  $2 \leq j_1, j_2 \leq \ell$ , so it implies that  $|x_{j_1}| = |x_{j_2}| = |x_1| + 1$ ,  $\Lambda = L \times (|x_1| + 2)$  and  $|s_{j-1}p_j| = L$  for all  $1 \leq j \leq \ell$ .

For all integers  $2 \leq q \leq \ell$ , we have  $|u_q| = |f(a_0x_1)p_1| = |f(x_\ell)p_\ell| = |s_{q-1}f(x_q)p_q|$  and so  $|s_{q-1}p_q| = |p_\ell|$ . It means that all the  $(|s_q|)_{q=1..l}$  are equal to 0 and that all the  $(|p_q|)_{q=1..l}$  are equal to  $L$ , i.e.,  $p_q = f(a_q)$ .

In particular, we get that  $f(a_1) = p_1 = p_\ell = f(a_\ell)$  and thus  $a_1 = a_\ell$ . It follows that  $w = a_0x_1a_1x_2a_2 \dots x_{\ell-1}a_{\ell-1}x_1a_1$  with  $|a_qx_q| = |x_1a_1|$  for all integers  $q \in \llbracket 2; \ell - 1 \rrbracket$ , i.e.,  $w$  is not a  $k$ -anti-power word: a contradiction.

• *Case 3* :  $|s_0| = |s_{\ell-1}|$

As previous case, we obtain, as a first step, that  $\Lambda$  is a multiple of  $L$ . Indeed, we have  $(\ell - 1) \times \Lambda = \left| \prod_{q=1}^{\ell-1} u_q \right| = |s_0 f(x_1 a_1 x_2 a_2 \dots x_{\ell-1}) p_{\ell-1}| = |f(x_1 a_1 x_2 a_2 \dots x_{\ell-1} a_{\ell-1})| = L \times |x_1 a_1 x_2 a_2 \dots x_{\ell-1} a_{\ell-1}| = L \times (\ell - 1) + L \times \sum_{q=1}^{\ell-1} |x_q|$ . Since  $\gcd(L, \ell - 1) = 1$ , we get that  $\sum_{q=1}^{\ell-1} |x_q|$  ( $\geq 1$ ) is a multiple of  $\ell - 1$ . So  $|x_j| = |x_1|$ ,  $\Lambda = L \times (1 + |x_1|)$  and  $|s_{j-1}p_j| = L$  for all  $1 \leq j \leq \ell$ .

As previous case, it means that all the  $(|s_q|)_{q=0..l}$  are equal and, consequently, all the  $(|p_q|)_{q=0..l}$  are also equal.

From the equalities  $s_0 f(x_1) p_1 = s_{\ell-1} f(x_\ell) p_\ell$  and  $|s_0| = |s_{\ell-1}|$ , we get  $s_{\ell-1} = s_0$  ( $\neq \varepsilon$ ). By Lemma 1.14, it also implies  $x_\ell = x_1$  and  $p_1 = p_\ell$  ( $\neq \varepsilon$ ).

In particular, since all the  $(|x_i|)_{i=1..l}$  are equal and since  $w = a_0x_1a_1x_2a_2 \dots x_{\ell-1}a_{\ell-1}x_1a_\ell$  is a  $k$ -anti-power word, we have  $a_0 \neq a_{\ell-1}$  and  $a_1 \neq a_\ell$ .

If  $a_0 = a_1$  then we get  $p_0 = p_1 (= p_\ell)$ ,  $f(a_0) = p_0 s_0$ ,  $f(a_{\ell-1}) = p_{\ell-1} s_{\ell-1} = p_{\ell-1} s_0$  and  $f(a_\ell) = p_\ell s_\ell = p_0 s_\ell$ . It means that  $f(a_{\ell-1} a_0 a_\ell)$  contains  $(s_0 p_0)^2$  with  $a_{\ell-1} a_0 a_\ell$  square-free since  $a_0 \neq a_{\ell-1}$  and  $a_0 = a_1 \neq a_\ell$ : a contradiction with the hypothesis that  $f$  is a square-free morphism.

In the same way, if  $a_{\ell-1} = a_\ell$ , we get that  $f(a_0 a_\ell a_1)$  contains  $(s_\ell p_\ell)^2$  with  $a_0 a_\ell a_1$  square-free. If  $a_0 = a_\ell$ , we get that  $f(a_{\ell-1} a_\ell a_1)$  contains  $(s_\ell p_\ell)^2$  with  $a_0 a_\ell a_1$  square-free. And, if  $a_1 = a_{\ell-1}$ , we get that  $f(a_0 a_1 a_\ell)$  contains  $(s_1 p_1)^2$  with  $a_0 a_1 a_\ell$  square-free. All these cases lead to the same contradiction.

Thus  $a_0, a_1, a_{\ell-1}$  and  $a_\ell$  are four different letters. It means that  $a_0 a_1 a_{\ell-1} a_\ell$  is a  $k$ -anti-power word of length lower than  $2p$ . But  $f(a_0 a_1 a_{\ell-1} a_\ell)$  contains the non-3-anti-power sequence  $s_0 p_1 s_1 p_{\ell-1} s_{\ell-1} p_\ell = s_0 p_1 s_1 p_{\ell-1} s_0 p_1$ : a (final) contradiction with the minimality of  $|w|$ .  $\square$

## References

- [1] J. Berstel. Axel Thue's papers on repetition in words: a translation. Technical Report 20, Laboratoire de Combinatoire et d'Informatique Mathématique, Université du Québec, Montréal, 1995.
- [2] M. Crochemore. Sharp characterizations of squarefree morphisms. *Theoretical Computer Science*, 18:221–226, 1982.
- [3] James D. Currie and Narad Rampersad. A proof of dejean's conjecture. *Mathematics of computation*, 80:1063–1070, 2011.
- [4] F. Dejean. Sur un théorème de Thue. *J. Comb. Theory*, 13:90–99, 1972. series A.
- [5] Gabriele Fici, Antonio Restivo, Manuel Silva, and Luca Q. Zamboni. Anti-powers in infinite words. *Journal of Combinatorial Theory, Series A*, 157:109 – 119, 2018.
- [6] V. Keränen. On the  $k$ -freeness of morphisms on free monoids. *Annales Academiae Scientiarum Fennicae* 61, Series A, 1986.
- [7] M. Leconte. *Codes sans répétition*. PhD thesis, LITP Université Paris 6, october 1985.
- [8] M. Lothaire. *Combinatorics on words*, volume 17 of *Encyclopedia of Mathematics*. Addison-Wesley, 1983. Reprinted in 1997 by Cambridge University Press in the Cambridge Mathematical Library, Cambridge, UK, 1997.
- [9] Michaël Rao. Last cases of dejean's conjecture. *Theoretical Computer Science*, 412(27):3010 – 3018, 2011. *Combinatorics on Words (WORDS 2009)*.
- [10] A. Thue. Über unendliche zeichenreihen. *Kristiania Videnskapsselskapets Skrifter Klasse I. Mat.-naturv*, 7:1–22, 1906.
- [11] A. Thue. Über die gegenseitige Lage gleicher Teile gewisser Zeichenreihen. *Kristiania Videnskapsselskapets Skrifter Klasse I. Mat.-naturv*, 1:1–67, 1912.